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In memory of Mario Lo Giudice

We show that the coarse moduli space $\mathcal{R}_5$ of étale double covers of curves of genus 5 over the complex numbers is unirational. We give two slightly different arguments, one purely geometric and the other more computational.

1. Introduction

The coarse moduli space $\mathcal{R}_g$ of étale double covers of genus g curves is sometimes referred to as the Prym moduli space. It can be thought of as the moduli space of curves $C \in \mathcal{M}_g$ equipped with a nontrivial line bundle $\mathcal{L}$ whose square is trivial. Thus $\mathcal{R}_g$ is also equipped with a morphism to $\mathcal{M}_g$, which is a finite cover of degree $2^{2g} - 1$. It has been extensively studied for small values of g. In particular Donagi [1984] showed that $\mathcal{R}_6$ is unirational. Other proofs of the unirationality of $\mathcal{R}_6$ were given by Verra [1984] and by Mori and Mukai [1983].

For $g \leq 6$ the Prym map $p_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$, which associates to an étale double cover $\tau : \tilde{C} \to C$ the Prym variety

$$P(\tau) = \operatorname{coker}(\tau^* \colon \operatorname{Jac} C \to \operatorname{Jac} \tilde{C}),$$

is dominant. It therefore follows from Donagi's result that the moduli space $\mathcal{A}_5$ of principally polarised abelian 5-folds is unirational.

Catanese [1983] showed that $\mathcal{R}_4$ is rational. The rationality of $\mathcal{R}_3$ follows from [Katsylo 1994; Bardelli and Del Centina 1990; Del Centina and Recillas 1989; Dolgachev 2008]. The rationality of $\mathcal{R}_2$ is also due to Dolgachev [2008]. Moreover $\mathcal{R}_1 = X_0(2)$ is rational.

Clemens [1983] showed that $\mathcal{A}_4$ is unirational, but using intermediate Jacobians, not Prym varieties. In the introduction to [Catanese 1983] it is stated that $\mathcal{R}_5$ is

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unirational and a reference is given to [Clemens 1983], then unpublished; but as far as we can determine no proof is given there or anywhere else.

In this paper we fill this gap by proving (Theorem 6.1) that $\mathcal{R}_5$ is indeed unirational. We work over an algebraically closed field $\mathbb{K}$ of characteristic different from 2 (except in Section 5).

The basic construction used in our proof is to be found in [Clemens 1983]. If X is a quartic surface in $\mathbb{P}^3$ with six ordinary double points at $P_0, \dots, P_5$ and no other singularities we define C_X to be the discriminant of the projection from P_0. Generically it is a 5-nodal plane sextic (hence of genus 5), with an everywhere tangent conic coming from the tangent cone to X at P_0. The quartic double solid branched along X has (after blowing up P_0) the structure of a conic bundle over $\mathbb{P}^2$ with discriminant curve C_X. Blowing up in the remaining five points yields a conic bundle over a degree 4 del Pezzo surface, and the discriminant is the canonical model $\widetilde{C}_X$ of C_X. This determines a connected étale double cover of $\widetilde{C}_X$.

The space $\mathfrak{Q}$ of quartic surfaces in $\mathbb{P}^3$ with six isolated ordinary double points, one of which is marked, is unirational. This is well known and quite easy to prove: see Proposition 3.1.

The construction above defines a morphism from the unirational variety $\mathfrak{Q}$ to $\mathcal{R}_5$, which is in turn endowed with a finite (in fact 1024-to-1) natural projection to $\mathcal{M}_5$. Since $\mathcal{R}_5$ is irreducible, to prove the unirationality it is now enough to prove that the map to $\mathcal{M}_5$ is generically surjective.

We present two different proofs that the map $\theta\colon \mathfrak{Q} \to \mathcal{M}_5$ is dominant. One method exploits the special geometry of the family, using ideas of Donagi as worked out in [Izadi 1995]. We show by a dimension count that the general genus 5 curve does have a plane model as a 5-nodal sextic with an everywhere tangent conic, and then show how to recover the quartic in $\mathbb{P}^3$ as a certain image of the double cover of $\mathbb{P}^2$ branched along the sextic.

The other approach, which was used in [Lo Giudice 2006], is computational, and applies to any family of 5-nodal sextics. It uses the fact that $\widetilde{C}_X$ is a canonical curve, and reduces the question of surjectivity of the Kodaira–Spencer map to computing the rank of a certain matrix. This can then be verified at a test point.

