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In 1998, Han and Yim proved that the Hopf vector fields, namely, the unit Killing vector fields, are the unique unit vector fields on the unit sphere S^3 that define harmonic maps from S^3 to $(T^1S^3, \widetilde{G}_s)$, where $\widetilde{G}_s$ is the Sasaki metric. In this paper, by using a different method, we get an analogue of Han and Yim's theorem for a Riemannian three-manifold with constant sectional curvature $k \neq 0$. An immediate consequence is that there does not exist a unit vector field on the hyperbolic three-space that defines a harmonic map. We also extend this result for Riemannian $(2n+1)$-manifolds (M, g) of constant sectional curvature $k > 0$ with $\pi_1(M) \neq 0$.

1. Introduction

The existence and explicit construction of harmonic mappings between two given Riemannian manifolds (M, g) and (M', g') are two of the most fundamental problems of the theory of harmonic mappings. If M is compact and M' has nonpositive sectional curvature, then any smooth map from M to M' can be deformed into a harmonic map using the heat flow method [Eells and Sampson 1964]. However, there is no general existence theory of harmonic mappings if the target manifold does not satisfy the nonpositivity curvature condition. This fact makes it interesting to find harmonic maps defined by vector fields and unit vector fields [Abbassi et al. 2007; 2008; Benyounes et al. 2007b; 2007a; Ishihara 1979; Nouhaud 1977; Perrone 2003; 2005; Rukimbira 2002; Tsukada and Vanhecke 2001].

Let $\mathfrak{X}^1(M)$ be the set of all smooth unit vector fields on (M, g), which we suppose to be nonempty, or, equivalently, we suppose the Euler–Poincaré characteristic of M vanishes. Let $(T_1M, \widetilde{G}_s)$ be the unit tangent sphere bundle equipped with the Sasaki metric $\widetilde{G}_s$. A unit vector field $V \in \mathfrak{X}^1(M)$ determines a map between (M, g) and $(T_1M, \widetilde{G}_s)$, and the energy of V is defined as the energy of the corresponding map $V : (M, g) \to (T_1M, \widetilde{G}_s)$. Of course the first candidates in

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the study of the harmonicity of unit vector fields are the Killing vector fields: on the unit sphere S^{2m+1} equipped with the canonical metric, the unit Killing vector fields are exactly the Hopf vector fields; see [Wiegmink 1995]. Then, Han and Yim [1998] proved that a unit Killing vector field ξ determines a harmonic map from S^{2m+1} to $(T_1 S^{2m+1}, \widetilde{G}_s)$. The same result holds if one considers a Riemannian manifold (M, g) of constant sectional curvature $\kappa > 0$. In dimension three they showed the following interesting result.

Theorem A [Han and Yim 1998]. *The unit vector fields that define harmonic maps on the unit sphere S^3, with respect to the Sasaki metric $\widetilde{G}_s$, are precisely the Hopf vector fields or, equivalently, the unit Killing vector fields.*

Recently, a very large family of metrics G on TM, called Riemannian g-natural metrics, has been considered and studied [Abbassi and Sarih 2005; Abbassi and Calvaruso 2007]. This family of metrics, which includes the Sasaki metric G_s, the Cheeger–Gromol metric and other well-known Riemannian metrics on TM, depends on six arbitrary smooth real functions defined on $[0, +\infty)$ [Abbassi and Sarih 2005]. The restrictions $\widetilde{G}$ of such metrics to the tangent sphere bundle $T_1 M$ possess a simpler form and globally depend on four real parameters a, b, c and d satisfying some inequalities (the parameters $a = 1$ and $b = c = d = 0$ define the Sasaki metric $\widetilde{G}_s$). In [Abbassi et al. 2008] the harmonicity of the map $M \to T_1 M$ defined by a unit vector field was studied in the case when the unit tangent sphere bundle $T_1 M$ is equipped with an arbitrary Riemannian g-natural metric $\widetilde{G}$.

Han and Yim [1998] showed their Theorem A by using the property that the Hopf fibration $S^3 \to \mathbb{CP}^1$ is the unique fibration of the round three-sphere by great circles such that the fibres are parallel (in the sense of having constance distance from another) [Escobales 1975]. In this paper, by using a different method, we extend Theorem A by replacing the unit sphere S^3 by a Riemannian three-manifold of constant sectional curvature $k \neq 0$ and the Sasaki metric $\widetilde{G}_s$ by an arbitrary Riemannian g-natural metric $\widetilde{G}_{a,d,c}$ that is a deformation depending on three real parameters of the Sasaki metric $\widetilde{G}_s$; such a deformation preserves the property (of the Sasaki metric) that horizontal and vertical lifts are orthogonal. We do not assume that M is compact. So, in particular, M may be an open (connected) subset of the sphere S^3. More precisely we get this:

Theorem 1.1. *Let (M, g) be a Riemannian three-manifold of constant sectional curvature $\kappa \neq 0$ and $T_1 M$ its unit tangent sphere bundle equipped with a Riemannian g-natural metric $\widetilde{G}_{a,d,c}$ with $d \neq -\kappa a$ (and $b = 0$). Let ξ be in $\mathfrak{X}^1(M)$. Then $\xi : (M, g) \to (T_1 M, \widetilde{G}_{a,d,c})$ is a harmonic map if and only if ξ is Killing and $\kappa > 0$.*

Theorem 1.1 has an immediate consequence:

Corollary 1.2. *Han and Yim's theorem is invariant under a three-parameter deformation of the Sasaki metric on $T_1 M$.*

In the case of the hyperbolic space $H^n(-k)$ for $k > 0$, it is an open question whether some unit vector field exists (of course, non-Killing) that defines a harmonic map from $H^n(-k)$ to $(T_1 H^n(-k), \widetilde{G}_s)$. From Theorem 1.1 we have the following nonexistence result in dimension three.

