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TWO CLASSES OF PSEUDOSYMMETRIC CONTACT METRIC 3-MANIFOLDS

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We classify the pseudosymmetric contact metric 3-manifolds that satisfy $\nabla_\xi \tau = 0$, and also the pseudosymmetric contact metric 3-manifolds of constant type satisfying $\nabla_\xi \tau = 2a\tau\phi$, where a is a smooth function.

1. Introduction

According to R. Deszcz [1992], a Riemannian manifold (M^m, g) is pseudosymmetric if the curvature tensor R satisfies the condition $R(X, Y)\cdot R = L\{(X\wedge Y)\cdot R\}$, where L is a smooth function, the endomorphism field $X \wedge Y$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y \tag{1-1}$$

for all vectors fields X, Y, Z on M, and the dot means that $R(X, Y)$ and $X \wedge Y$ act as derivations on R.

If L is constant, M said to be a pseudosymmetric manifold of constant type; if $L = 0$, then M is a semisymmetric manifold. Hence a pseudosymmetric manifold is a natural generalization of a semisymmetric manifold [Szabó 1982; 1985], which in turn is a generalization of a locally symmetric space, that is, one with $\nabla R = 0$; see [Takagi 1972].

Three-dimensional pseudosymmetric spaces of constant type have been studied by many researchers, beginning with O. Kowalski and M. Sekizawa [1996b; 1996a; 1997; 1998]. Later, N. Hashimoto and M. Sekizawa classified 3-dimensional, conformally flat pseudosymmetric spaces of constant type [2000], while G. Calvaruso gave the complete classification of conformally flat pseudosymmetric spaces of constant type for dimensions greater than two [2006]. J. T. Cho and J. Inoguchi studied pseudosymmetric contact homogeneous 3-manifolds [2005].

It is well known that in the geometry of a contact metric manifold, the tensors $\tau = L_\xi g$ and $\nabla_\xi \tau$, introduced by S. S. Chern and R. S. Hamilton [1985], play a fundamental role. The condition $\nabla_\xi \tau = 2a\tau\phi$, where a is a constant and $(\tau\phi)(X, Y)$ is

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interpreted as $\tau(\phi X, Y)$, is necessary for a contact metric 3-manifold to be homogeneous. We call a 3-dimensional contact metric manifold a 3-τ-a manifold if it satisfies $\nabla_\xi \tau = 2a\tau\phi$, where a is a smooth function; if $a = 0$, we call it a 3-τ manifold. The condition $\nabla_\xi \tau = 0$ appeared first in [Chern and Hamilton 1985] in the study of compact contact 3-manifolds, while Perrone [1990] proved that it is the critical point condition for the functional "integral of the scalar curvature" defined on the set of all metrics associated to the fixed contact form η. Moreover, this condition $\nabla_\xi \tau = 0$ is equivalent to the condition requiring equality of the sectional curvature of all planes at a given point and perpendicular to the contact distribution [Gouli-Andreou and Xenos 1998a].

This article studies contact metric 3-manifolds in which

(i) M is a pseudosymmetric manifold and $\nabla_\xi \tau = 0$, where $\tau = L_\xi g$; or

(ii) M is a pseudosymmetric manifold of constant type with $\nabla_\xi \tau = 2a\tau\phi$, where a is a smooth function on M.

2. Preliminaries

Let (M^m, g) for $m \geq 3$ be a connected Riemannian smooth manifold. We denote by ∇ the Levi-Civita connection of M^m and by R the corresponding Riemannian curvature tensor given by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

A Riemannian manifold (M^m, g) for $m \geq 3$ is said to be *pseudosymmetric* in the sense of R. Deszcz [1992] if at every point of M the curvature tensor satisfies the equation

$$(R(X, Y)\cdot R)(X_1, X_2, X_3) = L\{((X \wedge Y)\cdot R)(X_1, X_2, X_3)\}, \tag{2-1}$$

where

$$\begin{aligned}(R(X, Y)\cdot R)(X_1, X_2, X_3) = {}& R(X, Y)(R(X_1, X_2)X_3) - R(R(X, Y)X_1, X_2)X_3 \\ & - R(X_1, R(X, Y)X_2)X_3 - R(X_1, X_2)(R(X, Y)X_3),\end{aligned} \tag{2-2}$$

$$\begin{aligned}((X \wedge Y)\cdot R)(X_1, X_2, X_3) = {}& (X \wedge Y)(R(X_1, X_2)X_3) - R((X \wedge Y)X_1, X_2)X_3 \\ & - R(X_1, (X \wedge Y)X_2)X_3 - R(X_1, X_2)((X \wedge Y)X_3),\end{aligned} \tag{2-3}$$

and $X \wedge Y$ is given by (1-1). In particular, if $L = 0$, then M is semisymmetric. For details and examples of pseudosymmetric manifolds, see [Belkhelfa et al. 2002] and [Deszcz 1992].

A contact manifold is a differentiable manifold M^{2n+1} together with a global 1-form η (a *contact form*) such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Since $d\eta$ is of rank $2n$, there exists a unique vector field ξ on M^{2n+1} (the *Reeb* or *characteristic* *vector field* of the *contact structure* η) satisfying $\eta(\xi)=1$ and $d\eta(\xi, X)=0$ for all X. The distribution defined by the subspace $X \in T_pM : \eta(X)=0$ for $p \in M$ is called a *contact distribution*. Every contact manifold has an underlying *almost contact structure* (η, ϕ, ξ), where ϕ is a global tensor field of type $(1, 1)$, such that

$$\eta(\xi)=1, \qquad \phi\xi=0, \qquad \eta\circ\phi=0, \qquad \phi^2=-I+\eta\otimes\xi. \tag{2-4}$$

A Riemannian metric g (the *associated metric*) can be defined such that

$$\eta(X)=g(X,\xi) \quad \text{and} \quad d\eta(X,Y)=g(X,\phi Y) \tag{2-5}$$

for all vector fields X and Y on M^{2n+1}. We note that g and ϕ are not unique for a given contact form η, but g and ϕ are canonically related to each other. We refer to $(M^{2n+1}, \eta, \xi, \phi, g)$ as a *contact metric structure*.

We denote by S the Ricci tensor of type $(0, 2)$, by Q the corresponding Ricci operator satisfying $g(QX, Y) = S(X, Y)$, and by $r = \operatorname{Tr} Q$ the scalar curvature. We also define the tensor fields l, h and τ by the relations

$$l=R(\,\cdot\,,\xi)\xi, \quad h=\tfrac{1}{2}L_\xi\phi, \quad \tau=L_\xi g, \tag{2-6}$$

where L is the Lie differentiation. On every contact metric manifold M^{2n+1}, we have the important formulas

$$h\xi=l\xi=0, \quad \eta\circ h=0, \quad \operatorname{Tr}h=\operatorname{Tr}h\phi=0, \quad h\phi=-\phi h, \tag{2-7}$$

$$hX=\lambda X \quad \text{implies} \quad h\phi X=-\lambda\phi X, \tag{2-8}$$

$$\nabla_\xi\phi=0, \quad \nabla_X\xi=-\phi X-\phi hX, \quad \operatorname{Tr}l=g(Q\xi,\xi)=2n-\operatorname{Tr}h^2, \tag{2-9}$$

$$\tau=2g(\phi\cdot,h\cdot), \quad \nabla_\xi\tau=2g(\phi\cdot,\nabla_\xi h\cdot). \tag{2-10}$$

A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a Killing vector field, that is, for which $L_\xi g=0$, is called a K-contact manifold. A contact metric manifold is K-contact if and only if $\tau=0$ (or equivalently $h=0$).