2. Nodal quartics and nodal curves

The equation of a quartic surface $X \subset \mathbb{P}^3$ with an isolated ordinary double point at $P_0 = (0:0:0:1)$ is $F = u_2x_3^2 + 2u_3x_3 + u_4 = 0$, where u_d is a form of degree d in $\mathbb{K}[x_0, x_1, x_2]$ and the quadratic form u_2 is nondegenerate. The projection $\pi : X \setminus P_0 \to \mathbb{P}^2$ from P_0 is induced by the homomorphism

$$r : \mathbb{K}[x_0, x_1, x_2] \to \mathbb{K}[x_0, \dots, x_3]/(F),$$

sending x_i to $x_i + (F)$ for $i = 0, 1, 2$. If X is general, then any line in $\mathbb{P}^3$ through P_0 intersects X in at most two other points so π is a quasifinite morphism.

We define the plane curve C_X to be the locus of lines through P_0 tangent to X away from P_0. The following is easy to prove: see [Kreussler 1989, Lemma 5.1].

Proposition 2.1. *C_X is a plane curve of degree six given by $u_3^2 - u_2u_4 = 0$. Furthermore if $Q \in X$ is a singular point (different from P), then $\pi(Q)$ is a singular point of C_X.*

The locus of points in $\mathbb{P}^2$ whose reduced fibre under π consists of only one point is not irreducible. There are two components, the curve C_X and the conic $u_2 = 0$.

Definition 2.2. A conic $V \subset \mathbb{P}^2$ is called a contact conic of C_X if V cuts on X a divisor which is divisible by 2.

Corollary 2.3. *Let $\mathrm{Bl}_{P_0}(X) \to X$ be the blowup of the surface X at the point P_0. Then the unique morphism $\mathrm{Bl}_{P_0}(X) \to \mathbb{P}^2$ which extends π is a double cover of $\mathbb{P}^2$ branched along C_X. The image of the exceptional divisor in $\mathrm{Bl}_{P_0}(X)$ is the contact conic of C_X defined by the equation $u_2 = 0$.*

Proof. It is essentially enough to observe that the tangent cone of X at P_0 is defined by the equation $u_2 = 0$ in $\mathbb{P}^3$. The exceptional curve inside $\mathrm{Bl}_{P_0}(X)$ corresponds to the set of lines in the tangent cone. To see that the conic $u_2 = 0$ is a contact conic of C_X simply look at the ideal of the intersection of the conic with C_X: $(u_2, u_3^2 - u_2u_4) = (u_2, u_3^2)$ in $\mathbb{K}[x_0, x_1, x_2]$. In particular this means that the points of contact are given by $u_2 = u_3 = 0$. □

Next we describe the singular locus of C_X.

Lemma 2.4. *Let $Y_d \subset \mathbb{P}^3$ be the cone of vertex P_0 defined by the form u_d, and let $Q \in X$ be any point different from P_0 such that $\pi(Q) \in C_X$. Then $\pi(Q) \in \operatorname{Sing} C_X$ if and only if $Q \in (\operatorname{Sing} X) \cup (Y_2 \cap Y_3 \cap Y_4)$.*

Proof. If $\mathfrak{q}$ is the homogeneous ideal of Q, then $Q \in X$ means $F \in \mathfrak{q}$ and $\pi(Q) \in C_X$ means $u_3^2 - u_2u_4 \in \mathfrak{q}$. Then, from the equality $u_2F = (u_2x_3 + u_3)^2 - (u_3^2 - u_2u_4)$, we obtain immediately $u_2x_3 + u_3 \in \mathfrak{q}$. Now $\pi(Q)$ is a singular point if and only if $u_3^2 - u_2u_4 \in \mathfrak{q}^2$, and this happens if and only if $u_2F \in \mathfrak{q}^2$, which means that either $u_2 \in \mathfrak{q}$ or $F \in \mathfrak{q}^2$. In the latter case Q is a singular point of X; in the former we also have $u_3 \in \mathfrak{q}$, since $u_2x_3 + u_3 \in \mathfrak{q}$, and $u_4 \in \mathfrak{q}$ since $2u_3x_3 + u_4 \in \mathfrak{q}$. □

Proposition 2.5. *For a general quartic surface X with an isolated double point P_0, the singular points of C_X are all images of singular points of X.*

Proof. Let Q be a point of X mapping to a singular point of C_X and assume that $Q \in Y_2 \cap Y_3 \cap Y_4$ and $Q \neq P_0$. Then $\pi(Q) \in (u_2 = u_3 = u_4 = 0)$, which is empty for general (u_2, u_3, u_4). $\square$

If X is a quartic surface with at least one ordinary double point P_0, we let $p\colon \Lambda_X \to \mathbb{P}^3$ be the double cover branched along X and let $W_X = \mathrm{Bl}_{P_0}(\Lambda_X)$ be the blowup of Λ_X at P_0.

