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ALBERTO ELDUQUE

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ALBERTO ELDUQUE

We provide models of the exceptional simple modular Lie superalgebras in characteristic $p \geq 3$ that appeared in the 2007 classification by Bouarroudj, Grozman and Leites of Lie superalgebras with indecomposable symmetrizable Cartan matrices. The models relate these exceptional Lie superalgebras to some low-dimensional nonassociative algebraic systems.

Introduction

Finite-dimensional modular Lie superalgebras with indecomposable symmetrizable Cartan matrices over algebraically closed fields are classified in [Bouarroudj et al. 2007] under some extra technical hypotheses. Their results assert that, for characteristic ≥ 3, apart from Lie superalgebras obtained as the analogues of Lie superalgebras in the classification in characteristic 0 [Kac 1977], by reducing the Cartan matrices modulo p, there are the following exceptions that have to be added to the list of known simple Lie superalgebras:

(1) Two exceptions in characteristic 5: $\mathfrak{br}(2; 5)$ and $\mathfrak{el}(5; 5)$. (The superalgebra $\mathfrak{el}(5; 5)$ first appeared in [Elduque 2007b].)

(2) A family of exceptions given by the Lie superalgebras that appear in the *super-magic square* in characteristic 3 considered in [Cunha and Elduque 2007a; 2007b]. With the exception of $\mathfrak{g}(3, 6) = \mathfrak{g}(S_{1,2}, S_{4,2})$ these Lie superalgebras first appeared in [Elduque 2006b; 2007b].

(3) Another two exceptions in characteristic 3, similar to the ones in characteristic 5: $\mathfrak{br}(2; 3)$ and $\mathfrak{el}(5; 3)$.

The Lie superalgebra $\mathfrak{el}(5; 5)$ was shown in [Elduque 2007b] to be related to Kac's 10-dimensional exceptional Jordan superalgebra, by means of the Tits construction [1966] of Lie algebras in terms of alternative and Jordan algebras.

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The purpose of this paper is to provide models of the other three exceptions: $\mathfrak{br}(2;3)$ and $\mathfrak{el}(5;3)$ in characteristic 3, and $\mathfrak{br}(2;5)$ in characteristic 5.

Actually, the superalgebra $\mathfrak{br}(2;3)$ already appeared in [Elduque 2006b, Theorem 3.2(i)] related to a symplectic triple system of dimension 8. Here it will be shown to be related to a nice five-dimensional orthosymplectic triple system.

The Lie superalgebra $\mathfrak{el}(5;3)$ will be shown to be a maximal subalgebra of the Lie superalgebra

$$\mathfrak{g}(8,3)=\mathfrak{g}(S_8,S_{1,2})$$

in the supermagic square. Furthermore, it will be shown to be related to an orthogonal triple system defined on the direct sum of two copies of the octonions and, finally, it will be proved to be the Lie superalgebra of derivations of a specific orthosymplectic triple system, and this latter result will relate $\mathfrak{el}(5;3)$ to the Lie superalgebra

$$\mathfrak{g}(6,6)=\mathfrak{g}(S_{4,2},S_{4,2})$$

in the supermagic square.

Finally, a very explicit model of the Lie superalgebra $\mathfrak{br}(2;5)$ will be constructed.

The paper is organized as follows. The construction of the extended magic square (or supermagic square) in characteristic 3 in terms of composition superalgebras is recalled in Section 1. Then, in Section 2, the Lie superalgebra $\mathfrak{el}(5;3)$ (in characteristic 3) is shown to be a maximal subalgebra of the Lie superalgebra $\mathfrak{g}(S_8,S_{1,2})$ in the supermagic square. This gives a very concrete realization of $\mathfrak{el}(5;3)$ in terms of simple components: copies of the three-dimensional simple Lie algebra $\mathfrak{sl}_2$ and of its natural two-dimensional module. Orthogonal triple systems are reviewed in Section 3 and the Lie superalgebra $\mathfrak{el}(5;3)$ is shown to be isomorphic to the Lie superalgebra of an orthogonal triple system defined on the direct sum of two copies of the split Cayley algebra. Then the orthosymplectic triple systems, which extend both the orthogonal and symplectic triple systems, are recalled in Section 4. A very simple such system is defined on the set of trace zero elements of the $4|2$-dimensional composition superalgebra $B(4,2)$. (The dimension being $4|2$ means that the even part has dimension 4 and the odd part dimension 2.) The Lie superalgebra naturally attached to this orthosymplectic triple system is shown to be isomorphic to the Lie superalgebra $\mathfrak{br}(2;3)$. Section 5 deals with another distinguished orthosymplectic triple system, which lives inside the Lie superalgebra $\mathfrak{g}(S_8,S_{1,2})$ in the supermagic square. It turns out that the Lie superalgebra $\mathfrak{el}(5;3)$ is isomorphic to the Lie superalgebra of derivations of this system. This shows also how $\mathfrak{el}(5;3)$ embeds in the Lie superalgebra $\mathfrak{g}(S_{4,2},S_{4,2})$ of the supermagic square. Finally, Section 6 is devoted to give an explicit model of the Lie superalgebra $\mathfrak{br}(2;5)$ (in characteristic 5) in terms of two copies of $\mathfrak{sl}_2$ and of their natural modules.

All the vector spaces and superspaces considered in this paper will be assumed to be finite-dimensional over a ground field k of characteristic $\neq 2$. In dealing with elements of a superspace $V = V_{\bar 0} \oplus V_{\bar 1}$, an expression like $(-1)^{uv}$, for homogeneous elements u, v, is a shorthand for $(-1)^{p(u)p(v)}$, where p is the parity function.

1. The supermagic square in characteristic 3

Recall that an algebra C over a field k is said to be a *composition algebra* if it is endowed with a regular quadratic form q (that is, its polar form $\mathrm{b}(x, y) = q(x+y) - q(x) - q(y)$ is a nondegenerate symmetric bilinear form) such that $q(xy) = q(x)q(y)$ for any $x, y \in C$. The unital composition algebras will be termed *Hurwitz algebras*. On the other hand, a composition algebra is said to be *symmetric* in case the polar form is associative: $\mathrm{b}(xy, z) = \mathrm{b}(x, yz)$.

Hurwitz algebras are the well-known algebras that generalize the classical real division algebras: real and complex numbers, quaternions and octonions. Over any algebraically closed field k, there are exactly four of them: k, $k \times k$, $\mathrm{Mat}_2(k)$ and $C(k)$ (the split Cayley algebra), with dimensions 1, 2, 4 and 8.

Let us superize the above concepts.

A quadratic superform on a $\mathbb{Z}_2$-graded vector space

$$U = U_{\bar 0} \oplus U_{\bar 1}$$

over a field k is a pair $q = (q_{\bar 0}, \mathrm{b})$ where $q_{\bar 0} : U_{\bar 0} \to k$ is a quadratic form, and $\mathrm{b} : U \times U \to k$ is a supersymmetric even bilinear form such that $\mathrm{b}|_{U_{\bar 0} \times U_{\bar 0}}$ is the polar of $q_{\bar 0}$:

$$\mathrm{b}(x_{\bar 0}, y_{\bar 0}) = q_{\bar 0}(x_{\bar 0} + y_{\bar 0}) - q_{\bar 0}(x_{\bar 0}) - q_{\bar 0}(y_{\bar 0})$$

for any $x_{\bar 0}, y_{\bar 0} \in U_{\bar 0}$.

The quadratic superform $q = (q_{\bar 0}, \mathrm{b})$ is said to be *regular* if the bilinear form b is nondegenerate.

Then a superalgebra

$$C = C_{\bar 0} \oplus C_{\bar 1}$$

over k, endowed with a regular quadratic superform $q = (q_{\bar 0}, \mathrm{b})$, called the *norm*, is said to be a *composition superalgebra* (see [Elduque and Okubo 2002]) if

$$q_{\bar 0}(x_{\bar 0} y_{\bar 0}) = q_{\bar 0}(x_{\bar 0}) q_{\bar 0}(y_{\bar 0}), \tag{1-1a}$$

$$\mathrm{b}(x_{\bar 0} y, x_{\bar 0} z) = q_{\bar 0}(x_{\bar 0}) \mathrm{b}(y, z) = \mathrm{b}(y x_{\bar 0}, z x_{\bar 0}), \tag{1-1b}$$

$$\mathrm{b}(xy, zt) + (-1)^{xy+xz+yz} \mathrm{b}(zy, xt) = (-1)^{yz} \mathrm{b}(x, z) \mathrm{b}(y, t) \tag{1-1c}$$

for any $x_{\bar 0}, y_{\bar 0} \in C_{\bar 0}$ and homogeneous elements $x, y, z, t \in C$. Since the characteristic of the ground field is assumed not to be 2, (1-1c) already implies (1-1a) and (1-1b).

The unital composition superalgebras are termed *Hurwitz superalgebras*, while a composition superalgebra is said to be *symmetric* in case its bilinear form is associative, that is,

$$\mathrm{b}(xy, z) = \mathrm{b}(x, yz),$$

for any x, y, z.

Only over fields of characteristic 3 there appear nontrivial Hurwitz superalgebras (see [Elduque and Okubo 2002]):

- Let V be a two-dimensional vector space over a field k, endowed with a nonzero alternating bilinear form $\langle \cdot | \cdot \rangle$ (that is $\langle v|v\rangle = 0$ for any $v \in V$). Consider the superspace $B(1,2)$ (see [Shestakov 1997]) with

(1-2) $$B(1,2)_{\bar 0} = k1, \qquad \text{and} \qquad B(1,2)_{\bar 1} = V,$$

  endowed with the supercommutative multiplication given by

$$1x = x1 = x \qquad \text{and} \qquad uv = \langle u|v\rangle 1$$

  for any $x \in B(1,2)$ and $u, v \in V$, and with the quadratic superform $q = (q_{\bar 0}, \mathrm{b})$ given by

(1-3) $$q_{\bar 0}(1) = 1, \quad \mathrm{b}(u, v) = \langle u|v\rangle,$$

  for any $u, v \in V$. If the characteristic of k is equal to 3, then $B(1,2)$ is a Hurwitz superalgebra [Elduque and Okubo 2002, Proposition 2.7].

- Moreover, with V as before, let $f \mapsto \bar f$ be the associated symplectic involution on $\mathrm{End}_k(V)$ (so $\langle f(u)|v\rangle = \langle u|\bar f(v)\rangle$ for any $u, v \in V$ and $f \in \mathrm{End}_k(V)$). Consider the superspace $B(4,2)$ (see [Shestakov 1997]) with

(1-4) $$B(4,2)_{\bar 0} = \mathrm{End}_k(V), \qquad \text{and} \qquad B(4,2)_{\bar 1} = V,$$

  with multiplication given by the usual one (composition of maps) in $\mathrm{End}_k(V)$, and by

$$v \cdot f = f(v) = \bar f \cdot v \in V,$$
$$u \cdot v = \langle \cdot |u\rangle v \in \mathrm{End}_k(V)$$

  for any $f \in \mathrm{End}_k(V)$ and $u, v \in V$, where $\langle \cdot |u\rangle v$ denotes the endomorphism $w \mapsto \langle w|u\rangle v$, and with quadratic superform $q = (q_{\bar 0}, \mathrm{b})$ such that

$$q_{\bar 0}(f) = \det(f), \qquad \mathrm{b}(u, v) = \langle u|v\rangle,$$

  for any $f \in \mathrm{End}_k(V)$ and $u, v \in V$. If the characteristic is equal to 3, $B(4,2)$ is a Hurwitz superalgebra ([Elduque and Okubo 2002, Proposition 2.7]).

Given any Hurwitz superalgebra C with norm $q = (q_{\bar 0}, \mathrm{b})$, its standard involution is given by

$$x \mapsto \bar x = \mathrm{b}(x, 1)1 - x.$$

A new product can be defined on C by means of

$$x \bullet y = \bar x \bar y.$$

The resulting superalgebra, denoted by $\bar C$, is called the *para-Hurwitz superalgebra* attached to C, and it turns out to be a symmetric composition superalgebra.

Given a symmetric composition superalgebra S, its *triality Lie superalgebra* $\mathfrak{tri}(S) = \mathfrak{tri}(S)_{\bar 0} \oplus \mathfrak{tri}(S)_{\bar 1}$ is defined by

$$\begin{aligned}\mathfrak{tri}(S)_{\bar i} = \{(d_0, d_1, d_2) \in \mathfrak{osp}(S, q)^3_{\bar i} :\\ d_0(x \bullet y) = d_1(x) \bullet y + (-1)^{ix} x \bullet d_2(y) \text{ for all } x, y \in S_{\bar 0} \cup S_{\bar 1}\},\end{aligned}$$

where $\bar i = \bar 0, \bar 1$, and $\mathfrak{osp}(S, q)$ denotes the associated orthosymplectic Lie superalgebra. The bracket in $\mathfrak{tri}(S)$ is given componentwise.

Now, given two symmetric composition superalgebras S and S', one can form (see [Cunha and Elduque 2007a, §3], or [Elduque 2004] for the non-super situation) the Lie superalgebra

$$\mathfrak{g} = \mathfrak{g}(S, S') = \big(\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')\big) \oplus \left(\bigoplus_{i=0}^{2} \iota_i(S \otimes S')\right),$$

where $\iota_i(S \otimes S')$ is just a copy of $S \otimes S'$ ($i = 0, 1, 2$), with bracket given by

- the Lie bracket in $\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$, which thus becomes a Lie subalgebra of $\mathfrak{g}$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i(d_i(x) \otimes x')$,
- $[(d'_0, d'_1, d'_2), \iota_i(x \otimes x')] = (-1)^{d'_i x} \iota_i(x \otimes d'_i(x'))$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = (-1)^{x'y} \iota_{i+2}((x \bullet y) \otimes (x' \bullet y'))$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')]$
$$= (-1)^{xx'+xy'+yy'} \mathrm{b}'(x', y') \theta^i(t_{x,y}) + (-1)^{yx'} \mathrm{b}(x, y) \theta'^i(t'_{x',y'}),$$

for any $i = 0, 1, 2$ and homogeneous $x, y \in S$, $x', y' \in S'$, $(d_0, d_1, d_2) \in \mathfrak{tri}(S)$, and $(d'_0, d'_1, d'_2) \in \mathfrak{tri}(S')$. Here θ denotes the natural automorphism

$$\theta : (d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)$$

in $\mathfrak{tri}(S)$, while $t_{x,y}$ is defined by

$$t_{x,y} = \big(\sigma_{x,y}, \tfrac{1}{2}\mathrm{b}(x, y)1 - r_x l_y, \tfrac{1}{2}\mathrm{b}(x, y)1 - l_x r_y\big)$$

	S_1	S_2	S_4	S_8	$S_{1,2}$	$S_{4,2}$
S_1	$\mathfrak{sl}_2$	$\mathfrak{pgl}_3$	$\mathfrak{sp}_6$	$\mathfrak{f}_4$	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus (14)$
S_2		$\mathfrak{pgl}_3 \oplus \mathfrak{pgl}_3$	$\mathfrak{pgl}_6$	$\tilde{\mathfrak{e}}_6$	$(\mathfrak{pgl}_3 \oplus \mathfrak{sl}_2) \oplus (\mathfrak{psl}_3 \otimes (2))$	$\mathfrak{pgl}_6 \oplus (20)$
S_4			$\mathfrak{so}_{12}$	$\mathfrak{e}_7$	$(\mathfrak{sp}_6 \oplus \mathfrak{sl}_2) \oplus ((13) \otimes (2))$	$\mathfrak{so}_{12} \oplus \mathrm{spin}_{12}$
S_8				$\mathfrak{e}_8$	$(\mathfrak{f}_4 \oplus \mathfrak{sl}_2) \oplus ((25) \otimes (2))$	$\mathfrak{e}_7 \oplus (56)$
$S_{1,2}$					$\mathfrak{so}_7 \oplus 2\mathrm{spin}_7$	$\mathfrak{sp}_8 \oplus (40)$
$S_{4,2}$						$\mathfrak{so}_{13} \oplus \mathrm{spin}_{13}$

Table 1. Supermagic square (characteristic 3).

with $l_x(y) = x \bullet y$, $r_x(y) = (-1)^{xy} y \bullet x$, and

$$\sigma_{x,y}(z) = (-1)^{yz}\mathrm{b}(x, z)y - (-1)^{x(y+z)}\mathrm{b}(y, z)x \tag{1-5}$$

for homogeneous $x, y, z \in S$. Also θ' and $t'_{x',y'}$ denote the analogous elements for $\mathfrak{tri}(S')$.