If we take the product $M^{2n+1}\times\mathbb{R}$, then the contact structure on M^{2n+1} gives rise to an almost complex structure J on $M^{2n+1}\times\mathbb{R}$ given by

$$J\big(X, f\tfrac{d}{dt}\big)=\big(\phi X-f\xi, \eta(X)\tfrac{d}{dt}\big).$$

If this structure is integrable, then the contact structure is said to be normal and M^{2n+1} is called Sasakian. A contact metric manifold is Sasakian if and only if $R(X,Y)\xi=\eta(Y)X-\eta(X)Y$ for all vectors fields X, Y on the manifold. If $\dim M^{2n+1}=3$ then a K-contact manifold is Sasakian and the converse also holds. More details on contact manifolds are found in [Blair 2002].

Let (M, ϕ, ξ, η, g) be a contact metric 3-manifold. Let U be the open subset of points $p\in M$ such that $h\neq 0$ in a neighborhood of p. Let U_0 be the open subset of points $p\in M$ such that $h=0$ in a neighborhood of p. That h is a smooth function on M implies $U \cup U_0$ is an open and dense subset of M, so any property satisfied in $U_0 \cup U$ is also satisfied in M. For any point $p \in U \cup U_0$, there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p (this we call a ϕ-basis). On U, we put $he = \lambda e$, where λ is a nonvanishing smooth function assumed to be positive. From (2-8) we have $h\phi e = -\lambda \phi e$.

Lemma 2.1 [Calvaruso and Perrone 2002; Gouli-Andreou and Xenos 1998a]. *On the open set U we have*

$$\begin{aligned} &\nabla_\xi e = a\phi e, && \nabla_e e = b\phi e, && \nabla_{\phi e} e = -c\phi e + (\lambda - 1)\xi, \\ &\nabla_\xi \phi e = -ae, && \nabla_e \phi e = -be + (1+\lambda)\xi, && \nabla_{\phi e}\phi e = ce, \\ &\nabla_\xi \xi = 0, && \nabla_e \xi = -(1+\lambda)\phi e, && \nabla_{\phi e}\xi = (1-\lambda)e, \end{aligned}$$

$$\nabla_\xi h = -2ah\phi + (\xi \cdot \lambda)s \tag{2-11}$$

where a is a smooth function,

$$\begin{aligned} b &= \tfrac{1}{2\lambda}((\phi e \cdot \lambda) + A) \quad \text{with } A = \eta(Qe) = S(\xi, e), \\ c &= \tfrac{1}{2\lambda}((e \cdot \lambda) + B) \quad \text{with } B = \eta(Q\phi e) = S(\xi, \phi e), \end{aligned} \tag{2-12}$$

and s is the type $(1,1)$ tensor field defined by $s\xi = 0$, $se = e$ and $s\phi e = -\phi e$.

From Lemma 2.1 and the formula $[X, Y] = \nabla_X Y - \nabla_Y X$, we can prove that

$$\begin{aligned} [e, \phi e] &= \nabla_e \phi e - \nabla_{\phi e} e = -be + c\phi e + 2\xi, \\ [e, \xi] &= \nabla_e \xi - \nabla_\xi e = -(a + \lambda + 1)\phi e, \\ [\phi e, \xi] &= \nabla_{\phi e}\xi - \nabla_\xi \phi e = (a - \lambda + 1)e, \end{aligned} \tag{2-13}$$

and from (1-1) we estimate

$$\begin{aligned} &(e \wedge \phi e)e = -\phi e, && (e \wedge \xi)e = -\xi, && (\phi e \wedge \xi)\xi = \phi e, \\ &(e \wedge \phi e)\phi e = e, && (e \wedge \xi)\xi = e, && (\phi e \wedge \xi)\phi e = -\xi, \end{aligned} \tag{2-14}$$

while $(X \wedge Y)Z = 0$ whenever $X \neq Y \neq Z \neq X$ and $X, Y, Z \in \{e, \phi e, \xi\}$.

By direct computations, we calculate the nonvanishing independent components of the type $(1,3)$ Riemannian curvature tensor field R:

$$\begin{aligned} &R(\xi, e)\xi = -Ie - Z\phi e, && R(e, \phi e)e = -C\phi e - B\xi, \\ &R(\xi, \phi e)\xi = -Ze - D\phi e, && R(\xi, e)\phi e = -Ke + Z\xi, \\ &R(e, \phi e)\xi = Be - A\phi e, && R(\xi, \phi e)\phi e = He + D\xi, \\ &R(\xi, e)e = K\phi e + I\xi, && R(e, \phi e)\phi e = Ce + A\xi, \\ &R(\xi, \phi e)e = -H\phi e + Z\xi, \end{aligned} \tag{2-15}$$

where

$$
\begin{aligned}
& C=-b^2-c^2+\lambda^2-1+2a+(e\cdot c)+(\phi e\cdot b), && Z=\xi\cdot\lambda,\\
(2\text{-}16)\quad & H=b(\lambda-a-1)+(\xi\cdot c)+(\phi e\cdot a), && I=-2a\lambda-\lambda^2+1,\\
& K=c(\lambda+a+1)+(\xi\cdot b)-(e\cdot a), && D=2a\lambda-\lambda^2+1.
\end{aligned}
$$

Setting $X=e,\ Y=\phi e$ and $Z=\xi$ in the Jacobi identity $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$ and using (2-13), we get

$$
\begin{aligned}
b(a+\lambda+1)-(\xi\cdot c)-(\phi e\cdot\lambda)-(\phi e\cdot a)&=0,\\
c(a-\lambda+1)+(\xi\cdot b)+(e\cdot\lambda)-(e\cdot a)&=0,
\end{aligned} \tag{2-17}
$$

or equivalently $A=H$ and $B=K$.

We give the components of the Ricci operator Q with respect to a ϕ-basis:

$$
\begin{aligned}
Qe&=(\tfrac{1}{2}r-1+\lambda^2-2a\lambda)e+Z\phi e+A\xi,\\
Q\phi e&=Ze+(\tfrac{1}{2}r-1+\lambda^2+2a\lambda)\phi e+B\xi,\\
Q\xi&=Ae+B\phi e+2(1-\lambda^2)\xi,
\end{aligned} \tag{2-18}
$$

where

$$
r=\operatorname{Tr}Q=2(1-\lambda^2-b^2-c^2+2a+(e\cdot c)+(\phi e\cdot b)). \tag{2-19}
$$

The relations (2-16) and (2-19) yield

$$
C=-b^2-c^2+\lambda^2-1+2a+(e\cdot c)+(\phi e\cdot b)=2\lambda^2-2+r/2 \tag{2-20}
$$

Definition 2.2 [Gouli-Andreou et al. 2008]. Let M^3 be a 3-dimensional contact metric manifold. Let $h=\lambda h^+-\lambda h^-$ be the spectral decomposition of h on U. If

$$
\nabla_{h^-X}h^-X=[\xi,h^+X]
$$

for all vector fields X on M^3 and all points of an open subset W of U, and if $h=0$ on the points of M^3 that do not belong to W, then the manifold is said to be a *semi-K contact* manifold.