Proposition 2.6 [Clemens 1983, p. 222]. *Let X, P_0, Λ_X and W_X be as above. The unique morphism f that makes the diagram*

$$\begin{tikzcd} W_X \arrow[r] \arrow[drr, "f"'] & \Lambda_X \arrow[r, "p"] & \mathbb{P}^3 \arrow[d, dashed, "\pi"] \\ & & \mathbb{P}^2 \end{tikzcd}$$

commute is a conic bundle over $\mathbb{P}^2$, and the curve C_X is the locus of points whose fibre is a degenerate conic.

A more detailed version of the proof than Clemens' can be found in [Kreussler 2000, Section 2]. This latter paper also gives an explicit equation for W_X as a divisor inside a $\mathbb{P}^2$ bundle over $\mathbb{P}^2$. Put $\mathscr{E} = \mathscr{O} \oplus \mathscr{O}(-1) \oplus \mathscr{O}(-2)$ over $\mathbb{P}^2$, and consider $p\colon \mathbb{P}(\mathscr{E}) \to \mathbb{P}^2$. Here $\mathbb{P}(E)$ is the projective bundle of hyperplanes in the fibres of E. Let $z_k \in H^0(\mathbb{P}^2, \mathscr{E}(k))$, $k = 0,\,1,\,2$ be *constant* nonzero global sections and define the divisor $W_X \subset \mathbb{P}(\mathscr{E})$ by $-z_2^2 + z_1^2 u_2 + 2z_1 z_0 u_3 + z_0^2 u_4 = 0$ (the left-hand side is a section of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(2) \otimes p^*\mathscr{O}_{\mathbb{P}^2}(4)$).

Lemma 2.7. *Let C be a plane sextic curve whose only singularities are five nodes in linear general position. Let $\sigma_C\colon \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ denote the blowup of $\mathbb{P}^2$ in these five points. Then $\widetilde{\mathbb{P}}^2$ is a degree 4 Del Pezzo surface and the anticanonical embedding of $\widetilde{\mathbb{P}}^2$ in $\mathbb{P}^4$ realises the strict transform $\widetilde{C}$ of C as a smooth canonically embedded curve of genus 5.*

Proof. This is well known. That $\widetilde{C}$ is canonically embedded follows from a simple adjunction computation: if E is the exceptional divisor and H is the class of a line in $\mathbb{P}^2$, then in $\operatorname{Pic}\widetilde{\mathbb{P}}^2$ we have $\widetilde{C} = \sigma_C^*(6H) - 2E = -2K_{\widetilde{\mathbb{P}}^2}$. $\square$

Next we consider a more special case. We suppose $X \subset \mathbb{P}^3$ is a quartic surface with six isolated ordinary double points, $P_0, \ldots, P_5$ and no other singularities. We also assume that the P_i are in linear general position and that X is general with respect to P_0, in the precise sense of Proposition 2.5. Under these hypotheses C_X is a plane sextic with precisely five nodes, at $\overline{P}_i = \pi(P_i)$, $1 \le i \le 5$.

Let $f\colon W_X \to \mathbb{P}^2$ be the conic bundle as in Proposition 2.6, and let $\sigma_{C_X}\colon \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ and $\widetilde{C}_X$ be as in Lemma 2.7. Let Σ be the surface $f^{-1}(C_X) \subset W_X$ and put $S = \Sigma \times_{C_X} \widetilde{C}_X$, with $\tilde{f}\colon S \to \widetilde{C}_X$ the projection.

Let $\nu\colon \widetilde{S} \to S$ be the normalisation of S, and consider the Stein factorisation of $\tilde{f} \circ \nu\colon \widetilde{S} \to \widetilde{C}_X$

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\nu"] \arrow[d, "f'"'] & S \arrow[d, "\tilde{f}"] \\ \Gamma \arrow[r, "g"] & \widetilde{C}_X \end{tikzcd}$$

so f' is a projective morphism with connected fibres and g is a finite morphism.

Proposition 2.8. *For general X, the finite morphism $g\colon \Gamma \to \widetilde{C}_X$ is an étale degree 2 map between smooth connected curves, naturally associated with the pair (X, P_0).*

Proof. The cover g is unbranched of degree 2 because the restriction of the conic bundle W_X to the curve C_X consists of a fibration by pairs of distinct lines. To see this recall the equation for the conic bundle W_X. The preimage of a point $x = (x_1 : x_2 : x_3) \in \mathbb{P}^2$ is given by $-z_2^2 + z_1^2 u_2(x) + 2z_1 z_0 u_3(x) + z_0^2 u_4(x) = 0$, and this has rank 2 since u_2, u_3 and u_4 never vanish simultaneously.