Over a field k of characteristic 3, let S_r ($r = 1, 2, 4$ or 8) denote the para-Hurwitz algebra attached to the split Hurwitz algebra of dimension r (this latter algebra being either k, $k \times k$, $\mathrm{Mat}_2(k)$ or $C(k)$). Also, denote by $S_{1,2}$ the para-Hurwitz superalgebra $\overline{B(1, 2)}$, and by $S_{4,2}$ the para-Hurwitz superalgebra $\overline{B(4, 2)}$. Then the Lie superalgebras $\mathfrak{g}(S, S')$, where S, S' run over $\{S_1, S_2, S_4, S_8, S_{1,2}, S_{4,2}\}$, appear in Table 1, which has been obtained in [Cunha and Elduque 2007a].

Since the construction of $\mathfrak{g}(S, S')$ is symmetric, only the entries above the diagonal are needed. In Table 1, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ denote the simple exceptional classical Lie algebras, $\tilde{\mathfrak{e}}_6$ denotes a 78-dimensional Lie algebra whose derived Lie algebra is the 77-dimensional simple Lie algebra $\mathfrak{e}_6$ in characteristic 3. The even and odd parts of the nontrivial superalgebras in the table which have no counterpart in the classification in characteristic 0 [Kac 1977] are displayed, spin denotes the spin module for the corresponding orthogonal Lie algebra, while (n) denotes a module of dimension n, whose precise description is given in [Cunha and Elduque 2007a]. Thus, for example, $\mathfrak{g}(S_4, S_{1,2})$ is a Lie superalgebra whose even part is (isomorphic to) the direct sum of the symplectic Lie algebra $\mathfrak{sp}_6$ and of $\mathfrak{sl}_2$, while its odd part is the tensor product of a 13-dimensional module for $\mathfrak{sp}_6$ and the natural 2-dimensional module for $\mathfrak{sl}_2$.

In Table 2, a more precise description of the Lie superalgebras that appear in the supermagic square is given. This table displays the even parts and the highest weights of the odd parts. The numbering of the roots follows Bourbaki's conventions [1968]. The fundamental dominant weight for $\mathfrak{sl}_2$ will be denoted by ω, while the fundamental dominant weights for a Lie algebra with a Cartan matrix of order

	$S_{1,2}$	$S_{4,2}$
S_1	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus V(\omega_3)$ [5.3]
S_2	$(\mathfrak{pgl}_3 \oplus \mathfrak{sl}_2) \oplus \big(V(\omega_1+\omega_2) \otimes V(\omega)\big)$ [5.28]	$\mathfrak{pgl}_6 \oplus V(\omega_3)$ [5.16]
S_4	$(\mathfrak{sp}_6 \oplus \mathfrak{sl}_2) \oplus \big(V(\omega_2) \otimes V(\omega)\big)$ [5.24]	$\mathfrak{so}_{12} \oplus V(\omega_6)$ [5.5]
S_8	$(\mathfrak{f}_4 \oplus \mathfrak{sl}_2) \oplus \big(V(\omega_4) \otimes V(\omega)\big)$ [5.26]	$\mathfrak{e}_7 \oplus V(\omega_7)$ [5.8]
$S_{1,2}$	$\mathfrak{so}_7 \oplus 2V(\omega_3)$ [5.19]	$\mathfrak{sp}_8 \oplus V(\omega_3)$ [5.12]
$S_{4,2}$		$\mathfrak{so}_{13} \oplus V(\omega_6)$ [5.10]

Table 2. Even and odd parts in the supermagic square.

n will be denoted by $\omega_1, \ldots, \omega_n$. After each entry in the square brackets is the proposition number in [Cunha and Elduque 2007a] where the result can be found.

A precise description of these modules and of the superalgebras in Table 2 as Lie superalgebras with a Cartan matrix is given in the same reference. All inequivalent Cartan matrices for these simple Lie superalgebras are listed in [Bouarroudj et al. 2006].

With the exception of $\mathfrak{g}(S_{1,2}, S_{4,2})$, all these superalgebras have appeared previously [Elduque 2006b; 2007b]. Some relationships between the Lie superalgebras $\mathfrak{g}(S_{1,2}, S)$ and $\mathfrak{g}(S_{4,2}, S)$ to other algebraic structures have been considered in [Cunha and Elduque 2007b].

2. The Lie superalgebra $\mathfrak{el}(5;3)$

The aim of this section is to show how the Lie superalgebra $\mathfrak{el}(5;3)$ embeds in a nice way as a maximal subalgebra in the simple Lie superalgebra $\mathfrak{g}(S_8, S_{1,2})$ of the supermagic square.

Throughout this section the characteristic of the ground field k will be assumed to be 3.

The para-Hurwitz superalgebra $S_{1,2} = \overline{B(1,2)}$ is described as $S_{1,2} = k1 \oplus V$ (see (1-2) and (1-3)), where $(S_{1,2})_{\bar 0} = k1$ is a copy of the ground field, and $(S_{1,2})_{\bar 1} = V$ is a two-dimensional vector space equipped with a nonzero alternating bilinear form $\langle \cdot | \cdot \rangle$. The multiplication is given by

$$1 \bullet 1 = 1, \quad 1 \bullet u = -u = u \bullet 1, \quad u \bullet v = \langle u|v\rangle 1,$$

for any $u, v \in V$, and the norm $q = (q_{\bar 0}, \mathrm{b})$ is given by

$$q_{\bar 0}(1) = 1, \qquad \mathrm{b}(u, v) = \langle u|v\rangle,$$

for any $u, v \in V$.

Recall from [Elduque and Okubo 2002] or [Cunha and Elduque 2007a, Corollary 2.12] that the triality Lie superalgebra of $S_{1,2}$ is given by

$$\mathfrak{tri}(S_{1,2}) = \{(d, d, d) : d \in \mathfrak{osp}(S_{1,2}, q)\}, \tag{2-1}$$

and thus $\mathfrak{tri}(S_{1,2})$ can (and will) be identified with the Lie superalgebra

$$\mathfrak{b}_{0,1} = \mathfrak{sp}(V) \oplus V$$

(see [Cunha and Elduque 2007a, (2.18)]), with even part $\mathfrak{sp}(V)$ ($\cong \mathfrak{sl}_2$), odd part V, where $[\rho, v] = \rho(v)$ and $[u, v] = \gamma_{u,v}$ for any $\rho \in \mathfrak{sp}(V)$ and $u, v \in V$, with $\gamma_{u,v} = \langle u|\cdot\rangle v + \langle v|\cdot\rangle u$.

Besides, the action of $\mathfrak{b}_{0,1}$ on $S_{1,2}$ is given by

$$\rho : \begin{cases} 1 \mapsto 0, \\ u \mapsto \rho(u), \end{cases} \qquad\qquad u : \begin{cases} 1 \mapsto -u, \\ v \mapsto -\langle u|v\rangle 1, \end{cases}$$

for any $\rho \in \mathfrak{sp}(V)$ and $u, v \in V$ (see [Cunha and Elduque 2007a, (2.16)]).

Consider now the Lie superalgebra $\mathfrak{g}(S_8, S_{1,2})$ in the supermagic square:

$$\mathfrak{g}(S_8, S_{1,2}) = \big(\mathfrak{tri}(S_8) \oplus \mathfrak{tri}(S_{1,2})\big) \oplus \left(\bigoplus_{i=0}^{2} \iota_i(S_8 \otimes S_{1,2})\right).$$

This is $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded with

$$\begin{aligned}
\mathfrak{g}(S_8, S_{1,2})_{(0,0)} &= \mathfrak{tri}(S_8) \oplus \mathfrak{tri}(S_{1,2}),\\
\mathfrak{g}(S_8, S_{1,2})_{(1,0)} &= \iota_0(S_8 \otimes S_{1,2}),\\
\mathfrak{g}(S_8, S_{1,2})_{(0,1)} &= \iota_1(S_8 \otimes S_{1,2}),\\
\mathfrak{g}(S_8, S_{1,2})_{(1,1)} &= \iota_2(S_8 \otimes S_{1,2}),
\end{aligned}$$

and, therefore, the linear map τ, defined by

$$\tau = \begin{cases} \mathrm{id} & \text{on } \mathfrak{g}(S_8, S_{1,2})_{(0,0)} \oplus \mathfrak{g}(S_8, S_{1,2})_{(1,0)}, \\ -\mathrm{id} & \text{on } \mathfrak{g}(S_8, S_{1,2})_{(0,1)} \oplus \mathfrak{g}(S_8, S_{1,2})_{(1,1)}, \end{cases}$$

is a Lie superalgebra automorphism. On the other hand, the grading automorphism

$$\sigma = \begin{cases} \mathrm{id} & \text{on } \mathfrak{g}(S_8, S_{1,2})_{\bar 0}, \\ -\mathrm{id} & \text{on } \mathfrak{g}(S_8, S_{1,2})_{\bar 1}, \end{cases}$$

commutes with τ. Consider the order two automorphism $\xi = \sigma\tau = \tau\sigma$, which provides a $\mathbb{Z}_2$-grading of $\mathfrak{g}(S_8, S_{1,2})$ with even and odd components given by

$$\begin{aligned}
\mathfrak{g}(S_8, S_{1,2})_+ &= (\mathfrak{tri}(S_8) \oplus \mathfrak{sp}(V)) \oplus \iota_0(S_8 \otimes 1) \oplus \iota_1(S_8 \otimes V) \oplus \iota_2(S_8 \otimes V),\\
\mathfrak{g}(S_8, S_{1,2})_- &= V \oplus \iota_0(S_8 \otimes V) \oplus \iota_1(S_8 \otimes 1) \oplus \iota_2(S_8 \otimes 1).
\end{aligned} \tag{2-2}$$

Theorem 2.1. *In the situation above, the subalgebra* $\mathfrak{g}(S_8, S_{1,2})_+$ *of* $\mathfrak{g}(S_8, S_{1,2})$ *fixed by the automorphism* ξ *is a maximal subalgebra of* $\mathfrak{g}(S_8, S_{1,2})$ *isomorphic to the Lie superalgebra* $\mathfrak{el}(5; 3)$.

Proof. As a module for the subalgebra $\mathfrak{tri}(S_8) \oplus \mathfrak{sp}(V)$ of $\mathfrak{g}(S_8, S_{1,2})_+$, the odd component $\mathfrak{g}(S_8, S_{1,2})_-$ relative to the $\mathbb{Z}_2$-grading given by ξ decomposes as the direct sum of the nonisomorphic irreducible modules

$$V, \quad \iota_0(S_8 \otimes V), \quad \iota_1(S_8 \otimes 1), \quad \iota_2(S_8 \otimes 1).$$

Actually, identifying $\mathfrak{tri}(S_8)$ to the orthogonal Lie algebra $\mathfrak{so}_8$ through the projection onto the first component (this is possible because of the local principle of triality [Knus et al. 1998, §35]), $\iota_1(S_8 \otimes 1)$ and $\iota_2(S_8 \otimes 1)$ are the two half-spin representations of $\mathfrak{so}_8$, while $\iota_0(S_8 \otimes V)$ is the tensor product of the natural modules for $\mathfrak{so}_8$ and for $\mathfrak{sp}(V)$, so these four modules are indeed nonisomorphic. Therefore, any $\mathfrak{g}(S_8, S_{1,2})_+$-submodule of $\mathfrak{g}(S_8, S_{1,2})_-$ is a direct sum of some of them. But the definition of the Lie bracket in $\mathfrak{g}(S_8, S_{1,2})$ shows that any of these spaces generates $\mathfrak{g}(S_8, S_{1,2})_-$ as a module over $\mathfrak{g}(S_8, S_{1,2})_+$. Hence $\mathfrak{g}(S_8, S_{1,2})_-$ is an irreducible module for $\mathfrak{g}(S_8, S_{1,2})_+$ and, therefore, $\mathfrak{g}(S_8, S_{1,2})_+$ is a maximal subalgebra of $\mathfrak{g}(S_8, S_{1,2})$.

From now on, the proof relies heavily on the description of $\mathfrak{g}(S_8, S_{1,2})$ given in [Cunha and Elduque 2007a, §5.10] (which follows the ideas in [Elduque 2007a]). This description is obtained in terms of five vector spaces $V_1, \dots, V_5$ of dimension 2, endowed with nonzero alternating bilinear forms

$$\mathfrak{g}(S_8, S_{1,2}) = \bigoplus_{\sigma \in \mathcal{S}_{8,3}} V(\sigma), \tag{2-3}$$

with

$$\mathcal{S}_{8,3} = \big\{\varnothing, \{1,2,3,4\}, \{5\}, \{1,2\}, \{3,4\}, \{1,2,5\}, \{3,4,5\},\\ \{2,3\}, \{1,4\}, \{2,3,5\}, \{1,4,5\}, \{1,3\}, \{2,4\}, \{1,3,5\}, \{2,4,5\}\big\}.$$

Here $V(\varnothing) = \bigoplus_{i=1}^{5} \mathfrak{sp}(V_i)$, while for $\varnothing \neq \sigma = \{i_1, \dots, i_r\}$, $V(\sigma) = V_{i_1} \otimes \cdots \otimes V_{i_r}$. Also, any $\sigma \subseteq \{1,2,3,4,5\}$ can be thought of as an element in $\mathbb{Z}_2^5$ (for instance, $\{1,3,5\} = (\bar{1}, \bar{0}, \bar{1}, \bar{0}, \bar{1}) \in \mathbb{Z}_2^5$), so it makes sense to consider $\sigma + \tau$ for $\sigma, \tau \subseteq \{1,2,3,4,5\}$.

