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In this paper we prove that, given a compact four-dimensional smooth Riemannian manifold (M, g) with smooth boundary, there exists a metric conformal to g with constant T-curvature, zero Q-curvature and zero mean curvature under generic and conformally invariant assumptions. The problem amounts to solving a fourth-order nonlinear elliptic boundary value problem (BVP) with boundary conditions given by a third-order pseudodifferential operator and homogeneous Neumann operator. It has a variational structure, but since the corresponding Euler–Lagrange functional is in general unbounded from below, we look for saddle points. We do this by using topological arguments and min-max methods combined with a compactness result for the corresponding BVP.

1. Introduction

Recent years have seen intense study of conformally covariant differential (or even pseudodifferential) operators on compact smooth Riemannian manifolds, as well as their associated curvature invariants. This study seeks to understand the relationships between analytic and geometric properties of such objects.

A model example is the Laplace–Beltrami operator on compact closed surfaces (Σ, g), which governs the transformation law of the Gauss curvature. In fact under the conformal change of metric $g_u = e^{2u}g$, we have

$$\Delta_{g_u} = e^{-2u}\Delta_g \quad \text{and} \quad -\Delta_g u + K_g = K_{g_u}e^{2u}, \tag{1-1}$$

where Δ_g and K_g are the Laplace–Beltrami operator and the Gauss curvature of (Σ, g), and Δ_{g_u} and K_{g_u} are the corresponding objects for $(\Sigma_, g_u)$.

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Moreover we have the Gauss–Bonnet formula, which relates $\int_\Sigma K_g dV_g$ to the topology of Σ via

$$\int_\Sigma K_g dV_g = 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler–Poincaré characteristic of Σ. From this we deduce that $\int_\Sigma K_g dV_g$ is a topological invariant (and hence also a conformal one). Of particular interest is the classical uniformization theorem, which says the (Σ, g) carries a conformal metric with constant Gauss curvature.

Suppose (M, g) is a four-dimensional compact closed Riemannian manifold. On it, there exists a conformally covariant differential operator P_g called the Paneitz operator, to which is associated a natural concept of curvature. This operator, discovered by Paneitz in 1983 (see [2008]), and the corresponding Q-curvature introduced by Branson (see [Branson and Ørsted 1991]) are defined in terms of the Ricci tensor Ric_g and the scalar curvature R_g as

$$P_g\varphi = \Delta_g^2\varphi + \mathrm{div}_g\big(\tfrac{2}{3}R_g g - 2\,\mathrm{Ric}_g\big)d\varphi, \tag{1-2}$$

$$Q_g = -\tfrac{1}{12}(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2), \tag{1-3}$$

where φ is any smooth function on M.

The Laplace–Beltrami operator governs the transformation law of the Gauss curvature; the Paneitz operator does the same for the Q-curvature. Indeed under a conformal change of metric $g_u = e^{2u}g$, we have

$$P_{g_u} = e^{-4u}P_g \quad \text{and} \quad P_g u + 2Q_g = 2Q_{g_u}e^{4u}. \tag{1-4}$$

Apart from this analogy, we also have an extension of the Gauss–Bonnet formula, the Gauss–Bonnet–Chern formula

$$\int_M (Q_g + \tfrac{1}{8}|W_g|^2)dV_g = 4\pi^2\chi(M),$$

where W_g denotes the Weyl tensor of (M, g); see [Djadli and Malchiodi 2006]. Hence, from the pointwise conformal invariance of $|W_g|^2 dV_g$, it follows that the integral of Q_g over M is also conformally invariant.

In analogy to the uniformization theorem for (Σ, g), one can also ask if (M, g) carries a metric that is conformally related to the background metric with constant Q-curvature.

A first positive answer to this question was given by Chang and Yang [1995] under the assumptions that P_g is nonnegative, $\ker P_g \simeq \mathbb{R}$ and $\int_M Q_g dV_g < 8\pi^2$. Later Djadli and Malchiodi [2006] extended Chang and Yang's result to a large class of compact closed four-dimensional Riemannian manifolds assuming that P_g has no kernel and that $\int_M Q_g dV_g$ is not an integer multiple of $8\pi^2$.

One can consider analogous questions in dimensions higher than four, where there are there are higher-order analogues of the Laplace–Beltrami operator and of the Paneitz operator and also of the associated curvatures (called again Q-curvatures); see [Fefferman and Graham 2002; 1985; Graham et al. 1992].

For example, one can ask whether, for a compact closed Riemannian manifold of arbitrary dimension, there exists a constant Q-curvature conformal metric.

Using a geometric flow, Brendle [2003] has given a first affirmative answer in the even-dimensional case under the assumptions that the higher-dimensional Paneitz operator is nonnegative and has trivial kernel and that the total integral of the Q-curvature is less than $(n-1)!\omega_n$, where ω_n is the area of the unit sphere S^n of R^{n+1}. Brendle's result and that of Djadli and Malchiodi [2006] were extended to all dimensions in [Ndiaye 2007b]. However, some issues remain: Only the leading term of the operator is known, and in the odd case the operator is pseudodifferential.

Instead of compact closed Riemannian manifolds, one can consider compact smooth Riemannian manifolds with smooth boundary. Much work has already been done in studying their conformally covariant differential operators, their associated curvature invariants, the corresponding boundary operators and curvatures.

Suppose then (Σ, g) is a compact smooth surface with smooth boundary $\partial\Sigma$. Let Δ_g be the Laplace–Beltrami operator, and let $\partial/\partial n_g$ be the Neumann operator on $\partial\Sigma$. Under a conformal change of metric, the pair $(\Delta_g, \partial/\partial n_g)$ governs the transformation laws of the Gauss curvature K_g of (Σ, g) and the geodesic curvature k_g of $(\partial\Sigma, g)$. In fact, under the conformal change of metric $g_u = e^{2u}g$, we have

$$\Delta_{g_u} = e^{-2u}\Delta_g \quad \text{and} \quad \frac{\partial}{\partial n_{g_u}} = e^{-u}\frac{\partial}{\partial n_g},$$

and

$$\begin{aligned} -\Delta_g u + K_g &= K_{g_u}e^{2u} && \text{in } \Sigma, \\ \frac{\partial u}{\partial n_g} + k_g &= k_{g_u}e^{u} && \text{on } \partial\Sigma. \end{aligned}$$

We have the Gauss–Bonnet formula

$$\int_\Sigma K_g dV_g + \int_{\partial\Sigma} k_g dS_g = 2\pi\chi(\Sigma), \tag{1-5}$$

where $\chi(\Sigma)$ is the Euler–Poincaré characteristic of Σ, dV_g is the element area of Σ and dS_g is the line element of $\partial\Sigma$. Thus $\int_\Sigma K_g dV_g + \int_{\partial\Sigma} k_g dS_g$ is a topological invariant and hence a conformal invariant.

In this context, the question about the analogue of the classical uniformization theorem is, Does there exist a metric on Σ that is conformally related to g and with constant Gauss curvature and constant geodesic curvature? This problem has been solved by the following theorem; for a proof see [Brendle 2002].

Theorem 1.1. *Every compact smooth Riemannian surface with smooth boundary* (Σ, g) *carries a metric conformally related to* g *with constant Gauss curvature and constant geodesic curvature.*

Like compact closed four-dimensional Riemannian manifolds, 4-manifolds with boundary admit the Paneitz operator P_g^4 and the Q-curvature. These are also defined by formulas (1-2) and (1-3) and enjoy the same invariance properties as in the case without boundary; see (1-4).

Chang and Qing [1997a] discovered a boundary operator P_g^3 defined on the boundary of compact four-dimensional smooth Riemannian manifolds, and they associated to it a natural third-order curvature T_g. These are defined by

$$P_g^3\varphi = \frac{1}{2}\frac{\partial \Delta_g \varphi}{\partial n_g} + \Delta_{\hat{g}}\frac{\partial \varphi}{\partial n_g} - 2H_g\Delta_{\hat{g}}\varphi + (L_g)_{ab}(\nabla_{\hat{g}})_a(\nabla_{\hat{g}})_b + \nabla_{\hat{g}}H_g.\nabla_{\hat{g}}\varphi + (F - \tfrac{1}{3}R_g)\frac{\partial\varphi}{\partial n_g},$$

$$T_g = -\frac{1}{12}\frac{\partial R_g}{\partial n_g} + \tfrac{1}{2}R_gH_g - \langle G_g, L_g\rangle + 3H_g^3 - \tfrac{1}{3}Tr(L^3) + \Delta_{\hat{g}}H_g.$$

Here φ is any smooth function on M, and $\hat{g}$ is the metric induced by g on ∂M. Also $L_g = (L_g)_{ab} = -\frac{1}{2}\partial g_{ab}/\partial n_g$ is the second fundamental form of ∂M; from L_g is defined $H_g = \frac{1}{3}\operatorname{tr}(L_g) = \frac{1}{3}g^{ab}L_{ab}$, the mean curvature of ∂M. (Here the g^{ab} are the entries of the inverse g^{-1} of the metric g.) Finally $R^k{}_{bcd}$ is the Riemann curvature tensor, $F = R^a{}_{nan}$, $R_{abcd} = g_{ak}R^k{}_{bcd}$, and $\langle G_g, L_g\rangle = R_{anbn}(L_g)_{ab}$.