From Lemma 2.1 and the relations (2-13), the condition above for $X=e$ leads to $[\xi,e]=0$; for $X=\phi e$ it leads to $\nabla_{\phi e}\phi e=0$. Hence on a semi-K contact manifold we have $a+\lambda+1=c=0$. If we apply the deformation $e\to\phi e,\ \ \phi e\to e,\ \ \xi\to-\xi,\ \ \lambda\to-\lambda,\ \ b\to c$ and $c\to b$, then the contact metric structure remains the same. Hence the condition for a 3-dimensional contact metric manifold to be semi-K contact is equivalent to $a-\lambda+1=b=0$.

Remark 2.3. If $M^3=U_0$ (as in [Gouli-Andreou and Xenos 1998b]), Lemma 2.1 is expressed in a similar form, where $\lambda=0,\ e$ is a unit vector field belonging to the contact distribution, the Equation (2-11) is identically zero, and the functions A, B, D, H, I, K and Z satisfy

$$A = B = Z = H = K = 0, \qquad I = D = 1, \qquad C = \tfrac{1}{2}r - 2.$$

Proposition 2.4. *For a 3-dimensional contact metric manifold, we have*

$$\text{(2-21)} \quad Q\phi = \phi Q \quad \textit{if and only if} \quad \xi\cdot\lambda = 2b\lambda - (\phi e\cdot\lambda) = 2c\lambda - (e\cdot\lambda) = a\lambda = 0.$$

Proof. By (2-4), (2-12), (2-16) and (2-20), the relations (2-18) yield

$$(Q\phi - \phi Q)e = 2Ze + 4a\lambda\phi e + B\xi,$$
$$(Q\phi - \phi Q)\phi e = 4a\lambda e - 2Z\phi e - A\xi,$$
$$(Q\phi - \phi Q)\xi = Be - A\phi e.$$

Proposition 2.4 follows immediately. □

3. Pseudosymmetric contact metric 3-manifolds

Let (M, η, g, ϕ, ξ) be a contact metric 3-manifold. In the case $M = U_0$, that is, when (ξ, η, ϕ, g) is a Sasakian structure, M is a pseudosymmetric space of constant type [Cho and Inoguchi 2005]. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ-basis as in Lemma 2.1.

Lemma 3.1. *A contact metric 3-manifold (M, η, g, ϕ, ξ) is pseudosymmetric if and only if*

$$B(\xi\cdot\lambda) + (-2a\lambda - \lambda^2 + 1)A = LA,$$
$$A(\xi\cdot\lambda) + (2a\lambda - \lambda^2 + 1)B = LB,$$
$$(\xi\cdot\lambda)(\tfrac{1}{2}r + 2\lambda^2 - 2) + AB = L(\xi\cdot\lambda),$$

and

$$A^2 - |(\xi\cdot\lambda)|^2 + (2a\lambda - \lambda^2 + 1)(-2a\lambda - 3\lambda^2 + 3 - \tfrac{1}{2}r) = L(-2a\lambda - 3\lambda^2 + 3 - \tfrac{1}{2}r),$$
$$B^2 - |(\xi\cdot\lambda)|^2 + (-2a\lambda - \lambda^2 + 1)(2a\lambda - 3\lambda^2 + 3 - \tfrac{1}{2}r) = L(2a\lambda - 3\lambda^2 + 3 - \tfrac{1}{2}r),$$

where L is the function in the pseudosymmetry definition (2-1).

Proof. Setting $X_1 = e$, $X_2 = \phi e$ and $X_3 = \xi$ in Equation (2-1), we obtain

$$(R(X, Y)\cdot R)(e, \phi e, \xi) = L\big(((X\wedge Y)\cdot R)(e, \phi e, \xi)\big).$$

First we set $X = e$ and $Y = \phi e$. By virtue of (2-2), (2-3), (2-14) and (2-15) we obtain

$$(B(\xi\cdot\lambda) + (-2a\lambda - \lambda^2 + 1)A)e + (A(\xi\cdot\lambda) + (2a\lambda - \lambda^2 + 1)B)\phi e = L(Ae + B\phi e),$$

from which the first two equations of the lemma follow at once.

Similarly, setting $X=\phi e$ and $Y=\xi$ we obtain

$$\begin{aligned}&\big(A^2-|(\xi\cdot\lambda)|^2+(2a\lambda-\lambda^2+1)(-2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)\big)e\\&\quad+\big((\xi\cdot\lambda)(\tfrac{1}{2}r+2\lambda^2-2)+AB\big)\phi e=L\big((-2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)e+(\xi\cdot\lambda)\phi e\big),\end{aligned}$$

from which we get the next two equations.

Finally, setting $X=e$ and $Y=\xi$ we have

$$\begin{aligned}&\big(B^2-|(\xi\cdot\lambda)|^2+(-2a\lambda-\lambda^2+1)(2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)\big)\phi e\\&\qquad+\big((\xi\cdot\lambda)(\tfrac{1}{2}r+2\lambda^2-2)+AB\big)e=L\big((2a\lambda-3\lambda^2+2-\tfrac{1}{2}r)\phi e+(\xi\cdot\lambda)e\big),\end{aligned}$$

from which we obtain the last equation. □

Using Equations (2-16) and (2-20), the five equations take the more convenient form

$$\begin{aligned}&ZB+IA=LA,\\ &ZA+DB=LB, &&A^2-Z^2+D(I-C)=L(I-C),\\ &ZC+AB=LZ, &&B^2-Z^2+I(D-C)=L(D-C).\end{aligned}\tag{3-1}$$

Remark 3.2. If $L=0$, the manifold is semisymmetric and (3-1) is in accordance with [Calvaruso and Perrone 2002, Equations (3.1)–(3.5)].

Proposition 3.3. *Let M^3 be a 3-dimensional contact metric manifold satisfying $Q\phi=\phi Q$. Then M^3 is a pseudosymmetric space of constant type.*

Proof. In [2005], Cho and Inoguchi have proved that contact metric 3-manifolds that satisfy $Q\phi=\phi Q$ are pseudosymmetric. We can improve their result by proving that these manifolds are also pseudosymmetric of constant type. We know from [Blair et al. 1990] that for these manifolds the Ricci operator has the form $QX=\alpha X+\beta\eta(X)\xi$ or equivalently the Ricci tensor is given by the equation

$$S=\alpha g+\beta\eta\otimes\eta,\tag{3-2}$$

where $\alpha=(r-\operatorname{Tr} l)/2$, $\beta=(3\operatorname{Tr} l-r)/2$ and the ϕ-sectional curvature and $\operatorname{Tr} l$ are both constant functions. Also, from [Koufogiorgos 1995] we have that the ϕ-sectional curvature is given by the equation $r/2-\operatorname{Tr} l$, and hence in contact metric 3-manifolds with $Q\phi=\phi Q$, the function $r=\operatorname{Tr} Q$ is also constant; obviously the functions α and β in (3-2) are constant as well. The manifold is quasi-Einstein and hence pseudosymmetric, and because β is constant it is pseudosymmetric of constant type, that is, L is constant; see also [Cho and Inoguchi 2005]. □

Remark 3.4. If the manifold M^3 is Sasakian, we know from Cho and Inoguchi [2005] that M^3 is a pseudosymmetric space of constant type. Also, by using Remark 2.3, the system (3-1) is reduced to the equation $(C-1)(L-1)=0$. Hence a Sasakian 3-manifold satisfying the condition $R(X,Y)\cdot R = L((X\wedge Y)\cdot R)$ with $L\neq 1$ is a space of constant scalar curvature $r=6$, where L is some constant function on M^3.

