Pacific Journal of Mathematics

UNIRATIONAL SURFACES ON THE NOETHER LINE

CHRISTIAN LIEDTKE AND MATTHIAS SCHÜTT

Volume 239 No. 2 February 2009

PACIFIC JOURNAL OF MATHEMATICS
Vol. 239, No. 2, 2009

UNIRATIONAL SURFACES ON THE NOETHER LINE

CHRISTIAN LIEDTKE AND MATTHIAS SCHÜTT

We show that among simply connected surfaces of general type unirationality is a common feature, even when fixing the positive characteristic or numerical invariants. To do so, we construct unirational Horikawa surfaces in abundance.

1. Introduction

The Lüroth problem asks whether a variety that is dominated by a rational variety is rational itself, that is, whether unirationality implies rationality. This is known to be true for curves (Lüroth) and for complex surfaces (Castelnuovo). However, it is also known to be wrong in dimension ≥ 3 thanks to the counterexamples by Iskovskikh and Manin [1971] (going back to Fano) and by Clemens and Griffiths [1972]. Zariski [1958] constructed unirational but nonrational surfaces over fields of positive characteristic. Artin and Mumford [1972] gave unirational examples of dimension ≥ 3 in characteristic $\neq 2$ that cannot be rational if the characteristic is zero or the dimension is 3.

On the other hand, one can ask what kind of varieties can be unirational: over the complex numbers a unirational variety is always of Kodaira dimension $\kappa = -\infty$. Moreover, Serre [1959] has shown that the fundamental group is finite in general, and even trivial in characteristic zero.

This leads to the question among what kind of surfaces in positive characteristic, say with trivial fundamental group, we can find unirational ones. Clearly, the most "nonrational" classes of surfaces are those of general type, that is, of Kodaira dimension $\kappa = 2$.

Among them we pick a very prominent series, the so-called *Horikawa surfaces*. We recall that minimal surfaces of general type fulfill Noether's inequality

$$K^2 \geq 2p_g - 4.$$

MSC2000: 14M20, 14J29.
Keywords: unirationality, Lüroth problem, Noether–Horikawa surface.
Much of this work was done while the second author was visiting the Universität Düsseldorf. We thank the Mathematisches Institut and the DFG Forschergruppe *Classification of Algebraic Surfaces and Compact Complex Manifolds* for the kind hospitality and support. The second author gratefully acknowledges funding from DFG under research grant Schu 2266/2-2.

By definition, Horikawa surfaces are those for which the equality holds, which is why they are also known as *surfaces on the Noether line*. These surfaces have a particularly simple structure: their canonical maps exhibit their canonical models as double covers of minimal rational surfaces; see Theorem 2.2 and [Barth et al. 2004, Section VII.9].

These double covers are known to be simply connected. In characteristic 2 it can happen that the canonical map becomes inseparable. Then the surfaces are necessarily unirational; see [Liedtke 2008a, Section 5]. However, since the canonical map of a Horikawa surface is always of degree 2, this inseparability phenomenon can happen in characteristic 2 only.

Over fields of positive characteristic $p \geq 3$, we therefore face the following problems:

(1) Do there exist unirational Horikawa surfaces in characteristic $p \geq 3$?

(2) If so, is the characteristic bounded?

(3) If not, is the number of possible characteristics for unirational Horikawa surfaces with fixed p_g finite?

(4) Is the number of possible geometric genera p_g for unirational Horikawa surfaces with fixed characteristic finite?

Shioda showed that unirational surfaces are supersingular, that is, that their Picard numbers are equal to their second Betti numbers — even if $h^{0,2}$ or $h^{2,0}$ is nonzero. Moreover, he also conjectured the converse for simply connected surfaces; see [Shioda 1977] and Conjecture 2.3 below. So far, the Picard numbers of Horikawa surfaces have only been studied in the complex case by Persson [1982]; see also Section 2. Hence these questions can also be viewed as a characteristic p extension of Persson's work.

As to questions (1) and (2), we prove the existence of unirational Horikawa surfaces in 99.999985% of positive characteristics. More precisely:

Theorem 5.4. *For all primes p outside a set $\mathcal{P}$ of density $5 \cdot 2^{-25}$, there do exist unirational Horikawa surfaces in characteristic p.*

Hence question (2) has a negative answer, but even the refined question (3) has a negative answer:

Theorem 5.2. *Let g be an even integer. Then there exists an arithmetic progression P_g of primes such that for every $p \in P_g$ there exists a unirational Horikawa surface with $p_g = g$ in characteristic p.*

Using Delsarte surfaces, we obtain much more detailed results for Horikawa surfaces with $p_g = 3$ and $p_g = 6$; see Theorem 4.3 and Remark 4.5.