Just as the Laplace–Beltrami operator and the Neumann operator govern the transformation law of the Gauss curvature and the geodesic curvature on compact surfaces with boundary under conformal change of metrics, the pair (P_g^4, P_g^3) does the same for (Q_g, T_g) on compact four-dimensional smooth Riemannian manifolds with smooth boundary. In fact, after a conformal change of metric $g_u = e^{2u}g$ we have that

$$P_{g_u}^4 = e^{-4u}P_g^4 \quad \text{and} \quad P_{g_u}^3 = e^{-3u}P_g^3,$$

and

$$\begin{aligned} P_g^4u + 2Q_g &= 2Q_{g_u}e^{4u} && \text{in } M,\\ P_g^3u + T_g &= T_{g_u}e^{3u} && \text{on } \partial M. \end{aligned}$$

In addition to this analogy, we have also an extension of the Gauss–Bonnet formula (1-5), known as the Gauss–Bonnet–Chern formula, given by

$$\int_M (Q_g + \tfrac{1}{8}|W_g|^2)dV_g + \int_{\partial M}(T+Z)dS_g = 4\pi^2\chi(M), \tag{1-6}$$

where W_g denotes the Weyl tensor of (M, g) and ZdS_g is pointwise conformally invariant; for the definition of Z, see [Chang and Qing 1997a]. It turns out that Z

vanishes when the boundary is totally geodesic (by totally geodesic we mean that the boundary ∂M is umbilic and minimal). Setting

$$\kappa_{P_g^4} = \int_M Q_g \, dV_g \quad \text{and} \quad \kappa_{P_g^3} = \int_{\partial M} T_g \, dS_g,$$

we conclude from (1-6) and the pointwise conformal invariance of $W_g \, dV_g$ and $Z \, dS_g$ that the quantity $\kappa_{P_g^4} + \kappa_{P_g^3}$ is itself conformally invariant; we put

$$\kappa_{(P^4, P^3)} = \kappa_{P_g^4} + \kappa_{P_g^3}. \tag{1-7}$$

The celebrated Riemann mapping theorem says that an open, simply connected, proper subset of the plane is conformally diffeomorphic to the disk. One can ask if such a theorem remains true in four dimensions. Unfortunately, few four-dimensional regions are conformally diffeomorphic to the ball. However, in the spirit of the uniformization theorem (Theorem 1.1), one can still ask, On a given compact four-dimensional smooth Riemannian manifold with smooth boundary, does there exist a metric conformal to the background metric with zero Q-curvature, constant T-curvature and zero mean curvature? Escobar [1992] has asked related questions in the context of the Yamabe problem.

In this paper, we are interested in finding an analogue of the Riemann mapping theorem (in the spirit of Theorem 1.1) for compact four-dimensional smooth Riemannian manifolds with smooth boundary under generic and conformally invariant assumptions. Writing $g_u = e^{2u} g$, the problem is equivalent to solving the BVP

$$\begin{aligned} P_g^4 u + 2Q_g &= 0 && \text{in } M, \\ P_g^3 u + T_g &= \overline{T} e^{3u} && \text{on } \partial M, \\ \partial u / \partial n_g - H_g u &= 0 && \text{on } \partial M. \end{aligned}$$

Here $\overline{T}$ is a fixed real number, and $\partial / \partial n_g$ is the inward normal derivative with respect to g.

By a result by Escobar [1992], and by the fact that we are interested in the problem under assumptions of conformal invariance, it is not restrictive to assume that $H_g = 0$, since this can be always obtained through a conformal transformation of the background metric. Thus we are lead to solve the following BVP with Neumann homogeneous boundary condition:

$$\begin{aligned} P_g^4 u + 2Q_g &= 0 && \text{in } M, \\ P_g^3 u + T_g &= \overline{T} e^{3u} && \text{on } \partial M, \\ \partial u / \partial n_g &= 0 && \text{on } \partial M. \end{aligned} \tag{1-8}$$

Define $\{u \in H^2(M) : \partial u/\partial n_g = 0\}$, and define $P_g^{4,3}$ through

$$\langle P_g^{4,3} u, v\rangle_{L^2(M)} = \int_M \big(\Delta_g u \Delta_g v + \tfrac{2}{3} R_g \nabla_g u \nabla_g v\big) dV_g - 2\int_M \mathrm{Ric}_g(\nabla_g u, \nabla_g v) dV_g \\ -2\int_{\partial M} L_g(\nabla_{\hat g} u, \nabla_{\hat g} v) dS_g,$$

for every $u, v \in H_{\partial/\partial n}$. Then, by the regularity result in Lemma 2.3 below, the critical points in $H_{\partial/\partial n}$ of the functional

$$\mathrm{II}(u) = \langle P^{4,3} u, u\rangle_{L^2(M)} \\ + 4\int_M Q_g u\, dV_g + 4\int_{\partial M} T_g u\, dS_g - \tfrac{4}{3}\kappa_{(P^4,P^3)} \log \int_{\partial M} e^{3u} dS_g,$$

which are weak solutions of (1-8), are also smooth and hence strong solutions.

A similar problem is addressed in [Ndiaye 2007a], where constant Q-curvature metrics with zero T-curvature and zero mean curvature are found under generic and conformally invariant assumptions.

In [Ndiaye 2007c], heat flow methods are used to prove that if the operator $P_g^{4,3}$ is nonnegative, $\ker P_g^{4,3} \simeq \mathbb{R}$, and $\kappa_{(P^4,P^3)} < 4\pi^2$, then the problem (1-8) is solvable. Here we wish to extend this result under generic and conformally invariant assumptions. The following result is the main theorem of this paper:

Theorem 1.2. *Suppose* $\ker P_g^{4,3} \simeq \mathbb{R}$. *If* $\kappa_{(P^4,P^3)} \neq 4\pi^2 k$ *for* $k = 1, 2, \dots,$ *then* (M, g) *admits a conformal metric with constant T-curvature, zero Q-curvature, and zero mean curvature.*

Remark 1.3. Our assumptions are conformally invariant and generic, so the result applies to a large class of compact 4-dimensional Riemannian manifolds with boundary.

By the Gauss–Bonnet–Chern formula (1-6), Theorem 1.2 does *not* cover the case of locally conformally flat Riemannian manifolds with totally geodesic boundary and positive integer Euler–Poincaré characteristic.

Our assumptions include two cases:

Case 1.4. We have $\kappa_{(P^4,P^3)} < 4\pi^2$, or $P_g^{4,3}$ has $\bar{k}$ negative eigenvalues (counted with multiplicity).

Case 1.5. We have $\kappa_{(P^4,P^3)} \in (4\pi^2 k, 4(k+1)\pi^2)$ for some $k \in \mathbb{N}^*$, or $P_g^{4,3}$ has $\bar{k}$ negative eigenvalues (counted with multiplicity).

Remark 1.6. Case 1.4 includes the condition ($\bar{k} = 0$) under which the existence of solutions to (1-8) is proved in [Ndiaye 2007c]; hence we do not consider this case here. However due to a trace Moser–Trudinger type inequality (see Proposition 2.4 below) a solution can be found using direct methods of calculus of variation.

To simplify the exposition, we will prove Theorem 1.2 in Case 1.5 under the restriction that $\bar{k} = 0$ (that is, we assume $P_g^{4,3}$ is nonnegative). At the end of Section 4, a discussion to settle the general case will be made.