The brackets $V(\sigma) \times V(\tau) \to V(\sigma+\tau)$ are nonzero scalar multiples of the "contraction maps" $\varphi_{\sigma,\tau}$ in [Cunha and Elduque 2007a, (4.9)]. Under this description,

$$\mathfrak{g}(S_8, S_{1,2})_+ = \bigoplus_{\sigma \in \tilde{\mathcal{S}}_{8,3}} V(\sigma), \tag{2-4}$$

with

$$\tilde{\mathcal{S}}_{8,3} = \big\{\varnothing, \{1,2,3,4\}, \{1,2\}, \{3,4\}, \{2,3,5\}, \{1,4,5\}, \{1,3,5\}, \{2,4,5\}\big\}.$$

Thus, the even and odd degrees are

$$\Phi_{\bar 0} = \{\pm 2\epsilon_i : 1 \le i \le 5\} \cup \{\pm\epsilon_1 \pm\epsilon_2 \pm\epsilon_3 \pm\epsilon_4\} \cup \{\pm\epsilon_i \pm\epsilon_j : (i,j) \in \{(1,2),(3,4)\}\},$$
$$\Phi_{\bar 1} = \{\pm\epsilon_5\} \cup \{\pm\epsilon_i \pm\epsilon_j \pm\epsilon_5 : (i,j) \in \{(2,3),(1,4),(1,3),(2,4)\}\},$$

in the same notation of [Cunha and Elduque 2007a, §5]. With the lexicographic order given by $0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < \epsilon_4 < \epsilon_5$ in §5.10 of the same reference, the set of irreducible degrees is

$$\Pi = \{\alpha_1 = \epsilon_5 - \epsilon_2 - \epsilon_4,\ \alpha_2 = \epsilon_2 - \epsilon_1,\ \alpha_3 = 2\epsilon_1,\ \alpha_4 = \epsilon_4 - \epsilon_1 - \epsilon_2 - \epsilon_3,\ \alpha_5 = 2\epsilon_3\},$$

which is a $\mathbb{Z}$-linearly independent set with $\Phi = \Phi_{\bar 0} \cup \Phi_{\bar 1} \subseteq \mathbb{Z}\Pi$. The associated Cartan matrix is

$$\begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ -1 & 2 & -2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

which is equal to the second matrix in [Bouarroudj et al. 2007, §13.2] for $\mathfrak{el}(5;3)$ with the third and fourth rows and columns permuted. This shows that $\mathfrak{g}(S_8, S_{1,2})_+$ is isomorphic to $\mathfrak{el}(5;3)$. Note that the 4×4 submatrix on the lower right corner is the Cartan matrix of type B_4, and indeed it corresponds to the subalgebra $\mathfrak{tri}(S_8) \oplus \iota_0(S_8 \otimes 1)$, which is isomorphic to the orthogonal Lie algebra $\mathfrak{so}_9$ ($\mathfrak{tri}(S_8)$ being isomorphic to $\mathfrak{so}_8$ and S_8 to its natural module). □

3. Orthogonal triple systems and the Lie superalgebra $\mathfrak{el}(5;3)$

We now prove that $\mathfrak{el}(5;3)$ is the Lie superalgebra associated to a particular orthogonal triple system defined on the direct sum of two copies of the split octonions. Recall from [Okubo 1993] the notion of orthogonal triple systems:

Definition 3.1. Let T be a vector space over a field k endowed with a nonzero symmetric bilinear form $(\cdot\,|\,\cdot) : T \times T \to k$, and a triple product

$$T \times T \times T \to T : (x, y, z) \mapsto [xyz].$$

Then $(T, [\ldots], (\cdot\,|\,\cdot))$ is said to be an *orthogonal triple system* if it satisfies, for any elements $x, y, u, v, w \in T$,

(3-1a) $\quad [xxy] = 0,$

(3-1b) $\quad [xyy] = (x|y)y - (y|y)x,$

(3-1c) $\quad [xy[uvw]] = [[xyu]vw] + [u[xyv]w] + [uv[xyw]],$

(3-1d) $\quad ([xyu]|v) + (u|[xyv]) = 0.$

Equation (3-1c) shows that

$$\mathfrak{inder}\, T = \operatorname{span}\{[xy.] : x, y \in T\}$$

is a subalgebra (actually an ideal) of the Lie algebra $\mathfrak{der}\, T$ of derivations of T. The elements in $\mathfrak{inder}\, T$ are called *inner derivations*. Because of (3-1b), if $\dim T \geq 2$, then $\mathfrak{der}\, T$ is contained in the orthogonal Lie algebra $\mathfrak{so}(T, (\cdot\,|\,\cdot))$. Also note that (3-1d) is a consequence of (3-1b) and (3-1c) (see the comments after [Elduque 2006b, Definition 4.1]).

An ideal of an orthogonal triple system $(T, [\ldots], (\cdot\,|\,\cdot))$ is a subspace I such that $[ITT] + [TIT] + [TTI]$ is contained in I. The orthogonal triple system is said to be simple if it does not contain any proper ideal.

Some of the main properties of these systems are summarized in the next result, taken from [Elduque 2006b, Proposition 4.4, Theorem 4.5 and Theorem 5.1] (see also [Cunha and Elduque 2007b, Theorem 4.3]):

Proposition 3.2. *Let $(T, [\ldots], (\cdot\,|\,\cdot))$ be an orthogonal triple system of dimension ≥ 2. Then*

(1) *$(T, [\ldots], (\cdot\,|\,\cdot))$ is simple if and only if $(\cdot\,|\,\cdot)$ is nondegenerate.*

(2) *Let $(V, \langle\cdot\,|\,\cdot\rangle)$ be a two-dimensional vector space endowed with a nonzero alternating bilinear form. Let $\mathfrak{s}$ be a Lie subalgebra of $\mathfrak{der}\, T$ containing $\mathfrak{inder}\, T$. Define the superalgebra $\mathfrak{g} = \mathfrak{g}(T, \mathfrak{s}) = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ with*

$$\begin{cases} \mathfrak{g}_{\bar 0} = \mathfrak{sp}(V) \oplus \mathfrak{s}, \\ \mathfrak{g}_{\bar 1} = V \otimes T, \end{cases}$$

and superanticommutative multiplication given by

- *the multiplication on $\mathfrak{g}_{\bar 0}$ coincides with its bracket as a Lie algebra (the direct sum of the ideals $\mathfrak{sp}(V)$ and $\mathfrak{s}$);*
- *$\mathfrak{g}_{\bar 0}$ acts naturally on $\mathfrak{g}_{\bar 1}$, that is,*

$$[s, v \otimes x] = s(v) \otimes x, \qquad [d, v \otimes x] = v \otimes d(x),$$

for any $s \in \mathfrak{sp}(V)$, $d \in \mathfrak{s}$, $v \in V$, and $x \in T$;
- *for any $u, v \in V$ and $x, y \in T$,*

$$[u \otimes x, v \otimes y] = -(x|y)\gamma_{u,v} + \langle u|v\rangle d_{x,y} \tag{3-2}$$

where $\gamma_{u,v} = \langle u|\cdot\rangle v + \langle v|\cdot\rangle u$ and $d_{x,y} = [xy.]$.

Then $\mathfrak{g}(T, \mathfrak{s})$ is a Lie superalgebra. Moreover, $\mathfrak{g}(T, \mathfrak{s})$ is simple if and only if $\mathfrak{s}$ coincides with $\mathfrak{inder}\, T$ and T is a simple orthogonal triple system.

Conversely, given a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ *with*

$$\begin{cases} \mathfrak{g}_{\bar 0} = \mathfrak{sp}(V) \oplus \mathfrak{s} & (\textit{direct sum of ideals}), \\ \mathfrak{g}_{\bar 1} = V \otimes T & (\textit{as a module for } \mathfrak{g}_{\bar 0}), \end{cases}$$

where T *is a module over* $\mathfrak{s}$, *by* $\mathfrak{sp}(V)$*-invariance of the Lie bracket*, (3-2) *is satisfied for a symmetric bilinear form* $(\cdot \mid \cdot) : T \times T \to k$ *and an antisymmetric bilinear map* $d_{\cdot,\cdot} : T \times T \to \mathfrak{s}$. *Then, if* $(\cdot \mid \cdot)$ *is not* 0 *and a triple product on* T *is defined by means of* $[xyz] = d_{x,y}(z)$, T *becomes an orthogonal triple system and the image of* $\mathfrak{s}$ *in* $\mathfrak{gl}(T)$ *under the given representation is a subalgebra of* $\mathfrak{Der}\, T$ *containing* $\mathfrak{inDer}\, T$.

(3) *If the characteristic of the ground field* k *is equal to* 3, *define the* $\mathbb{Z}_2$*-graded algebra* $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(T) = \tilde{\mathfrak{g}}_{\bar 0} \oplus \tilde{\mathfrak{g}}_{\bar 1}$, *with*

$$\tilde{\mathfrak{g}}_{\bar 0} = \mathfrak{inDer}(T), \qquad \tilde{\mathfrak{g}}_{\bar 1} = T,$$

and anticommutative multiplication given by:

- *the multiplication on* $\tilde{\mathfrak{g}}_{\bar 0}$ *coincides with its bracket as a Lie algebra*;
- $\tilde{\mathfrak{g}}_{\bar 0}$ *acts naturally on* $\tilde{\mathfrak{g}}_{\bar 1}$, *that is*, $[d, x] = d(x)$ *for any* $d \in \mathfrak{inDer}(T)$ *and* $x \in T$;
- $[x, y] = d_{x,y} = [xy.]$, *for any* $x, y \in T$.

Then $\tilde{\mathfrak{g}}(T)$ *is a Lie algebra. Moreover*, T *is a simple orthogonal triple system if and only if* $\tilde{\mathfrak{g}}(T)$ *is a simple* $\mathbb{Z}_2$*-graded Lie algebra.*

The Lie superalgebra

$$\mathfrak{g}(T) = \mathfrak{g}(T, \mathfrak{inDer}(T))$$

in item (2) above will be called the *Lie superalgebra of the orthogonal triple system* T and, if the characteristic is 3, the Lie algebra $\tilde{\mathfrak{g}}(T)$ will be called the *Lie algebra of the orthogonal triple system* T.

The classification of the simple finite-dimensional orthogonal triple systems in characteristic 0 appears in [Elduque 2006b, Theorem 4.7]. In characteristic 3, there appears at least one new family of simple orthogonal triple systems, which are attached to degree 3 Jordan algebras (see Examples 4.20 of the same reference):

Let $J = \mathcal{J}ord(n, 1)$ be the Jordan algebra of a nondegenerate cubic form n with basepoint 1, over a field k of characteristic 3, and assume that $\dim_k J \geq 3$. Then any $x \in J$ satisfies a cubic equation [McCrimmon 2004, II.4]

$$x^{\circ 3} - t(x)x^{\circ 2} + s(x)x - n(x)1 = 0, \tag{3-3}$$

where t is its trace linear form, $s(x)$ is the spur quadratic form and the multiplication in J is denoted by $x \circ y$. For our purposes it is enough to consider the Jordan algebras in (3-5) below.

Let $J_0 = \{x \in J : t(x) = 0\}$ be the subspace of trace zero elements. Since $\operatorname{char} k = 3$, $t(1) = 0$, so that $k1 \in J_0$. Consider the quotient space $\hat{J} = J_0/k1$. For any $x \in J_0$, we have $s(x) = -\frac{1}{2}t(x^{\circ 2})$ and, by linearization of (3-3), we get, for any $x, y \in J_0$,

$$
\begin{aligned}
y^{\circ 2} \circ x - (x \circ y) \circ y &\equiv -2t(x, y)y - t(y, y)x \quad \bmod k1, \\
&\equiv t(x, y)y - t(y, y)x \quad \bmod k1.
\end{aligned} \tag{3-4}
$$

Let us denote by $\hat{x}$ the class of x modulo $k1$. Since J_0 is the orthogonal complement of $k1$ relative to the trace bilinear form $t(a, b) = t(a \circ b)$, t induces a nondegenerate symmetric bilinear form on $\hat{J}$ defined by $t(\hat{x}, \hat{y}) = t(x, y)$ for any $x, y \in J_0$. Now, for any $x, y \in J_0$ consider the inner derivation of J given by

$$D_{x,y} : z \mapsto x \circ (y \circ z) - y \circ (x \circ z)$$

(see [Jacobson 1968]). Since the trace form is invariant under the Lie algebra of derivations, $D_{x,y}$ leaves J_0 invariant, and obviously satisfies $D_{x,y}(1) = 0$, so it induces a map

$$d_{x,y} : \hat{J} \to \hat{J}, \ \hat{z} \mapsto \widehat{D_{x,y}(z)}$$

and a well-defined bilinear map

$$(\cdot\,,\cdot) : \hat{J} \times \hat{J} \to \mathfrak{gl}(\hat{J}), \ (\hat{x}, \hat{y}) \mapsto d_{x,y}.$$

Consider now the triple product $[\dots]$ on $\hat{J}$ defined by

$$[\hat{x}\hat{y}\hat{z}] = d_{x,y}(\hat{z})$$

for any $x, y, z \in J_0$. This is well-defined and satisfies (3-1a), because of the antisymmetry of $d_{\cdot,\cdot}$. Also, (3-4) implies that

$$[\hat{x}\hat{y}\hat{y}] = d_{x,y}(\hat{y}) = t(x, y)\hat{y} - t(y, y)\hat{x} = t(\hat{x}, \hat{y})\hat{y} - t(\hat{y}, \hat{y})\hat{x},$$

so (3-1b) is satisfied too, relative to the trace bilinear form. Since $D_{x,y}$ is a derivation of J for any $x, y \in J$, (3-1c) follows immediately, while (3-1d) is a consequence of $D_{x,y}$ being a derivation and the trace t being associative.

Therefore, by nondegeneracy of the trace form, $(\hat{J}, [\dots], t(\cdot\,,\cdot))$ is a simple orthogonal triple system over k [Elduque 2006b, Examples 4.20].

Now, let $e \neq 0, 1$ be an idempotent ($e^{\circ 2} = e$) of such a Jordan algebra. Changing e by $1 - e$ if necessary, it can be assumed that $t(e) = 1$. Consider the Peirce 1-space

$$J_1(e) = \{x \in J : e \circ x = \tfrac{1}{2}x\}.$$

Note that $J_1(e)$ is contained in J_0, because for any $x \in J_1(e)$, we have

$$t(x) = 2t(e \circ x) = 2t((e \circ e) \circ x)) = 2t(e \circ (e \circ x)) = \tfrac{1}{2}t(x),$$

so $t(x)=0$, and since $1\in J_0(e)\oplus J_2(e)$, $J_1(e)$ embeds in $\hat{J}=J_0/k1$. Besides, since $J_1(e)\circ J_1(e)\subseteq J_0(e)\oplus J_2(e)$ and $(J_0(e)\oplus J_2(e))\circ J_1(e)\subseteq J_1(e)$ (see [McCrimmon 2004, II.8]), it follows that $J_1(e)$ is an orthogonal triple subsystem of the orthogonal triple system $\hat{J}$ above.