4. Pseudosymmetric contact metric 3-manifolds with $\nabla_\xi\tau = 0$

Theorem 4.1. *Let M^3 be a 3-dimensional pseudosymmetric contact metric manifold satisfying $\nabla_\xi\tau=0$. Then M^3 is of constant type, and it is either Sasakian, flat, or locally isometric to either* SU(2) *or* SL(2, R), *where these two Lie groups are equipped with a left invariant metric.*

Proof. We consider the open subsets

$$(4\text{-}1)\qquad \begin{aligned} U_0 &= \{p\in M : \lambda = 0 \text{ in a neighborhood of } p\},\\ U &= \{p\in M : \lambda \neq 0 \text{ in a neighborhood of } p\}\end{aligned}$$

of M. Suppose $M=U_0$, that is, (ξ,η,ϕ,g) is a Sasakian structure. In [2005], Cho and Inoguchi proved that M is a pseudosymmetric space of constant type.

If U is not empty, let $\{e,\phi e,\xi\}$ be a ϕ-basis. In contact metric 3-manifolds, the assumption $\nabla_\xi\tau=0$ is equivalent to $a=Z=0$; see [Gouli-Andreou and Xenos 1998a]. Hence (3-1) becomes

$$\begin{aligned} &AB=0\\ &B(D-L)=0, \qquad A^2+D(D-C)=L(D-C),\\ &A(D-L)=0, \qquad B^2=A^2,\end{aligned}$$

or equivalently

$$(4\text{-}2)\qquad \begin{aligned} Z=A=B=a&=0,\\ (D-L)(D-C)&=0,\end{aligned}$$

where the functions A, B, C, D, I and Z are given by (2-12) and (2-16). Using Proposition 2.4 we obtain $Q\phi=\phi Q$, and hence the manifold is pseudosymmetric of constant type (by [Cho and Inoguchi 2005] and Proposition 3.3). The equation $(D-L)(D-C)=0$ does not contribute any further information to our problem, and hence the proof is completed by [Blair et al. 1990, Theorem 3.3] and [Blair and Chen 1992, Main Theorem]. □

5. Pseudosymmetric contact metric 3-manifolds of constant type with $\nabla_\xi\tau = 2a\tau\phi$

Theorem 5.1. *Let M^3 be a 3-dimensional pseudosymmetric contact metric manifold of constant type satisfying the condition $\nabla_\xi\tau = 2a\tau\phi$, where a is a smooth function. Then the functions a and* $\operatorname{Tr} l$ *are constant on M^3, and M^3 is either*

- *Sasakian*;
- *flat*;
- *locally isometric to either* SU(2) *or* SL(2, R), *where both Lie groups are equipped with a left invariant metric*;
- *semi-K contact of constant type* $L=-4$, *with* $\operatorname{Tr} l=0$ *and with constant scalar curvature* $r=-4$;
- *semi-K contact of constant type* $L=a^2$, *with* $\operatorname{Tr} l=-2a(2+a)$;
- *of constant scalar curvature* $r=4a$ *and of constant type* $L=2a$, *with* $\operatorname{Tr} l=0$;
- *of constant scalar curvature* $r=2a(4-a)$ *and of constant type* $L=a^2$, *with* $\operatorname{Tr} l=2a(2-a)$;
- *of constant scalar curvature* $r=2a(4-a)$ *and of constant type* $L=\operatorname{Tr} l-a^2$, *with* $\operatorname{Tr} l=2a(2-a)$.

Proof. We consider again the open subsets of (4-1). If $M=U_0$, then M is Sasakian and hence it is a pseudosymmetric space of constant type [Cho and Inoguchi 2005]. Next, assume that U is not empty, and let $\{e, \phi e, \xi\}$ be a ϕ-basis. Using (2-10) and (2-11), we can prove that the assumption $\nabla_\xi \tau=2a\tau\phi$ is equivalent to $Z=\xi\cdot\lambda=0$, and hence the system (3-1) becomes

$$
\begin{aligned}
AB&=0, &&\\
DB&=LB, & A^2+D(I-C)&=L(I-C),\\
IA&=LA, & B^2+I(D-C)&=L(D-C),
\end{aligned}
$$

or using (2-16), it is

$$
\begin{aligned}
0&=AB,\\
0&=B(2a\lambda-\lambda^2+1-L),\\
0&=A(-2a\lambda-\lambda^2+1-L),\\
0&=A^2+(-2a\lambda-2\lambda^2+2+b^2+c^2-2a-(e\cdot c)-(\phi e\cdot b))\\
&\qquad\qquad\times(2a\lambda-\lambda^2+1-L),\\
0&=B^2+(2a\lambda-2\lambda^2+2+b^2+c^2-2a-(e\cdot c)-(\phi e\cdot b))\\
&\qquad\qquad\times(-2a\lambda-\lambda^2+1-L),\\
0&=Z=\xi\cdot\lambda
\end{aligned}
$$

To study this system we consider these open subsets of U:

$$
\begin{aligned}
U'&=\{p\in U: A=2b\lambda-(\phi e\cdot\lambda)=0 \text{ in a neighborhood of } p\},\\
U_3&=\{p\in U: A\neq 0 \text{ in a neighborhood of } p\},
\end{aligned}
$$

where $U' \cup U_3$ is open and dense in the closure of U. In U', we have

$$
\begin{aligned}
&0 = (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(2a\lambda - \lambda^2 + 1 - L),\\
&0 = B(2a\lambda - \lambda^2 + 1 - L),\\
&0 = B^2 + (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))\\
&\qquad\qquad \times (-2a\lambda - \lambda^2 + 1 - L),\\
&0 = Z = \xi \cdot \lambda.
\end{aligned}
$$

We consider these open subsets of U':

$$
\begin{aligned}
U_1 &= \{p \in U' : B = 2c\lambda - (e \cdot \lambda) = 0 \text{ in a neighborhood of } p\},\\
U_2 &= \{p \in U' : B \neq 0 \text{ in a neighborhood of } p\},
\end{aligned}
$$

where $U_1 \cup U_2$ is open and dense in the closure of U'. Because $B \neq 0$ in U_2, we have $2a\lambda - \lambda^2 + 1 - L = 0$ there. Hence U_1 can also be described as the set of $p \in U$ satisfying

$$
\begin{aligned}
&0 = 2b\lambda - (\phi e \cdot \lambda), \qquad 0 = 2c\lambda - (e \cdot \lambda) = 0, \qquad 0 = \xi \cdot \lambda,\\
&0 = (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(-2a\lambda - \lambda^2 + 1 - L),\\
&0 = (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))(2a\lambda - \lambda^2 + 1 - L),
\end{aligned}
$$

in a neighborhood of p, whereas U_2 is the set of $p \in U$ satisfying

$$
\begin{aligned}
&0 = 2b\lambda - (\phi e \cdot \lambda), \qquad 0 = 2a\lambda - \lambda^2 + 1 - L, \qquad 0 = \xi \cdot \lambda,\\
&0 = (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))\\
&\qquad\qquad \times (-2a\lambda - \lambda^2 + 1 - L) + B^2
\end{aligned}
$$

in a neighborhood of p. In U_3 we have $A \neq 0$ (or equivalently $B = 2c\lambda - (e \cdot \lambda) = 0$) and the system becomes

$$
\begin{aligned}
&0 = -2a\lambda - \lambda^2 + 1 - L,\\
&0 = A^2 + (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))\\
&\qquad\qquad \times (2a\lambda - \lambda^2 + 1 - L),\\
&0 = Z = \xi \cdot \lambda.
\end{aligned}
$$

The set U_3 is also described as the set of $p \in U$ for which there is a neighborhood satisfying

$$
\begin{aligned}
&0 = 2c\lambda - (e \cdot \lambda), \qquad 0 = -2a\lambda - \lambda^2 + 1 - L, \qquad 0 = \xi \cdot \lambda,\\
&0 = A^2 + (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b))\\
&\qquad\qquad \times (2a\lambda - \lambda^2 + 1 - L).
\end{aligned}
$$

We shall study the initial system at each U_i for $i = 1, 2, 3$.