To prove Theorem 1.2, we look for critical points of II. Unless $\kappa_{(P^4,P^3)} < 4\pi^2$ and $\bar{k} = 0$, this Euler–Lagrange functional is unbounded from above and below (see Section 4), so it is necessary to find extremals that are possibly saddle points. To do this we will use a min-max method: By classical arguments in critical point theory, the scheme yields a *Palais–Smale sequence* (or PS sequence), namely a sequence $(u_l)_l \in H_{\partial/\partial n}$ satisfying the properties

$$\mathrm{II}(u_l) \to c \in \mathbb{R} \quad \text{and} \quad \mathrm{II}'(u_l) \to 0 \text{ as } l \to +\infty.$$

Then, as is usual in min-max theory, one should recover existence by proving that the *Palais–Smale condition* holds, that is, by proving every Palais–Smale sequence has a converging subsequence or by proving a similar compactness criterion. Since we do not know if the Palais–Smale condition holds, we will employ Struwe's monotonicity method [1988], which is also used in [Djadli and Malchiodi 2006; Ndiaye 2007b]. The latter yields existence of solutions for arbitrary small perturbations of the given equation, so to consider the original problem one is led to study compactness of solutions to perturbations of (1-8). Precisely, we consider

$$\begin{aligned} P_g^4 u_l + 2Q_l &= 0 && \text{in } M, \\ P_g^3 u_l + T_l &= \overline{T}_l e^{3u_l} && \text{on } \partial M, \\ \partial u_l/\partial n_g &= 0 && \text{on } \partial M, \end{aligned} \tag{1-9}$$

where

$$\overline{T}_l \to \overline{T}_0 > 0 \text{ in } C^2(\partial M), \quad T_l \to T_0 \text{ in } C^2(\partial M), \quad Q_l \to Q_0 \text{ in } C^2(M). \tag{1-10}$$

Remark 1.7. It follows from the Green representation formula given in Lemma 2.2 below that if u_l is a sequence of solutions to (1-9), then u_l satisfies

$$\begin{aligned} u_l(x) = -2\int_M G(x,y)Q_l(y)dV_g - 2\int_{\partial M} G(x,y)T_l(y)dS_g(y) \\ + 2\int_{\partial M} G(x,y)\overline{T}_l(y)e^{3u_l(y)}dS_g(y) + \bar{u}_{\partial M,l}. \end{aligned}$$

Therefore under the assumption (1-10), if $\sup_{\partial M} u_l \le C$, then u_l is bounded in $C^{4+\alpha}$ for every $\alpha \in (0,1)$.

In this context, we may say by Remark 1.7 that a sequence (u_l) of solutions to (1-9) *blows up* if

$$\text{there exists an } x_l \in \partial M \text{ such that } u_l(x_l) \to +\infty \text{ as } l \to +\infty, \tag{1-11}$$

and we prove the following compactness result.

Theorem 1.8. *Suppose* $\ker P_g^{4,3} \simeq \mathbb{R}$ *and that* (u_l) *is a sequence of solutions to* (1-9) *with* $\overline{T}_l$, T_l *and* Q_l *satisfying* (1-10). *If* $(u_l)_l$ *blows up* (*in the sense of* (1-11)) *and*

$$\int_M Q_0 \, dV_g + \int_{\partial M} T_0 \, dS_g + o_l(1) = \int_{\partial M} \overline{T}_l e^{3u_l} \, dS_g, \tag{1-12}$$

then there exists an $N \in \mathbb{N} \setminus \{0\}$ *such that*

$$\int_M Q_0 \, dV_g + \int_{\partial M} T_0 \, dS_g = 4N\pi^2.$$

From this we derive a corollary which will be used to ensure compactness of some solutions to a sequence of approximate BVPs produced by the topological argument combined with Struwe's monotonicity method. Its proof is a trivial application of Theorem 1.8 and Lemma 2.3 below.

Corollary 1.9. *Suppose* $\ker P_g^{4,3} \simeq \mathbb{R}$.

(a) *Let* (u_l) *be a sequence of solutions to* (1-9) *with* $\overline{T}_l$, T_l *and* Q_l *satisfying* (1-10). *Assume also that*

$$\int_M Q_0 \, dV_g + \int_{\partial M} T_0 \, dS_g + o_l(1) = \int_{\partial M} \overline{T}_l e^{3u_l} \, dS_g$$

and

$$k_0 = \int_M Q_0 \, dV_g + \int_{\partial M} T_0 \, dS_g \neq 4\pi^2 k \quad \text{for } k = 1, 2, 3, \dots.$$

Then $(u_l)_l$ *is bounded in* $C^{4+\alpha}(M)$ *for any* $\alpha \in (0, 1)$.

(b) *Let* (u_l) *be a sequence of solutions to* (1-8) *for a fixed value of the constant* $\overline{T}$. *Assume also that* $\kappa_{(P^4, P^3)} \neq 4\pi^2 k$. *Then* $(u_l)_l$ *is bounded in* $C^m(M)$ *for every positive integer* m.

(c) *Let* (u_{ρ_k}) *with* $\rho_k \to 1$ *be a family of solutions to* (1-8) *with* T_g *replaced by* $\rho_k T_g$, *with* Q_g *replaced by* $\rho_k Q_g$, *and with* $\overline{T}$ *replaced by* $\rho_k \overline{T}$ *for a fixed value of the constant* $\overline{T}$. *Assume also that* $\kappa_{(P^4, P^3)} \neq 4\pi^2 k$. *Then* $(u_{\rho_k})_k$ *is bounded in* $C^m(M)$ *for every positive integer* m.

(d) *If* $\kappa_{(P^4, P^3)} \neq 4\pi^2 k$ *for* $k = 1, 2, 3, \dots$, *then the set of metrics conformal to* g *with constant* T*-curvature, with zero* Q*- and mean curvature, and with unit boundary volume is compact in* $C^m(M)$ *for every positive integer* m.

We are going to describe the main ideas needed to prove the above results. Since the proof of Theorem 1.2 relies on the compactness result of Theorem 1.8 (see Corollary 1.9), it is convenient to discuss first the latter. We use a strategy related to that in [Druet and Robert 2006], but in our case, due to the Green representation formula (see Lemma 2.2), we have to consider blow-ups at the boundary; see Remark 1.7. In [Ndiaye 2007b; 2007a] a variant of this method was used that relies strongly on the Green representation formula, transforming (1-8) into an integral equation. Here we will employ a similar strategy, since for the BVP one has also the existence of a Green representation formula; see Lemma 2.2. We point out that in the present case, the nonlinearity appears only in the boundary term of the integral representation. We consider the same scaling as in [Druet and Robert 2006] and [Ndiaye 2007b; 2007a]. As already remarked, we have to consider only boundary blow-up points. When dealing with the boundary blow-up phenomenon, we adopt the same strategy as in [Ndiaye 2007b; 2007a] to conclude that the limit function V_0 describing the profile near the blow-up points satisfies the integral equation

$$\tilde{V}_0(x) = \int_{\mathbb{R}^3} \sigma_3 \log\Big(\frac{|z|}{|x-z|}\Big) e^{3\tilde{V}_0(z)} dz - \tfrac{1}{4}\log(k_3). \tag{1-13}$$

for some constants σ_3 and k_3. Recalling that we are looking for boundary quantization, we have therefore only to understand the behavior of the singularity V_0 at the boundary ∂M. For this, we follow an argument in [Ndiaye 2007b; 2007a], which is based on a classification result of X. Xu [2005], and we deduce that the restriction of V_0 on $\mathbb{R}^3$ is a *standard bubble* (on $\mathbb{R}^3$) and the local volume is $4\pi^2$. At this stage we finish by arguing as in [Ndiaye 2007b; 2007a] to show that the residual volume tends to zero, and we obtain the desired quantization.

With this compactness result in hand, we can describe the proof of Theorem 1.2 assuming Case 1.5 and that $P^{4,3}_g$ is nonnegative (that is, $\bar{k}=0$). In [Djadli and Malchiodi 2006; Ndiaye 2007b] the existence theorem was proved considering the *formal barycenters* of the manifold M, which we will recall, together with the differences with the present case. The arguments in [Djadli and Malchiodi 2006] can be summarized as follows. First, from $\kappa_{P^4} \in (k8\pi^2, (k+1)8\pi^2)$ and considerations coming from an improvement of a Moser–Trudinger type inequality, it follows that if $\mathrm{II}(u)$ attains large negative values, then e^{4u} must concentrate near at most k points of M. This means that, if we normalize u so that $\int_M e^{4u}\, dV_g = 1$ (which is possible because the functional is invariant under translation by a constant), then naively

$$e^{4u} \simeq \textstyle\sum_{i=1}^k t_i \delta_{x_i} \quad \text{for } x_i \in M \text{ and } t_i \geq 0 \text{ with } \sum_{i=1}^k t_i = 1.$$

Such a convex combination of Dirac deltas is called a formal barycenter of M of order k (see [Djadli and Malchiodi 2006, Section 2]) and is denoted by M_k. With further analysis (see [Djadli and Malchiodi 2006, Proposition 3.1]), it is possible to show that the sublevel $\{\mathrm{II} < -L\}$ for large L has the same homology as M_k. The existence of solutions was found using this fact and the noncontractibility of M_k (which is a crucial ingredient).