It remains to check that g is nontrivial, that is, that Γ is connected. In characteristic zero this follows from the fact that the Prym variety $P(g)$ is isomorphic to the intermediate Jacobian of the conic bundle [Beauville 1977b, Section 2]. If the double cover were trivial, then $P(g)$ would have dimension 5, which is impossible. To extend this to the case of characteristic $p \neq 2$, we observe that the quartic equation lifts to characteristic zero and in that case the double cover is nontrivial, as we have just seen. Therefore the double cover in positive characteristic is connected. □

3. Moduli of curves

The functor $\mathbf{r}_g$ ($g \geq 2$) given by families of smooth projective curves of genus g with a connected étale double cover is coarsely represented by an irreducible quasiprojective scheme $\mathcal{R}_g$: see [Beauville 1989, §6]. The dimension of this moduli space is $3g-3$, the same as the dimension of $\mathcal{M}_g$, the extra data being one of the $2^{2g}-1$ nontrivial 2-torsion points in the Jacobian of C. So forgetting this defines a natural transformation between $\mathbf{r}_g$ and $\mathbf{m}_g$, and thus a morphism $\mathcal{R}_g \to \mathcal{M}_g$ with finite fibres.

Fix five points $P_1, \dots, P_5$ in $\mathbb{P}^3$ in linear general position. Let $\mathfrak{Q}$ be the space of quartic surfaces in $\mathbb{P}^3$ with ordinary double points at the P_i, one additional ordinary double point distinct from the P_i, and no other singularity.

Proposition 3.1. *$\mathfrak{Q}$ is an irreducible locally closed subscheme of the Hilbert scheme of quartic surfaces in $\mathbb{P}^3$ and hence inherits a universal family of quartics from the Hilbert scheme. Furthermore $\mathfrak{Q}$ is unirational of dimension* 13.

Proof. The Hilbert scheme of quartic surfaces in $\mathbb{P}^3$ is

$$\mathrm{Hilb}_{2m^2+2}(\mathbb{P}^3) = \mathbb{P}\big(H^0(\mathbb{P}^3, \mathcal{O}(4))\big).$$

Let $\mathfrak{p}_i$ be the homogeneous ideal of P_i in $\mathbb{P}^3$. The set of quartic surfaces in $\mathbb{P}^3$ with five double points at the P_i is the closed subscheme of the Hilbert scheme given by $\mathbb{P}(I)$ where $I = \bigcap_{i=0}^{4} \mathfrak{p}_i^2$ is the ideal of the five double points. In other words it is $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{I}(4))$, where we denote $\mathcal{I}$ the sheaf of ideals defined by I. Note that $h^0(\mathbb{P}^3, \mathcal{I}(4)) = 15$.

Going on, we ask now for a sixth double point. We take the product $\mathbb{P}(I) \times \mathbb{P}^3$ and consider the closed subscheme B_0 defined as

$$B_0 := \{(F, \mathfrak{p}_0) | F \in \mathfrak{p}_0^2\} = \left\{(F, P_0) \;\middle|\; F(P_0) = \frac{\partial F}{\partial x_i}(P_0) = 0, \text{ for all } i\right\}.$$

The projection

$$B_0 \to \mathbb{P}^3$$

is surjective and the fibres are linear spaces. Indeed, having a singular point at $P_0 \in \mathbb{P}^3$ defines four linear conditions on the linear space $H^0(\mathbb{P}^3, \mathcal{I}(4))$. Over an open subset $U \subset \mathbb{P}^3$ these conditions are independent and define a vector bundle E of rank 11. To see that U is not empty it is enough to fix a sixth point in $\mathbb{P}^3$ and compute the Hilbert function of the ideal $J = \bigcap_{i=0}^{5} \mathfrak{p}_i^2$, which we may easily do with *Macaulay*. The projective space bundle B_1 over U associated to E is a rational variety embedded in B_0 as a dense open subset. Hence B_0 is irreducible and rational.

Projecting onto $\mathbb{P}(I)$, we consider the scheme theoretic image of B_0, which is a closed subscheme B of $\mathbb{P}(I)$, and observe that in the universal factorisation

$$\begin{tikzcd} B_0 \arrow[r, "\beta"] \arrow[dr] & B \arrow[d] \\ & \mathbb{P}(I) \end{tikzcd}$$

the dominant morphism β is also proper, because it is the external morphism of a composition which is proper. As a consequence β is surjective and the scheme theoretic image of B_0 coincides with the set theoretic image. Observe also that B is irreducible. However B contains all the possible degenerations of a quartic surface with six double points, while we are interested in those surfaces with ordinary double points and no other singularities. This is clearly an open condition, so we have proved that $\mathcal{Q}$ is an open subset of an irreducible closed subset of the Hilbert scheme.

The unirationality of $\mathcal{Q}$ follows from the fact that an open dense subset of the rational variety B_0 maps onto it.