Corollary 1.3. *Let $H^3(-k)$ be the hyperbolic three-space. Then there does not exist a unit vector field that defines a harmonic map between the Riemannian manifolds $H^3(-k)$ and $(T_1 H^3(-k), \widetilde{G}_s)$. Such a result is invariant under a three-parameter deformation of the Sasaki metric on $T_1 H^3(-k)$.*

On a flat three-space and on the sphere S^3 equipped with a metric of nonconstant sectional curvature we give examples of unit vector fields that are not Killing but define harmonic maps; see Example 3.1.

In their paper Han and Yim [Han and Yim 1998, page 84] posed the question of whether Theorem A is true for higher dimensions spheres. Here, we consider the following question (which generalizes the Han and Yim's question): *Let (M, g) be a real space form of constant sectional curvature $\kappa > 0$. Are the unit Killing vector fields on M the only unit vector fields that define harmonic maps from (M, g) to $(T_1 M, \widetilde{G})$?* If M is not homeomorphic to the sphere, we get a positive answer:

Theorem 1.4. *Let (M, g) be a real space form of constant positive sectional curvature κ, with $\dim M = 2m+1$. Suppose that M is not homeomorphic to the sphere S^{2m+1}. Let $(T^1 M, \widetilde{G})$ be its unit tangent sphere bundle equipped with a Riemannian g-natural metric $\widetilde{G} = \widetilde{G}_{a,b,c,d}$ with $b \neq 0$ and $d \neq -ka$. Let $\xi \in \mathfrak{X}^1(M)$. Then*

(i) *$\xi : (M, g) \to (T^1 M, \widetilde{G})$ is a harmonic map if and only if ξ is Killing*;

(ii) *if ξ is a solenoidal (that is, a divergence free) unit vector field, then*

$$\xi : (M, g) \to (T^1 M, \widetilde{G})$$

is a harmonic map if and only if ξ has minimum energy $E_{\widetilde{G}} : \mathfrak{X}^1(M) \to \mathbb{R}$.

Remark 1.5. Brito and Salvai [2004, Proposition 1] proved that if M is a compact Riemannian manifold and ξ is a unit Killing vector field eigenvector of the Ricci operator and is of minimum Ricci curvature, then ξ has minimum energy among all solenoidal unit vector fields.

In the final Section 5, we remark that the main results of [Brito 2000; Perrone 2008], related to the energy restricted to $\mathfrak{X}^1(M)$, are invariant under a four-parameter deformation of the Sasaki metric on $T_1 M$.

2. Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold and ∇ its Levi-Civita connection. We denote by R the Riemannian curvature tensor of (M, g) with the sign convention

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z \quad \text{for all } X, Y, Z \in \mathfrak{X}(M).$$

We denote by Ric the Ricci tensor of type $(0, 2)$, by Q the corresponding endomorphism field, and by r the scalar curvature.

At any point (x, u) of the tangent bundle TM, the tangent space of TM splits into horizontal and vertical subspaces with respect to ∇:

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}.$$

The *Riemannian g-natural metrics* form a large family of Riemannian metrics on TM. These metrics depend on several smooth functions from $\mathbb{R}^+ = [0, +\infty)$ to $\mathbb{R}$, and, as their name suggests, they arise from a natural construction starting from a Riemannian metric g over M; see [Abbassi and Sarih 2005; Kolář et al. 1993]. Given an arbitrary g-natural metric G on the tangent bundle TM of a Riemannian manifold (M, g), there are six smooth functions $\alpha_i,\ \beta_i : \mathbb{R}^+ \to \mathbb{R}$ for $i = 1, 2, 3$ such that for every $u, X, Y \in M_x$, we have

$$\text{(2-1)} \qquad \begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1+\alpha_3)(r^2) g_x(X,Y) \\ \qquad\qquad\qquad\qquad + (\beta_1+\beta_3)(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) \\ \qquad\qquad\qquad = \alpha_2(r^2) g_x(X,Y) + \beta_2(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2) g_x(X,Y) + \beta_1(r^2) g_x(X,u) g_x(Y,u), \end{cases}$$

where $r^2 = g_x(u, u)$. Put

$$\phi_i(t) = \alpha_i(t) + t\beta_i(t), \qquad \alpha(t) = \alpha_1(t)(\alpha_1+\alpha_3)(t) - \alpha_2^2(t),$$
$$\phi(t) = \phi_1(t)(\phi_1+\phi_3)(t) - \phi_2^2(t)$$

for all $t \in \mathbb{R}^+$. Then, a g-natural metric G on TM is Riemannian if and only if the inequalities

$$\text{(2-2)} \qquad \begin{aligned} \alpha_1(t) > 0, &\qquad \phi_1(t) > 0, \\ \alpha(t) > 0, &\qquad \phi(t) > 0 \end{aligned}$$

hold for all $t \in \mathbb{R}^+$. Notice that the Sasaki metric G_s, the Cheeger–Gromoll metric and other Riemannian metrics on TM belong to the class of Riemannian g-natural metrics on TM for which horizontal and vertical distributions are mutually orthogonal (that is, $\alpha_2 = \beta_2 = 0$).

Next, the *unit tangent sphere bundle* over a Riemannian manifold (M, g) is the hypersurface $T_1M = \{(x, u) \in TM \mid g_x(u, u) = 1\}$. The tangent space of T_1M at a point $(x,u) \in T_1M$ is given by

$$(T_1M)_{(x,u)} = \{X^h + Y^v : X \in M_x,\ Y \in \{u\}^\perp \subset M_x\}. \tag{2-3}$$