The present case differs in that M_k might be contractible and also boundary concentration can appear, so the same arguments cannot be applied. However, due to a trace Moser–Trudinger type inequality and its improvement, we are able to derive that if $\kappa_{(P^4,P^3)} \in (k4\pi^2, (k+1)4\pi^2)$, then the fact that $\mathrm{II}(u)$ attains large negative values implies that $e^{3u}|_{\partial M}$ must concentrate near at most k points of ∂M. This means that, if we normalize u so that $\int_{\partial M} e^{3u} ds_g = 1$ (which is also possible in this case because II is invariant under translation by a constant), then naively

$$e^{3u}|_{\partial M} \simeq \textstyle\sum_{i=1}^k t_i\delta_{x_i} \quad \text{for } x_i \in \partial M \text{ and } t_i \geq 0 \text{ with } \textstyle\sum_{i=1}^k t_i = 1.$$

Such a convex combination of Dirac deltas is called a formal barycenter of ∂M of order k (see Section 2) and will be denoted by ∂M_k. It is therefore natural to use the set ∂M_k to describe the homology of very large negative sublevels of the functional II. Indeed, with a further analysis (see Proposition 4.10), it is possible to show that the sublevel $\{\mathrm{II} < -L\}$ for large L has the same homology as ∂M_k. Using the noncontractibility of ∂M_k, we define a min-max scheme for a perturbed functional II_ρ with ρ close to 1, and we find a PS sequence at some level c_ρ. Applying the monotonicity procedure of Struwe, we can show existence of critical points of II_ρ for almost all ρ, which means that the assumptions of Corollary 1.9 are satisfied.

The structure of the paper is as follows. In Section 2, we present notation and some preliminaries, such as the existence of the Green function for (P_g^4, P_g^3) with homogeneous Neumann condition, a regularity result for BVPs of the type (1-8), and a trace Moser–Trudinger-type inequality. In Section 3, we prove Theorem 1.8, from which the proof of Corollary 1.9 becomes a trivial application. Section 4, in which we prove Theorem 1.2, has four subsections. The first concerns an improvement of the trace Moser–Trudinger-type inequality and its applications. The second deals with the existence of a nontrivial global projection from negative sublevels of II onto ∂M_k. The third concerns mapping ∂M_k into negative sublevels, and the last deals with the min-max scheme.

2. Notations and preliminaries

In this brief section we first fix some useful notations. We then state a lemma giving the existence of the Green function of the operator (P_g^4, P_g^3) with homogeneous Neumann boundary condition, find its asymptotics near the singularity, and present a trace analogue of the well-known Moser–Trudinger inequality for the operator $P_g^{4,3}$ when it is nonnegative (that is, when $\bar{k} = 0$).

Let $B_p(r)$ be the metric ball of radius r and center p. We let $B_p^+(r) = B_p(r) \cap M$ if $p \in \partial M$, but sometimes we use $B_p^+(r)$ to denote $B_p(r) \cap M$ even if $p \notin \partial M$. We let $B^x(r)$ be the Euclidean ball of center x and radius r. We let $B_+^x(r) = B^x(r) \cap \mathbb{R}_+^4$ if $x \in \partial\mathbb{R}_+^4$, but again we use $B_+^x(r)$ to denote $B^x(r) \cap \mathbb{R}_+^4$ even if $x \notin \partial\mathbb{R}_+^4$.

We denote by $d_g(x, y)$ the metric distance between $x, y \in M$ and by $d_{\hat{g}}(x, y)$ the intrinsic distance between $x, y \in \partial M$. Here $\hat{g}$ is the metric on ∂M induced by g. Given a point $x \in \partial M$ and $r > 0$, let $B_x^{\partial M}(r)$ be the ball in ∂M centered at x and with radius r with respect to the (intrinsic) distance $d_{\hat{g}}(\cdot, \cdot)$.

Let $H^2(M)$ be the usual Sobolev space of functions on M of class H^2 in each coordinate system.

We use C to denote a large positive constant, and the value of C is allowed to vary from formula to formula and also within the same line.

Denote by M^2 the cartesian product $M \times M$ and by $\mathrm{diag}(M)$ the diagonal of M^2.

For $u \in L^1(\partial M)$, denote by $\bar{u}_{\partial M}$ its average on ∂M, that is,

$$\bar{u}_{\partial M} = \frac{1}{\mathrm{vol}_{\hat{g}}(\partial M)} \int_{\partial M} u(x)\, dS_g(x), \quad \text{where } \mathrm{vol}_{\hat{g}}(\partial M) = \textstyle\int_{\partial M} dS_g.$$

As usual, $\mathbb{N}$ is the natural numbers, and $\mathbb{N}^*$ is the set of positive integers.

$A_l = o_l(1)$ means that $A_l \to 0$ as the integer $l \to +\infty$.

$A_\epsilon = o_\epsilon(1)$ means that $A_\epsilon \to 0$ as the real number $\epsilon \to 0$.

$A_\delta = o_\delta(1)$ means that $A_\delta \to 0$ as the real number $\delta \to 0$.

$A_l = O(B_l)$ means that $A_l \le C B_l$ for some constant C independent of l.

We denote by dV_g the Riemannian measure associated to the metric g, by dS_g the Riemannian measure associated to $\hat{g}$, and by $d\sigma_{\hat{g}}$ the surface measure on the boundary of balls of ∂M. We let $|\cdot|_{\hat{g}}$ be the norm associated to $\hat{g}$.

The notation $f = f(a, b, c, \dots)$ means that f is a quantity that depends only on $a, b, c, \dots$.

Define a family of formal sums by

$$\partial M_k = \left\{ \textstyle\sum_{i=1}^k t_i \delta_{x_i} : t_i \ge 0,\ \sum_{i=1}^k t_i = 1,\ x_i \in \partial M \right\}; \tag{2-1}$$

the set ∂M_k is called the formal set of barycenters relative to ∂M of order k. We recall that ∂M_k is a stratified set, namely, a union of sets of various dimension with maximum dimension equal to $4k - 1$.

We recall the following result (see [Djadli and Malchiodi 2006, Lemma 3.7]), which is necessary in order to carry out the topological argument below.

Lemma 2.1 (well known). *For any $k \ge 1$, one has $H_{4k-1}(\partial M_k; \mathbb{Z}_2) \ne 0$. As a consequence, ∂M_k is noncontractible.*

If $\varphi \in C^1(\partial M)$ and if $\sigma \in \partial M_k$, the action of σ on φ is $\langle \sigma, \varphi \rangle = \sum_{i=1}^k t_i \varphi(x_i)$, or we may write $\sigma = \sum_{i=1}^k t_i \delta_{x_i}$.

Moreover, if f is a nonnegative L^1 function on ∂M with $\int_{\partial M} f\,ds_g = 1$, we can define a distance of f from ∂M_k in through

$$d(f, \partial M_k) = \inf_{\sigma \in \partial M_k} \sup\left\{ \left| \int_{\partial M} f\varphi\, dS_g - \langle \sigma, \varphi\rangle \right| : \|\varphi\|_{C^1(\partial M)} = 1 \right\}. \tag{2-2}$$

We also define the set

$$\mathcal{D}_{\varepsilon,k} = \left\{ f \in L^1(\partial M) : f \ge 0,\ \|f\|_{L^1(\partial M)} = 1,\ d(f, \partial M_k) < \varepsilon \right\}. \tag{2-3}$$

Now we state a lemma which asserts the existence of the Green function of (P_g^4, P_g^3) with homogeneous Neumann condition. Its proof is the same as that of [Ndiaye 2007a, Proposition 2.3].

Lemma 2.2. *Suppose* $\ker P_g^{4,3} \simeq \mathbb{R}$. *Then the Green function* $G(x, y)$ *of* (P_g^4, P_g^3) *exists in the following sense*:

(a) *For all functions* $u \in C^2(M)$ *with* $\partial u/\partial n_g = 0$, *we have, for* $x \in M$,

$$u(x) - \bar{u}_{\partial M} = \int_M G(x, y) P_g^4 u(y)\, dV_g(y) + 2\int_{\partial M} G(x, y') P_g^3 u(y')\, dS_g(y').$$

(b) $G(x, y) = H(x, y) + K(x, y)$ *is smooth on* $M^2 \setminus \operatorname{diag}(M^2)$, K *extends to a* $C^{2+\alpha}$ *function on* M^2, *and*

$$H(x, y) = \begin{cases} (8\pi^2)^{-1} f(r) \log(1/r) & \text{if } B_\delta(x) \cap \partial M = \varnothing, \\ (8\pi^2)^{-1} f(r)(\log(1/r) + \log(1/\bar{r})) & \text{otherwise.} \end{cases}$$

Here $r = d_g(x, y)$ *and* $\bar{r} = d_g(x, \bar{y})$. *Also* $f(x) = 1$ *for* $x \in [-\delta/2, \delta/2]$ *and* $f \in C_0^\infty(-\delta, \delta)$, *where* $\delta \le \frac{1}{2}\min\{\delta_1, \delta_2\}$. *In the latter,* δ_1 *is the injectivity radius of* M *in an extension* $\tilde{M}$ *of* M, *and* $\delta_2 = \delta_0/2$.