In particular, let C be a Hurwitz algebra over the field k of characteristic 3 with norm q and polar form b, and consider the Jordan algebra $J=H_3(C,*)$ of hermitian 3×3 matrices (where $(a_{ij})^*=(\bar{a}_{ji})$) under the symmetrized product

$$x\circ y=\tfrac{1}{2}(xy+yx).$$

Let S be the associated para-Hurwitz algebra. Then,

$$\begin{aligned} J=H_3(C,*)&=\left\{\begin{pmatrix}\alpha_0&\bar{a}_2&a_1\\ a_2&\alpha_1&\bar{a}_0\\ \bar{a}_1&a_0&\alpha_2\end{pmatrix}:\alpha_0,\alpha_1,\alpha_2\in k,\ a_0,a_1,a_2\in S\right\}\\ &=\left(\bigoplus_{i=0}^{2}ke_i\right)\oplus\left(\bigoplus_{i=0}^{2}\iota_i(S)\right), \end{aligned}\tag{3-5}$$

where

$$e_0=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad \iota_0(a)=2\begin{pmatrix}0&0&0\\0&0&\bar{a}\\0&a&0\end{pmatrix},$$
$$e_1=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad \iota_1(a)=2\begin{pmatrix}0&0&a\\0&0&0\\\bar{a}&0&0\end{pmatrix},$$
$$e_2=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},\quad \iota_2(a)=2\begin{pmatrix}0&\bar{a}&0\\a&0&0\\0&0&0\end{pmatrix},$$

for any $a\in S$. Then J is the Jordan algebra of the nondegenerate cubic form n with basepoint 1, where

$$n(x)=\alpha_0\alpha_1\alpha_2+\mathrm{b}(a_0a_1a_2,1)-\sum_{i=0}^{2}\alpha_iq(a_i),$$

for

$$x=\sum_{i=0}^{2}\alpha_ie_i+\sum_{i=0}^{2}\iota_i(a_i).$$

Here the trace form t is the usual trace: $t(x)=\sum_{i=0}^{2}\alpha_i$.

Identify $ke_0\oplus ke_1\oplus ke_2$ with k^3 by means of

$$\alpha_0e_0+\alpha_1e_1+\alpha_2e_2\simeq(\alpha_0,\alpha_1,\alpha_2).$$

Then the Jordan product becomes

$$\begin{cases} (\alpha_0,\alpha_1,\alpha_2)\circ(\beta_1,\beta_2,\beta_3)=(\alpha_0\beta_0,\alpha_1\beta_1,\alpha_2\beta_2),\\ (\alpha_0,\alpha_1,\alpha_2)\circ\iota_i(a)=\frac{1}{2}(\alpha_{i+1}+\alpha_{i+2})\iota_i(a),\\ \iota_i(a)\circ\iota_{i+1}(b)=\iota_{i+2}(a\bullet b),\\ \iota_i(a)\circ\iota_i(b)=2\mathrm{b}(a,b)(e_{i+1}+e_{i+2}),\end{cases}$$

for any $\alpha_i,\beta_i\in k$, $a,b\in S$, where $i=0,1,2$, and indices are taken modulo 3.

Now, $e=e_0$ is an idempotent of trace 1 and the Peirce 1-space is $\iota_1(S)\oplus\iota_2(S)$. Denote by T_{2S} this orthogonal triple system. Then, in case $S=S_8$, T_{2S_8} is an orthogonal triple system defined on the direct sum of two copies of the split octonions, and we obtain:

Theorem 3.3. *Let k be a field of characteristic 3. Then the Lie superalgebra $\mathfrak{g}(T_{2S_8})$ of the orthogonal triple system T_{2S_8} is isomorphic to $\mathfrak{el}(5;3)$.*

Proof. Let C be the split Cayley algebra over k, whose associated para-Hurwitz algebra is S_8, and let J be the degree three simple Jordan algebra $H_3(C,*)$ considered above. Then, as vector spaces, T_{2S_8} coincides with the Peirce 1-space $J_1(e_0)$. The decomposition in (3-5) is a grading over $\mathbb{Z}_2\times\mathbb{Z}_2$ of the Jordan algebra J, and thus the Lie algebra of derivations of J is also $\mathbb{Z}_2\times\mathbb{Z}_2$-graded as follows (see [Cunha and Elduque 2007b, (3.12)]):

$$\mathfrak{Der}\,J=D_{\mathfrak{tri}(S_8)}\oplus\Bigl(\bigoplus_{i=0}^{2}D_i(S_8)\Bigr),$$

where, for $(d_0,d_1,d_2)\in\mathfrak{tri}(S_8)$,

$$\begin{cases} D_{(d_0,d_1,d_2)}(e_i)=0,\\ D_{(d_0,d_1,d_2)}(\iota_i(a))=\iota_i(d_i(a))\end{cases}$$

for any $i=0,1,2$ and $a\in S_8$ (see [Cunha and Elduque 2007b, (3.6)]), while

$$D_i(a)=2[L_{\iota_i(a)},L_{e_{i+1}}]$$

(indices modulo 3) for $0\le i\le 2$ and $a\in S_8$, where L_x denotes the left multiplication by x.

Note that $D_{\mathfrak{tri}(S_8)}\oplus D_0(S_8)$ leaves

$$J_1(e_0)=\iota_1(S_8)\oplus\iota_2(S_8)$$

invariant, and therefore embeds naturally in $\mathfrak{Der}\,T_{2S_8}$.

Besides, the Lie superalgebra of the orthogonal triple system $\hat{J}$ is (see [Cunha and Elduque 2007b, §4])

$$\mathfrak{g}(J)=(\mathfrak{sp}(V)\oplus\mathfrak{Der}\,J)\oplus(V\otimes\hat{J}),$$

which is shown in [Cunha and Elduque 2007b, Theorem 4.9] to be isomorphic to $\mathfrak{g}(S_8, S_{1,2})$. Under this isomorphism $V \otimes T_{2S_8}$ corresponds to

$$\iota_1(S_8 \otimes V) \oplus \iota_2(S_8 \otimes V)$$

inside $\mathfrak{g}(S_8, S_{1,2})$ which, under the isomorphism in Theorem 2.1, corresponds to the odd part of $\mathfrak{el}(5; 3)$, and this odd part generates $\mathfrak{el}(5; 3)$ as a Lie superalgebra. Therefore, the Lie superalgebra generated by $V \otimes T_{2S_8}$ corresponds to the subalgebra $\mathfrak{g}(S_8, S_{1,2})_+$ (isomorphic to $\mathfrak{el}(5; 3)$). Using the isomorphism in [Cunha and Elduque 2007b, Theorem 4.9], this proves that the subalgebra generated by $V \otimes T_{2S_8}$ in $\mathfrak{g}(J)$ is

$$\bigl(\mathfrak{sp}(V) \oplus (D_{\mathfrak{tri}(S_8)} \oplus D_0(S_8))\bigr) \oplus (V \otimes T_{2S_8}).$$

Since this is a simple Lie superalgebra, by Proposition 3.2 (2) it follows that it is isomorphic to the Lie superalgebra of the orthogonal triple system T_{2S_8}. □

Remark 3.4. Proposition 3.2 (3) shows that $\tilde{\mathfrak{g}}(T_{2S_8})$ is a simple Lie algebra. By the proof above, it is $\mathbb{Z}_2$-graded with even component isomorphic to $D_{\mathfrak{tri}(S_8)} \oplus D_0(S_8)$, which is isomorphic to the orthogonal Lie algebra $\mathfrak{so}_9$, and with odd component (in the $\mathbb{Z}_2$-grading) given by T_{2S_8}, which is the spin module for the even component. It follows that $\tilde{\mathfrak{g}}(T_{2S_8})$ is the exceptional Lie algebra of type F_4.

4. Orthosymplectic triple systems and the Lie superalgebra $\mathfrak{br}(2; 3)$

Orthosymplectic triple systems are the superversion of the orthogonal triple systems. They unify both orthogonal and symplectic triple systems. The definition was given in [Cunha and Elduque 2007b, Definition 6.2]:

Definition 4.1. Let $T = T_{\bar{0}} \oplus T_{\bar{1}}$ be a vector superspace endowed with an even nonzero supersymmetric bilinear form

$$(\cdot\,|\,\cdot) : T \times T \to k$$

(that is, $(T_{\bar{0}}|T_{\bar{1}}) = 0$, $(\cdot\,|\,\cdot)$ is symmetric on $T_{\bar{0}}$ and alternating on $T_{\bar{1}}$) and a triple product

$$[\ldots] : T \times T \times T \to T, \ (x, y, z) \mapsto [xyz]$$

($[x_i y_j z_k] \in T_{i+j+k}$ for any $x_i \in T_i$, $y_j \in T_j$, $z \in T_k$, where $i, j, k = \bar{0}$ or $\bar{1}$).

Then T is said to be an *orthosymplectic triple system* if it satisfies, for any homogeneous elements $x, y, u, v, w \in T$,

$$[xyz] + (-1)^{xy}[yxz] = 0, \tag{4-1a}$$

$$[xyz] + (-1)^{yz}[xzy] = (x|y)z + (-1)^{yz}(x|z)y - 2(y|z)x, \tag{4-1b}$$

$$\begin{aligned}[xy[uvw]] = [[xyu]vw] &+ (-1)^{(x+y)u}[u[xyv]w] \\ &+ (-1)^{(x+y)(u+v)}[uv[xyw]],\end{aligned} \tag{4-1c}$$

$$([xyu]|v) + (-1)^{(x+y)u}(u|[xyv]) = 0. \tag{4-1d}$$

Remark 4.2. If $T_{\bar{1}} = 0$, this is just the definition of an orthogonal triple system, while if $T_{\bar{0}} = 0$, then it reduces to a symplectic triple system.

As for orthogonal triple systems, the subspace

$$\mathfrak{inder}\, T = \text{span}\,\{[xy.] : x, y \in T\}$$

is a subalgebra (actually an ideal) of the Lie superalgebra $\mathfrak{der}\, T$ of derivations of T, whose elements are called *inner derivations*.

Proposition 4.3. *Let T be a simple orthosymplectic triple system. Then its supersymmetric bilinear form $(\cdot\,|\,\cdot)$ is nondegenerate. The converse is valid unless the characteristic of* k *is* 3, $T = T_{\bar{1}}$ *and* $\dim T = 2$.

Proof. Given an orthosymplectic triple system, the kernel of its supersymmetric bilinear form: $T^\perp = \{x \in T : (x|T) = 0\}$, satisfies $[TTT^\perp] \subseteq T^\perp$ because of (4-1d), while (4-1a) and (4-1b) show that

$$[TT^\perp T] = [T^\perp TT] \subseteq [TTT^\perp] + T^\perp,$$

so $T^\perp$ is an ideal of T. Thus, if T is simple, then $(\cdot\,|\,\cdot)$ is nondegenerate.

Conversely, assume $T^\perp = 0$ and let $I = I_{\bar{0}} \oplus I_{\bar{1}}$ be a proper ideal of T. For homogeneous elements $x, y, z \in T$, (4-1b) shows that the element

$$(x|y)z + (-1)^{yz}(x|z)y - 2(y|z)x$$

belongs to I if at least one of x, y, z is in I. For $x \in I$ we obtain

$$(x|y)z + (-1)^{yz}(x|z)y \in I,$$

while for $y \in I$, after permuting x and y,

$$(x|y)z - 2(-1)^{yz}(x|z)y \in I,$$

for homogeneous $x \in I$, $y, z \in T$. If the characteristic of k is not 3, it follows that $(I|T)T \subseteq I$, so $I = T$, a contradiction. But, even if the characteristic is 3, it follows that the codimension 1 subspace

$$(kx)^\perp = \{y \in T : (x|y) = 0\}$$

is contained in I for any homogeneous element $x\in I$, and $I=T$ unless $\dim T=2$. In the latter case, either $T=T_{\bar 0}$ or $T=T_{\bar 1}$. But for $T=T_{\bar 0}$, $(x|y)y\in I$ for any homogeneous $x\in I$ and $y\in T$, and hence also $\{y\in T:(x|y)\neq 0\}$ is contained in I, so $I=T$. Thus $T=T_{\bar 1}$. □

Remark 4.4. The two-dimensional symplectic triple system in [Elduque 2006b, Proposition 2.7(i)] shows that there are indeed nonsimple orthosymplectic triple systems of superdimension $0|2$ (that is, $\dim T_{\bar 0}=0$, $\dim T_{\bar 1}=2$).

Proposition 4.5 [Cunha and Elduque 2007b, Theorem 6.3]. *Let $(T,[\ldots],(\cdot\mid\cdot))$ be an orthosymplectic triple system and let $(V,\langle\cdot\mid\cdot\rangle)$ be a two-dimensional vector space endowed with a nonzero alternating bilinear form. Let $\mathfrak{s}$ be a Lie subsuperalgebra of $\mathfrak{Der}\,T$ containing $\mathfrak{inder}\,T$. Define the $\mathbb{Z}_2$-graded superalgebra $\mathfrak{g}=\mathfrak{g}(T,\mathfrak{s})=\mathfrak{g}(0)\oplus\mathfrak{g}(1)$ with*

$$\begin{cases}\mathfrak{g}(0)=\mathfrak{sp}(V)\oplus\mathfrak{s} & (\textit{so } \mathfrak{g}(0)_{\bar 0}=\mathfrak{sp}(V)\oplus\mathfrak{s}_{\bar 0},\ \mathfrak{g}(0)_{\bar 1}=\mathfrak{s}_{\bar 1}),\\ \mathfrak{g}(1)=V\otimes T & (\textit{with } \mathfrak{g}(1)_{\bar 0}=V\otimes T_{\bar 1},\ \mathfrak{g}(1)_{\bar 1}=V\otimes T_{\bar 0},\ V \textit{ is odd!})\end{cases}$$

and superanticommutative multiplication given by:

- *the multiplication on $\mathfrak{g}(0)$ coincides with its bracket as a Lie superalgebra*;
- *$\mathfrak{g}(0)$ acts naturally on $\mathfrak{g}(1)$*:

$$[s,v\otimes x]=s(v)\otimes x,\qquad [d,v\otimes x]=(-1)^d v\otimes d(x),$$

  for any $s\in\mathfrak{sp}(V)$, $v\in V$, and homogeneous elements $d\in\mathfrak{s}$ and $x\in T$;
- *for any $u,v\in V$ and homogeneous $x,y\in T$*:

$$[u\otimes x,v\otimes y]=(-1)^x\bigl(-(x|y)\gamma_{u,v}+\langle u|v\rangle d_{x,y}\bigr) \tag{4-2}$$

  where $\gamma_{u,v}=\langle u|\cdot\rangle v+\langle v|\cdot\rangle u$ and $d_{x,y}=[xy.]$.