Finally, one may check by computing the differential of the projection at one point that the dimension of $\mathfrak{Q}$ is 13 (it is irreducible because it is an open subset of an irreducible variety). □

Proposition 3.2. *There exists a morphism of schemes $\varrho\colon \mathfrak{Q} \to \mathcal{R}_5$ given by the constructions above that associates to any nodal quartic surface in $\mathbb{P}^3$ a nodal sextic plane curve with a double cover.*

Proof. We must globalise our earlier constructions. This is a standard gluing argument. Suppose first that the base scheme is $B = \operatorname{Spec} A$ for some ring A. Associate to the scheme

$$\mathcal{X} = \operatorname{Proj} A[x_0, \ldots, x_3]/(u_2x_3^2 + 2u_3x_3 + u_4)$$

the plane curve over A defined by the equation $u_3^2 - u_2u_4$. This association is natural, in that it commutes with pull-backs. Indeed for any homomorphism of rings $A \to A'$ the pull-back of $\mathcal{X}$ is given by $\operatorname{Proj}(A \otimes A')[x_0, x_1, x_2]/(u_3^2 - u_2u_4)$, and this is the same graded ring one would obtain by first pulling back the family of surfaces and then applying the correspondence. □

We want to use the morphism ϱ to prove the unirationality of $\mathcal{R}_5$. To do so we must show that ϱ is dominant. We can simplify the problem by taking advantage of the irreducibility of $\mathcal{R}_g$ and $\mathcal{M}_g$. We have a commutative diagram

$$\begin{tikzcd} \mathfrak{Q} \arrow[r, "\varrho"] \arrow[dr, "\theta"'] & \mathcal{R}_5 \arrow[d, "\eta"] \\ & \mathcal{M}_5 \end{tikzcd}$$

where η forgets the double cover.

Lemma 3.3. *The morphism ϱ is dominant if and only if θ is dominant.*

Proof. Since η is a dominant morphism between irreducible spaces, it is immediate that θ is dominant if ϱ is. Conversely, if ϱ is not dominant then the scheme theoretic image $\varrho(\mathfrak{Q}) \subset \mathcal{R}_5$ has dimension less than $\dim \mathcal{R}_5 = 12$, because $\mathcal{R}_5$ is irreducible, so $\dim \theta(\mathfrak{Q}) < 12$ also, so θ is not dominant. □

4. Reconstructing the double solid

In this section we give a proof that $\theta\colon \mathfrak{Q} \to \mathcal{M}_5$ is dominant by making use of the special geometry of the family $\mathfrak{Q}$. We show how to reconstruct the quartic surface from a suitable plane sextic model of a sufficiently general genus 5 curve.

Lemma 4.1. *For a general $C \in \mathcal{M}_5$ there exists a birational map $C \to \overline{C} \subset \mathbb{P}^2$ to a plane 5-nodal sextic $\overline{C}$, such that $\overline{C}$ admits a contact conic $V \subset \mathbb{P}^2$ meeting $\overline{C}$ at six distinct smooth points of $\overline{C}$. Furthermore, a general $C \in \mathcal{M}_5$ has a one-parameter family of such birational plane models.*

Proof. By the Kempf–Kleiman–Laksov theorem [Arbarello et al. 1985, V (1.1)], a general $C \in \mathcal{M}_5$ has a 2-dimensional family $G^2_6(C)$ of g^2_6s (linear systems of degree 6 and dimension 2) and hence of birational models $\overline{C}$ as a plane sextic. The general such sextic, for any given general C, has five nodes.

For fixed general C, the image of the map

$$|\mathcal{O}_{\mathbb{P}^2}(2)| \times G^2_6(C) \cong \mathbb{P}^5 \times \mathbb{P}^2 \to \mathrm{Hilb}_{12}\,\mathbb{P}^1$$

given by $(V, \overline{C}) \mapsto V \cap \overline{C} \subset V \cong \mathbb{P}^1$ intersects the codimension 6 locus consisting of subschemes with multiplicity at least 2 at each point in a variety of dimension 1. In particular this intersection is nonempty. Indeed, if we take $\overline{C}$ to be the discriminant curve of the projection from a node of a general 6-nodal quartic surface in $\mathbb{P}^3$, that is, we take F as in Section 2 with u_d general, and $V = (u_2 = 0)$, we obtain a pair $(V, \overline{C})$ whose intersection is six distinct smooth points.

Thus for a general genus 5 curve C, there is a 1-dimensional family of plane 5-nodal sextic models $\overline{C}$ of C, each having a contact conic meeting $\overline{C}$ at six distinct smooth points of $\overline{C}$. □

Proposition 4.2. *Given $\overline{C}$ as in Lemma 4.1 with contact conic V, there exists a quartic surface X with* 6 *nodes such that $\overline{C}$ arises as the discriminant locus of the projection of X from one of its nodes and V is the projection of the tangent cone of X at the same node.*

Proof. We follow the construction on [Izadi 1995, pages 104–105]. We take the double cover $\psi : Y \to \mathbb{P}^2$ branched along $\overline{C}$ and map it to $\mathbb{P}^3$ by a linear system determined by V.