Question (4) asked what happens when fixing the characteristic:

Theorem 5.3. *Let $p \geq 3$ be a prime with $p \not\equiv 1 \mod 15$. Then there exists an infinite set G_p of positive integers such that there exist unirational Horikawa surfaces in characteristic p with $p_g = g$ for all $g \in G_p$.*

These results show that unirational Horikawa surfaces exist in abundance — even when fixing the characteristic p or the geometric genus p_g. In Section 6 we show moreover that the unirationality cannot be explained by singular fibrations arising from pluricanonical systems.

As an interesting byproduct we construct a five-dimensional family of supersingular Horikawa surfaces with $p_g = 3$ in characteristic $p = 7$ where all surfaces have Artin invariants $3 \leq \sigma \leq 4$. Unlike supersingular K3 surfaces, this implies that the moduli space of supersingular Horikawa surfaces with $p_g = 3$ in characteristic 7 does not admit a stratification with moduli dimension equalling Artin invariant minus one.

The paper is organised as follows: Section 2 recalls Enriques' and Horikawa's classification and the basic invariants and notions. Section 3 introduces some families of unirational Horikawa surfaces that arise naturally. In Sections 4 and 5, we construct the surfaces to answer questions (1)–(4). Section 6 discusses whether unirationality can be explained through singular fibrations arising from pluricanonical systems.

2. Enriques' and Horikawa's classification

In this section, we recall the main results about Horikawa surfaces and in particular Enriques' and Horikawa's classification.

Definition 2.1. A minimal surface of general type is called *Horikawa surface* if it lies on the Noether line, that is,

$$K^2 = 2p_g - 4.$$

The most classical example for an Horikawa surface may be a double octic: the double cover of $\mathbb{P}^2$ branched along an octic curve C. Here we can allow the curve C to have isolated ade-singularities. Then we have to consider the minimal resolution of the resulting ADE-singularities on the double cover.

The same ideas apply to the construction of the other Horikawa surfaces. Over the complex numbers, they were classified by Enriques and Horikawa [1976]. The techniques were carried over to positive characteristic by the first author in [Liedtke 2008a]. We denote by $\mathbb{F}_d$ the d-th Hirzebruch surface.

Theorem 2.2 (Enriques–Horikawa). *Let X be a Horikawa surface. Via the canonical map ϕ_1, X is the double cover of a rational surface $S := \phi_1(X)$ in $\mathbb{P}^{p_g-1}$ (possibly singular). The following cases occur*:

(1) *If S is a smooth surface, then we have three possibilities*:

  (i) $S \cong \mathbb{P}^2$ *and* $p_g = 3$ (*double octic*),
  (ii) $S \cong \mathbb{P}^2$ *and* $p_g = 6$ (*the branch locus is a curve of degree* 10), *and*
  (iii) $S \cong \mathbb{F}_d$, $p_g \geq \max\{d+4, 2d-2\}$ *and* $p_g - d$ *is even.*

(2) *If S is not smooth, then it is the cone over the rational normal curve of degree $d := p_g - 2$. The minimal desingularisation of S is the Hirzebruch surface $\mathbb{F}_d$ and $4 \leq p_g \leq 6$.*

It follows from this description that complex Horikawa surfaces are topologically simply connected [Horikawa 1976, Theorem 3.4] and that they are algebraically simply connected with reduced Picard schemes in positive characteristic [Liedtke 2008a, Proposition 3.7].

Given the geometric genus p_g of a complex Horikawa surface, we compute the remaining Hodge number

$$h^{1,1} = 8p_g + 14.$$

Persson [1982] investigated complex Horikawa surfaces with maximal Picard number $\rho = h^{1,1}$. He proved existence whenever $p_g \not\equiv 5 \mod 6$ or $p_g = 5$.

In positive characteristic, however, Igusa showed that the Picard number can be as big as b_2. Such surfaces are often called *supersingular*.