Next we give a regularity result for boundary value problems of type (1-8), and also give high order *a priori* estimates for sequences of solutions to BVPs like (1-9) when the solutions are bounded from above. The proof is a trivial adaptation of the arguments of [Ndiaye 2007a, Proposition 2.4].

Lemma 2.3. *Let* $u \in H_{\partial/\partial n}$ *be a weak solution to*

$$P_g^4 u = h \quad \text{in } M \quad \text{and} \quad P_g^3 u + f = \bar{f} e^{3u} \quad \text{on } \partial M$$

with $f \in C^\infty(\partial M)$, $h \in C^\infty(M)$ *and* $\bar{f}$ *a real constant. Then* $u \in C^\infty(M)$.

Let $u_l \in H_{\partial/\partial n}$ *be a sequence of weak solutions to*

$$P_g^4 u_l = h_l \quad \text{in } M \quad \text{and} \quad P_g^3 u_l + f_l = \bar{f}_l e^{3u_l} \quad \text{on } \partial M.$$

with $f_l \to f_0$ *in* $C^k(\partial M)$, $\bar{f}_l \to \bar{f}_0$ *in* $C^k(\partial M)$ *and* $h_l \to h_0$ *in* $C^k(M)$ *for some fixed* $k \in \mathbb{N}^*$. *Assuming* $\sup_{\partial M} u_l \le C$ *we have that* $\|u_l\|_{C^{k+3+\alpha}(M)} \le C$ *for any* $\alpha \in (0, 1)$.

Now we present a proposition giving a trace Moser–Trudinger-type inequality when the operator $P_g^{4,3}$ is nonnegative (that is, when $\bar{k}=0$) with trivial kernel.

Proposition 2.4. *Assume $P_g^{4,3}$ is a nonnegative operator with* $\ker P_g^{4,3} \simeq \mathbb{R}$. *Then for all $\alpha < 12\pi^2$, there exists a constant $C = C(M, g, \alpha)$ such that*

$$\int_{\partial M} \exp\left(\frac{\alpha(u-\bar{u}_{\partial M})^2}{\langle P_g^{4,3}u, u\rangle_{L^2(M,g)}}\right) dS_g \le C, \tag{2-4}$$

for all $u \in H_{\partial/\partial n}$. Hence

$$\log \int_{\partial M} e^{3(u-\bar{u})}\, dS_g \le C + \tfrac{9}{4\alpha}\langle P_g^{4,3}u, u\rangle_{L^2(M,g)} \quad \textit{for all } u \in H_{\partial/\partial n}. \tag{2-5}$$

Proof. Without loss of generality we can assume $\bar{u}_{\partial M} = 0$. Following the same argument as in [Chang and Qing 1997b, Lemma 2.2], we get that for all $\beta < 16\pi^2$, there exists a $C = C(\beta, M)$ such that

$$\int_M \exp\left(\frac{\beta v^2}{\int_M |\Delta_g v|^2 dV_g}\right) dV_g \le C \quad \text{for all } v \in H_{\partial/\partial n} \text{ with } \bar{v}_{\partial M} = 0.$$

From this, using the same reasoning as in [Ndiaye 2007a, Proposition 2.7], we derive

$$\int_M \exp\left(\frac{\beta v^2}{\langle P_g^{4,3}v, v\rangle_{L^2(M)}}\right) dV_g \le C \quad \text{for all } v \in H_{\partial/\partial n} \text{ with } \bar{v}_{\partial M} = 0. \tag{2-6}$$

Now let X be a vector field extending the outward normal at the boundary ∂M. Using the divergence theorem we obtain

$$\begin{aligned} \int_{\partial M} e^{\alpha u^2}\, dS_g &= \int_M \operatorname{div}_g(X e^{\alpha u^2})\, dV_g \\ &= \int_M (\operatorname{div}_g X + 2u\alpha \nabla_g u \nabla_g X) e^{\alpha u^2}\, dV_g, \end{aligned} \tag{2-7}$$

where we have used in the second line the formula for the divergence of the product of a vector field and a function. Now we suppose $\langle P_g^{4,3}u, u\rangle_{L^2(M)} \le 1$, then since the vector field X is smooth we have

$$\left|\int_M \operatorname{div}_g(X e^{\alpha u^2})\, dV_g\right| \le C \tag{2-8}$$

by (2-6). Next let us show that

$$\left|\int_M 2\alpha u \nabla_g u \nabla_g X e^{\alpha u^2}\, dV_g\right| \le C.$$

To do so, let ϵ be small and positive, and set $p_1 = 4/(3-\epsilon)$, $p_2 = 4$ and $p_3 = 4/\epsilon$. It is easy to check that $(p_1)^{-1} + (p_2)^{-1} + (p_3)^{-1} = 1$. Using Young's inequality we obtain

$$\left|\int_M 2\alpha u\nabla_g u\nabla_g X e^{\alpha u^2}\,dV_g\right| \le C\|u\|_{L^{4/\epsilon}}\|\nabla_g u\|_{L^4}\left(\int_M e^{\alpha u^2 4/(3-\epsilon)}\,dV_g\right)^{(3-\epsilon)/4}.$$

On the other hand, [Ndiaye 2007a, Lemma 2.8] and the Sobolev embedding theorem imply $\|u\|_{L^{\frac{4}{\epsilon}}} \le C$ and $\|\nabla_g u\|_{L^4} \le C$. Furthermore, from the fact that $\alpha < 12\pi^2$, by taking ϵ sufficiently small and using (2-4), we obtain

$$\left(\int_M e^{\alpha u^2 4/(3-\epsilon)}\,dV_g\right)^{(3-\epsilon)/4} \le C.$$

Thus we arrive at

$$\left|\int_M 2\alpha u\nabla_g u\nabla_g X e^{\alpha u^2}\,dV_g\right| \le C. \tag{2-9}$$

Hence (2-7), (2-8) and (2-9) imply $\int_{\partial M} e^{\alpha u^2}\,dS_g \le C$, as desired. So the first part of the lemma is proved.

Now, using the algebraic inequality $3ab \le 3\gamma^2 a^2 + 3b^2/4\gamma^2$, the second part follows directly from the first one, and we are done. □

3. Proof of Theorem 1.8

Here we use a strategy related to those in [Ndiaye 2007b; 2007a]. Hence we will only sketch many steps, referring to the corresponding arguments there for details. However, in contrast to the situation in [Ndiaye 2007a], due to Remark 1.7, we need only take care of the behavior of the restriction of the sequence u_l to the boundary ∂M of M.

First we recall a particular case of a result of X. Xu.

Theorem 3.1 [Xu 2005, Theorem 1.2]. *There exists a dimensional constant $\sigma_3 > 0$ such that if $u \in C^1(\mathbb{R}^3)$ is a solution to the integral equation*

$$u(x) = \int_{\mathbb{R}^3} \sigma_3 \log\left(\frac{|y|}{|x-y|}\right) e^{3u(y)}\,dy + c_0,$$

where c_0 is a real number, then $e^u \in L^3(\mathbb{R}^3)$ implies that there exists $\lambda > 0$ and $x_0 \in \mathbb{R}^3$ such that $u(x) = \log(2\lambda/(\lambda^2 + |x-x_0|^2))$.

Now, if σ_3 is as in Theorem 3.1, we set $k_3 = 2\pi^2\sigma_3$ and $\gamma_3 = 2(k_3)^3$.

We divide the proof into five claims as in [Ndiaye 2007b; 2007a].

Claim 1. *For some $N \in \mathbb{N}^*$, there exist N converging points $(x_{i,l}) \subset \partial M$ and N sequences $(\mu_{i,l})$ of positive real numbers converging to 0 (both sets are indexed by $i = 1, \dots, N$) such that the following hold:*

(a) $d_g(x_{i,l}, x_{j,l})/\mu_{i,l} \to +\infty$ *for* $i, j = 1, \dots, N$ *with* $i \neq j$ *and*

$$\overline{T}_l(x_{i,l})\mu_{i,l}^3 e^{3u_l(x_{i,l})} = 1.$$

(b) *For every* i,

$$v_{i,l}(x) = u_l(\exp_{x_{i,l}}(\mu_{i,l}x)) - u_l(x_{i,l}) - \tfrac{1}{3}\log(k_3) \to V_0(x) \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^4_+),$$

where $V_0(x) := \log(4\gamma_3/(4\gamma_3^2 + |x|^2))$ *for* $x \in \partial\mathbb{R}^4_+$, *and*

$$\lim_{R\to+\infty}\lim_{l\to+\infty}\int_{B^+_{x_{i,l}}(R\mu_{i,l})\cap\partial M} \overline{T}_l(y)e^{3u_l(y)}ds_g(y) = 4\pi^2.$$

(c) *There exists a* $C > 0$ *such that* $\inf_{i=1,\dots,N} d_g(x_{i,l}, x)^3 e^{3u_l(x)} \le C$ *for all* $x \in \partial M$ *and for all* $l \in \mathbb{N}$.