We call *g-natural metrics on* T_1M the restrictions of g-natural metrics of TM to its hypersurface T_1M. These metrics possess a simpler form. Precisely, taking in account of (2-3), a Riemannian g-natural metric $\widetilde{G}$ on T_1M is induced by a Riemannian g-natural G on TM of the form

$$\begin{cases} G_{(x,u)}(X_1^h, X_2^h) = (\alpha_1+\alpha_3)(r^2)g_x(X_1,X_2) \\ \qquad\qquad\qquad\qquad + (\beta_1+\beta_3)(r^2)g_x(X_1,u)g_x(X_2,u), \\ G_{(x,u)}(X_1^h, Y_1^v) = \alpha_2(r^2)g_x(X_1,Y_1), \\ G_{(x,u)}(Y_1^v, Y_2^v) = \alpha_1(r^2)g_x(Y_1,Y_2), \end{cases} \tag{2-4}$$

for all $X_1, X_2 \in M_x$ and $Y_1, Y_2 \in \{u\}^\perp$; see [Abbassi and Sarih 2005; Abbassi and Calvaruso 2007] and references therein. In other words, $\widetilde{G}$ on T_1M is necessarily induced by a Riemannian g-natural G on TM of the form (2-1) with

$$\alpha_1 = a, \quad \alpha_2 = b, \quad \alpha_3 = c, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 = \beta, \tag{2-5}$$

where a, b, c are three real constants and $\beta : [0,\infty) \to \mathbb{R}$ is a smooth function. Such a metric $\widetilde{G}$ on T_1M only depends on the value $d := \beta(1)$ of β at 1. From (2-2) and (2-5), it follows that $\widetilde{G}$ is Riemannian if and only if the constants a, b, c and d satisfy the inequalities

$$a > 0, \quad \alpha := a(a+c) - b^2 > 0, \quad \phi := a(a+c+d) - b^2 > 0. \tag{2-6}$$

We denote by $\widetilde{G}_{a,b,c,d}$ such a Riemannian g-natural metric on T_1M and define $\widetilde{G}_{a,c,d} = \widetilde{G}_{a,0,c,d}$. We can consider $\widetilde{G}_{a,b,c,d}$ to be a deformation on four parameters of the Sasaki metric $\widetilde{G}_s = \widetilde{G}_{1,0,0,0}$. It should be noted that, by (2-4), horizontal and vertical lifts are orthogonal with respect to $\widetilde{G}$ if and only if $b = 0$.

Now let (M, g) be a compact Riemannian manifold of dimension n. A unit vector field V defines a mapping from (M, g) to T_1M equipped with a Riemannian g-natural metric $\widetilde{G}$. The energy functional $E_{\widetilde{G}} : \mathfrak{X}^1(M) \to \mathbb{R}$ is defined by $E_{\widetilde{G}}(V) = \int_M e(V)\,dv_g$, where $e(V)$ is the energy density of V and is given by [Abbassi et al. 2008]

$$2e(V) = n(a+c) + d + a\|\nabla V\|^2 + 2b \operatorname{div} V.$$

So, integrating over M we get

$$E_{\widetilde{G}}(V) = \tfrac{1}{2}(n(a+c)+d)\operatorname{vol}(M,g) + \frac{a}{2}\int_M \|\nabla V\|^2 dv_g. \tag{2-7}$$

In [Abbassi et al. 2008], it was proved that $V:(M,g)\to(T_1M,\widetilde{G})$ is a harmonic map if and only if

$$\text{(2-8)}\qquad \begin{aligned}&\bar{\Delta}V \text{ is collinear to } V, \text{ and}\\ &b\,QV + a\operatorname{tr}(R(\nabla.V,V)\cdot) = (b\|\nabla V\|^2 - d\operatorname{div}V)V + d\nabla_V V,\end{aligned}$$

where $\bar{\Delta}V = -\operatorname{tr}\nabla^2 V$ is the *rough Laplacian* at V. Such conditions have a tensorial character; hence they may also be considered to define a harmonic map on noncompact manifolds. In the special case of the Sasaki metric $\widetilde{G}_s$, that is, $a=1$ and $b=c=d=0$, (2-8) gives a result of [Han and Yim 1998]. If $V\in\mathfrak{X}^1(M)$ is a unit Killing vector field, then [Poor 1981, page 169]

$$\nabla_V V = 0,\quad \operatorname{div}V = 0,\quad \bar{\Delta}V = QV,$$

and, V being a unit vector field, $g(QV,V) = g(\bar{\Delta}V,V) = \|\nabla V\|^2$. Then, since $a>0$, (2-8) gives this:

Proposition 2-9. *If $V\in\mathfrak{X}^1(M)$ is a unit Killing vector field, then the harmonicity of the map $V:(M,g)\to(T_1M,\tilde{G}_s)$ implies that of $V:(M,g)\to(T_1M,\widetilde{G})$ for any $\widetilde{G}$.*

3. Proof of Theorem 1.1

Let (M,g) be a Riemannian three-manifold of constant sectional curvature $\kappa\neq 0$ and T_1M its unit tangent sphere bundle equipped with a Riemannian g-natural metric $\widetilde{G}$.

Assume that ξ is a unit Killing vector field. Since M has constant sectional curvature, $\xi:(M,g)\to(T_1M,\tilde{G}_s)$ is a harmonic map, and hence, by Proposition 2-9, we obtain that $\xi:(M,g)\to(T_1M,\widetilde{G})$ is a harmonic map for any $\widetilde{G}$.

Conversely, we suppose that $\xi:(M,g)\to(T^1M,\widetilde{G}_{a,d,c})$ is a harmonic map, where $\widetilde{G}_{a,d,c}$ is a Riemannian g-natural metric with $d\neq-\kappa a$ (and $b=0$). By (2-8), we obtain

$$\text{(3-1a)}\qquad \bar{\Delta}\xi = \|\xi\|^2\xi,$$

$$\text{(3-1b)}\qquad \operatorname{tr}(R(\nabla.\xi,\xi)\cdot) = -\frac{d}{a}((\operatorname{div}\xi)\xi - \nabla_\xi\xi).$$

Now $\operatorname{tr}(R(\nabla.\xi,\xi)\cdot) = k((\operatorname{div}\xi)\xi-\nabla_\xi\xi)$ since M has constant sectional curvature k. Then $k\neq -d/a$ and condition (3-1a) imply

$$\text{(3-2)}\qquad \operatorname{div}\xi = 0 \quad\text{and}\quad \nabla_\xi\xi = 0.$$