Conjecture 2.3 [Shioda 1977]. *Let k be an algebraically closed field of positive characteristic. Let X be an algebraically simply connected surface over k. Then X is unirational if and only if it is supersingular.*

The "only if"-implication of the conjecture holds almost trivially true; see [Shioda 1977]. However, in practice it is often easier to verify that a given surface is supersingular than unirational. For instance, one can start by computing the characteristic polynomial of Frob_p on the second étale cohomology and apply the Tate conjecture [1965]. We pursued this approach without success for the reductions of several families of complex Horikawa surfaces of maximal Picard number borrowed from Persson's construction. In the following sections, we thus present surfaces where we can prove unirationality in a direct and systematic way.

As a side remark, we note a very different behaviour for K3 surfaces. We regard them close to Horikawa surfaces as they often can be realised as double sextics. Complex K3 surfaces of maximal Picard number 20 are called singular. They are necessarily defined over a number field. The supersingularity of the reductions does depend only on the splitting behaviour in an imaginary quadratic field. The interested reader is referred to [Shioda and Inose 1977] for details.

3. Families of unirational Horikawa surfaces

In this section, we give an *ad hoc* example of a family of unirational Horikawa surfaces with $p_g = 3$ in characteristic 7. It mimics a construction of Pho and Shimada [Pho and Shimada 2006] for unirational K3 surfaces in characteristic 5.

The Horikawa surfaces are realised as double octics over an algebraically closed field k of characteristic 7. The branch curve C is affinely given in $\mathbb{A}^2$ by

$$y^7 = f(x). \tag{3-1}$$

Here $f \in k[x]$ is a polynomial of degree 8 such that its formal derivative has no multiple roots in k. Consequently, C is smooth outside seven a_6 singularities where the formal derivative of f vanishes.

Let Y be the double cover of $\mathbb{P}^2$ branched along C. Denote the minimal resolution of the seven A_6 singularities of Y by X. By construction, X is a unirational Horikawa surface with $p_g = 3$ for any such polynomial f. As we can rescale and apply Möbius transformations, we obtain a five-dimensional family of unirational Horikawa surfaces.

Recall that every Horikawa surface with $p_g = 3$ arises as a double cover of $\mathbb{P}^2$ branched along an octic curve with at worst ade-singularities. Hence the moduli space of all such surfaces is of dimension 36.

Proposition 3.1. *Let X be a Horikawa surface as above. Then for some $3 \le \sigma \le 4$,*

$$\operatorname{disc} NS(X) = -7^{2\sigma}.$$

Proof. The existence of such an invariant $\sigma \ge 1$ was worked out in [Liedtke 2008b] and [Schütt and Schweizer 2006]. It is not difficult to see that the minimal desingularisation X of Y for every Y in our family lifts over the Witt ring. In particular, the Frölicher spectral sequence degenerates at E_1 and the Hodge numbers in characteristic $p = 7$ and zero coincide. Hence $h^0(\Omega^1_X) = 0$ and since $\operatorname{Pic}(X)$ is reduced, it follows from [Illusie 1979, Proposition II.5.16] that the second crystalline cohomology of X is torsion free. From [Illusie 1979, Remark II.5.21] or [Liedtke 2008b, Proposition 6.1] we obtain $\sigma \ge p_g = 3$. We will prove $\sigma \le 4$ by exhibiting a suitable sublattice of finite index in $NS(X)$.

The exceptional curves of the minimal resolution $X \to Y$ generate a sublattice

$$N = 7A_6(-1) \subset NS(X).$$

As $b_2(X) = 10p_g(X) + 14 = 44$, we shall exhibit another sublattice L of $NS(X)$ of rank two which is orthogonal to N. For this, consider the line ℓ at infinity with respect to the chart of C in (3-1). Then ℓ meets C in a single point of multiplicity 8. Hence the pullback of ℓ under the composition $X \to Y \to \mathbb{P}^2$ splits into two

curves:

$$\pi^*\ell = \ell_1 + \ell_2,$$

which meet in a single point with multiplicity 4. Since $(\pi^*\ell)^2 = 2$, it follows that $\ell_i^2 = -3$. Hence the curves ℓ_i span an indefinite sublattice L of $NS(X)$ with intersection form

$$\begin{pmatrix} -3 & 4 \\ 4 & -3 \end{pmatrix}.$$

As C is smooth along ℓ, L is orthogonal to N. Hence $N + L$ is a sublattice of $NS(X)$ of finite index and discriminant -7^8. Thus $\sigma \leq 4$. □

Remark 3.2. As we have exhibited a five-dimensional family of (unirational) Horikawa surfaces with these discriminants, we deduce that the moduli space of supersingular Horikawa surfaces with $p_g = 3$ in characteristic 7 does not admit a stratification with moduli dimension equalling invariant σ minus one.