Then $\mathfrak{g}(T,\mathfrak{s})$ is a $\mathbb{Z}_2$-graded Lie superalgebra. Moreover, $\mathfrak{g}(T,\mathfrak{s})$ is simple if and only if $\mathfrak{s}$ coincides with $\mathfrak{inder}\,T$ and $(\cdot\mid\cdot)$ is nondegenerate.

Conversely, given a $\mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{g}=\mathfrak{g}(0)\oplus\mathfrak{g}(1)$ with

$$\begin{cases}\mathfrak{g}(0)=\mathfrak{sp}(V)\oplus\mathfrak{s},\\ \mathfrak{g}(1)=V\otimes T,\end{cases}$$

where T is an $\mathfrak{s}$-module and V is considered as an odd vector space, by $\mathfrak{sp}(V)$-invariance of the bracket, (4-2) is satisfied for an even supersymmetric bilinear form $(\cdot\mid\cdot):T\times T\to k$ and a superantisymmetric bilinear map $d_{.,.}:T\times T\to\mathfrak{s}$. Then, if $(\cdot\mid\cdot)$ is not 0 and a triple product on T is defined by means of $[xyz]=d_{x,y}(z)$, T becomes an orthosymplectic triple system and the image of $\mathfrak{s}$ in $\mathfrak{gl}(T)$ under the given representation is a subalgebra of $\mathfrak{Der}\,T$ containing $\mathfrak{inder}\,T$.

The Lie superalgebra $\mathfrak{g}(T) = \mathfrak{g}(T, \mathfrak{inder}(T))$ is called the *Lie superalgebra of the orthosymplectic triple system T*.

If the characteristic of the ground field k is equal to 3, then for any homogeneous elements x, y, z in an orthosymplectic triple system, we have:

$$
\begin{aligned}
&[xyz] + (-1)^{x(y+z)}[yzx] + (-1)^{(x+y)z}[zxy] \\
&\qquad = [xyz] + (-1)^{x(y+z)}[yzx] - 2(-1)^{(x+y)z}[zxy] \\
&\qquad = ([xyz] + (-1)^{yz}[xzy]) \\
&\qquad\qquad\quad - (-1)^{xy+xz+yz}([zyx] + (-1)^{xy}[zxy]) \quad \text{(by (4-1a))} \\
&\qquad = ((x|y)z + (-1)^{yz}(x|z)y - 2(y|z)x) \\
&\qquad\qquad\quad - (-1)^{xy+xz+yz}((z|y)x + (-1)^{xy}(z|x)y - 2(y|x)z) \quad \text{(by (4-1b))} \\
&\qquad = 3((x|y)z - (y|z)x) = 0,
\end{aligned}
$$

so that, as in [Elduque 2006b, Theorem 5.1]:

Proposition 4.6. *Let $(T, [\ldots], (\cdot|\cdot))$ be an orthosymplectic triple system over a field k of characteristic 3. Define the $\mathbb{Z}_2$-graded superalgebra*

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(T) = \tilde{\mathfrak{g}}(T)_+ \oplus \tilde{\mathfrak{g}}(T)_-,$$

with $\tilde{\mathfrak{g}}_+ = \mathfrak{inder}(T)$, $\tilde{\mathfrak{g}}_- = T$ and superanticommutative multiplication given by:

- *the multiplication on $\tilde{\mathfrak{g}}_+$ coincides with its bracket as a Lie superalgebra*;
- *$\tilde{\mathfrak{g}}_+$ acts naturally on $\tilde{\mathfrak{g}}_-$, that is, $[d, x] = d(x)$ for any $d \in \mathfrak{inder}(T)$ and $x \in T$*;
- *$[x, y] = d_{x,y} = [xy.]$, for any $x, y \in \tilde{\mathfrak{g}}_- = T$.*

Then $\tilde{\mathfrak{g}}$ is a $\mathbb{Z}_2$-graded Lie superalgebra, with the even part $\tilde{\mathfrak{g}}_{\bar{0}} = \mathfrak{inder}(T)_{\bar{0}} \oplus T_{\bar{0}}$ and the odd part $\tilde{\mathfrak{g}}_{\bar{1}} = \mathfrak{inder}(T)_{\bar{1}} \oplus T_{\bar{1}}$. Moreover, T is a simple orthosymplectic triple system if and only if $\tilde{\mathfrak{g}}$ is simple as a $\mathbb{Z}_2$-graded Lie superalgebra.

Now, let C be a Hurwitz superalgebra of dimension > 1 over a field k of characteristic $\neq 2$, with norm $q = (q_{\bar{0}}, \mathrm{b})$, and standard involution $x \mapsto \bar{x}$. For any homogeneous elements x, y, z, the following holds (see [Elduque and Okubo 2002]):

$$
\begin{aligned}
&\mathrm{b}(xy, z) = (-1)^{xy}\mathrm{b}(y, \bar{x}z) = (-1)^{yz}\mathrm{b}(x, z\bar{y}), \\
&x\bar{y} + (-1)^{xy}y\bar{x} = \mathrm{b}(x, y)1 = \bar{x}y + (-1)^{xy}\bar{y}x, \\
&\bar{x}(yz) + (-1)^{xy}\bar{y}(xz) = \mathrm{b}(x, y)z = (zx)\bar{y} + (-1)^{xy}(zy)\bar{x}.
\end{aligned}
$$

Consider the subspace of trace zero elements,

$$C^0 = \{x \in C : \mathrm{b}(1, x) = 0\} = \{x \in C : \bar{x} = -x\}.$$

Then, for any homogeneous elements $x, y \in C^0$, we have

$$xy + (-1)^{xy} yx = -(x\bar{y} + (-1)^{xy} y\bar{x}) = -\mathrm{b}(x, y)1,$$

while $xy - (-1)^{xy} yx = [x, y]$. Thus

$$xy = \tfrac{1}{2}(-\mathrm{b}(x, y)1 + [x, y]). \tag{4-3}$$

Also, for any homogeneous elements $x, y, z \in C^0$, we have

$$\begin{aligned}\mathrm{b}([x, y], z) &= \mathrm{b}(xy - (-1)^{xy} yx, z) = \mathrm{b}(x, (-1)^{yz} z\bar{y} - \bar{y}z)\\ &= \mathrm{b}(x, yz - (-1)^{yz} zy) = \mathrm{b}(x, [y, z]),\end{aligned}$$

so

$$\mathrm{b}([x, y], z) = \mathrm{b}(x, [y, z]) \tag{4-4}$$

for any $x, y, z \in C^0$. Using (4-3) and (4-4) we obtain:

$$\begin{aligned}&[[x, y], z] + (-1)^{yz}[[x, z], y]\\ &\qquad = \mathrm{b}([x, y], z)1 + 2[x, y]z + (-1)^{yz}(\mathrm{b}([x, z], y)1 + 2[x, z]y)\\ &\qquad = 2([x, y]z + (-1)^{yz}[x, z]y)\\ &\qquad = 2\big(\mathrm{b}(x, y)z + 2(xy)z + (-1)^{yz}(\mathrm{b}(x, z)y + 2(xz)y)\big)\\ &\qquad = 2(\mathrm{b}(x, y)z + (-1)^{yz}\mathrm{b}(x, z)y) - 4((xy)\bar{z} + (-1)^{yz}(xz)\bar{y})\\ &\qquad = 2\mathrm{b}(x, y)z + 2(-1)^{yz}\mathrm{b}(x, z)y - 4\mathrm{b}(y, z)x,\end{aligned}$$

for any homogeneous $x, y, z \in C^0$. Therefore, with $(x|y) = 2\mathrm{b}(x, y)$, for any $x, y, z \in C^0$ we have:

$$[[x, y], z] + (-1)^{yz}[[x, z], y] = (x|y)z + (-1)^{yz}(x|z)y - 2(y|z)x. \tag{4-5}$$

Now, if the characteristic of the ground field k is equal to 3, for any homogeneous $x, y, z \in C^0$ we have:

$$\begin{aligned}&[[x, y], z] + (-1)^{x(y+z)}[[y, z], x] + (-1)^{(x+y)z}[[z, x], y]\\ &\qquad = [[x, y], z] + (-1)^{x(y+z)}[[y, z], x] - 2(-1)^{(x+y)z}[[z, x], y]\\ &\qquad = [[x, y], z] + (-1)^{yz}[[x, z], y]\\ &\qquad\qquad - (-1)^{xy+xz+yz}([[z, y], x] + (-1)^{xy}[[z, x], y])\\ &\qquad = ((x|y)z + (-1)^{yz}(x|z)y - 2(y|z)x)\\ &\qquad\qquad - (-1)^{xy+xz+yz}((z|y)x + (-1)^{xy}(z|x)y - 2(y|x)z)\\ &\qquad = 3((x|y)z - (y|z)x) = 0.\end{aligned}$$

Thus, $(C^0, [\cdot\,,\cdot])$ is a Lie superalgebra, and then (4-4) and (4-5) show the validity of the first assertion in the following result:

Theorem 4.7. *Let C be a Hurwitz superalgebra of dimension ≥ 2 over a field k of characteristic 3 with norm $q = (q_{\bar{0}}, \mathrm{b})$. Then, with the triple product $[xyz] = [[x, y], z]$ and the supersymmetric bilinear form $(\cdot\,|\,\cdot) = 2\mathrm{b}(\cdot\,,\cdot)$, C^0 becomes an orthosymplectic triple system. Moreover, if the dimension of C is ≤ 3, then the triple product is trivial, otherwise the inner derivation algebra $\mathfrak{inder}(C^0)$ equals ad_{C^0}, the linear span of the adjoint maps $\mathrm{ad}_x : y \mapsto [x, y]$ for any $x \in C^0$.*

Proof. Only the last assertion needs to be verified. If the dimension of C is at most 3, then C is supercommutative, so $[C^0, C^0] = 0$. However, if the dimension of C is at least 4 (hence either 4, 6 or 8) then $[C^0, C^0] = C^0$. □

Corollary 4.8. *Let C be a Hurwitz superalgebra of dimension ≥ 4 over a field k of characteristic 3. Let V be a two-dimensional vector space endowed with a nonzero alternating bilinear form $\langle\cdot\,|\,\cdot\rangle$. Consider the anticommutative superalgebra*

$$\mathfrak{g} = (\mathfrak{sp}(V) \oplus C^0) \oplus (V \otimes C^0),$$

with $\mathfrak{g}_{\bar{0}} = (\mathfrak{sp}(V) \oplus (C^0)_{\bar{0}}) \oplus (V \otimes (C^0)_{\bar{1}})$ and $\mathfrak{g}_{\bar{1}} = (C^0)_{\bar{1}} \oplus (V \otimes (C^0)_{\bar{0}})$, and multiplication given by:

- *the usual Lie bracket in the direct sum of the Lie algebra $\mathfrak{sp}(V)$ and the Lie superalgebra C^0,*
- *$[\gamma, v \otimes x] = \gamma(v) \otimes x$, for any $\gamma \in \mathfrak{sp}(V)$, $v \in V$ and $x \in C^0$,*
- *$[x, v \otimes y] = (-1)^x v \otimes [x, y]$, for any homogeneous $x, y \in C^0$ and $v \in V$,*
- *$[u \otimes x, v \otimes y] = (-1)^x(-(x|y)\gamma_{u,v} + \langle u|v\rangle[x, y]) \in \mathfrak{sp}(V) \oplus C^0$, for any $u, v \in V$ and homogeneous $x, y \in C^0$ (where, as before, $(x|y) = 2\mathrm{b}(x, y)$ and $\gamma_{u,v} = \langle u|\cdot\rangle v + \langle v|\cdot\rangle u$).*

Then $\mathfrak{g}$ is a Lie superalgebra.

Proof. It suffices to note that the Lie superalgebra $\mathfrak{g}$ is just the Lie superalgebra

$$\mathfrak{g}(C^0, \mathfrak{inder}(C^0))$$

in Proposition 4.5 of the orthosymplectic triple system $(C^0, [\ldots], (\cdot\,|\,\cdot))$ after the natural identification of $\mathfrak{inder}(C^0) = \mathrm{ad}_{C^0}$ with C^0. □

If the dimension of C in Corollary 4.8 is 4 (and hence C is a quaternion algebra), it is not difficult to see that the Lie superalgebra $\mathfrak{g}$ is a form of the orthosymplectic Lie superalgebra $\mathfrak{osp}_{3,2}$. Also, if the dimension of C is 8, so that C is an algebra of octonions, then $\mathfrak{g}$ is a form of the Lie superalgebra that appears in [Elduque

2006b, Theorem 4.22(i)], which is the derived subalgebra of the Lie superalgebra $\mathfrak{g}(S_2, S_{1,2})$ in the supermagic square (see [Cunha and Elduque 2007b, Corollary 4.10(ii)] and [Elduque 2007c, §3]). Also note that if the characteristic is not 3, then C^0 is still an orthogonal triple system, but its associated Lie superalgebra is a simple Lie superalgebra of type $G(3)$ (see [Elduque 2006b, Theorem 4.7 (G-type)]).

We are left with the $4|2$-dimensional Hurwitz superalgebra $C = B(4, 2)$ in (1-4) over a field k of characteristic 3. The Lie bracket of elements in C^0 is given by:

- The usual bracket $[f, g] = fg - gf$ in $\mathfrak{sl}(V) = \mathfrak{sp}(V)$;
- $[f, u] = f \cdot u - u \cdot f = -2f(u) = f(u)$ for any $f \in \mathfrak{sp}(V)$ and $u \in V$;
- $[u, v] = u\cdot v-(-1)^{uv}v\cdot u = u\cdot v+v\cdot u = \mathrm{b}(\,\cdot\,, u)v+\mathrm{b}(\,\cdot\,, v)u = (u|\cdot)v+(v|\cdot)u$ for any $u, v \in V$ (recall that $(\cdot|\cdot) = 2\mathrm{b}(\cdot\,,\cdot) = -\mathrm{b}(\cdot\,,\cdot)$).

Proposition 4.9. *The Lie superalgebra $B(4, 2)^0$ is isomorphic to the orthosymplectic Lie superalgebra $\mathfrak{osp}_{1,2}$.*

Proof. The orthosymplectic Lie superalgebra $\mathfrak{osp}_{1,2}$ is the subalgebra of the general Lie superalgebra $\mathfrak{gl}(1, 2)$ given by:

$$\mathfrak{osp}_{1,2} = \left\{ \left(\begin{array}{c|cc} 0 & -\nu & \mu \\ \hline \mu & \alpha & \beta \\ \nu & \gamma & -\alpha \end{array} \right) : \alpha, \beta, \gamma, \mu, \nu \in k \right\}.$$

Fix a basis $\{u, v\}$ of V with $(u|v) = 1$, and consider the linear map:

$$C^0 = \mathfrak{sp}(V) \oplus V \to \mathfrak{osp}_{1,2},$$

$$f \in \mathfrak{sp}(V) \mapsto \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \alpha & \beta \\ 0 & \gamma & -\alpha \end{array} \right) \quad \text{with} \quad \begin{cases} f(u) = \alpha u + \gamma v, \\ f(v) = \beta u - \alpha v, \end{cases}$$

$$\mu u + \nu v \in V \mapsto \left(\begin{array}{c|cc} 0 & -\nu & \mu \\ \hline \mu & 0 & 0 \\ \nu & 0 & 0 \end{array} \right).$$

This is checked to be an isomorphism of Lie algebras by straightforward computations. □

Also note that for $f \in \mathfrak{sp}(V)$, $f^2 = -\det(f)1$ (by the Cayley–Hamilton equation) and $q_{\bar 0}(f) = \det(f)$ and $\operatorname{tr}(f^2) = -2\det(f) = \det(f)$, so $q_{\bar 0}(f) = \operatorname{tr}(f^2)$, $\mathrm{b}(f, g) = 2\operatorname{tr}(fg)$, and $(f|g) = \operatorname{tr}(fg)$ for any $f, g \in \mathfrak{sp}(V) = (C^0)_{\bar 0}$. (Here tr denotes the usual trace in $\operatorname{End}_k(V) = B(4, 2)_{\bar 0}$).