Proof. First let $x_l \in \partial M$ be such that $u_l(x_l) = \max_{x\in\partial M} u_l(x)$. Then the fact that u_l blows up implies $u_l(x_l) \to +\infty$. Now since ∂M is compact, we can assume without loss of generality that $x_l \to \bar{x} \in \partial M$.

Next let $\mu_l > 0$ be such that $\overline{T}_l(x_l)\mu_l^3 e^{3u_l(x_l)} = 1$. Since $\overline{T}_l \to \overline{T}_0$ in $C^1(\partial M)$, $\overline{T}_0 > 0$ and $u_l(x_l) \to +\infty$, we have $\mu_l \to 0$.

Let $B^0_+(\delta\mu_l^{-1})$ be the Euclidean half-ball of center 0 and radius $\delta\mu_l^{-1}$ for some positive, fixed, and small δ. For $x \in B^0_+(\delta\mu_l^{-1})$, we set

$$v_l(x) = u_l(\exp_{x_l}(\mu_l x)) - u_l(x_l) - \tfrac{1}{3}\log(k_3); \tag{3-1}$$

$$\tilde{Q}_l(x) = Q_l(\exp_{x_l}(\mu_l x)); \tag{3-2}$$

$$\tilde{\overline{Q}}_l(x) = \overline{Q}_l(\exp_{x_l}(\mu_l x)); \tag{3-3}$$

$$g_l(x) = (\exp^*_{x_l} g)(\mu_l x). \tag{3-4}$$

Now from the Green representation formula we have

$$u_l(x) - \bar{u}_{\partial M,l} = \int_M G(x,y)P^4_g u_l(y)\,dV_g(y) + 2\int_{\partial M} G(x,y')P^3_g u_l(y')\,dS_g(y') \quad \text{for all } x \in M, \tag{3-5}$$

where G is the Green function of (P^4_g, P^3_g) (see Lemma 2.2).

Next using Equation (1-9) and differentiating (3-5) with respect to x, we obtain for $k = 1, 2$ that

$$|\nabla^k u_l|_g(x) \le \int_{\partial M} |\nabla^k G(x,y)|_g \overline{T}_l(y)e^{3u_l(y)}\,dV_g + O(1),$$

since $T_l \to T_0$ in $C^1(\partial M)$ and $Q_l \to Q_0$ in $C^1(M)$.

Now let $y_l \in B^+_{x_l}(R\mu_l)$ for some fixed positive R. By the same argument as in [Ndiaye 2007b, formula 43, page 11], we get

$$\int_{\partial M} |\nabla^k G(y_l, y)|_g e^{3u_l(y)} dV_g(y) = O(\mu_l^{-k}). \tag{3-6}$$

Hence we have

$$|\nabla^k v_l|_g(x) \le C. \tag{3-7}$$

Furthermore from the definition of v_l (see (3-1)), we obtain

$$v_l(x) \le v_l(0) = -\tfrac{1}{3}\log(k_3) \quad \text{for all } x \in \mathbb{R}^4_+. \tag{3-8}$$

Thus we infer that $(v_l)_l$ is uniformly bounded in $C^2(K)$ for all compact subsets K of $\mathbb{R}^4_+$. Hence by the Arzelà–Ascoli theorem we derive that

$$v_l \to V_0 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^4_+), \tag{3-9}$$

On the other hand, (3-8) and (3-9) imply that

$$V_0(x) \le V_0(0) = -\tfrac{1}{3}\log(k_3) \quad \text{for all } x \in \mathbb{R}^4_+. \tag{3-10}$$

Moreover from (3-7) and (3-9) we have that V_0 is Lipschitz.

Now, using again the Green's representation formula for (P^4_g, P^3_g), we obtain for $x \in \mathbb{R}^4_+$ fixed and for R big enough such that $x \in B^0_+(R)$ that

$$\begin{aligned} u_l(\exp_{x_l}(\mu_l x)) - \bar{u}_{\partial M,l} = \int_M & G(\exp_{x_l}(\mu_l x), y) P^4_g u_l(y)\, dV_g(y) \\ & + 2\int_{\partial M} G(\exp_{x_l}(\mu_l x), y') P^3_g u_l(y')\, dS_g(y'). \end{aligned} \tag{3-11}$$

Next let us set

$$\mathrm{I}_l(x) = 2 \int_{B^+_{x_l}(R\mu_l)\cap\partial M} \big(G(\exp_{x_l}(\mu_l x), y') - G(\exp_{x_l}(0), y')\big)\, \overline{T}_l(y') e^{3u_l(y')}\, dS_g(y'),$$

$$\mathrm{II}_l(x) = 2 \int_{\partial M\setminus(B^+_{x_l}(R\mu_l)} \big(G(\exp_{x_l}(\mu_l x), y') - G(\exp_{x_l}(0), y')\big)\, \overline{T}_l(y') e^{3u_l(y')}\, dS_g(y'),$$

$$\mathrm{III}_l(x) = 2 \int_{\partial M} \big(G(\exp_{x_l}(\mu_l x), y') - G(\exp_{x_l}(0), y')\big)\, T_l(y')\, dS_g(y'),$$

$$\mathrm{IIII}_l(x) = 2 \int_{M} \big(G(\exp_{x_l}(\mu_l x), y) - G(\exp_{x_l}(0), y)\big)\, Q_l(y)\, dV_g(y).$$

Using arguments of [Ndiaye 2007b, formulas (45)–(51)], we get

$$v_l(x) = \mathrm{I}_l(x) + \mathrm{II}_l(x) - \mathrm{III}_l(x) - \mathrm{IIII}_l(x) - \tfrac{1}{4}\log(3). \tag{3-12}$$

By following the methods of [Ndiaye 2007b, formulas (53)–(62)], we get

$$
\begin{aligned}
&\lim_l \mathrm{I}_l(x) = \int_{B_+^0(R)\cap\partial R_+^4} \sigma_3 \log\Big(\frac{|z|}{|x-z|}\Big) e^{3V_0(z)}\,dz,\\
&\limsup_l \mathrm{II}_l(x) = o_R(1), \quad \mathrm{III}_l(x) = o_l(1), \quad \mathrm{IIII}_l(x) = o_l(1).
\end{aligned} \tag{3-13}
$$

Hence from (3-9), (3-12), and (3-13) we may let l and R tend to infinity to obtain that $V_0|_{\mathbb{R}^3}$ (for simplicity we will still write this as V_0) satisfies the conformally invariant integral equation

$$
V_0(x) = \int_{\mathbb{R}^3} \sigma_3 \log\Big(\frac{|z|}{|x-z|}\Big) e^{3V_0(z)}\,dz - \tfrac{1}{3}\log(k_3) \tag{3-14}
$$

on $\mathbb{R}^3$. Now since V_0 is Lipschitz, the theory of singular integral operators gives that $V_0 \in C^1(\mathbb{R}^3)$.

On the other hand, by using the change of variable $y = \exp_{x_l}(\mu_l x)$, one can check that

$$
\lim_{l\to+\infty} \int_{B_{x_l}^+(R\mu_l)\cap\partial M} \overline{T}_l e^{3u_l}\,dV_g = k_3 \int_{B_0^+(R)\cap\partial R_+^4} e^{3V_0}\,dx. \tag{3-15}
$$

Hence (1-12) implies that $e^{V_0} \in L^3(\mathbb{R}^3)$.

Furthermore by a classification result by X. Xu (see Theorem 3.1 for the solutions of (3-14)), we derive that

$$
V_0(x) = \log\Big(\frac{2\lambda}{\lambda^2 + |x-x_0|^2}\Big) \tag{3-16}
$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^3$.