This result is contrary to the situation for supersingular K3 surfaces in arbitrary characteristic as investigated by Artin [1974]. For supersingular K3 surfaces, σ is called Artin invariant and ranges from 1 to 10, giving the moduli dimension $\sigma - 1$.

The analogous construction can be applied to double covers of $\mathbb{P}^2$ branched along a curve of degree 10. This gives rise to a seven-dimensional family of unirational Horikawa surfaces of geometric genus $p_g = 6$ in characteristic 3.

4. Unirational Horikawa surfaces with $p_g = 3$

In this section we shall only construct single Horikawa surfaces over $\mathbb{Q}$. However, we will show that their reductions are unirational for all primes in some arithmetic progression. This gives a negative answer to question (2) stated in Section 1. The surfaces will have geometric genus $p_g = 3$, but the same approach works also for $p_g = 6$.

The main idea is as follows: We will exhibit all those Horikawa surfaces which are at the same time double octics and Delsarte surfaces. After [Shioda 1986], we shall then use that Delsarte surfaces are dominated by Fermat surfaces. Thanks to the work by Shioda and Katsura [1979], we gain complete knowledge about unirationality.

Definition 4.1. A surface $X \subset \mathbb{P}^3$ is called *Delsarte surface* if it is defined by a homogeneous polynomial of four monomials

$$\sum_{i=0}^{3} x_0^{a_{i0}} x_1^{a_{i1}} x_2^{a_{i2}} x_3^{a_{i3}} = 0$$

satisfying the following conditions:

(i) For each j, there is some i with $a_{ij} = 0$.

(ii) $\det(a_{ij})_{i,j} \neq 0$.

Shioda considered Delsarte surfaces because they are always dominated by some Fermat surface

$$S = \{y_0^m + y_1^m + y_2^m + y_3^m = 0\} \subset \mathbb{P}^3.$$

Here the appropriate degree m of the Fermat surface can be computed purely in terms of the matrix $A = (a_{ij})_{i,j}$. Let $A^* = (a^*_{i,j})_{i,j}$ denote the cofactor matrix of A, that is, $A \cdot A^* = (\det A) \cdot \mathbf{1}$. Set $\delta = \gcd(a^*_{i,j})$ and $d = (\det A)/\delta$. Then the Delsarte surface X given by A is dominated by the Fermat surface of degree d. Write $b_{i,j} = a^*_{i,j}/\delta$. The covering map is given as

$$\phi : S \to X, \ (y_i) \mapsto \left(x_i = \prod_{j=0}^{3} y_j^{b_{i,j}}\right).$$

Sometimes a smaller degree suffices for the covering Fermat surface. This issue will be important for Theorem 5.4. We will investigate the question of the precise degree for one example in the sequel of Theorem 4.3. A general statement can be found in [Shioda 1987, Definition 1 and Lemma 2].

Fermat surfaces are very well understood. The study of their arithmetic goes back to Weil who computed the ζ-function over finite fields in terms of Jacobi sums. Shioda and Katsura [1979] proved Conjecture 2.3 for Fermat surfaces:

Theorem 4.2 (Shioda–Katsura). *The Fermat surface of degree m is unirational in characteristic p if and only if it is supersingular. Equivalently, there is some $\nu \in \mathbb{N}$ with*

$$p^\nu \equiv -1 \mod m. \tag{4-1}$$

One prominent feature of Fermat surfaces is the motivic decomposition of their cohomology in terms of some character group. This decomposition allows the explicit calculation of Picard number $\rho(S)$ and Lefschetz number

$$\lambda(S) = b_2(S) - \rho(S)$$

over any algebraically closed field of characteristic coprime to the degree m.

A Delsarte surface arises from the dominating Fermat surface as the quotient by some group action. Hence, the transcendental part of the cohomology of the quotient surface can be identified with the transcendental part of the original surface that is invariant under the group. Thus we can calculate the Lefschetz number of any Delsarte surface and consequently also the Picard number. Clearly, the Delsarte surface is unirational if the Fermat surface is.