In U_1, we have

$$(\phi e \cdot \lambda) = 2b\lambda, \tag{5-1}$$

$$(e \cdot \lambda) = 2c\lambda, \tag{5-2}$$

$$0 = \xi \cdot \lambda, \tag{5-3}$$

$$0 = (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \times (-2a\lambda - \lambda^2 + 1 - L), \tag{5-4}$$

$$0 = (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \times (2a\lambda - \lambda^2 + 1 - L). \tag{5-5}$$

Differentiating the equations (5-1) and (5-2) with respect to e and ϕe, respectively, and subtracting we get $[e, \phi e]\lambda = 2b(e \cdot \lambda) + 2\lambda(e \cdot b) - 2c(\phi e \cdot \lambda) - 2\lambda(\phi e \cdot c)$ or, because of (2-13), (5-1), (5-2) and (5-3),

$$e \cdot b = \phi e \cdot c. \tag{5-6}$$

Differentiating the equations (5-1), (5-3) with respect to ξ and ϕe, respectively, and subtracting, we get $[\xi, \phi e]\lambda = 2\lambda(\xi \cdot b)$ or, because of (2-13), (2-17) and (5-2),

$$\xi \cdot b = c(\lambda - a - 1), \tag{5-7}$$

$$e \cdot a = 2c\lambda. \tag{5-8}$$

Differentiating the equations (5-2) and (5-3) with respect to ξ and e, respectively, and subtracting, we get $[\xi, e]\lambda = 2\lambda(\xi \cdot c)$ or, because of (2-13), (2-17) and (5-1),

$$\xi \cdot c = b(\lambda + a + 1), \tag{5-9}$$

$$\phi e \cdot a = -2b\lambda. \tag{5-10}$$

In order to study the system of (5-4) and (5-5) we consider these open subsets of U_1:

$$V = \{p \in U_1 : 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \text{ in a neighborhood of } p\},$$

$$V' = \{p \in U_1 : 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) \neq 0 \text{ in a neighborhood of } p\},$$

where $V \cup V'$ is open and dense in the closure of U_1. In the set V, the Equation (5-5) also holds; hence we consider these open subsets of V:

$$V_1 = \{p \in V : \; -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \; 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0 \text{ in a neighborhood of } p\}$$

and

$$V_2 = \{p \in V : 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ 2a\lambda - \lambda^2 + 1 - L = 0 \quad \text{in a neighborhood of } p\},$$

where $V_1 \cup V_2$ is open and dense in the closure of V. Similarly for V', where $-2a\lambda - \lambda^2 + 1 - L = 0$, we consider the open subsets

$$V_3 = \{p \in V' : -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ -2a\lambda - \lambda^2 + 1 - L = 0 \quad \text{in a neighborhood of } p\},$$

$$V_4 = \{p \in V' : -2a\lambda - \lambda^2 + 1 - L = 0, \\ 2a\lambda - \lambda^2 + 1 - L = 0 \quad \text{in a neighborhood of } p\},$$

where $V_3 \cup V_4$ is open and dense in the closure of V' and the set $\bigcup V_i$ is open and dense in the closure of U_1. We shall prove that the functions λ and a are constant at every V_i for $i = 1, 2, 3, 4$

Now

$$\text{in } V_1, \qquad \begin{cases} -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0. \end{cases}$$

Subtracting these two equations, we deduce that $a = 0$ in $V_1 \subset U$. Hence from (5-8) and (5-10) we have $c = b = 0$, and from (5-1) and (5-2) we have $\phi e \cdot \lambda = e \cdot \lambda = 0$. These, together with (5-3), give $\lambda =$ constant in V_1. Moreover, if we put $a = b = c = 0$ in one of the equations of the set V_1, we finally get $\lambda^2 = 1$ and the structure is flat.

Next,

$$\text{(5-11)} \quad \text{in } V_2 \qquad \begin{cases} 2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ 2a\lambda - \lambda^2 + 1 - L = 0, \end{cases}$$

$$\text{(5-12)} \quad \text{in } V_3 \qquad \begin{cases} -2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b) = 0, \\ -2a\lambda - \lambda^2 + 1 - L = 0. \end{cases}$$

In V_2 we differentiate the equation $2a\lambda - \lambda^2 + 1 - L = 0$ with respect to ξ, ϕe and e, and because of (5-3), (5-8), (5-10) we obtain respectively

$$\text{(5-13)} \qquad \xi \cdot a = 0,$$

$$\text{(5-14)} \qquad b(a - 2\lambda) = 0,$$

$$\text{(5-15)} \qquad ac = 0,$$

while in V_3 from $-2a\lambda - \lambda^2 + 1 - L = 0$ we obtain Equation (5-13), and

$$\text{(5-16)} \qquad ba = 0, \qquad c(a + 2\lambda) = 0.$$

Differentiating the relations (5-8) and (5-13) with respect to ξ and e, respectively, and subtracting, we get $[\xi, e]a = 2\lambda(\xi \cdot c)$ or, because of (2-13), (5-9) and (5-10),

$$b(\lambda + a + 1) = 0. \tag{5-17}$$

Similarly, differentiating (5-10) and (5-13) with respect to ξ and ϕe, respectively, and subtracting, we have $[\xi, \phi e]a = -2\lambda(\xi \cdot b)$ or, because of (2-13), (5-7) and (5-8),

$$c(\lambda - a - 1) = 0. \tag{5-18}$$

To study the system of (5-17) and (5-18), we consider these open subsets of V_2:

$$G = \{p \in V_2 : b = 0 \text{ in a neighborhood of } p\},$$
$$G' = \{p \in V_2 : b \neq 0 \text{ in a neighborhood of } p\},$$

where $G \cup G'$ is open and dense in the closure of V_2.

In G we have $c(\lambda - a - 1) = 0$, hence we consider these open subsets of G:

$$G_1 = \{p \in G : c = 0 \text{ in a neighborhood of } p\},$$
$$G_2 = \{p \in G : c \neq 0 \text{ in a neighborhood of } p\},$$

where $G_1 \cup G_2$ is open and dense in the closure of G. These sets are described more specifically as

$$G_1 = \{p \in G \subset V_2 : b = c = 0 \text{ in a neighborhood of } p\},$$
$$G_2 = \{p \in G \subset V_2 : b = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\}.$$

The set G' (where $b \neq 0$ or equivalently $\lambda + a + 1 = 0$) is decomposed similarly as

$$G_3 = \{p \in G' : c = 0 \text{ in a neighborhood of } p\},$$
$$G_4 = \{p \in G' : c \neq 0 \text{ in a neighborhood of } p\},$$

where $G_3 \cup G_4$ is open and dense in the closure of G'. These can also be written

$$G_3 = \{p \in G' \subset V_2 : c = \lambda + a + 1 = 0 \text{ in a neighborhood of } p\},$$
$$G_4 = \{p \in G' \subset V_2 : \lambda + a + 1 = \lambda - a - 1 = 0 \text{ in a neighborhood of } p\}.$$

We have $V_2 \subset U$ where $\lambda \neq 0$; hence $G_4 = \varnothing$.