To define the linear system, take the desingularisation $\sigma : \widetilde{Y} \to Y$. The inverse image $\sigma^{-1}(V)$ consists of two components, V_+ and V_-. We consider the linear systems $H_\pm = |(\psi\sigma)^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\widetilde{Y}}(V_\pm)|$. Either of these linear systems (and no other) maps the K3 surface $\widetilde{Y}$ onto a quartic surface $\overline{Y}$, unique up to projective equivalence, with six nodes: five of these nodes are the images of the exceptional curves of σ, corresponding to nodes of Y, and the sixth is the image of $V_\pm$. The discriminant curve for the projection from this sixth node is $\overline{C}$ up to projective equivalence. We refer to the proof of [Izadi 1995, Theorem 2.1.1] for the computations of the degree and dimension of $H_\pm$ needed to justify these assertions. □

Corollary 4.3. *The map $\varrho : \mathfrak{Q} \to \mathcal{R}_5$ is dominant.*

Proof. Immediate from Proposition 4.2. □

5. Families of canonical curves

In this section we give an alternative proof that θ, and hence ϱ, is dominant. The method is to check directly, by computation, that the Kodaira–Spencer map is locally surjective at a test point. It does not rely on the special geometry of $\mathcal{Q}$: it is a method of checking computationally that a given family of 5-nodal sextics is general in the sense of moduli of genus 5 curves. For simplicity we assume in this section that $\mathbb{K}$ is of characteristic zero.

We first write down a local condition for θ to be dominant, that is, generically surjective.

Lemma 5.1. *Let $u\colon X \to X'$ be a morphism, with X' irreducible. Then $u(X)$ is Zariski dense in X' if and only if there exists a smooth point $P \in X$ such that the differential $du_P\colon T_{X,P} \to T_{X',u(P)}$ is surjective.*

Proof. Since X' is irreducible the closure of $u(X)$ is X' if and only if the dimension of one of its irreducible components is equal to $\dim X'$. Now it is enough to recall that the dimension of the irreducible component of $\overline{u(X)}$ containing a regular point $u(P)$ is given by the rank of the differential. □

The tangent space to any scheme X at a closed point P is the set of maps from $D = \operatorname{Spec} \mathbb{K}[\varepsilon]/(\varepsilon^2)$ to X centred at P. For any morphism $u\colon X \to X'$ and any closed regular point $P \in X$ the differential $du_P\colon T_{X,P} \to T_{X',u(P)}$ is given by $\varphi \mapsto u \circ \varphi$.

Let C be a canonically embedded curve of genus five, which we assume to be given by the complete intersection of three quadrics in $\mathbb{P}^4$ (that is, by Petri's Theorem, nontrigonal). Two canonically embedded curves of genus g are isomorphic if and only if they are projectively equivalent.

We put $R_2 = H^0(\mathcal{O}_{\mathbb{P}^4}(2))$, the degree 2 part of $\mathbb{K}[x_0, \ldots, x_4]$, which we identify with the space of 4×4 symmetric matrices over $\mathbb{K}$.

The set of all canonical curves in $\mathbb{P}^4$ is an open subset of the Grassmannian $\operatorname{Gr}(3, R_2)$. Projective equivalence is then given by the action of the group PGL(5) on $\mathbb{P}^4$.

But the Grassmannian itself can be realised as an orbit space, this time under the action of GL(3), as follows. Let V be the open set inside the 45-dimensional vector space $R_2 \times R_2 \times R_2$ where the three components span a 3-dimensional subspace of R_2, and consider the action of GL(3) whose orbits are all the possible bases for a given subspace. This is the action

$$M({}^t\underline{x}Q_1\underline{x}, {}^t\underline{x}Q_2\underline{x}, {}^t\underline{x}Q_3\underline{x})_j = \sum_{i=1}^{3} m_{ji}\, {}^t\underline{x}Q_i\underline{x}, \qquad 1 \le j \le 3,$$

where $M \in \mathrm{GL}(3)$ and ${}^t\underline{x}$ is the row vector $(x_0, \ldots, x_4)$, so ${}^t\underline{x}Q_i\underline{x} \in R_2$ if Q_i is a symmetric matrix.

The action of $N \in \mathrm{PGL}(5)$ is given by

$$N({}^t\underline{x}Q_1\underline{x}, {}^t\underline{x}Q_2\underline{x}, {}^t\underline{x}Q_3\underline{x})_j = {}^t\underline{x}\, {}^tN Q_j N\underline{x}, \qquad 1 \le j \le 3.$$

The two actions commute and we can regard one as acting on the orbit space of the other.