With this knowledge, we find unirational Horikawa surfaces as double octics which at the same time are Delsarte surfaces:

Theorem 4.3. *Let X be the minimal resolution of the double cover of $\mathbb{P}^2$ branched along one of the following octic curves C.*

C	m
$x^8+y^8+z^8=0$	8
$x^7y+y^7z+z^7x=0$	86
$x(x^7+y^7+z^7)=0$	14
$x(x^6y+y^6z+z^6x)=0$	62
$xy(x^6+y^6+z^6)=0$	12
$xy(x^5y+y^5z+z^5x)=0$	42
$xyz(x^5+y^5+z^5)=0$	10
$xyz(x^4y+y^4z+z^4x)=0$	26
$x^8+zy^7+xyz^6=0$	82
$x(x^7+zy^6+xyz^5)=0$	58
$z(x^6z+y^5x^2+z^5y^2)=0$	44
$x^7z+y^7x+z^6y^2=0$	74
$xz(x^5z+y^5x+z^4y^2)=0$	34

Then X is dominated by the Fermat surface of degree m as in the table. In particular, X is unirational in characteristic p if and only if (4-1) holds for the respective degree m.

Over $\mathbb{C}$, all curves in Theorem 4.3 have only isolated ade-singularities. Hence we obtain Horikawa surfaces. In positive characteristic, there are only finitely many exceptions where the singularities of a given curve degenerate. In the cases at hand, C is dominated by the Fermat curve of degree m (see the example below). Since this Fermat curve has good reduction outside the primes dividing m, the same holds for C. These primes are excluded by the condition (4-1).

For one Horikawa surface in Theorem 4.3, we will now sketch how to obtain the given degree of the covering Fermat surface. Consider the second octic curve

$$C=\{x^7y+y^7z+z^7x=0\}\subset\mathbb{P}^2.$$

We start by determining the covering Fermat curve along the lines of Definition 4.1. Let A be the matrix of exponents. Then A has determinant 344. Its cofactor matrix is

$$A^*=\begin{pmatrix}49&-7&1\\1&49&-7\\-7&1&49\end{pmatrix},$$

so C is covered by the Fermat curve of degree 344. Since we work in projective space, the rational map ϕ stays the same when we add constants to the columns of A^*. After adding 7 to all columns, the coefficients become divisible by 8. Division

yields the matrix

$$B = \begin{pmatrix} 7 & 0 & 1 \\ 1 & 7 & 0 \\ 0 & 1 & 7 \end{pmatrix}.$$

The corresponding rational map ϕ takes the Fermat curve of degree $43 = 344/8$ to C. One can easily complement ϕ to a rational map from a Fermat surface to the Horikawa surface associated to C. Since the Horikawa surface is exhibited as a double cover, the dominating Fermat surface has degree 86. Expressed through the matrix of exponents, the covering map corresponds to

$$\begin{pmatrix} 43 & 7 & 7 & 7 \\ 0 & 14 & 0 & 2 \\ 0 & 2 & 14 & 0 \\ 0 & 0 & 2 & 14 \end{pmatrix}.$$

Remark 4.4. There are further octic curves with only ade-singularities which are defined by a three-term polynomial. They give rise to more unirational Horikawa surfaces as double cover. However, the degrees of the dominating Fermat surface are not essentially new in the sense that the degree is always a multiple of some degree in the previous table or of $m = 6$ which will turn up as Fermat degree in the next section (see Theorem 5.2). The precise degrees occurring will be relevant for Theorem 5.4.

Remark 4.5. The same construction applies to Horikawa surfaces as double covers of $\mathbb{P}^2$ branched along a curve of degree 10. Allowing ade-singularities, we find many more unirational Horikawa surfaces. There is one essentially new degree for the dominating Fermat surfaces: $m = 146$. It is achieved by the curve

$$\{x^9 y + y^9 z + z^9 x = 0\} \subset \mathbb{P}^2$$

of degree 10.

5. Unbounded invariants

In the previous section we have seen unirational Horikawa surfaces with $p_g = 3$ for a large density of characteristics. We now construct unirational Horikawa surfaces in arithmetic progressions of characteristics for every even value of p_g.

Lemma 5.1. *Let a, b be positive integers and set $m := \mathrm{lcm}(a, b)$. Then the double cover $X_{a,b}$ of $\mathbb{A}^2$*

$$z^2 = (x^a - 1) \cdot (y^b - 1)$$

defines a unirational surface in every characteristic $p \geq 3$ for which there exists a $\nu \in \mathbb{N}$ with $p^\nu \equiv -1 \mod m$.