In G_1, $b = c = 0$. From (5-1), (5-2), (5-3), (5-8), (5-10) and (5-13), we can conclude that λ and a are constant in G_1, and from the first of (5-11), we have $(\lambda - 1)(a - \lambda - 1) = 0$. We consider these open subsets of G_1:

$$K_1 = \{p \in G_1 : \lambda = 1 \text{ in a neighborhood of } p\},$$
$$K_2 = \{p \in G_1 : \lambda \neq 1 \text{ in a neighborhood of } p\}.$$

In K_1, we get $\operatorname{Tr} l = 0$, $L = 2a$, and, from (2-19), $r = 4a$. In K_2, we have $a - \lambda - 1 = 0$, $\operatorname{Tr} l = 2a(2 - a)$, $L = a^2$, and $r = 2a(4 - a)$.

In G_2, $b=\lambda-a-1=0$. Using this to eliminate a from the second equation of (5-11), we obtain $\lambda^2-2\lambda+1-L=0$. We suppose that there is a point p in G_2 at which $e\cdot\lambda\neq 0$. Then there is a neighborhood S of p in which $e\cdot\lambda\neq 0$. We differentiate $\lambda^2-2\lambda+1-L=0$ with respect to e twice and obtain $e\cdot\lambda=0$ in S, a contradiction. Hence $e\cdot\lambda=0$ (and $c=0$), and from (5-1) and (5-3), we conclude that λ is constant in G_2 and similarly $a=\lambda-1$. In particular, from (5-11), we obtain $\lambda=1$ and $a=0$; hence the structure is flat.

We have proved that λ and a are constant on G_1 and G_2. The set $G_1\cup G_2$ is open and dense in the closure of G. Hence λ and a are constant everywhere in G.

In G_3, $c=\lambda+a+1=0$. Using this to eliminate a from the second equation of (5-11), we obtain $-3\lambda^2-2\lambda+1-L=0$. If we assume that there is a point p in G_3 at which $\phi e\cdot\lambda\neq 0$, then there is a neighborhood S of p in which $\phi e\cdot\lambda\neq 0$. We differentiate $-3\lambda^2-2\lambda+1-L=0$ with respect to ϕe twice and obtain $\phi e\cdot\lambda=0$ in S, a contradiction. Thus $\phi e\cdot\lambda=0$ everywhere in G_3, which gives $b=0$. We note that $\xi\cdot\lambda=\phi e\cdot\lambda=e\cdot\lambda=0$, so λ is constant in G_3, and obviously $a=-\lambda-1$. Moreover, if we put $b=c=0$ and $a=-\lambda-1$ in the system (5-11), we get $\lambda^2=1$ (hence $\operatorname{Tr} l=0$), $a=0$ or -2, and $L=-2a(a+1)$. If $a=0$, we obtain a flat structure, while if $a=-2$, we have a semi-K contact structure with constant scalar curvature $r=4a=-8$.

The functions λ and a are constant in G and G'. The set $G\cup G'$ is open and dense in the closure of V_2; hence λ and a are constant in V_2, and Equations (5-14) and (5-15) are satisfied because $b=c=0$.

We similarly consider these open subsets of V_3:

$$G'_1=\{p\in V_3 : b=c=0 \text{ in a neighborhood of } p\},$$
$$G'_2=\{p\in V_3 : b=\lambda-a-1=0 \text{ in a neighborhood of } p\},$$
$$G'_3=\{p\in V_3 : c=\lambda+a+1=0 \text{ in a neighborhood of } p\},$$
$$G'_4=\{p\in V_3 : \lambda+a+1=\lambda-a-1=0 \text{ in a neighborhood of } p\}.$$

The set $\bigcup G'_i$ is open and dense subset of V_3 and $G'_4=\varnothing$.

In G'_1, $b=c=0$. From (5-1), (5-2), (5-3), (5-8), (5-10) and (5-13) we can conclude that λ and a are constant in G'_1. From the first of (5-12) we have $(\lambda+1)(a+\lambda-1)=0$. We consider these open subsets of G'_1:

$$K'_1=\{p\in G'_1 : \lambda=-1 \text{ in a neighborhood of } p\},$$
$$K'_2=\{p\in G'_1 : \lambda\neq -1 \text{ in a neighborhood of } p\}.$$

In K'_1, we get $\operatorname{Tr} l=0$, $L=2a$, and, from (2-19), $r=4a$. In K'_2 we have $a+\lambda-1=0$, $\operatorname{Tr} l=2a(2-a)$, $L=a^2$, and $r=2a(4-a)$.

In G'_2, $b=\lambda-a-1=0$. Using this to eliminate a from the second equation of (5-12), we obtain $-3\lambda^2+2\lambda+1-L=0$. If we assume $e\cdot\lambda\neq 0$, we may differentiate this equation twice with respect to e, obtaining $e \cdot \lambda = 0$, a contradiction. Hence $e \cdot \lambda = 0$ (and $c = 0$), and from (5-1) and (5-3) we can conclude that λ is constant in G_2' and similarly $a = \lambda - 1$. In particular, from the system (5-12), we have $\lambda^2 = 1$ (hence $\operatorname{Tr} l = 0$), $a = 0$ or -2, and $L = -2a(a+1)$. If $a = 0$ we obtain a flat structure, and if $a = -2$ we have a semi-K contact structure with constant scalar curvature $r = 4a = -8$.

In G_3', $c = \lambda + a + 1 = 0$. Using this to eliminate a from the second equation of V_3, we get $\lambda^2 + 2\lambda + 1 - L = 0$. If we assume $\phi e \cdot \lambda \neq 0$, we may differentiate this equation with twice respect to ϕe, obtaining $\phi e \cdot \lambda = 0$, a contradiction. Thus $\phi e \cdot \lambda = 0$ everywhere in G_3', which gives $b = 0$. We note that $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$. Thus λ is constant in G_3', and obviously $a = -\lambda - 1$. Moreover, if we put $b = c = 0$ and $a = -\lambda - 1$ in the system (5-12), we get $\lambda = -1$ and $a = 0$; hence the structure is flat.

As in case of V_2, we have that the functions λ and a are constant in V_3, and the equations (5-16) are satisfied because $b = c = 0$.

In V_4, $2a\lambda - \lambda^2 + 1 - L = 0$ and $-2a\lambda - \lambda^2 + 1 - L = 0$. Working as in the set V_1, we find that $a = 0$ (hence Theorem 4.1 applies), $b = c = 0$, and λ is constant in V_4, that is, $\lambda^2 = 1 - L \geq 0$. Also, from (2-19), $r = \operatorname{Tr} l = 2L$.

The functions λ and a are constant in each V_i for $i = 1, 2, 3, 4$, and the set $\bigcup V_i$ is open and dense in the closure of U_1. Hence λ and a are constant in U_1.