Moreover from $V_0(x) \le V_0(0) = -\frac{1}{3}\log(k_3)$ for all $x \in \mathbb{R}^3$, we have $\lambda = 2k_3$ and $x_0 = 0$, so that $V_0(x) = \log(4\gamma_3/(4\gamma_3^2 + |x|^2))$. On the other hand, by letting R tend to infinity in (3-15), we obtain

$$
\lim_{R\to+\infty}\ \lim_{l\to+\infty} \int_{B_{x_l}^+(R\mu_l)\cap\partial R_+^4} \overline{T}_l(y) e^{3u_l(y)}\,dS_g(y) = k_3 \int_{\mathbb{R}^3} e^{3V_0}\,dx. \tag{3-17}
$$

By a generalized Pohozaev type identity of X. Xu [2005, Theorem 1.1], we get $\sigma_3 \int_{\mathbb{R}^3} e^{3V_0(y)}\,dy = 2$; hence using (3-17), we derive that

$$
\lim_{R\to+\infty}\ \lim_{l\to+\infty} \int_{B_{x_l}^+(R\mu_l)\cap\partial M} \overline{T}_l(y) e^{3u_l(y)}\,dS_g(y) = 4\pi^2.
$$

Now for $k \ge 1$ we say that (H_k) holds if there exist k converging sequences of points $(x_{i,l})_l \subset \partial M$ and k sequences $(\mu_{i,l})$ of positive real numbers converging to 0 (both of which are indexed by $i = 1, \dots, k$) such that

(A_k^1) $d_{\hat g}(x_{i,l}, x_{j,l})/\mu_{i,l} \to +\infty$ for $i, j = 1, \dots, k$ with $i \neq j$ and

$$\overline{T}_l(x_{i,l})\mu_{i,l}^3 e^{3u_l(x_{i,l})} = 1;$$

(A_k^2) for every $i \in \{1, \dots, k\}$,

$$v_{i,l}(x) = u_l(\exp_{x_{i,l}}(\mu_{i,l}x)) - u_l(x_{i,l}) - \tfrac{1}{3}\log(k_3) \to V_0(x) \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^4_+),$$

where $V_{0|}\partial\mathbb{R}^4_+ := \log(4\gamma_3/(4\gamma_3^2 + |x|^2))$ and

$$\lim_{R\to+\infty}\ \lim_{l\to+\infty}\int_{B^+_{x_{i,l}}(R\mu_{i,l})\cap\partial M} \overline{T}_l(y)e^{3u_l(y)} = 4\pi^2.$$

Clearly (H_1) holds by the arguments above. We let now $k \geq 1$ and assume that (H_k) holds. We also assume that

$$\sup_{\partial M} R_{k,l}(x)^3 e^{3u_l(x)} \to +\infty \quad \text{as } l \to +\infty, \tag{3-18}$$

where $R_{k,l}(x) = \min_{i=1,\dots,k} d_g(x_{i,l}, x)$. Now, by using the arguments of [Druet and Robert 2006; Ndiaye 2007b], one can easily see that (H_{k+1}) also holds. Hence, since (A_k^1) and (A_k^2) of (H_k) imply that

$$\int_{\partial M} \overline{T}_l(y)e^{3u_l(y)}\,dS_g(y) \geq k4\pi^2 + o_l(1),$$

if follows from (1-12) that there exists a maximal k with

$$1 \leq k \leq \frac{1}{4\pi^2}\Big(\int_M Q_0(y)\,dV_g(y) + \int_{\partial M} T_0(y')\,dS_g(y')\Big)$$

such that (H_k) holds. Upon arriving at this maximal k, we conclude that (3-18) cannot hold. Hence setting $N = k$, the proof of Claim 1 is completed. □

Claim 2. *There exists a constant $C > 0$ such that*

$$R_l(x)|\nabla_g u_l|_g(x) \leq C \quad \textit{for all } x \in \partial M \textit{ and } l \in N, \tag{3-19}$$

where $R_l(x) = \min_{i=1,..,N} d_g(x_{i,l}, x)$, and the $x_{i,l}$ are as in Claim 1.

Proof. First, using the Green representation formula for (P_g^4, P_g^3) (see Lemma 2.2), we obtain

$$\begin{aligned} u_l(x) - \bar u_{\partial M,l} &= \int_M G(x,y)P_g^4 u_l(y)\,dV_g(y) + 2\int_{\partial M} G(x,y')P_g^3 u_l(y')\,dS_g(y') \\ &= -2\int_M G(x,y)Q_l\,dV_g(y) - 2\int_{\partial M} G(x,y')T_l(y')u_l(y')\,dS_g(y') \\ &\qquad + 2\int_{\partial M} G(x,y)\overline{T}_l(y')e^{3u_l(y')}\,dS_g(y'), \end{aligned}$$

where the second equality was obtained using the BVP (1-8). Thus differentiating this equation with respect to x and using the facts that $Q_l \to Q_0$, $\overline{Q}_l \to \overline{Q}_0$ and $T_l \to T_0$ in C^2, we have for $x_l \in \partial M$

$$|\nabla_g u_l(x_l)|_g = O\Big(\int_{\partial M} \frac{1}{d_g(x_l, y)} e^{3u_l(y)}\, dS_g(y)\Big) + O(1).$$

By following the arguments of the proof of [Ndiaye 2007b, Theorem 1.3, Step 2], we obtain

$$\int_{\partial M} \frac{1}{(d_g(x_l, y))} e^{3u_l(y)}\, dV_g(y) = O\Big(\frac{1}{R_l(x_l)}\Big).$$

Since x_l is arbitrary, the proof of Claim 2 is finished. □

Claim 3. *Set* $R_{i,l} = \min_{i\neq j} d_g(x_{i,l}, x_{j,l})$.

(a) *There exists a constant* $C > 0$ *such that for all* $r \in (0, R_{i,l}]$ *and* $s \in (\frac{r}{4}, r]$

(3-20) $$\big|u_l(\exp_{x_{i,l}}(rx)) - u_l(\exp_{x_{i,l}}(sy))\big| \le C$$

for all $x, y \in \partial\mathbb{R}^4_+$ *such that* $|x|, |y| \le 3/2$.

(b) *If* $d_{i,l}$ *is such that* $0 < d_{i,l} \le R_{i,l}/2$ *and* $d_{i,l}/\mu_{i,l} \to +\infty$, *then provided*

(3-21) $$\int_{B^+_{x_{i,l}}(d_{i,l})\cap\partial M} \overline{T}_l(y) e^{3u_l(y)}\, dS_g(y) = 4\pi^2 + o_l(1),$$

we have

$$\int_{B^+_{x_{i,l}}(2d_{i,l})\cap\partial M} \overline{T}_l(y) e^{3u_l(y)} ds_g(y) = 4\pi^2 + o_l(1).$$

(c) *Suppose, for* R *large and fixed, that* $d_{i,l} > 0$ *satisfies* $d_{i,l} \to 0$, $d_{i,l}/\mu_{i,l} \to +\infty$, *and* $d_{i,l} < R_{i,l}/(4R)$. *Also suppose*

$$\int_{B^+_{x_{i,l}}(d_{i,l}/(2R))\cap\partial M} \overline{Q}_l(y) e^{3u_l(y)}\, dS_g(y) = 4\pi^2 + o_l(1).$$

Set $\tilde{u}_l(x) = u_l(\exp_{x_{i,l}}(d_{i,l}x))$ *for* $x \in A^+_{2R} := (B^0_+(2R) \setminus B^0_+(1/(2R))) \cap \partial\mathbb{R}^4_+$. *Then*

$$\|d^3_{i,l} e^{3\tilde{u}_l}\|_{C^\alpha(A^+_R)} \to 0 \quad \textit{as } l \to +\infty$$

for some $\alpha \in (0, 1)$, *where* $A^+_R = (B^0_+(R) \setminus B^0_+(1/R)) \cap \partial\mathbb{R}^4_+$.

Proof. Part (a) follows immediately from Claim 2 and the definition of $R_{i,l}$. In fact we can join rx to sy by a curve whose length is bounded by a constant proportional to r.

We turn to part (b). By $d_{i,l}/\mu_{i,l} \to +\infty$, Claim 1(c) and (3-21), we have

(3-22) $$\int_{B^+_{x_{i,l}}(d_{i,l})\cap\partial M \setminus B^+_{x_{i,l}}(d_{i,l}/2)\cap\partial M} e^{3u_l(y)}\, dS_g(y) = o_l(1).$$

Thus, using (3-20) with $s = r/2$ and $r = 2d_{i,l}$, we get

$$\int_{B^+_{x_{i,l}}(2d_{i,l})\cap\partial M\setminus B^+_{x_{i,l}}(d_{i,l})\cap\partial M} e^{3u_l(y)}\,dS_g(y) \le C \int_{B^+_{x_{i,l}}(d_{i,l})\cap\partial M\setminus B^+_{x_{i,l}}(d_{i,l}/2)\cap\partial M} e^{3u_l(y)}\,dS_g(y).$$

Hence

$$\int_{B^+_{x_{i,l}}(2d_{i,l})\cap\partial M\setminus B^+_{x_{i,l}}(d_{i,l})\cap\partial M} e^{3u_l(y)}\,dS_g(y) = o_l(1).$$

This proves part (b). One proves the last part (c) by following straightforwardly the proof of [Ndiaye 2007b, point 3 in Step 3 of Theorem 1.3]. □

Claim 4. *There exists a positive constant C independent of l and i such that*

$$\int_{B^+_{x_{i,l}}(R_{i,l}/C)\cap\partial M} \overline{T}_l(y)e^{3u_l(y)}\,dS_g(y) = 4\pi^2 + o_l(1).$$

Proof. The proof is an adaptation of the arguments proving [Ndiaye 2007b, Step 4], but for convenience we provide full details.