Theorem 4.10. *The Lie superalgebra of the orthosymplectic triple system $B(4, 2)^0$ is isomorphic to the Lie superalgebra $\mathfrak{br}(2; 3)$.*

Proof. Since there are two vector spaces of dimension 2 involved here, let us denote them by V_1 and V_2, whose nonzero alternating bilinear forms will be both denoted by $\langle\cdot|\cdot\rangle$. Then consider the Hurwitz superalgebra $C = B(4,2) = \mathrm{End}_k(V_2) \oplus V_2$, as defined in (1-4). The Lie superalgebra associated to the orthosymplectic triple system C^0 is given, up to isomorphism, in Corollary 4.8:

$$\mathfrak{g} = (\mathfrak{sp}(V_1) \oplus C^0) \oplus (V_1 \otimes C^0).$$

Its even part is

$$\mathfrak{g}_{\bar{0}} = (\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)) \oplus (V_1 \otimes V_2),$$

with multiplication given by the natural Lie bracket in the direct sum $\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$, the natural action of this subalgebra on $V_1 \otimes V_2$, and by

$$[a \otimes u, b \otimes v] = (u|v)\gamma_{a,b} - \langle a|b\rangle \gamma_{u,v},$$

for any $a, b \in V_1$ and $u, v \in V_2$, where $(\cdot|\cdot) = 2\mathrm{b}(\cdot,\cdot)$. Here $\gamma_{a,b} = \langle a|\cdot\rangle b + \langle b|\cdot\rangle a$, while $\gamma_{u,v} = (u|\cdot)v + (v|\cdot)u$. This Lie algebra is precisely the Lie algebra $L(1)$ of Kostrikin [1970] (see also [Elduque 2006b, Proposition 2.12]).

On the other hand, its odd part is

$$\mathfrak{g}_{\bar{1}} = V_2 \oplus (V_1 \otimes \mathfrak{sp}(V_2)).$$

Since C^0 is a simple orthosymplectic triple system, the Lie superalgebra $\mathfrak{g}$ is simple (Proposition 4.5). Fix bases $\{a_i, b_i\}$ of V_i $(i = 1, 2)$ with $\langle a_1|b_1\rangle = 1 = (a_2|b_2)$, and let $h_i, e_i, f_i \in \mathfrak{sp}(V_i)$ be given by

$$\begin{aligned} h_i(a_i) &= a_i, & h_i(b_i) &= -b_i, \\ e_i(a_i) &= 0, & e_i(b_i) &= a_i, \\ f_i(a_i) &= b_i, & f_i(b_i) &= 0. \end{aligned} \tag{4-6}$$

Then span $\{h_1, h_2\}$ is a Cartan subalgebra of $\mathfrak{g}$, and it is the (0, 0)-component of the $\mathbb{Z}\times\mathbb{Z}$-grading of $\mathfrak{g}$ obtained by assigning $\deg(a_i) = \epsilon_i$, $\deg(b_i) = -\epsilon_i$ for $i = 1, 2$, where $\{\epsilon_1, \epsilon_2\}$ is the canonical basis of $\mathbb{Z}\times\mathbb{Z}$. The set of nonzero degrees is

$$\Phi = \{\pm 2\epsilon_1, \pm 2\epsilon_2, \pm\epsilon_1 \pm \epsilon_2, \pm\epsilon_2, \pm\epsilon_1, \pm\epsilon_1 \pm 2\epsilon_2\}.$$

Consider the elements

$$\begin{aligned} E_1 &= a_1 \otimes f_2, & F_1 &= b_1 \otimes e_2, & H_1 &= [E_1, F_1] = h_1 - h_2, \\ E_2 &= a_2, & F_2 &= -b_2, & H_2 &= [E_2, F_2] = h_2. \end{aligned}$$

Then we have that the subspace span $\{H_1, H_2\}$ = span $\{h_1, h_2\}$ is the previous Cartan subalgebra of $\mathfrak{g}$, E_1 belongs to the homogeneous component $\mathfrak{g}_{\epsilon_1 - 2\epsilon_2}$ in the

$\mathbb{Z}\times\mathbb{Z}$-grading, and similarly $F_1 \in \mathfrak{g}_{-\epsilon_1+2\epsilon_2}$, $E_2 \in \mathfrak{g}_{\epsilon_2}$, and $F_2 \in \mathfrak{g}_{-\epsilon_2}$. Also, the elements E_1, E_2, F_1, F_2 generate the Lie superalgebra $\mathfrak{g}$. Besides,

$$\begin{aligned}
&[H_1, E_1] = h_1(a_1)\otimes f_2 - a_1\otimes[h_2, f_2] = a_1\otimes f_2 + 2a_1\otimes f_2 = 0,\\
&[H_1, E_2] = (h_1 - h_2)(a_2) = -a_2,\\
&[H_2, E_1] = a_1\otimes[h_2, f_2] = -2a_1\otimes f_2,\\
&[H_2, E_2] = h_2(a_2) = a_2,
\end{aligned}$$

and similarly for the action of the H_i's on the F_j's. It follows, with the same arguments as in [Cunha and Elduque 2007a, §4], that $\mathfrak{g}$ is the Lie superalgebra with Cartan matrix

$$\begin{pmatrix} 0 & -1\\ -2 & 1\end{pmatrix},$$

which is the first Cartan matrix of the Lie superalgebra $\mathfrak{br}(2;3)$ given in [Bouarroudj et al. 2007, §10.1]. □

In this way, the Lie superalgebra $\mathfrak{br}(2;3)$, of superdimension $10|8$ is completely determined by the orthosymplectic triple system $B(4,2)^0$ (that is, by the orthosymplectic triple system obtained naturally from the Lie superalgebra $\mathfrak{osp}_{1,2}$) of superdimension $3|2$.

5. Orthosymplectic triple systems and the Lie superalgebra $\mathfrak{el}(5;3)$

In this section, the characteristic of the ground field k will always be assumed to be 3, since we will be dealing with the superalgebras $S_{1,2}$ and $\mathfrak{el}(5;3)$, which only make sense in this characteristic.

Equation (2-2), together with Theorem 2.1, which allows us to identify the Lie superalgebra $\mathfrak{el}(5;3)$ with the maximal subalgebra $\mathfrak{g}(S_8, S_{1,2})_+$, show that there is a decomposition of $\mathfrak{g}(S_8, S_{1,2})$ into the direct sum ($\mathbb{Z}_2$-grading)

$$\mathfrak{g}(S_8, S_{1,2}) = \mathfrak{el}(5;3)\oplus T,$$

where

$$\begin{aligned}
\mathfrak{el}(5;3) &= (\mathfrak{tri}(S_8)\oplus\mathfrak{sp}(V))\oplus\iota_0(S_8\otimes 1)\oplus(\iota_1(S_8\otimes V)\oplus\iota_2(S_8\otimes V)),\\
T &= (\iota_1(S_8\otimes 1)\oplus\iota_2(S_8\otimes 1))\oplus(V\oplus\iota_0(S_8\otimes V)),
\end{aligned}\tag{5-1}$$

where V is a two-dimensional vector space endowed with a nonzero alternating bilinear form.

This section will show that T is an orthosymplectic triple system, with the triple product given by $[xyz] = [[x,y],z]$ and a suitable supersymmetric bilinear form, and that $\mathfrak{el}(5;3)$ is isomorphic to the Lie superalgebra of derivations of this orthosymplectic triple system.

A few preliminary results are needed.

Lemma 5.1. *There exists a unique supersymmetric associative bilinear form*

$$B : \mathfrak{g}(S_8, S_{1,2}) \times \mathfrak{g}(S_8, S_{1,2}) \to k$$

such that

$$B(\iota_i(x \otimes u), \iota_j(y \otimes v)) = \delta_{ij}\mathrm{b}(x, y)\mathrm{b}(u, v), \tag{5-2}$$

for any $i, j = 0, 1, 2$, $x, y \in S_8$ and $u, v \in S_{1,2}$. Here δ_{ij} is the usual Kronecker delta and b *denotes the polar form of the norm in both S_8 and $S_{1,2}$.*

Proof. This is proved as in [Elduque 2006a, Corollary 4.9]. First there is a unique invariant supersymmetric bilinear form $B_{1,2}$ on the orthosymplectic Lie superalgebra $\mathfrak{osp}(S_{1,2}, q)$ such that

$$B_{1,2}(d, \sigma_{u,v}) = \mathrm{b}(d(u), v)$$

for any $u, v \in S_{1,2}$ and $d \in \mathfrak{osp}(S_{1,2}, q)$, where $\sigma_{u,v}$ is defined in (1-5). Actually, $B_{1,2}$ is given by $B_{1,2}(d, d') = -\frac{1}{2}\,\mathrm{str}(dd')$, where str denotes the supertrace. Note that

$$\mathrm{str}(\sigma_{x,y}\sigma_{u,v}) = -2\big((-1)^{yu}\mathrm{b}(x, u)\mathrm{b}(y, v) - (-1)^{(y+u)v}\mathrm{b}(x, v)\mathrm{b}(y, u)\big).$$

Also, in [Elduque 2006a] it is proved that there is a unique invariant symmetric bilinear form B_8 on $\mathfrak{tri}(S_8)$ such that

$$B_8((d_0, d_1, d_2), \theta^i(t_{x,y})) = \mathrm{b}(d_i(x), y)$$

for any $x, y \in S_8$ and $(d_0, d_1, d_2) \in \mathfrak{tri}(S_8)$.

Then the supersymmetric invariant bilinear form B required is defined by imposing the following conditions:

- The restriction of B to $\mathfrak{tri}(S_{1,2})$ is given by $B_{1,2}$ (after identifying $\mathfrak{tri}(S_{1,2})$ with $\mathfrak{osp}(S_{1,2}, q)$ because of (2-1)).
- The restriction of B to $\mathfrak{tri}(S_8)$ is given by B_8.
- The restriction of B to $\bigoplus_{i=0}^{2} \iota_i(S_8 \otimes S_{1,2})$ is given by (5-2). □

Note that $\mathfrak{g}(S_8, S_{1,2})$ is then the orthogonal direct sum, relative to B, of the subspaces $\mathfrak{tri}(S_8)$, $\mathfrak{tri}(S_{1,2})$ and $\iota_i(S_8 \otimes S_{1,2})$ for $i = 0, 1, 2$.

Now, the description of $\mathfrak{g}(S_8, S_{1,2})$ in the proof of Theorem 2.1 becomes quite useful in the proof of the next result:

Lemma 5.2. *Any derivation of the Lie superalgebra $\mathfrak{g}(S_8, S_{1,2})$ is inner.*

Proof. As in [Cunha and Elduque 2007a], take five two-dimensional vector spaces $V_1, \dots, V_5$ endowed with nonzero alternating bilinear forms $\langle \cdot \mid \cdot \rangle$. Take symplectic bases $\{v_i, w_i\}$ of V_i for any $i = 1, \dots, 5$ with $\langle v_i \mid w_i \rangle = 1$ and the basis $\{h_i, e_i, f_i\}$

of $\mathfrak{sp}(V_i)$ given by

$$h_i = \gamma_{v_i,w_i}, \quad e_i = \gamma_{w_i,w_i}, \quad f_i = -\gamma_{v_i,v_i},$$

which satisfy

$$[h_i, e_i] = 2e_i, \quad [h_i, f_i] = -2f_i, \quad \text{and} \quad [e_i, f_i] = h_i.$$

Consider the description of $\mathfrak{g}(S_8, S_{1,2})$ in (2-3):

$$\mathfrak{g}(S_8, S_{1,2}) = \bigoplus_{\sigma \in \mathcal{S}_{8,3}} V(\sigma).$$

This shows that $\mathfrak{g}(S_8, S_{1,2})$ is $\mathbb{Z}^5$-graded, by assigning

$$\deg w_i = \epsilon_i, \qquad \deg v_i = -\epsilon_i,$$

where $\{\epsilon_1, \dots, \epsilon_5\}$ is the canonical basis of $\mathbb{Z}^5$. The vector subspace

$$\mathfrak{h} = \operatorname{span}\{h_1, \dots, h_5\}$$

is a Cartan subalgebra of $\mathfrak{g}(S_8, S_{1,2})$. Consider the $\mathbb{Z}$-linear map

$$R : \mathbb{Z}^5 \to \mathfrak{h}^*, \quad \epsilon_i \mapsto R(\epsilon_i) : h_j \mapsto \delta_{ij}.$$

The set of nonzero degrees of $\mathfrak{g}(S_8, S_{1,2})$ in the $\mathbb{Z}^5$-grading is given by

$$\Phi = \{\pm 2\epsilon_i : i = 1, \dots, 5\} \\ \cup \bigl\{\pm\epsilon_{i_1} \pm \cdots \pm \epsilon_{i_r} : 1 \le i_1 < \cdots < i_r \le 5, \{i_1, \dots, i_r\} \in \mathcal{S}_{8,3} \setminus \{\varnothing\}\bigr\}.$$

The set $R(\Phi)$ is the set of roots of $\mathfrak{g}(S_8, S_{1,2})$ relative to the Cartan subalgebra $\mathfrak{h}$. Note that the restriction of R to Φ fails to be one-to-one only because $\{\pm 2\epsilon_5, \pm\epsilon_5\}$ is contained in Φ, and $R(\pm 2\epsilon_5) = R(\mp\epsilon_5)$, as the characteristic is equal to 3.