Proof. For every integer $n \geq 3$ we consider the hyperelliptic curve C_n:

$$v^2 = 1 - u^n,$$

together with its involution $\iota : (u, v) \mapsto (u, -v)$. For $m = \operatorname{lcm}(a, b)$ the product $C_m \times C_m$ dominates $C_a \times C_b$. By abuse of notation we also denote by ι the involution on these products given by $(u_1, v_1, u_2, v_2) \mapsto (u_1, -v_1, u_2, -v_2)$. Then,

$$Y_m := (C_m \times C_m)/\iota$$

dominates $(C_a \times C_b)/\iota$ and Y_m is unirational in characteristic p if and only if there exists a ν such that $p^\nu \equiv -1 \mod m$ by [Shioda 1977, Lemma 1.2]. In particular, $(C_a \times C_b)/\iota$ is unirational in these characteristics. An elementary calculation shows that this surface is birational to $X_{a,b}$. □

Theorem 5.2. *Let g be an even integer. Then there exists an arithmetic progression P_g of primes such that for every $p \in P_g$ there exists a unirational Horikawa surface with $p_g = g$ in characteristic p.*

Proof. We define

$$M_g := \{\operatorname{lcm}(5, g+1), \operatorname{lcm}(5, g+2), \operatorname{lcm}(6, g+1), \operatorname{lcm}(6, g+2)\},$$

$$P_g := \{p \in \mathbb{N} \mid p \geq 3 \text{ a prime such that there exist } \nu \in \mathbb{N}, m \in M_g : p^\nu \equiv -1 \mod m\},$$

and note that P_g is the union of primes in arithmetic progressions.

We use the surfaces $X_{a,b}$ from Lemma 5.1. Let a' (respectively, b') be the smallest even integer bigger than or equal to a (respectively, b). We think of $\mathbb{A}^1 \times \mathbb{A}^1$ as contained in $\mathbb{P}^1 \times \mathbb{P}^1$. If we consider

$$(x^a - 1)(y^b - 1)$$

as a global section of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a', b')$ then its zero set is the union of $a' + b'$ lines intersecting transversally. Hence a double cover branched over this divisor is a surface with only Du Val singularities of Type A_1.

In case $a' = 6$ (that is, $a = 5$ or $a = 6$) and $b' = g + 2$ (that is, $b = g + 1$ or $b = g + 2$) this surface is the canonical model of a Horikawa surface with $p_g = g$; see [Barth et al. 2004, Section VII.9] or [Liedtke 2008a, Theorem 3.3]. By Lemma 5.1 this surface is unirational in every characteristic p for which there exists a ν with

$$p^\nu \equiv -1 \mod \operatorname{lcm}(a, b).$$

Hence for every $p \in P_g$ we have constructed a unirational Horikawa surface with $p_g = g$ in characteristic p. □

Theorem 5.3. *Let $p \geq 3$ be a prime with $p \not\equiv 1 \mod 15$. Then there exists an infinite set G_p of positive integers such that there exist unirational Horikawa surfaces in characteristic p with $p_g = g$ for all $g \in G_p$.*

Proof. If p is a prime as in the statement then there exists an infinite number of ν such that $p^\nu \equiv -1 \mod 5$ or $p^\nu \equiv -1 \mod 6$. For every such ν, we obtain a unirational Horikawa surface with

$$p_g = (p^\nu + 1) - 2$$

in characteristic p using the examples constructed in the proof of Theorem 5.2. □

Theorem 5.4. *For all primes p outside a set $\mathcal{P}$ of density $5 \cdot 2^{-25}$, there do exist unirational Horikawa surfaces in characteristic p.*

Proof. Consider all the surfaces constructed in this and the preceding section. Their unirationality modulo p does depend only on the respective Fermat degree m. Condition (4-1) does not distinguish between odd modulus m and $2m$, so we list odd moduli whenever applicable. We constructed unirational Horikawa surfaces for the moduli

$$\mathcal{M} = \{3, 5, 7, 8, 13, 17, 29, 31, 37, 41, 43, 44, 73\}.$$

Hence the set $\mathcal{P}$ can be defined as

$$\mathcal{P} = \{p \mid \text{for all } m \in \mathcal{M}, \nu \in \mathbb{N} : \ p^\nu \not\equiv -1 \mod m\}.$$

Working out the congruence conditions for each modulus $m \in \mathcal{M}$, one easily computes the claimed density. □

Remark 5.5. We have thus constructed unirational Horikawa surfaces for a density of 99.999985% of characteristics. The smallest characteristic for which we have no example is $p = 67665781$.