In U_2,

(5-1) $$\phi e \cdot \lambda = 2b\lambda,$$

(5-3) $$0 = \xi \cdot \lambda,$$

(5-19) $$0 = 2a\lambda - \lambda^2 + 1 - L,$$

(5-20) $$0 = B^2 + (2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \times (-2a\lambda - \lambda^2 + 1 - L).$$

We differentiate (5-19) with respect to ξ, ϕe, and e and because of (5-1) and (5-3) we obtain respectively

(5-21) $$0 = \xi \cdot a,$$

(5-22) $$\phi e \cdot a = 2b\lambda - 2ab,$$

(5-23) $$0 = (a - \lambda)(e \cdot \lambda) + \lambda(e \cdot a) = 0.$$

We differentiate Equations (5-1) and (5-3) with respect to ξ and ϕe respectively and subtract. Then, because of (5-3), (2-13) and (2-17), we obtain

(5-24) $$(3\lambda - a - 1)(e \cdot \lambda) - 2\lambda(e \cdot a) = 2\lambda c(\lambda - a - 1)$$

We consider the system of (5-23) and (5-24) with unknown functions the derivatives of $e \cdot \lambda$ and $e \cdot a$. We consider these open subsets of U_2:

$$F_1 = \{p \in U_2 \subset U : -a - \lambda + 1 \neq 0 \text{ in a neighborhood of } p\},$$
$$F_2 = \{p \in U_2 \subset U : -a - \lambda + 1 = 0 \text{ in a neighborhood of } p\}.$$

In the neighborhood F_1 solving the system of (5-23) and (5-24), we obtain

$$(-a - \lambda + 1)(e \cdot \lambda) = -2\lambda c(\lambda - a - 1). \tag{5-25}$$

From (2-17), (5-1) and (5-22), we obtain

$$\xi \cdot c = -b(3\lambda - 3a - 1). \tag{5-26}$$

We differentiate (5-3), (5-25) with respect to e and ξ, respectively, and subtract. Then using (5-3), (5-21), we obtain $(-a - \lambda + 1)[e, \xi]\lambda = 2\lambda(\lambda - a - 1)(\xi \cdot c)$ or, by (2-13), (5-1), (5-26), and $\lambda \neq 0$ (as $F_1 \subset U$), we finally obtain

$$b(a^2 + \lambda^2 - a\lambda + a - \lambda) = 0 \tag{5-27}$$

We work in the open subset F_1 and suppose that there is a point p in F_1 at which $b \neq 0$ (or equivalently, by (5-1), $\phi e \cdot \lambda \neq 0$). The function b is smooth, and because of its continuity there is an open neighborhood S of p such that $S \subset F_1$ and $b \neq 0$ everywhere in S. Hence from (5-27) we have in S

$$a^2 + \lambda^2 - a\lambda + a - \lambda = 0. \tag{5-28}$$

We differentiate this equation with respect to ϕe, use (5-1) and (5-22), and with the assumption $b \neq 0$ obtain $2a^2 - 2a\lambda - \lambda^2 + a = 0$. This last equation together with (5-28) gives $-3\lambda^2 - a + 2\lambda = 0$. We differentiate this equation with respect to ϕe. Then using (5-1) and (5-22), we get $2b(-6\lambda^2 + a + \lambda) = 0$ or equivalently $-6\lambda^2 + a + \lambda = 0$. The equations $-3\lambda^2 - a + 2\lambda = 0$ and $-6\lambda^2 + a + \lambda = 0$ give $-9\lambda^2 + 3\lambda = 0$. We differentiate this equation twice with respect to ϕe and obtain $\phi e \cdot \lambda = 0$ or equivalently $b = 0$ everywhere in S, a contradiction. Hence, from (5-27) we deduce that $b = 0$ everywhere in F_1.

Equation (5-1) and $b = 0$ give $\phi e \cdot \lambda = 0$, which together with (5-3) gives $[\xi, \phi e]\lambda = 0$, or because of (2-13), $(\lambda - a - 1)(e \cdot \lambda) = 0$. Let's suppose that there is a point $q \in F_1$ at which $e \cdot \lambda \neq 0$. Then, there is a neighborhood Y of q in which $e \cdot \lambda \neq 0$. In Y we then have $\lambda - a - 1 = 0$, and hence from (5-25) we have $(-a - \lambda + 1)(e \cdot \lambda) = 0$, which in $Y \subset F_1$ gives $e \cdot \lambda = 0$, a contradiction. Hence $e \cdot \lambda = 0$ everywhere in F_1. Then $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$ implies that λ is constant in F_1. From (5-21), (5-22) and (5-23) we obtain that a is constant in F_1 and from

(5-25) we have $c(\lambda - a - 1) = 0$. We consider these two open subsets of F_1:

$$J_1 = \{p \in F_1 : c = 0 \text{ in a neighborhood of } p\},$$
$$J_2 = \{p \in F_1 : c \neq 0 \text{ in a neighborhood of } p\}.$$

In J_1, Equation (5-20) yields $a(\lambda - 1)(a - \lambda - 1) = 0$, where a and λ are constant and $b = 0$. We consider these two open subsets of J_1:

$$H_1 = \{p \in J_1 : a = 0 \text{ in a neighborhood of } p\},$$
$$H_2 = \{p \in J_1 : a \neq 0 \text{ in a neighborhood of } p\}.$$

In H_1, we have $\nabla_\xi \tau = 0$ (hence Theorem 4.1 applies) and from (2-19) $r = 2L$. In H_2 we have $(\lambda - 1)(a - \lambda - 1) = 0$, and hence we consider these two open subsets of H_2:

$$H_3 = \{p \in J_1 : \lambda = 1 \text{ in a neighborhood of } p\},$$
$$H_4 = \{p \in J_1 : \lambda \neq 1 \text{ in a neighborhood of } p\}$$

In H_3, we have $\operatorname{Tr} l = 0$, $L = 2a$ by (5-19), and $r = 4a$ by (2-19). In H_4 we obtain $a - \lambda - 1 = 0$, $\operatorname{Tr} l = 2a(2 - a)$, $L = a^2$, and $r = 2a(4 - a)$.

In J_2, we have $b = a - \lambda + 1 = 0$ (a semi-K contact structure with constant a and λ); hence $\operatorname{Tr} l = -2a(2 + a)$, $L = a^2$ by (5-19), and $c^2 + a(e \cdot c) + 4a^2 = 0$ by (5-20). The set $J_1 \cup J_2$ is open and dense inside the closure of F_1; hence we conclude that $b = 0$ and that a and λ are constant in F_1.

In the open set F_2 we have $-a - \lambda + 1 = 0$, which together with (5-19) gives $-3\lambda^2 + 2\lambda + 1 - L = 0$. If we assume $e \cdot \lambda \neq 0$, we may differentiate this equation twice with respect to e and obtain $e \cdot \lambda = 0$, a contradiction. Hence $e \cdot \lambda = 0$. Similarly we can deduce that $\phi e \cdot \lambda = 0$ (so (5-1) implies $b = 0$), and hence λ is constant in F_2. Obviously $a = -\lambda + 1$ is constant in F_2. The system of (5-23) and (5-24) gives $ca = 0$. We consider these two open subsets of F_2:

$$Q_1 = \{p \in F_2 : c = 0 \text{ in a neighborhood of } p\},$$
$$Q_2 = \{p \in F_2 : c \neq 0 \text{ in a neighborhood of } p\}.$$

In Q_1, (5-19) implies $L = \operatorname{Tr} l - a^2$, where $\operatorname{Tr} l = 2a(2 - a)$ and $r = 2a(4 - a)$. In Q_2, we have a 3-τ manifold structure with $\operatorname{Tr} l = L = 0$.

We have proved that λ and a are constant in F_1 and F_2. Since $F_1 \cup F_2$ is open and dense inside the closure of U_2, we conclude that λ and a are constant in U_2.