The Lie superalgebra of derivations of $\mathfrak{g} = \mathfrak{g}(S_8, S_{1,2})$ inherits the $\mathbb{Z}^5$-grading, so in order to prove Lemma 5.2 it is enough to prove that homogeneous derivations (in this $\mathbb{Z}^5$-grading) are inner. Thus, assume that $d \in \mathfrak{Der}(\mathfrak{g})_\nu$, with $\nu \in \mathbb{Z}^5$:

(1) If $\nu \neq 0$ and $d(\mathfrak{h}) = 0$ (note that the Cartan subalgebra $\mathfrak{h}$ is just the 0-component in this grading), then d preserves the eigenspaces (root spaces) of $\mathfrak{h}$, and hence $d(\mathfrak{g}_\mu) = 0$ for any $\mu \in \Phi \setminus \{\pm 2\epsilon_5, \pm\epsilon_5\}$, as $d(\mathfrak{g}_\mu)$ must simultaneously be contained in $\mathfrak{g}_{\mu+\nu}$ and in the root space of root $R(\mu)$. But the subspaces $\mathfrak{g}_\mu$, with $\mu \in \Phi \setminus \{\pm 2\epsilon_5, \pm\epsilon_5\}$ generate the Lie superalgebra $\mathfrak{g}$. (This can be checked easily, but it also follows from [Cunha and Elduque 2007a, Proposition 5.25].) Hence $d = 0$, which is trivially inner.

(2) If $\nu \neq 0$ and $d(\mathfrak{h}) \neq 0$, then $d(\mathfrak{h})$ is contained in $\mathfrak{g}_\nu$, which has dimension at most 1. Thus $\mathfrak{g}_\nu = kx_\nu$ for some x_ν. Then for any $h \in \mathfrak{h}$, $d(h) = f(h)x_\nu$ for some $f \in \mathfrak{h}^*$. Then, for any $h, h' \in \mathfrak{h}$,

$$0 = d([h, h']) = [d(h), h'] + [h, d(h')] = \bigl(-R(\nu)(h')f(h) + R(\nu)(h)f(h')\bigr)x_\nu.$$

As $R(\nu) \neq 0$, it follows that there is a scalar $\alpha \in k$ with

$$f = \alpha R(\nu) \quad \text{and} \quad d' = d - \alpha \operatorname{ad} x_\nu$$

is another derivation in $\mathfrak{Der}(\mathfrak{g})_\nu$ with $d'(\mathfrak{h}) = 0$, so d' must be 0 by the previous case, and hence d is inner.

(3) Finally, if $\nu = 0$, then $d(e_i) \in \mathfrak{g}_{2\epsilon_i} = ke_i$, so $d(e_i) = \alpha_i e_i$ for any i. Also, $d(f_i) = \beta_i f_i$ for any i ($\alpha_i, \beta_i \in k$). As $ke_i + kf_i + kh_i$ is a Lie subalgebra isomorphic to $\mathfrak{sl}_2$, it follows at once that $\alpha_i + \beta_i = 0$. Then the derivation $d' = d - \frac{1}{2}\operatorname{ad}_{\alpha_1 h_1 + \cdots + \alpha_5 h_5}$ satisfies $d'(e_i) = 0 = d'(f_i)$ for any i, so $d'(h_i) = 0$, and hence $d'(\mathfrak{sp}(V_i)) = 0$ for any i. As d' preserves degrees, it preserves each subspace $V(\sigma)$, for $\varnothing \neq \sigma \in \mathcal{S}_{8,3}$, which is an irreducible module for $\bigoplus_{i=1}^5 \mathfrak{sp}(V_i)$. By Schur's Lemma, there is a scalar $\alpha_\sigma \in k$ such that the restriction of d' to any $V(\sigma)$ is α_σ id. But

$$0 \neq [V(\sigma), V(\sigma)] \subseteq \bigoplus_{i=1}^5 \mathfrak{sp}(V_i),$$

so $2\alpha_\sigma = 0$ for any such σ and $d' = 0$. Thus d is inner in this case, too. □

Consider now the triple product on the subspace T (the odd component in the $\mathbb{Z}_2$-grading of $\mathfrak{g}(S_8, S_{1,2})$ considered so far) inherited from the Lie bracket in $\mathfrak{g}(S_8, S_{1,2})$:

$$T \otimes T \otimes T \to T, \quad X \otimes Y \otimes Z \mapsto [XYZ] = [[X, Y], Z].$$

As T is the odd component of $\mathfrak{g}(S_8, S_{1,2})$, it is a Lie triple supersystem. Therefore $(T, [\dots])$ satisfies equations (4-1a) and (4-1c).

Also, if we consider the supersymmetric bilinear form $(\cdot\,|\,\cdot)$ on T given by the restriction of the bilinear form B given in Lemma 5.1, the invariance of B immediately shows that $(T, [\dots], (\cdot\,|\,\cdot))$ also satisfies (4-1d).

Theorem 5.3. $(T, [\dots], (\cdot\,|\,\cdot))$ *is an orthosymplectic triple system whose Lie superalgebra of derivations is isomorphic to* $\mathfrak{el}(5;3)$. *Moreover, the associated Lie superalgebra* $\mathfrak{g}(T)$ *is isomorphic to the Lie superalgebra* $\mathfrak{g}(S_{4,2}, S_{4,2})$ *in the supermagic square.*

Proof. It is enough to check (4-1b).

Take a symplectic basis $\{a, b\}$ of the two-dimensional vector space V in (5-1) (that is, $\langle a|b\rangle = 1$), then T is generated, as a module over $\mathfrak{el}(5;3)$ by $\iota_0(S_8 \otimes a)$ or by $\iota_0(S_8\otimes b)$. Also, $T\otimes T$ is generated by $\iota_0(S_8\otimes a)\otimes\iota_0(S_8\otimes b)$. Both the left and right sides of (4-1b) are given by $\mathfrak{el}(5;3)$-invariant trilinear maps $T\otimes T\otimes T\to T$. Therefore, it is enough to prove that

$$\begin{aligned}[][X\,\iota_0(y\otimes a)\,\iota_0(z\otimes b)]-[X\,\iota_0(z\otimes b)\,\iota_0(y\otimes a)]\\ =(X|\iota(y\otimes a))\iota_0(z\otimes b)-(X|\iota_0(z\otimes b))\iota_0(y\otimes a)+\mathrm{b}(y,z)X\end{aligned}$$

for any $X\in T$ and $y,z\in S_8$.

- For $X=u\in V\simeq\mathfrak{tri}(S_{1,2})_{\bar 1}$, since $\{a,b\}$ is a symplectic basis, $u=\langle u|b\rangle a-\langle u|a\rangle b$, so
$$\begin{aligned}[][u\,\iota_0(y\otimes a)\,\iota_0(z\otimes b)]&=-\langle u|a\rangle[\iota_0(y\otimes 1),\iota_0(z\otimes b)]\\&=-\langle u|a\rangle\mathrm{b}(y,z)\,t_{1,b}=-\langle u|a\rangle\mathrm{b}(y,z)\,b,\end{aligned}$$
where, as before, V is identified with $\mathfrak{tri}(S_{1,2})_{\bar 1}$. Thus,
$$\begin{aligned}[][X\,\iota_0(y\otimes a)\,\iota_0(z\otimes b)]-[X\,\iota_0(z\otimes b)\,\iota_0(y\otimes a]\\ =-\langle u|a\rangle\mathrm{b}(y,z)\,b+\langle u|b\rangle\mathrm{b}(y,z)\,a=\mathrm{b}(y,z)u=\mathrm{b}(y,z)X.\end{aligned}$$
Since $(X|\iota_0(y\otimes a))=0=(X|\iota_0(z\otimes b))$, the result follows in this case.
- For $X=\iota_0(x\otimes u)$ with $x\in S_8$ and $u\in V$, we have
$$\begin{aligned}[][\iota_0(x\otimes u)\,\iota_0(y\otimes a)\,\iota_0(z\otimes b)]\\ &=[\langle u|a\rangle t_{x,y}+\mathrm{b}(x,y)t_{u,a},\iota_0(z\otimes b)]\\ &=\langle u|a\rangle\iota_0(\sigma_{x,y}(z)\otimes b)+\mathrm{b}(x,y)\iota_0(z\otimes\sigma_{u,a}(b))\\ &=\langle u|a\rangle\iota_0\big(b\otimes(\mathrm{b}(x,z)y-\mathrm{b}(y,z)x)\big)-\mathrm{b}(x,y)\iota_0\big(z\otimes(\langle u|b\rangle a+\langle a|b\rangle u)\big).\end{aligned}$$
Thus,
$$\begin{aligned}[][X\,\iota_0(y\otimes a)\,\iota_0(z\otimes b)]-[X\,\iota_0(z\otimes b)\,\iota_0(y\otimes a)]\\ &=-\mathrm{b}(x,y)\iota_0\big(z\otimes(\langle u|b\rangle a+\langle a|b\rangle u+\langle u|b\rangle a)\big)\\ &\quad+\mathrm{b}(x,z)\iota_0\big(y\otimes(\langle u|a\rangle b+\langle b|a\rangle u+\langle u|a\rangle b)\big)\\ &\quad+\mathrm{b}(y,z)\iota_0\big(x\otimes(-\langle u|a\rangle b+\langle u|b\rangle a)\big)\\ &=\mathrm{b}(x,y)\langle u|a\rangle\iota_0(z\otimes b)-\mathrm{b}(x,z)\langle u|b\rangle\iota_0(y\otimes a)+\mathrm{b}(y,z)\iota_0(x\otimes u)\\ &=(X|\iota_0(y\otimes a))\iota_0(z\otimes b)-(X|\iota_0(z\otimes b))\iota_0(y\otimes a)+\mathrm{b}(y,z)X.\end{aligned}$$
- For $X=\iota_1(x\otimes 1)$, we have
$$[\iota_1(x\otimes 1)\,\iota_0(y\otimes a)\,\iota_0(z\otimes b)]=[\iota_2(y\bullet x\otimes a),\iota_0(z\otimes b)]=\iota_1((y\bullet x)\bullet z\otimes 1)$$
as $a\bullet 1=-a$ and $a\bullet b=1$ and
$$[\iota_1(x\otimes 1)\iota_0(z\otimes b)\iota_0(y\otimes a)]=[\iota_2(z\bullet x\otimes b),\iota_0(y\otimes a)]-\iota_1((z\bullet x)\bullet y\otimes 1)$$

as $b \bullet 1 = -b$, $b \bullet a = -1$. Thus,

$$[X\,\iota_0(y\otimes a)\,\iota_0(z\otimes b)] - [X\,\iota_0(z\otimes b)\,\iota_0(y\otimes a)] \\ = \iota_1\big(((y\bullet x)\bullet z + (z\bullet x)\bullet y)\otimes 1\big) = \mathrm{b}(y,z)X,$$

because the associativity of the bilinear form b in a symmetric composition algebra is equivalent to the condition

$$(x\bullet y)\bullet x = q(x)y = x\bullet(y\bullet x)$$

(see [Knus et al. 1998, (34.1)]) and hence it follows that

$$(y\bullet x)\bullet z + (z\bullet x)\bullet y = \mathrm{b}(y,z)x$$

by linearization.

- For $X = \iota_2(x\otimes 1)$ the situation is similar.

Therefore, $(T, [\,\dots], (\cdot\,|\,\cdot))$ is an orthosymplectic triple system and, by its own construction, its Lie superalgebra of inner derivations is isomorphic to $\mathfrak{el}(5;3)$, as $[TT\cdot] = \mathrm{ad}_{[T,T]} = \mathrm{ad}_{\mathfrak{el}(5;3)}$. Thus, the Lie superalgebra $\tilde{\mathfrak{g}}(T)$ in Proposition 4.6 is isomorphic to the Lie superalgebra $\mathfrak{g}(S_8, S_{1,2})$.

But any derivation $d \in \mathfrak{Der}\,T$ extends to a derivation of $\tilde{\mathfrak{g}}(T)$ which is inner (by Lemma 5.2). It follows that $\mathfrak{Der}\,T = \mathfrak{inDer}\,T$ is isomorphic to $\mathfrak{el}(5;3)$, as required.

The associated Lie superalgebra (see Proposition 4.5) is

$$\mathfrak{g} = (\mathfrak{sp}(V)\oplus\mathfrak{el}(5;3))\oplus(V\otimes T). \tag{5-3}$$

Consider again the description of $\mathfrak{g}(S_8, S_{1,2})$ in (2-3):

$$\mathfrak{g}(S_8,S_{1,2}) = \bigoplus_{\sigma\in\mathcal{S}_{8,3}} V(\sigma).$$

Then, as in (2-4),

$$\mathfrak{el}(5,3) = \bigoplus_{\sigma\in\mathcal{S}_+} V(\sigma), \qquad T = \bigoplus_{\sigma\in\mathcal{S}_-} V(\sigma),$$

with

$$\mathcal{S}_+ = \{\varnothing, \{1,2,3,4\}, \{1,2\}, \{3,4\}, \{2,3,5\}, \{1,4,5\}, \{1,3,5\}, \{2,4,5\}\},$$
$$\mathcal{S}_- = \{\{5\}, \{1,2,5\}, \{3,4,5\}, \{2,3\}, \{1,4\}, \{1,3\}, \{2,4\}\}.$$

Now, assign the index 6 to the new copy of V in (5-3). Then,

$$\mathfrak{g} = \bigoplus_{\sigma\in\tilde{\mathcal{S}}} V(\sigma),$$

with $\tilde{\mathcal{S}} \subseteq 2^{\{1,2,3,4,5,6\}}$ given by

$$\tilde{\mathcal{S}} = \big\{\varnothing, \{1,2,3,4\}, \{1,2\}, \{3,4\}, \{2,3,5\}, \{1,4,5\}, \{1,3,5\}, \{2,4,5\}\\ \{5,6\}, \{1,2,5,6\}, \{3,4,5,6\}, \{2,3,6\}, \{1,4,6\}, \{1,3,6\}, \{2,4,6\}\big\}.$$

We now write

$$\bar{1} = 1, \quad \bar{2} = 3, \quad , \bar{3} = 5, \quad \bar{4} = 2, \quad \bar{5} = 4, \quad \bar{6} = 6$$

and obtain

$$\tilde{\mathcal{S}} = \big\{\varnothing, \{\bar{1},\bar{2},\bar{3},\bar{5}\}, \{\bar{1},\bar{4}\}, \{\bar{2},\bar{5}\}, \{\bar{2},\bar{3},\bar{4}\}, \{\bar{1},\bar{3},\bar{5}\}, \{\bar{1},\bar{2},\bar{3}\}, \{\bar{3},\bar{4},\bar{5}\}\\ \{\bar{3},\bar{6}\}, \{\bar{1},\bar{3},\bar{4},\bar{6}\}, \{\bar{2},\bar{3},\bar{5},\bar{6}\}, \{\bar{2},\bar{4},\bar{6}\}, \{\bar{1},\bar{5},\bar{6}\}, \{\bar{1},\bar{2},\bar{6}\}, \{\bar{4},\bar{5},\bar{6}\}\big\},$$

and this coincides with $\mathcal{S}_{S_{4,2},S_{4,2}}$ in [Cunha and Elduque 2007a, §5.4]. Hence this superalgebra is a Lie superalgebra with the same Cartan matrix $A_{S_{4,2},S_{4,2}}$ in [Cunha and Elduque 2007a, §5.4], thus proving that $\mathfrak{g}$ is isomorphic to the Lie superalgebra $\mathfrak{g}(S_{4,2}, S_{4,2})$ in the supermagic square. □

Remark 5.4. Theorem 5.3 shows that the Lie superalgebra $\mathfrak{el}(5;3)$ lives inside $\mathfrak{g}(S_{4,2}, S_{4,2})$, and that, in fact, $\mathfrak{g}(S_{4,2}, S_{4,2})$ contains a maximal subalgebra isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{el}(5;3)$.