Put $\tau := \mathscr{L}_\xi g$, where $\mathscr{L}_\xi$ denotes the Lie derivative, that is,

$$\tau(X,Y) = g(\nabla_X\xi,Y) + g(X,\nabla_Y\xi).$$

Since τ is a symmetric tensor of type $(0, 2)$, we can define the corresponding symmetric $(1, 1)$ tensor h through $\tau(X, Y) = g(hX, Y)$. Then, from (3-2),

$$g(h\xi, Y) = \tau(\xi, Y) = g(\nabla_\xi \xi, Y) + g(\xi, \nabla_Y \xi) = \tfrac{1}{2} Y(\|\xi\|^2) = 0,$$

and hence $h\xi = 0$. So, we can consider a local orthonormal basis $\{e_1, e_2, e_3 = \xi\}$ of eigenvectors of h, with $h\xi = 0$, $he_1 = \lambda_1 e_1$ and $he_2 = \lambda_2 e_2$. Since

$$\begin{aligned} \operatorname{div} \xi &= g(\nabla_\xi \xi, \xi) + g(\nabla_{e_1} \xi, e_1) + g(\nabla_{e_2} \xi, e_2) \\ &= \tfrac{1}{2} g(he_1, e_1) + \tfrac{1}{2} g(he_2, e_2) = \tfrac{1}{2}(\lambda_1 + \lambda_2), \end{aligned}$$

from (3-2) we get $\lambda_2 = -\lambda_1$. Therefore ξ is Killing if and only if $\lambda_1 = 0$. Put

$$\begin{aligned} f_1 &= \tfrac{1}{2}\lambda_1, \quad & f_2 &= g(\nabla_{e_1}\xi, e_2), \quad & f_3 &= g(\nabla_{e_1} e_2, e_1), \\ & & f_4 &= g(\nabla_{e_2} e_2, e_1), \quad & f_5 &= g(\nabla_\xi e_1, e_2). \end{aligned}$$

Then, we have the following list of covariant derivatives:

$$\begin{aligned} \nabla_{e_1}\xi &= f_1\, e_1 + f_2\, e_2, \\ \nabla_\xi \xi &= 0, \\ \nabla_{e_2}\xi &= g(\nabla_{e_2}\xi, e_1) e_1 + g(\nabla_{e_2}\xi, e_2) e_2 \\ &= (g(he_1, e_2) - g(\nabla_{e_1}\xi, e_2)) e_1 + \tfrac{1}{2} g(he_2, e_2) e_2 = -f_2\, e_1 - f_1\, e_2, \\ \nabla_{e_1} e_1 &= g(\nabla_{e_1} e_1, \xi)\xi + g(\nabla_{e_1} e_1, e_2) e_2 \\ &= -g(\nabla_{e_1}\xi, e_1)\xi - g(\nabla_{e_1} e_2, e_1) e_2 = -f_1\, \xi - f_3\, e_2, \\ \nabla_{e_2} e_2 &= g(\nabla_{e_2} e_2, \xi)\xi + g(\nabla_{e_2} e_2, e_1) e_1 \\ &= -g(\nabla_{e_2}\xi, e_2)\xi + f_4\, e_1 = -\tfrac{1}{2} g(he_2 e_2)\xi + f_4\, e_1 = f_1\, \xi + f_4\, e_1, \\ \nabla_{e_1} e_2 &= g(\nabla_{e_1} e_2, \xi)\xi + g(\nabla_{e_1} e_2, e_1) e_1 \\ &= -g(\nabla_{e_1}\xi, e_2)\xi + f_3\, e_1 = -f_2\, \xi + f_3\, e_1, \\ \nabla_{e_2} e_1 &= g(\nabla_{e_2} e_1, \xi)\xi + g(\nabla_{e_2} e_1, e_2) e_2 = -g(\nabla_{e_2}\xi, e_1)\xi - g(\nabla_{e_2} e_2, e_1) e_2 \\ &= -(g(he_1, e_2) - g(\nabla_{e_1}\xi, e_2))\xi - f_4 e_2 = f_2\, \xi - f_4\, e_2, \\ \nabla_\xi e_1 &= g(\nabla_\xi e_1, \xi)\xi + g(\nabla_\xi e_1, e_2) e_2 = -g(\nabla_\xi \xi, e_1)\xi + f_5\, e_2 = f_5\, e_2, \\ \nabla_\xi e_2 &= g(\nabla_\xi e_2, \xi)\xi + g(\nabla_\xi e_2, e_1) e_1 = -g(\nabla_\xi \xi, e_2)\xi - f_5\, e_1 = -f_5\, e_1. \end{aligned}$$

Moreover,

$$\begin{aligned} &[e_1, \xi] = f_1 e_1 + (f_2 - f_5) e_2, \\ &[e_2, \xi] = (f_5 - f_2) e_1 - f_1 e_2, \qquad [e_1, e_2] = -2 f_2 \xi + f_3 e_1 + f_4 e_2. \end{aligned}$$

By using the list of covariant derivatives, we get

$$
\tag{3-3}
\begin{cases}
R(e_1,\xi)\xi = -\nabla_{e_1}\nabla_\xi\xi + \nabla_\xi\nabla_{e_1}\xi + \nabla_{[e_1,\xi]}\xi \\
\qquad = (f_1^2 - f_2^2 + \xi(f_1)g)e_1 + (2f_1f_5 + \xi(f_2))e_2, \\
R(e_2,\xi)\xi = -\nabla_{e_2}\nabla_\xi\xi + \nabla_\xi\nabla_{e_2}\xi + \nabla_{[e_2,\xi]}\xi \\
\qquad = (2f_1f_5 - \xi(f_2))e_1 + (f_1^2 - f_2^2 - \xi(f_1))e_2, \\
R(e_1,e_2)\xi = -\nabla_{e_1}\nabla_{e_2}\xi + \nabla_{e_2}\nabla_{e_1}\xi + \nabla_{[e_1,e_2]}\xi \\
\qquad = (e_1(f_2) + e_2(f_1) + 2f_1f_3)e_1 \\
\qquad\qquad + (e_1(f_1) + e_2(f_2) - 2f_1f_4)e_2, \\
R(e_1,e_2)e_1 = -\nabla_{e_1}\nabla_{e_2}e_1 + \nabla_{e_2}\nabla_{e_1}e_1 + \nabla_{[e_1,e_2]}e_1 \\
\qquad = -(e_1(f_2) + e_2(f_1) + 2f_1f_3)\xi \\
\qquad\quad + (f_1^2 - f_2^2 + e_1(f_4) - e_2(f_3) - 2f_2f_5 - f_3^2 - f_4^2)e_2.
\end{cases}
$$