In order to investigate properties of a smooth family of deformations of a canonical genus 5 curve C it is enough to consider the case in which the base scheme is the spectrum of $A = \mathbb{K}[t_0, \dots, t_n]/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal generated by $t_0, \dots, t_n$, corresponding (as a point of Spec A) to the curve C. Then the family is the scheme

$$\mathscr{C} = \operatorname{Proj} A[x_0, \dots, x_4]/(F_1, F_2, F_3),$$

where the coefficients of F_i depend linearly on the parameters t_i:

$$F_i = H_i + \sum_{j=0}^{n} t_j H_{ij},$$

where H_i, $H_{ij} \in R_2$. The n triples of quadrics (H_{1j}, H_{2j}, H_{3j}) generate the linear subspace of $\mathbb{A}^{45} = R_2 \times R_2 \times R_2$ tangent to the family $\mathscr{C}$.

We want to compare this linear space with the tangent space to $\mathscr{M}_5$ at C. Our strategy is to work inside the tangent space to V at s: we construct a basis for all the trivial deformations using the fact that they are those given by the actions of PGL(5) and GL(3), and then check how many of the above triples lie inside this linear space.

Around any point

$$v = ({}^t\underline{x} Q_1 \underline{x}, {}^t\underline{x} Q_2 \underline{x}, {}^t\underline{x} Q_3 \underline{x}) \in V$$

the action of the two groups is linearised by the action of the corresponding Lie algebras, so a system of generators for the linear space tangent to the orbit passing through v is simply determined by applying a basis for the Lie algebra to it. The Lie algebra $\mathfrak{gl}(3)$ is simply the whole space of three-by-three matrices and its action is the same as the action of GL(3) so we obtain a first set of trivial deformations given by the nine vectors

$$(H_1, 0, 0), (H_2, 0, 0), \dots, (0, 0, H_2), (0, 0, H_3).$$

The algebra $\mathfrak{sl}(5)$ (which is the tangent space to PGL(5)) is the space of traceless 5×5 matrices, and its action is determined as follows:

$$\begin{aligned} {}^t(N\underline{x}) Q_i (N\underline{x}) &= {}^t\underline{x}\,{}^t(I + \varepsilon\Delta) Q_i (I + \varepsilon\Delta)\underline{x} = {}^t\underline{x}(Q_i + \varepsilon({}^t\Delta Q_i + Q_i\Delta))\underline{x} \\ &= {}^t\underline{x} Q_i \underline{x} + \varepsilon \underline{x}({}^t\Delta Q_i + Q_i \Delta)\underline{x}. \end{aligned}$$

Letting Δ vary among a basis for $\mathfrak{sl}(5)$ we get another set of trivial deformations given by the 24 vectors

$$({}^t\Delta H_1 + Q_1\Delta, {}^t\Delta H_2 + Q_2\Delta, {}^t\Delta H_3 + Q_3\Delta).$$

Now, given an n-dimensional family $\mathcal{F}$ centred at C, we construct a matrix

$$M_{\mathcal{F}} := \begin{pmatrix} H_{11} & H_{21} & H_{31} \\ H_{12} & H_{22} & H_{32} \\ \vdots & \vdots & \vdots \\ H_{1n} & H_{2n} & H_{3n} \\ H_1 & 0 & 0 \\ H_2 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & H_2 \\ 0 & 0 & H_3 \\ D_{21}H_1 + H_1 D_{12} & D_{21}H_2 + H_2 D_{12} & D_{21}H_3 + H_3 D_{12} \\ D_{31}H_1 + H_1 D_{13} & D_{31}H_2 + H_2 D_{13} & D_{31}H_3 + H_3 D_{13} \\ \vdots & \vdots & \vdots \\ D_{55}H_1 + H_1 D_{55} & D_{55}H_2 + H_2 D_{55} & D_{55}H_3 + H_3 D_{55} \end{pmatrix}$$

The first n rows are given by the family $\mathcal{F}$: they are tangent vectors at the central point $s = (H_1, H_2, H_3)$. The second set of 9 rows is given by the tangent vectors to the orbit of the GL(3)-action, and the last 24 rows are the tangent vectors to the orbits of the PGL(5)-action described above. We have chosen a vector space basis D_{ij} for $\mathfrak{sl}(5)$, for example $D_{ij} = \delta_{ij}$ for $i \neq j$ and $D_{ii} = \delta_{11} - \delta_{ii}$ for $1 < i \leq 4$.

The linear space generated by the rows of $M_{\mathcal{F}}$ is the span inside the tangent space to V of the three linear spaces tangent respectively to the given family and to each of the two orbits through s. To determine the dimension of this span we now need to compute the rank of $M_{\mathcal{F}}$.