6. The Lie superalgebra $\mathfrak{br}(2;5)$

In this section a model of the simple Lie superalgebra $\mathfrak{br}(2;5)$ is explicitly built. To this aim, consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded vector space

$$\mathfrak{g} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,1)},$$

with

$$\begin{aligned} \mathfrak{g}_{(0,0)} &= \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2),\\ \mathfrak{g}_{(1,0)} &= \mathfrak{sp}(V_1) \otimes V_2,\\ \mathfrak{g}_{(0,1)} &= V_1 \otimes \mathfrak{sp}(V_2),\\ \mathfrak{g}_{(1,1)} &= V_1 \otimes V_2, \end{aligned}$$

where, as usual, V_1 and V_2 are two-dimensional vector spaces endowed with nonzero alternating bilinear forms denoted by $\langle \cdot \mid \cdot \rangle$.

This vector space becomes a superspace with

$$\begin{aligned} \mathfrak{g}_{\bar{0}} &= \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)},\\ \mathfrak{g}_{\bar{1}} &= \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}. \end{aligned}$$

Now, define a superanticommutative product on $\mathfrak{g}$ by means of the natural Lie bracket on $\mathfrak{g}_{(0,0)}$, the natural action of $\mathfrak{g}_{(0,0)}$ on each $\mathfrak{g}_{(i,j)}$ (V_i is the natural module for $\mathfrak{sp}(V_i)$, while $\mathfrak{sp}(V_i)$ is its adjoint module), and by

$$
\begin{aligned}
[f\otimes u, g\otimes v] &= \langle u|v\rangle[f,g] + 2\operatorname{tr}(fg)\gamma_{u,v},\\
[a\otimes p, b\otimes q] &= -(2\operatorname{tr}(pq)\gamma_{a,b} + \langle a|b\rangle[p,q]),\\
[a\otimes u, b\otimes v] &= \langle u|v\rangle\gamma_{a,b} + \langle a|b\rangle\gamma_{u,v},\\
[f\otimes u, a\otimes p] &= f(a)\otimes p(u),\\
[f\otimes u, a\otimes v] &= f(a)\otimes\gamma_{u,v},\\
[a\otimes p, b\otimes v] &= -\gamma_{a,b}\otimes p(v),
\end{aligned}
\tag{6-1}
$$

for any

$$a,b\in V_1,\quad u,v\in V_2,\quad f,g\in\mathfrak{sp}(V_1)=\mathfrak{sl}(V_1)\quad\text{and}\quad p,q\in\mathfrak{sp}(V_2)=\mathfrak{sl}(V_2).$$

Here, as before,

$$\gamma_{a,b} = \langle a|\cdot\rangle + \langle b|\cdot\rangle a$$

and similarly for $\gamma_{u,v}$.

This multiplication converts $\mathfrak{g}$ into a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded anticommutative superalgebra.

Theorem 6.1. *Let k be a field of characteristic* 5. *Then the superalgebra $\mathfrak{g}$ above is a Lie superalgebra isomorphic to* $\mathfrak{br}(2;5)$.

Proof. It is clear that all the products in (6-1) are invariant under the action of

$$\mathfrak{sp}(V_1)\oplus\mathfrak{sp}(V_2).$$

Several instances of the Jacobi identity have to be checked. To do so, it is harmless to assume that the ground field k is infinite (extend scalars otherwise) and hence, Zariski topology can be used.

First, for elements $a,b,c\in V_1$ and $u,v,w\in V_2$, to check that the Jacobian

$$
\begin{aligned}
&J(a\otimes u, b\otimes v, c\otimes w)\\
&\qquad = [[a\otimes u, b\otimes v], c\otimes w] + [[b\otimes v, c\otimes w], a\otimes u] + [[c\otimes w, a\otimes u], b\otimes v]
\end{aligned}
$$

is 0, it can be assumed, by Zariski density, that $\langle a|b\rangle\neq 0$ and $\langle u|v\rangle\neq 0$. (Note that the set $\{(a,b)\in V\times V : \langle a|b\rangle\neq 0\}$ is a nonempty open set in the Zariski topology of $V\times V$, and hence it is dense.) Moreover, scaling now b and v if necessary, it can be assumed that $\langle a|b\rangle = 1 = \langle u|v\rangle$, that is, $\{a,b\}$ is a symplectic basis of V_1 and $\{u,v\}$ is a symplectic basis of V_2. Now

$$c = \alpha a + \beta b \qquad\text{and}\qquad w = \mu u + \nu v$$

for some $\alpha, \beta, \mu, \nu \in k$. Then

$$
\begin{aligned}
J(a\otimes u, b\otimes v, c\otimes w)&\\
&= [[a\otimes u, b\otimes v], c\otimes w] + [[b\otimes v, c\otimes w], a\otimes u] + [[c\otimes w, a\otimes u], b\otimes v]\\
&= \langle u|v\rangle \gamma_{a,b}(c)\otimes w + \langle a|b\rangle c\otimes \gamma_{u,v}(w) + \langle v|w\rangle \gamma_{b,c}(a)\otimes u + \langle b|c\rangle a\otimes \gamma_{v,w}(u)\\
&\qquad + \langle w|u\rangle \gamma_{c,a}(b)\otimes v + \langle c|a\rangle b\otimes \gamma_{w,u}(v)\\
&= (\beta b - \alpha a)\otimes(\mu u + \nu v) + (\alpha a + \beta b)\otimes(\nu v - \mu u)\\
&\qquad + \mu(\alpha a + 2\beta b)\otimes u + \alpha a\otimes(\mu u + 2\nu v)\\
&\qquad - \nu(2\alpha a + \beta b)\otimes v - \beta b\otimes(2\mu u + \nu v)\\
&= 0.
\end{aligned}
$$

Hence, $\mathfrak{g}_{\bar 0} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)}$ is a Lie algebra, which can be easily checked to be isomorphic to the symplectic Lie algebra $\mathfrak{sp}(V_1 \perp V_2) \simeq \mathfrak{sp}_4$.

Now, for elements $f, g, h \in \mathfrak{sp}(V_1)$ and $u, v, w \in V_2$, it can be assumed as before that $\langle u|v\rangle = 1$ and that $w = \mu u + \nu v$. Then,

$$
\begin{aligned}
[[f\otimes u, g\otimes v], h\otimes w] &= [\langle u|v\rangle [f,g] + 2\operatorname{tr}(fg)\gamma_{u,v}, h\otimes w]\\
&= \langle u|v\rangle [[f,g],h]\otimes w + 2\operatorname{tr}(fg) h\otimes \gamma_{u,v}(w),
\end{aligned}
$$

so that

$$
\begin{aligned}
J(f\otimes u, g\otimes v, h\otimes w)&\\
&= [[f\otimes u, g\otimes v], h\otimes w] + [[g\otimes v, h\otimes w], f\otimes u] + [[h\otimes w, f\otimes u], g\otimes w]\\
&= \langle u|v\rangle [[f,g],h]\otimes w + 2\operatorname{tr}(fg) h\otimes \gamma_{u,v}(w)\\
&\qquad + \langle v|w\rangle [[g,h],f]\otimes u + 2\operatorname{tr}(gh) f\otimes \gamma_{v,w}(u)\\
&\qquad + \langle w|u\rangle [[h,f],g]\otimes v + 2\operatorname{tr}(hf) g\otimes \gamma_{w,u}(v)\\
&= \mu\big([[f,g],h] - 2\operatorname{tr}(fg)h - [[g,h],f] - 2\operatorname{tr}(gh)f + 4\operatorname{tr}(hf)g\big)\otimes u\\
&\qquad + \nu\big([[f,g],h] + 2\operatorname{tr}(fg)h - 2\operatorname{tr}(gh)f - [[h,f],g] + 2\operatorname{tr}(hf)g\big)\otimes v.
\end{aligned}
\tag{6-2}
$$

But for any $f \in \mathfrak{sl}(V_1)$, $f^2 = -\det(f) = \frac{1}{2}\operatorname{tr}(f^2)\,\mathrm{id}$. Hence $fg + gf = \operatorname{tr}(fg)\,\mathrm{id}$ for any $f, g \in \mathfrak{sl}(V_1)$, and thus

$$fgf = \operatorname{tr}(fg)f - gf^2 = \operatorname{tr}(fg)f - \tfrac{1}{2}\operatorname{tr}(f^2)g.$$

Hence,

$$[[g,f],f] = gf^2 + f^2g - 2fgf = 2\operatorname{tr}(f^2)g - 2\operatorname{tr}(fg)f$$

and

$$[[g,f],h] + [[g,h],f] = 4\operatorname{tr}(fh)g - 2\operatorname{tr}(fg)h - 2\operatorname{tr}(gh)f.$$

This shows that the Jacobian in (6-2) is trivial. Therefore, the subspace $\mathfrak{g}_{(0,0)}\oplus\mathfrak{g}_{(1,0)}$ is a Lie superalgebra. The same happens to the subspace $\mathfrak{g}_{(0,0)}\oplus\mathfrak{g}_{(0,1)}$.

Take now elements $a,b\in V_1$, $f\in\mathfrak{sp}(V_1)$ and $u,v,w\in V_2$. Then it can be assumed that $\langle a|b\rangle=1=\langle u|v\rangle$, and $w=\mu u+\nu v$. In this situation,

$$\begin{aligned}
&J(a\otimes u,b\otimes v,f\otimes w)\\
&=[[a\otimes u,b\otimes v],f\otimes w]+[[b\otimes v,f\otimes w],a\otimes u]+[[f\otimes w,a\otimes u],b\otimes v]\\
&=\langle u|v\rangle[\gamma_{a,b},f]\otimes w+\langle a|b\rangle f\otimes\gamma_{u,v}(w)+\gamma_{a,f(b)}\otimes\gamma_{v,w}(u)-\gamma_{f(a),b}\otimes\gamma_{u,w}(v)\\
&=\mu([\gamma_{a,b},f]-f-\gamma_{a,f(b)}-2\gamma_{f(a),b})\otimes u\\
&\qquad\qquad\qquad\qquad+\nu([\gamma_{a,b},f]+f-2\gamma_{a,f(b)}-\gamma_{f(a),b})\otimes v.
\end{aligned}$$

But, since the bilinear map $(c,d)\mapsto\gamma_{c,d}$ is $\mathfrak{sp}(V_1)$-invariant,

$$[f,\gamma_{a,b}]=\gamma_{f(a),b}+\gamma_{a,f(b)}.$$

Hence,

$$\begin{aligned}
\text{(6-3)}\quad &J(a\otimes u,b\otimes v,f\otimes w)\\
&\qquad=-\mu(f+3\gamma_{f(a),b}+2\gamma_{a,f(b)})\otimes u+\nu(f-2\gamma_{f(a),b}-3\gamma_{a,f(b)}).
\end{aligned}$$

Also, by taking the coordinate matrix of f in the symplectic basis $\{a,b\}$, it is checked at once that $f=-\frac{1}{2}\gamma_{f(a),b}+\frac{1}{2}\gamma_{a,f(b)}$. Since the characteristic of k is equal to 5, this proves that the Jacobian in (6-3) is trivial.

The other instances of the Jacobi identity are checked in a similar way.

Finally, fix symplectic bases $\{a_i,b_i\}$ of V_i $(i=1,2)$. Then $\mathfrak{g}$ is $\mathbb{Z}\times\mathbb{Z}$-graded by assigning

$$\deg(a_i)=\epsilon_i,\qquad \deg(b_i)=-\epsilon_i,$$

where $\{\epsilon_1,\epsilon_2\}$ denotes the canonical $\mathbb{Z}$-basis of $\mathbb{Z}\times\mathbb{Z}$. Let $\{h_i,e_i,f_i\}$ be the basis of $\mathfrak{sp}(V_i)$ defined as in (4-6). Then span $\{h_1,h_2\}$ is a Cartan subalgebra of $\mathfrak{g}$, and coincides with the $(0,0)$-component in the $\mathbb{Z}\times\mathbb{Z}$-grading. The set of nonzero degrees is

$$\Phi=\{\pm2\epsilon_1,\pm2\epsilon_2,\pm\epsilon_1\pm\epsilon_2,\pm\epsilon_2,\pm2\epsilon_1\pm\epsilon_2,\pm\epsilon_1,\pm\epsilon_1\pm2\epsilon_2\}.$$

Consider the elements

$$\begin{aligned}
&E_1=a_1\otimes f_2, &&F_1=-b_1\otimes e_2, &&H_1=[E_1,F_1]=-2h_1-h_2,\\
&E_2=h_1\otimes a_2, &&F_2=h_1\otimes b_2, &&H_2=[E_2,F_2]=h_2.
\end{aligned}$$

Then, span $\{H_1,H_2\}$ coincides with the previous Cartan subalgebra span $\{h_1,h_2\}$ of $\mathfrak{g}$, E_1 belongs to the homogeneous component $\mathfrak{g}_{\epsilon_1-2\epsilon_2}$ in the $\mathbb{Z}\times\mathbb{Z}$-grading, and similarly $F_1 \in \mathfrak{g}_{-\epsilon_1+2\epsilon_2}$, $E_2 \in \mathfrak{g}_{\epsilon_2}$, and $F_2 \in \mathfrak{g}_{-\epsilon_2}$. The elements E_1, E_2, F_1, F_2 generate the Lie superalgebra $\mathfrak{g}$. Besides,

$$[H_1, E_1] = -2h_1(a_1) \otimes f_2 - a_1 \otimes [h_2, f_2] = -2a_1 \otimes f_2 + 2a_1 \otimes f_2 = 0,$$
$$[H_1, E_2] = h_1 \otimes (-h_2)(a_2) = -h_1 \otimes a_2,$$
$$[H_2, E_1] = a_1 \otimes [h_2, f_2] = -2a_1 \otimes f_2,$$
$$[H_2, E_2] = h_1 \otimes h_2(a_2) = h_1 \otimes a_2,$$

and similarly for the action of the H_i's on the F_j's. It follows, with the same arguments as in [Cunha and Elduque 2007a, §4], that $\mathfrak{g}$ is the Lie superalgebra with Cartan matrix $\left(\begin{smallmatrix} 0 & -1 \\ -2 & 1 \end{smallmatrix}\right)$, which is the first Cartan matrix of the Lie superalgebra $\mathfrak{br}(2; 5)$ given in [Bouarroudj et al. 2007, §12]. □

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ALBERTO ELDUQUE
DEPARTAMENTO DE MATEMÁTICAS
E INSTITUTO UNIVERSITARIO DE MATEMÁTICAS Y APLICACIONES
UNIVERSIDAD DE ZARAGOZA
50009 ZARAGOZA
SPAIN

elduque@unizar.es
http://www.unizar.es/matematicas/algebra/elduque