Remark 5.6. For all surfaces in this and the previous section, we can compute the zeta function by motivic decomposition. For the dominating Fermat varieties, this decomposition is due to Weil [1949]. Then one computes the submotive invariant under the automorphism group corresponding to the covering map to obtain the essential factor of the zeta function.

6. A remark on singular fibrations

An interesting question is what *causes* unirationality. Clearly, unirational surfaces are covered by possibly singular rational curves. On the other hand, if a surface admits a fibration onto a curve whose generic fibre is a possibly singular geometrically rational curve then the surface in question is uniruled. In case the base of

this fibration is $\mathbb{P}^1$ (in our case this would be automatic since our surfaces have $b_1 = 0$), the surface is even unirational.

For surfaces of general type, it is natural to investigate their pluricanonical systems. Maybe the unirationality of Horikawa surfaces, say with fixed p_g, can be explained by such singular fibrations arising from pluricanonical systems.

Proposition 6.1. *Let X be a Horikawa surface in characteristic p and $n \in \mathbb{N}$. Assume that some subsystem of $|nK_X|$ defines a (possibly rational) map with 1-dimensional image whose generic fibre is a singular and geometrically rational curve. Then*

$$p \leq 2 + (n^2 + n)(p_g(X) - 2).$$

Proof. Let $f : X \dashrightarrow C$ be a rational map with 1-dimensional image defined by some subsystem of $|nK_X|$. In particular, if F denotes the generic fibre of this fibration then

$$A := nK_X - F$$

is effective. Since K_X is nef and A is effective, we have $K_X F \leq nK_X^2$. Moreover, we compute

$$F^2 = (nK_X - A)^2 = n^2K_X^2 - A(2nK_X - A) = n^2K_X^2 - A(nK_X + F) \leq n^2K_X^2$$

using numerical connectedness of pluricanonical divisors. Using the adjunction formula and the definition of a Horikawa surface we get

$$p_a(F) = 1 + \tfrac{1}{2}(F^2 + K_X F) \leq 1 + \tfrac{1}{2}(n^2 + n)K_X^2 = 1 + \tfrac{1}{2}(n^2 + n)(2p_g(X) - 4).$$

We assumed the generic fibre of f to be a singular curve, that is, that this curve is regular but not smooth over the function field of C. By Tate's theorem on genus change in inseparable field extensions $p - 1$ divides $p_a(L)$, since we assumed L to be geometrically rational. In particular, $p_a(L)$ must be larger than or equal to $p - 1$. □

However, we have seen that even for fixed p_g there may be an infinite number of characteristics in which there exist unirational Horikawa surfaces with this given p_g. Proposition 6.1 tells us that (possibly rational) fibrations with geometrically rational fibres arising from $|nK_X|$ with n bounded can only exist in a finite number of characteristics. Hence the unirationality of Horikawa surfaces in arbitrary large characteristics is not related to the existence of such singular fibrations arising from a finite number of pluricanonical systems.

Acknowledgement

We thank the referee for helpful comments.

References

[Artin 1974] M. Artin, "Supersingular $K3$ surfaces", *Ann. Sci. École Norm. Sup.* (4) **7** (1974), 543–567. MR 51 #8116 Zbl 0322.14014

[Artin and Mumford 1972] M. Artin and D. Mumford, "Some elementary examples of unirational varieties which are not rational", *Proc. London Math. Soc.* (3) **25** (1972), 75–95. MR 48 #299 Zbl 0244.14017

[Barth et al. 2004] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3 **4**, Springer, Berlin, 2004. MR 2004m:14070 Zbl 1036.14016

[Clemens and Griffiths 1972] C. H. Clemens and P. A. Griffiths, "The intermediate Jacobian of the cubic threefold", *Ann. of Math.* (2) **95** (1972), 281–356. MR 46 #1796 Zbl 0245.14010

[Horikawa 1976] E. Horikawa, "Algebraic surfaces of general type with small c_1^2, I", *Ann. of Math.* (2) **104**:2 (1976), 357–387. MR 54 #12789 Zbl 0339.14024