In U_3,

(5-2) $$2c\lambda = (e \cdot \lambda),$$

(5-3) $$0 = \xi \cdot \lambda,$$

(5-29) $$0 = -2a\lambda - \lambda^2 + 1 - L,$$

(5-30) $$0 = A^2 + (-2a\lambda - 2\lambda^2 + 2 + b^2 + c^2 - 2a - (e \cdot c) - (\phi e \cdot b)) \times (2a\lambda - \lambda^2 + 1 - L).$$

We differentiate the relation (5-29) with respect to ξ, e and ϕe, and because of (5-2) and (5-3) we obtain respectively (5-21),

(5-31) $$e \cdot a = -2ac - 2c\lambda,$$

(5-32) $$0 = (a + \lambda)(\phi e \cdot \lambda) + \lambda(\phi e \cdot a).$$

We differentiate (5-2) and (5-3) with respect to ξ and e respectively and subtract. Then by (5-21), we obtain $[\xi, e]\lambda = 2\lambda(\xi \cdot c)$, or because of (2-13) and (2-17),

(5-33) $$(3\lambda + a + 1)(\phi e \cdot \lambda) + 2\lambda(\phi e \cdot a) = 2\lambda b(\lambda + a + 1).$$

Meanwhile from (2-17), (5-2) and (5-31), we obtain

(5-34) $$\xi \cdot b = -c(3a + 3\lambda + 1).$$

Now consider the system of (5-32) and (5-33) and these two open subsets of U_3:

$$F_1' = \{p \in U_3 \subset U : a - \lambda - 1 \neq 0 \text{ in a neighborhood of } p\},$$
$$F_2' = \{p \in U_3 \subset U : a - \lambda - 1 = 0 \text{ in a neighborhood of } p\}.$$

In the open set F_1' of p in which $a - \lambda - 1 \neq 0$, we may solve the system of (5-32) and (5-33) to obtain

(5-35) $$(a - \lambda - 1)(\phi e \cdot \lambda) = -2\lambda b(\lambda + a + 1).$$

We differentiate (5-3) and (5-35) with respect to ϕe and ξ respectively and subtract. Then by (5-3) and (5-21) we obtain $(a - \lambda - 1)[\phi e, \xi]\lambda = 2\lambda(\lambda + a + 1)(\xi \cdot b)$ or, because of (2-13), $(a - \lambda - 1)(a - \lambda + 1)(e \cdot \lambda) = 2\lambda(\lambda + a + 1)(\xi \cdot b)$. Then, using (5-2), (5-34) and $\lambda \neq 0$ (in $F_1' \subset U$), we get

$$c(a^2 + \lambda^2 + a\lambda + a + \lambda) = 0.$$

As in the case of Equation (5-27) we can deduce in F_1' that $c = 0$. Equation (5-2), because $c = 0$, gives $e \cdot \lambda = 0$. This together with (5-3) gives $[e, \xi]\lambda = 0$ or, because of (2-13), $(a + \lambda + 1)(\phi e \cdot \lambda) = 0$. Suppose that there is a point $q \in F_1'$ at which $\phi e \cdot \lambda \neq 0$. Then there is a neighborhood S of q in which $\phi e \cdot \lambda \neq 0$, and hence $a + \lambda + 1 = 0$. Because $a + \lambda + 1 = 0$, Equation (5-35) gives $(a - \lambda - 1)(\phi e \cdot \lambda) = 0$.

Working in $S \subset F_1'$, where $a-\lambda-1 \neq 0$, we can conclude that $\phi e \cdot \lambda = 0$, a contradiction. Hence, $\phi e \cdot \lambda = 0$ everywhere in F_1'. In the neighborhood F_1', we have $\xi \cdot \lambda = \phi e \cdot \lambda = e \cdot \lambda = 0$ or equivalently λ is constant in F_1'. From (5-21), (5-31) and (5-33), we obtain that a is constant in F_1', and from (5-32) we have $b(\lambda+a+1) = 0$. We consider these two open subsets of F_1':

$$J_1' = \{p \in F_1' : b = 0 \text{ in a neighborhood of } p\},$$
$$J_2' = \{p \in F_1' : b \neq 0 \text{ in a neighborhood of } p\}.$$

In J_1', we obtain from (5-30) that $a(\lambda+1)(a+\lambda-1) = 0$, where a and λ are constant and $c = 0$. We consider these two open subsets of J_1':

$$H_1' = \{p \in J_1' : a = 0 \text{ in a neighborhood of } p\},$$
$$H_2' = \{p \in J_1' : a \neq 0 \text{ in a neighborhood of } p\}.$$

In H_1', we obtain $\nabla_\xi \tau = 0$ (hence Theorem 4.1 applies). and $r = 2L$. In H_2', we have $(\lambda+1)(a+\lambda-1) = 0$. We consider these two open subsets of H_2':

$$H_3' = \{p \in J_1 : \lambda = -1 \text{ in a neighborhood of } p\},$$
$$H_4' = \{p \in J_1 : \lambda \neq -1 \text{ in a neighborhood of } p\}.$$

In H_3', we have $\operatorname{Tr} l = 0$, $L = 2a$ by (5-29), and $r = 4a$ by (2-19). In H_4', we obtain $a+\lambda-1 = 0$, $\operatorname{Tr} l = 2a(2-a)$, $L = a^2$ and $r = 2a(4-a)$.

In J_2' we have $c = a+\lambda+1 = 0$ (a semi-K contact structure with constant a and λ) and hence $\operatorname{Tr} l = -2a(2+a)$. Then $L = a^2$ from (5-29), and from (5-30) we obtain $b^2 + a(\phi e \cdot b) + 4a^2 = 0$ with a a constant. The set $J_1' \cup J_2'$ is open and dense inside the closure of F_1'; hence we can conclude that $c = 0$ and that a and λ are constant in F_1'.

In the open set F_2' we have $a-\lambda-1 = 0$, which together with (5-29) gives $-3\lambda^2 - 2\lambda + 1 - L = 0$. If we assume $\phi e \cdot \lambda \neq 0$, we may differentiate this equation twice with respect to e, obtaining $\phi e \cdot \lambda = 0$, a contradiction. Hence $\phi e \cdot \lambda = 0$. Similarly we find that $e \cdot \lambda = 0$, and hence λ is constant in F_2'. Obviously $a = \lambda + 1$ is constant in F_2'. The system of (5-32) and (5-33) gives $ba = 0$. We consider these two open subsets of F_2':

$$Q_1' = \{p \in F_2' : b = 0 \text{ in a neighborhood of } p\},$$
$$Q_2' = \{p \in F_2' : b \neq 0 \text{ in a neighborhood of } p\}.$$

In Q_1', (5-29) implies $L = \operatorname{Tr} l - a^2$, where $\operatorname{Tr} l = 2a(2-a)$ and $r = 2a(4-a)$. In Q_2', we have a 3-τ manifold structure with $\operatorname{Tr} l = L = 0$.

We have proved that λ and a are constant in F_1' and F_2', while $F_1' \cup F_2'$ is open and dense inside the closure of U_3. Hence we conclude that λ and a are constant in U_3.

Finally because λ and a are constant in each U_i for $i = 1, 2, 3$, and because the set $\bigcup U_i$ is open and dense inside of the closure of U, we conclude that λ and a are constant in U. Then by (2-9), $\operatorname{Tr} l = 2(1-\lambda^2)$ is also constant in U and obviously on M^3. □

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