Proposition 5.2. *Let C be a smooth complete intersection of three linearly independent quadrics in $\mathbb{P}^4$, and let $\mathcal{F}$ be an n-dimensional family of deformations of C as above. Suppose that $n \geq 12$. If the rank of $M_{\mathcal{F}}$ is maximal then the Kodaira–Spencer map of $\mathcal{F}$ at C is surjective.*

Proof. First observe that in the matrix $M_{\mathcal{F}}$ there are 45 columns, and under our assumptions there are at least 45 rows. When the rank of the matrix $M_{\mathcal{F}}$ is maximal the span of the three vector spaces, the two corresponding to trivial deformations and the one given by the family, is the whole of the tangent space to V at the point s. Thus we are guaranteed the existence of enough linearly independent deformations, namely 12, to fill the tangent space to $\mathcal{M}_5$. □

Corollary 5.3. *The map $\varrho\colon \mathfrak{Q} \to \mathcal{R}_5$ is dominant.*

Proof. This is now a straightforward computation of the rank of $M_{\mathcal{F}}$ in one particular case. We carried it out using *Macaulay*, with points defined over a finite field (we chose $\mathbb{F}_{101}$, for no special reason). This is enough because if the rank is generically maximal after reduction modulo p it is also maximal in characteristic 0.

We chose the test point of $\mathfrak{Q}(\mathbb{F}_{101})$ given by

$$\begin{aligned}
u_2 &= 19x_0^2 - 33x_0x_1 + 50x_1^2 - 13x_0x_2 + 50x_1x_2 - 15x_2^2\\
u_3 &= -2x_0^2x_1 - 35x_0x_1^2 - 18x_0^2x_2 - 8x_0x_1x_2 - 36x_1^2x_2 - 4x_0x_2^2 + 45x_1x_2^2\\
u_4 &= -38x_0^2x_1^2 - 32x_0^2x_1x_2 - 32x_0x_1^2x_2 - 6x_0^2x_2^2 - 38x_0x_1x_2^2 + 2x_1^2x_2^2.
\end{aligned}$$

We arrived at this by first selecting six points

$$P_0 = (0:0:0:1), \ldots, P_3 = (1:0:0:0),\ P_4 = (1:1:1:1),\ P_5 = (1:2:3:4) \in \mathbb{P}^3,$$

with ideals $\mathfrak{p}_0, \ldots, \mathfrak{p}_5$, to be the prescribed nodes of a quartic and then choosing at random a quartic $F \in \mathfrak{p}_0^2 \cap \cdots \cap \mathfrak{p}_5^2$. Then u_2, u_3 and u_4 are defined by $F = u_2x_3^2 + u_3x_3 + u_4$, and the 6-nodal quartic surface is given by $F = 0$.

Having chosen F at random one must check that it is suitably general, namely that X has no other singular points and that the singularity of X at each P_i is a simple node.

The 5-nodal plane sextic in this example is given by $u_3^2 = u_2u_4$, with nodes at

$$(0:0:1),\ (0:1:0),\ (1:0:0),\ (1:1:1) \text{ and } (1:2:3).$$

We construct the blowup of $\mathbb{P}^2$ in these five points by considering the linear system of cubics passing through them and from this we can easily compute the canonical curve $\widetilde{C}$ as the intersection of three quadrics H_1, H_2 and H_3. These at once give us the last 33 rows of $M_{\mathfrak{Q}}$. To compute the first 13 rows one must know the family $\mathfrak{Q}$, that is B_0, near X. We can obtain local coordinates on B_0 from the coordinates on $\mathbb{P}(I) \times \mathbb{P}^3$ by computing a Gröbner basis for the ideal of B_0. Then we compute the quadrics defining the canonical curve $\widetilde{C}_j$ exactly as we did for $\widetilde{C}$ for each first-order deformation X_j corresponding to a local coordinate t_j. These quadrics are $H_i + t_jH_{ij}$ (with a correct choice of coordinates) and we have computed $M_{\mathfrak{Q}}$. □

6. Conclusions

Our main result now follows easily from those of Sections 4 or 5.

Theorem 6.1. *The moduli space $\mathcal{R}_5$ of étale double covers of curves of genus five is unirational.*

Proof. This follows from Corollary 4.3 or Corollary 5.3. □

Theorem 6.1 provides a slightly different proof of a theorem of Clemens [1983].

Corollary 6.2. *$\mathcal{A}_4$ is unirational.*

This follows from Theorem 6.1 because the Prym map $p_5\colon \mathcal{R}_5 \to \mathcal{A}_4$ is dominant (see for instance [Beauville 1977a]). The original proof of Clemens also starts from quartic double solids: Clemens exhibits the general principally polarised abelian 4-fold as an intermediate Jacobian rather than a Prym variety. Note that since $\mathcal{R}_5$ has dimension 12 and $\mathcal{A}_4$ has dimension 10, the dominance of the intermediate Jacobian map from $\mathfrak{Q}$ to $\mathcal{A}_4$ does *not* imply the dominance of the map $\rho\colon \mathfrak{Q} \to \mathcal{R}_5$.

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