Besides, $R(X,Y)Z = \kappa(g(X,Z)Y - g(Y,Z)X)$ gives

$$
\tag{3-4}
\begin{aligned}
R(e_1,\xi,e_1,\xi) = R(e_2,\xi,e_2,\xi) = R(e_1,e_2,e_1,e_2) = \kappa, \\
R(e_1,\xi)e_2 = R(e_1,e_2)\xi = R(e_2,\xi)e_1 = 0.
\end{aligned}
$$

From (3-3) and (3-4), we obtain

$$
\tag{3-5}
f_2^2 - f_1^2 - \xi(f_1) = \kappa, \qquad 2f_1f_5 + \xi(f_2) = 0,
$$

$$
\tag{3-6}
f_2^2 - f_1^2 + \xi(f_1) = \kappa, \qquad 2f_1f_5 - \xi(f_2) = 0,
$$

$$
\tag{3-7}
e_1(f_2) + e_2(f_1) + 2f_1f_3 = 0, \qquad e_1(f_1) + e_2(f_2) - 2f_1f_4 = 0,
$$

$$
\tag{3-8}
f_1^2 - f_2^2 + e_1(f_4) - e_2(f_3) - 2f_2f_5 - f_3^2 - f_4^2 = \kappa.
$$

From (3-5) and (3-6), we get

$$
\tag{3-9}
f_2^2 - f_1^2 = \kappa, \quad f_1f_5 = 0, \quad \xi(f_1) = \xi(f_2) = 0.
$$

Using (3-9), (3-8) becomes

$$
\tag{3-10}
e_1(f_4) - e_2(f_3) - 2f_2f_5 - f_3^2 - f_4^2 = 2k.
$$

Now we compute $\bar{\Delta}\xi$. Since

$$
\begin{aligned}
\nabla_{e_1}\nabla_{e_1}\xi &= \nabla_{e_1}(f_1e_1 + f_2e_2) = e_1(f_1)e_1 + f_1\nabla_{e_1}e_1 + e_1(f_2)e_2 + f_2\nabla_{e_1}e_2 \\
&= -(f_1^2 + f_2^2)\xi + (e_1(f_1) + f_2f_3)e_1 + (e_1(f_2) - f_1f_3)e_2, \\
\nabla_{e_2}\nabla_{e_2}\xi &= \nabla_{e_2}(-f_2e_1 - f_1e_2) = -e_2(f_2)e_1 - f_2\nabla_{e_2}e_1 - e_2(f_1)e_2 - f_1\nabla_{e_2}e_2 \\
&= -(f_1^2 + f_2^2)\xi - (e_2(f_2) + f_1f_4)e_1 - (e_2(f_1) - f_2f_4)e_2,
\end{aligned}
$$

and

$$-\nabla_{\nabla_{e_1}e_1}\xi = f_3\nabla_{e_2}\xi = -f_3(f_2\, e_1 + f_1\, e_2),$$
$$-\nabla_{\nabla_{e_2}e_2}\xi = -f_4\nabla_{e_1}\xi = -f_4(f_1\, e_1 + f_2\, e_2),$$

we have

$$\begin{aligned} -\bar{\Delta}\xi &= \operatorname{tr}\nabla^2\xi \\ &= \nabla_{e_1}\nabla_{e_1}\xi + \nabla_{e_2}\nabla_{e_2}\xi + \nabla_{\xi}\nabla_{\xi}\xi - \nabla_{\nabla_{e_1}e_1}\xi - \nabla_{\nabla_{e_2}e_2}\xi - \nabla_{\nabla_{\xi}\xi}\xi \\ &= -\|\nabla\xi\|^2\,\xi + (e_1(f_1) - e_2(f_2) - 2f_1f_4)e_1 \\ &\qquad + (e_1(f_2) - e_2(f_1) - 2f_1f_3)e_2, \end{aligned} \tag{3-11}$$

where $\|\nabla\xi\|^2 = 2\big(f_1^2 + f_2^2\big)$. Because of (3-1a), from (3-11) we get

$$e_1(f_1) - e_2(f_2) = 2f_1f_4 \quad \text{and} \quad e_1(f_2) - e_2(f_1) = 2f_1f_3. \tag{3-12}$$

Combining (3-7) and (3-12), we have

$$e_1(f_2) = e_2(f_2) = 0.$$

Moreover, (3-9) implies $\xi(f_2) = 0$ and $f_1^2 = f_2^2 - \kappa$. So f_1 and f_2 are constant. If $f_1 = \text{const} \neq 0$, from (3-5) and (3-7) we have $f_3 = f_4 = f_5 = 0$ which, by (3-8), imply $\kappa = 0$. Since $\kappa \neq 0$, we conclude that $f_1 = 0$ and hence ξ is Killing and, by (3-6), we have $\kappa = f_2^2 > 0$. □.

Example 3.1 (Non-Killing unit vector fields that define harmonic maps). Let $\mathfrak{g}$ be a three-dimensional Lie algebra. Introduce a basis (e_1, e_2, e_3) for $\mathfrak{g}$, and for real numbers λ_1, λ_2 and λ_3, define the Lie bracket by

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3. \tag{3-13}$$

On the associated Lie group $\mathcal{G}$, define a metric g by left translation of the basis (e_1, e_2, e_3), taken as orthogonal at the identity. Suppose $\mathcal{G}$ to be simply connected; otherwise we consider its universal covering. Let $(\vartheta^1, \vartheta^2, \vartheta^3)$ be the metric dual one-forms of (e_1, e_2, e_3). If $\lambda_i \neq 0$, then ϑ^i is a contact form, that is, $\vartheta^i \wedge \mathrm{d}\vartheta^i \neq 0$, and e_i is the corresponding Reeb vector field. Assuming $\lambda_1 = 2$ and defining φ by $\varphi(e_1) = 0$, $\varphi(e_2) = e_3$ and $\varphi(e_3) = -e_2$, we have $d\vartheta^1 = g(\cdot\,, \varphi\,\cdot)$. Then $(\eta = \vartheta^1, g, \varphi, \xi = e_1)$ is a contact metric structure on $\mathcal{G}$; see [Blair 2002]. Such a structure is a (κ, μ)-structure, and it is Sasakian, that is, ξ is Killing if and only if $\lambda_2 = \lambda_3$; see [Perrone 1998], which also gives that