[Illusie 1979] L. Illusie, "Complexe de de Rham–Witt et cohomologie cristalline", *Ann. Sci. École Norm. Sup.* (4) **12**:4 (1979), 501–661. MR 82d:14013 Zbl 0436.14007

[Iskovskih and Manin 1971] V. A. Iskovskih and J. I. Manin, "Three-dimensional quartics and counterexamples to the Lüroth problem", *Mat. Sb.* (*N.S.*) **86(128)** (1971), 140–166. In Russian; translated in *Math. USSR-Sb.* **15** (1971), 141–166. MR 45 #266

[Liedtke 2008a] C. Liedtke, "Algebraic surfaces of general type with small c_1^2 in positive characteristic", *Nagoya Math. J.* **191** (2008), 111–134. arXiv math.AG/0702548

[Liedtke 2008b] C. Liedtke, "Uniruled surfaces of general type", *Math. Z.* **259**:4 (2008), 775–797. MR 2403741

[Persson 1982] U. Persson, "Horikawa surfaces with maximal Picard numbers", *Math. Ann.* **259**:3 (1982), 287–312. MR 84d:14018 Zbl 0466.14010

[Pho and Shimada 2006] D. T. Pho and I. Shimada, "Unirationality of certain supersingular $K3$ surfaces in characteristic 5", *Manuscripta Math.* **121**:4 (2006), 425–435. MR 2007i:14036 Zbl 1108.14036

[Schütt and Schweizer 2006] M. Schütt and A. Schweizer, "Davenport–Stothers inequalities and elliptic surfaces in positive characteristic", preprint, 2006. To appear in *Quarterly J. Math.* arXiv math.NT/0608427

[Serre 1959] J.-P. Serre, "On the fundamental group of a unirational variety", *J. London Math. Soc.* **34** (1959), 481–484. MR 22 #43 Zbl 0097.36301

[Shioda 1977] T. Shioda, "Some results on unirationality of algebraic surfaces", *Math. Ann.* **230**:2 (1977), 153–168. MR 58 #28004 Zbl 0343.14021

[Shioda 1986] T. Shioda, "An explicit algorithm for computing the Picard number of certain algebraic surfaces", *Amer. J. Math.* **108**:2 (1986), 415–432. MR 87g:14033 Zbl 0602.14033

[Shioda 1987] T. Shioda, "Supersingular $K3$ surfaces with big Artin invariant", *J. Reine Angew. Math.* **381** (1987), 205–210. MR 88j:14047 Zbl 0618.14014

[Shioda and Inose 1977] T. Shioda and H. Inose, "On singular $K3$ surfaces", pp. 119–136 in *Complex analysis and algebraic geometry*, edited by W. L. Baily, Jr., Iwanami Shoten, Tokyo, 1977. MR 56 #371 Zbl 0374.14006

[Shioda and Katsura 1979] T. Shioda and T. Katsura, "On Fermat varieties", *Tôhoku Math. J.* (2) **31**:1 (1979), 97–115. MR 80g:14033 Zbl 0415.14022

[Tate 1965] J. T. Tate, "Algebraic cycles and poles of zeta functions", pp. 93–110 in *Arithmetical Algebraic Geometry* (Purdue University, West Lafayette, IN, 1963), edited by O. F. G. Schilling, Harper & Row, New York, 1965. MR 37 #1371 Zbl 0213.22804

[Weil 1949] A. Weil, "Numbers of solutions of equations in finite fields", *Bull. Amer. Math. Soc.* **55** (1949), 497–508. MR 10,592e Zbl 0032.39402

[Zariski 1958] O. Zariski, "On Castelnuovo's criterion of rationality $p_a = P_2 = 0$ of an algebraic surface", *Illinois J. Math.* **2** (1958), 303–315. MR 20 #6426 Zbl 0085.36203

Received June 16, 2008. Revised September 10, 2008.

CHRISTIAN LIEDTKE
MATHEMATISCHES INSTITUT
UNIVERSITÄT DÜSSELDORF
UNIVERSITÄTSSTRASSE 1
40225 DÜSSELDORF
GERMANY

liedtke@math.uni-duesseldorf.de

MATTHIAS SCHÜTT
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF COPENHAGEN
UNIVERSITETSPARK 5
2100 COPENHAGEN
DENMARK

mschuett@math.ku.dk