(a) if $\lambda_2 > 0$, $\lambda_3 > 0$ and $\lambda_2 \neq \lambda_3$, then the group $\mathcal{G}$ is the three-sphere group $SU(2)$, ξ is not Killing, and the metric has nonconstant sectional curvature;

(b) if $\lambda_2 = 0$ and $\lambda_3 > 0$, then $\mathcal{G}$ is $\widetilde{E}_2$ the universal covering of the group of rigid motions of Euclidean 2-space, ξ is not Killing, and the Ricci tensor is given by

$\mathrm{Ric}_{ij} = 0$ for $i \neq j$, $\mathrm{Ric}_{11} = -\mathrm{Ric}_{33} = (4-\lambda_3^2)/2$ and $\mathrm{Ric}_{22} = -(\lambda_3 - 2)^2/2$; hence the scalar curvature $r \leq 0$, and the metric is flat if and only if $\lambda_3 = 2$.

In both cases from [Perrone 2003, Theorem 1.1], we have that ξ defines a harmonic map between the Riemannian manifolds $(\mathcal{G}, g)$ and $(T_1\mathcal{G}, \widetilde{G}_s)$, that is, $Q\xi$ is collinear to ξ and $\mathrm{tr}(R(\nabla.\xi, \xi)\cdot) = 0$. On the other hand, in [Abbassi et al. 2008, Theorem 7] we proved this:

Theorem 3.2. *Let (M^{2m+1}, η, g) be a contact metric manifold and $\widetilde{G}$ an arbitrary Riemannian g-natural metric on T_1M. Then $\xi : (M, g) \to (T_1M, \widetilde{G})$ is a harmonic map if and only if*

$$a\,\mathrm{tr}(R(\nabla.\xi, \xi)\cdot) = -2b(\|\nabla\xi\|^2 - 2m)\xi \quad \textit{and} \quad Q\xi \textit{ is collinear to } \xi.$$

Since $\|\nabla\xi\|^2 - 2m = 0$ if and only if ξ is Killing [Blair 2002, Lemma 6.2, page 67], from Theorem 3.2 we deduce that $\xi : (\mathcal{G}, g) \to (T_1\mathcal{G}, \widetilde{G})$ defines a harmonic map if and only if $b = 0$. Thus $\xi : (\mathcal{G}, g) \to (T_1\mathcal{G}, \widetilde{G}_{a,d,c})$ is a harmonic map.

4. Proof of Theorem 1.4

Let (M, g) be a real space form of positive constant sectional curvature $\kappa > 0$, with $\pi_1(M) \neq 0$. Then (M, g) is isometric to the spherical space form $(S^{2m+1}/\Gamma, g)$, where $\Gamma \neq \{Id\}$ is a finite subgroup of $O(2m+2)$ in which only the identity element has $+1$ as an eigenvalue, and g is the Riemannian metric on the quotient space S^{2m+1}/Γ induced by the canonical metric.

We first prove Theorem 1.4(i). If ξ is a unit Killing vector field on M, then as in the 3-dimensional case, $\xi : (M, g) \to (T_1M, \tilde{G}_s)$ is a harmonic map and, by Proposition 2-9, $\xi : (M, g) \to (T_1M, \widetilde{G})$ is a harmonic map for any $\widetilde{G}$.

Vice versa, let V be in $\mathfrak{X}^1(M)$. Suppose that $V : (M, g) \to (T^1M, \widetilde{G})$ is a harmonic map, where $\widetilde{G}$ is a Riemannian g-natural metric on T^1M with $b \neq 0$ and $d + ak \neq 0$. Then, by (2-8), $\bar{\Delta}V = \|\nabla V\|^2 V$ and

$$bQV = -a(\mathrm{tr}\, R(\nabla.V, V)\cdot) + b\|\nabla V\|^2 V + d(\nabla_V V - (\mathrm{div}\, V)V). \tag{4-1}$$

Since M has constant sectional curvature κ, (4-1) becomes

$$b(2m\kappa - \|\nabla V\|^2)V = (d + a\kappa)(\nabla_V V - (\mathrm{div}\, V)V). \tag{4-2}$$

This formula implies $b(\|\nabla V\|^2 - 2m\kappa) = (d + a\kappa)\,\mathrm{div}\, V$, and hence

$$b\int_M (\|\nabla V\|^2 - 2m\kappa)dv_g = 0. \tag{4-3}$$

Recall that, if $f : (M, g) \to (N, h)$ is a harmonic map, the Hessian form of the energy E at f is defined by the second variation formula [Smith 1975]:

$$(\text{Hess } E)_f(X, X) = \frac{d^2}{dt^2} E(f_t)\Big|_{t=0} = \int_M h(X, J_f X) dv_g, \tag{4-4}$$

where X is a vector field along f. The operator J_f, called the *Jacobi operator* of f, is a second order self adjoint elliptic differential operator acting on the space $\Gamma(f^{-1}TN)$ of the vector fields along f, and is defined by

$$J_f X = \bar{\Delta}_f X - \text{Ric}_f X. \tag{4-5}$$

The operator $\bar{\Delta}_f$, called the *rough Laplacian* along f, is defined by

$$\bar{\Delta}_f X = -\sum_{i=1}^{n} \big(\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} X - \overline{\nabla}_{\nabla_{e_i} e_i} X\big) \quad \text{for } X \in \Gamma(f^{-1}TN),$$

where $\overline{\nabla}$ is the connection (on the vector bundle $f^{-1}TN$) induced by the Levi-Civita connection of (N, h), and $\{e_i\}_{i=1,\dots,n}$ is a local orthonormal frame on M. Moreover, denoting by R_h the curvature tensor of (N, h),

$$\text{Ric}_f X = \sum_{i=1}^{n} R_h(f_* e_i, X) f_* e_i. \tag{4-6}$$

A harmonic map f is said to be *stable* if $(\text{Hess } E)_f$ is semidefinite positive or, equivalently, if the eigenvalues of the Jacobi operator are nonnegative. The identity map $Id : (M, g) \to (M, g)$ is a trivial example of a harmonic map. From (4-4)–(4-6) we readily deduce that the second variation formula of the energy for *Id* is given by

$$(\text{Hess } E)_{Id}(X, X) = \int_M g(J_{Id} X, X) dv_g = \int_M g(\bar{\Delta} X - QX, X) dv_g \tag{4-7}$$

for $X \in \mathfrak{X}(M)$. Since $J_{Id} V = \bar{\Delta} V - QV$, we get

$$g(J_{Id} V, V) = g(\bar{\Delta} V, V) - g(QV, V) = \|\nabla V\|^2 - 2mk.$$

Then, from (4-3) we have

$$b \int_M g(J_{Id} V, V) dv_g = b \int_M \big(\|\nabla V\|^2 - 2mk\big) dv_g = 0. \tag{4-8}$$

Smith [1975] proved that *if (M, g) is a compact Einstein manifold of dimension n, then Id is stable if and only if* $\lambda_1 \geq 2r/n$, where λ_1 is the first eigenvalue of the Laplace–Beltrami operator acting on functions and r is the scalar curvature. On the other hand, Urakawa [1987, page 572] proved that the first eigenvalue λ_1 of the Laplace–Beltrami operator on $(S^{2m+1}/\Gamma, g)$, when $\Gamma \neq \{Id\}$, is bigger than or equal to $4m\kappa$, that is, in such case the identity map Id is stable. Therefore, $g(J_{Id}X, X) \geq 0$ for any $X \in \mathfrak{X}(M)$. Now take $X = V$ and $b \neq 0$. Then (4-8) gives

$$\|\nabla V\|^2 = 2mk = g(QV, V). \tag{4-9}$$

Moreover, (4-2) becomes $(d + a\kappa)(\nabla_V V - (\operatorname{div} V)V) = 0$, from which, since $\nabla_V V \perp V$ and $(d + a\kappa) \neq 0$, we have

$$\nabla_V V = 0 \quad \text{and} \quad \operatorname{div} V = 0. \tag{4-10}$$

From (4-9) and (4-10), applying [Abbassi et al. 2008, Section 6, Proposition 1], we deduce that V is Killing.

We now prove Theorem 1.4(ii). Since Id is stable, from (4-7) we get

$$(\operatorname{Hess} E)_{Id}(X, X) = \int_M \big(\|\nabla X\|^2 - g(QX, X)\big) dv_g \geq 0 \tag{4-11}$$

for all $X \in \mathfrak{X}(M)$. Then, applying (2-7) and (4-11), the energy $E_{\widetilde{G}} : \mathfrak{X}^1(M) \to \mathbb{R}$ satisfies

$$\begin{aligned} E_{\widetilde{G}}(V) &= \tfrac{1}{2}((2m+1)(a+c)+d)\operatorname{vol}(M, g) + \frac{a}{2}\int_M \|\nabla V\|^2 dv_g \\ &\geq \tfrac{1}{2}((2m+1)(a+c)+d+2ma\kappa)\operatorname{vol}(M, g) \end{aligned} \tag{4-12}$$

for all $V \in \mathfrak{X}^1(M)$. Let ξ be a solenoidal unit vector field (that is, $\operatorname{div}\xi = 0$). If $\xi : (M, g) \to (T_1M, \widetilde{G})$ is a harmonic map, then ξ is Killing by Theorem 1.4(i), and hence $\|\nabla\xi\|^2 = g(Q\xi, \xi) = 2m\kappa$; see for example [Poor 1981, page 169]. Then, by (4-12), we have $E_{\widetilde{G}}(V) \geq E_{\widetilde{G}}(\xi)$ for all $V \in \mathfrak{X}^1(M)$. Vice versa, let ξ be a unit vector field that minimizes the energy, that is,

$$E_{\widetilde{G}}(\xi) = \tfrac{1}{2}((2m+1)(a+c)+d+2ma\kappa)\operatorname{vol}(M, g).$$

Then, by (4-12), $\int_M \|\nabla\xi\|^2 dv_g = 2m\kappa\operatorname{vol}(M, g)$, and thus, by (4-11), we get $(\operatorname{Hess} E)_{Id}(\xi, \xi) = 0$. Since Id is stable, we can expand ξ into the infinite sum $\xi = \sum_{i=1}^{\infty} E_i$ with $J_{Id}E_i = a_iE_i$ and $\int_M g(E_i, E_j)dv_g = 0$ for all $i \neq j$, where $a_i \geq 0$. Then we have $J_{Id}\xi = \sum_{i=p+1}^{\infty} a_iE_i$, where $p = \dim\ker J_{Id}$ and $a_i > 0$ for all $i \geq p+1$. Thus

$$0 = (\operatorname{Hess} E)_{Id}(\xi, \xi) = \int_M g(J_{Id}\xi, \xi)dv_g = \sum_{i=p+1}^{\infty} a_i \int_M g(E_i, E_i)dv_g$$

implies that $E_i = 0$ for any $i \geq p+1$. So, $J_{Id}\xi = 0$, that is, $\bar{\Delta}\xi = Q\xi$. Moreover, ξ is a solenoidal unit vector field, that is, $\operatorname{div}\xi = 0$; then it is easy to get that ξ is a Killing vector field (see for example [Poor 1981, page 171]), and hence, by Theorem 1.4(i), $\xi : (M, g) \to (T^1M; \widetilde{G})$ is a harmonic map. □

5. A remark about the energy of unit vector fields

In Section 4 we characterized the harmonicity of a unit vector field using the energy restricted to $\mathfrak{X}^1(M)$. About this energy, Brito [2000] proved the following.

Theorem 5.1. *The unit vector fields of minimum energy (with respect to the Sasaki metric) on the unit sphere S^3 are precisely the unit Killing vector fields.*

Brito proved the uniqueness part of his theorem by applying the uniqueness part of Gluck and Ziller's theorem [1986]. On the other hand, the unit sphere S^3 is a Sasakian three-manifold with constant Webster scalar curvature $w = 1$. By a direct method, Perrone [2008] proved the following generalization of Brito's theorem:

Theorem 5.2. *Let (M, g, ξ, η) be a compact Sasakian three-manifold with Webster scalar curvature $w \geq 1$. Then, the Reeb vector field ξ minimizes the energy, $E_{\tilde{G}_s}(\xi) = \frac{5}{2}vol(M)$, and the unit vector fields of minimum energy are precisely the unit Killing vector fields V that are eigenvectors of the Ricci operator with eigenvalue* 2*.*

The Ricci operator of a compact Sasakian three-manifold is given by (see for example [Blair 2002, page 105 and 171])

$$Q = 2(2w-1)\, I + 4(1-w)\, \eta \otimes \xi.$$

Thus, V is a unit vector field eigenvector of the Ricci operator with eigenvalue 2 if and only if $(1-w)V = (1-w)\eta(V)\xi$. Then Theorem 5.2 has a direct consequence:

Corollary 5.3. *Let (M, g, ξ, η) be a compact Sasakian three-manifold. If the Webster scalar curvature w is greater than* 1*, the Reeb vector field ξ is, up to sign, the only minimizer of the energy.*

Regarding the Webster scalar curvature, the main result of Chern and Hamilton [1985] says that a compact contact three-manifold (M, η) admits a contact metric g whose Webster scalar curvature w is either greater than 0 or is a nonpositive constant. Now, let (M, g, ξ, η) be a compact Sasakian three-manifold with Webster scalar curvature $w > 0$. Consider the D-homothetic deformation

$$g_t = tg + (t^2 - t)\eta \otimes \eta, \quad \eta_t = t\eta, \quad \xi_t = (1/t)\xi,$$

where $0 < t \leq c_0 = \inf\{w(p) : p \in M\} > 0$. Then (g_t, η_t, ξ_t) is also a Sasakian structure with Webster scalar curvature w_t given by (see [Blair 2002, page 173])

$$w_t = (1/t)w \geq (1/t)c_0 \geq 1.$$

Then (M, g_t, η_t, ξ_t) is a compact Sasakian three-manifold satisfying the condition of Theorem 5.2. In the special case of a compact Sasakian three-manifold

(M, g, ξ, η) with constant Webster scalar curvature $w > 0$, the universal covering $\widetilde{M}$ is homothetic to a Berger's sphere. In fact, $\widetilde{M}$ is the Sasakian manifold $(S^3, g = g_t, \xi = \xi_t, \eta = \eta_t)$, where the structure (g_t, ξ_t, η_t) is obtained from the standard Sasakian structure (g_0, ξ_0, η_0) on S^3 by a D-homothetic deformation with $t = 1/w > 0$ [Blair 2002]. Since $\eta_0 = g_0(\xi_0, \cdot)$, the metric

$$\bar{g}_t = (1/t) g_t = g_0 + (t-1)\eta_0 \otimes \eta_0$$

satisfies

$$\bar{g}_t|_{\xi_0^\perp} = g_0|_{\xi_0^\perp}, \quad \bar{g}_t(\xi_0, \xi_0) = t\, g_0(\xi_0, \xi_0), \quad \bar{g}_t(\xi_0, X) = 0 \quad \text{for } X \in \xi_0^\perp,$$

where $\xi_0^\perp$ denotes the orthogonal with respect to the metric g_0. The Riemannian manifold $(S^3, \bar{g}_t)$ is known as a Berger's sphere [Besse 1987, page 252], and $\bar{\xi}_t = \sqrt{t}\xi = (1/\sqrt{t})\xi_0$ is also called a Hopf vector field [Gil-Medrano and Hurtado 2005]. But under the homothetic transformation $\bar{g}_t = (1/t)g_t = (1/t)g$, the energy behaves as follows: if $\overline{V} = \sqrt{t}V$ and $V \in \mathfrak{X}^1(M, g)$, then

$$\begin{aligned} E_{\widetilde{G}_s}(\overline{V}, \bar{g}_t) &= (1/\sqrt{t})\left(\tfrac{3}{2}(t^{-1}-1)\mathrm{vol(M, g)}\right) + (1/\sqrt{t}) E_{\widetilde{G}_s}(V, g) \\ &\geq (1/\sqrt{t})\left(\tfrac{3}{2}(t^{-1}-1)\,\mathrm{vol}(M, g)\right) + (1/\sqrt{t}) E_{\widetilde{G}_s}(\xi, g) = E_{\widetilde{G}_s}(\bar{\xi}_t, \bar{g}_t). \end{aligned}$$

Hence Theorem 5.2 includes the special case of Hopf vector fields $\bar{\xi} g_t = (1/\sqrt{t})\xi_0$ of the Berger's spheres $(S^3, \bar{g}_t)$; this special case was first studied in [Gil-Medrano and Hurtado 2005].

We now consider the general compact Sasakian three-manifold (M, g, ξ, η) with Webster scalar curvature $w \geq 1$. Then, by Theorem 5.2 and (2-7), we have for any $V \in \mathfrak{X}^1(M)$ that

$$\begin{aligned} E_{\widetilde{G}}(V) &= \tfrac{1}{2}[3c + d]\,\mathrm{vol}(M, g) + a E_{\widetilde{G}_s}(V) \\ &\geq \tfrac{1}{2}[3c + d]\,\mathrm{vol}(M, g) + a E_{\widetilde{G}_s}(\xi) = E_{\widetilde{G}}(\xi), \end{aligned}$$

that is, $E_{\widetilde{G}}(V) \geq E_{\widetilde{G}}(\xi)$, where the equality holds if and only if V is Killing and $QV = 2V$. Therefore we get the following:

Theorems 5.1 and 5.2 and Corollary 5.3 are invariant under a four-parameter deformation of the Sasaki metric $\widetilde{G}_s$ on $T_1 M$.

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