First fix $1/3 < \nu < 2/3$, and for $i = 1, \dots, N$, set

$$\bar{u}_{i,l}(r) = \mathrm{vol}_g(\partial B^+_{x_{i,l}}(r)\cap\partial M)^{-1}\int_{\partial B^+_{x_{i,l}}(r)\cap\partial M} u_l(x)\,d\sigma_{\hat{g}}(x),$$
$$\varphi_{i,l}(r) = r^{3\nu}\exp(\bar{u}_{i,l}(r))$$

for all $0 \le r < \min\{\mathrm{inj}_g(M), \mathrm{inj}_{\hat{g}}(\partial M)\}$. By Claim 1(b), there exists R_ν such that, for all $R \ge R_\nu$,

(3-23) $$\varphi'_{i,l}(R\mu_{i,l}) < 0 \quad \text{for all } l \text{ sufficiently large (depending on } R).$$

Now we define $r_{i,l}$ by

(3-24) $$r_{i,l} = \sup\{R_\nu\mu_{i,l} \le r \le R_{i,l}/2 \mid \varphi'_{i,l}(r) < 0 \text{ for } r \in [R_\nu, r)\}.$$

Hence (3-23) implies that

(3-25) $$r_{i,l}/\mu_{i,l} \to +\infty \text{ as } l \to +\infty.$$

Now to prove the claim it suffices to show that $R_{i,l}/r_{i,l} \not\to +\infty$ as $l \to +\infty$.

Indeed if $R_{i,l}/r_{i,l} \not\to +\infty$, there exists a positive constant C independent of l and i such that

(3-26) $$R_{i,l}/C \le r_{i,l}.$$

On the other hand, from the Harnack-type inequality (3-20), Claim 1(b), and (3-24) there exists for any $\eta > 0$ an $R_\eta > 0$ such that for any $R > R_\eta$,

(3-27) $$d_g(x, x_{i,l})^{4\nu}e^{4u_l} \le \eta\mu_{i,l}^{4(\nu-1)} \quad \text{for all } x \in (B^+_{x_{i,l}}(r_{i,l}) \setminus B^+_{x_{i,l}}(R\mu_{i,l}))\cap\partial M.$$

Since $r_{i,l}/\mu_{i,l} \to +\infty$ by (3-25) and $R_{i,l}/2 \geq r_{i,l}$ by (3-24), $R_{i,l}/C\mu_{i,l} \to +\infty$; hence Claim 1(b), (3-27) and (3-26) imply that

$$\int_{B^+_{x_{i,l}}(R_{i,l}/C)\cap\partial M} \overline{T}_l e^{3u_l} = 4\pi^2 + o_l(1).$$

By continuity and by the definition of $r_{i,l}$, it follows that $\varphi'_{i,l}(r_{i,l}) = 0$. Let us assume by contradiction that $R_{i,l}/r_{i,l} \to +\infty$. We will show next that $\varphi'_{i,l}(r_{i,l}) < 0$ for large l, thus contradicting the previous equality. To do so we will study the function $\bar{u}_{i,l}$.

First let us remark that since M is compact $R_{i,l}/r_{i,l} \to +\infty$ implies that $r_{i,l} \to 0$. From Green's representation formula for u_l, we have

$$\begin{aligned} u_l(x) &= \int_M G(x,y)P^4_g u_l(y)\,dV_g(y) + \bar{u}_{\partial M,l} + 2\int_M G(x,y,)P^3_g u_l(y')\,dS_g(y') \\ &= 2\int_{\partial M} G(x,y)\overline{T}_l(y)e^{3u_l(y)}\,dS_g(y) \\ &\qquad - 2\int_M G(x,y)Q_l(y)\,dV_g(y) - 2\int_{\partial M} G(x,y')T_l(y')\,dS_g(y') + \bar{u}_{\partial M,l}. \end{aligned}$$

Hence

$$\begin{aligned} \bar{u}_{i,l}(r) &= 2(\mathrm{vol}_g(\partial B^+_{x_{i,l}}(r)\cap\partial M))^{-1}\iint_{\partial M} G(x,y)\overline{T}_l(y)e^{3u_l(y)}\,dVS_g(y)\,d\sigma_g(x) \\ &\qquad - 2(\mathrm{vol}_g(\partial B^+_{x_{i,l}}(r)\cap\partial M))^{-1}\iint_M G(x,y)Q_l(y)\,dV_g(y)\,d\sigma_g(x) \\ &\qquad - 2(\mathrm{vol}_g(\partial B^+_{x_{i,l}}(r)\cap\partial M))^{-1}\iint_{\partial M} G(x,y)T_l(y)\,dS_g(y)\,d\sigma_g(x) + \bar{u}_{\partial M,l}, \end{aligned}$$

where here and below the first integration is over $\partial B^+_{x_{i,l}}(r)\cap\partial M$. Setting

$$\begin{aligned} F_{i,l}(r) &= 2(\mathrm{vol}_g(\partial B^+_{x_{i,l}}(r)\cap\partial M))^{-1}\iint_M G(x,y)Q_l(y)\,dV_g(y)\,d\sigma_g(x) \\ &\qquad + 2(\mathrm{vol}_g(\partial B^+_{x_{i,l}}(r)\cap\partial M))^{-1}\iint_{\partial M} G(x,y)T_l(y)\,dS_g(y)\,d\sigma_g(x), \end{aligned}$$

we obtain

$$\begin{aligned} \bar{u}_{i,l} &= 2(\mathrm{vol}_g(\partial B^+_{x_{i,l}}(r)\cap\partial M)^{-1}\iint_{\partial M} G(x,y)\overline{T}_l(y)e^{3u_l(y)}\,dS_g(y)\,d\sigma_g(x) \\ &\qquad + \bar{u}_{\partial M,l} - F_{i,l}(r). \end{aligned}$$

Since $Q_l \to Q_0$ in $C^1(M)$ and $T_l \to T_0$ in $C^1(\partial M)$, it follows that $F_{i,l}$ is of class C^1 for all i, l. Also

$$|F'_{i,l}(r)| \leq C \quad \text{for all } r \in (0, \min\{\mathrm{inj}_g(M)/4, \mathrm{inj}_{\hat{g}}(\partial M)/4\}). \tag{3-28}$$

Now fix A such that

$$\min\{\tfrac{1}{4}\mathrm{inj}_g(M), \tfrac{1}{4}\mathrm{inj}_{\hat{g}}(\partial M)\} < A < \min\{\tfrac{1}{2}\mathrm{inj}_g(M), \tfrac{1}{2}\mathrm{inj}_{\hat{g}}(\partial M)\}.$$

We have

$$\int_{\partial M} G(x,y)\overline{T}_l(y)e^{3u_l(y)}\,dS_g(y) = \int_{B^+_{x_{i,l}}(A)\cap\partial M} G(x,y)\overline{T}_l e^{3u_l(y)}\,dS_g(y) + \int_{\partial M\setminus B^+_{x_{i,l}}(A)} G(x,y)\overline{T}_l e^{3u_l(y)}\,dS_g(y).$$

So

$$\begin{aligned}\bar{u}_{i,l}(r) = 2\,\mathrm{vol}_g(\partial B^+_{x_{i,l}}(r)\cap\partial M)^{-1} \times &\int_{\partial B^+_{x_{i,l}}(r)\cap\partial M}\int_{B^+_{x_{i,l}}(A)\cap\partial M}(G(x,y)-K(x,y))\,\overline{T}_l(y)e^{3u_l(y)}\,dS_g(y)d\sigma_g(x)\\ &+\bar{u}_{\partial M,l} - F_{i,l}(r) + H_{i,l}(r);\end{aligned}$$

with

$$\begin{aligned}H_{i,l}(r) = 2\,\mathrm{vol}_g(\partial B^+_{x_{i,l}}(r)\cap\partial M)^{-1} \times &\biggl(\int_{\partial B^+_{x_{i,l}}(r)\cap\partial M}\int_{\partial M\setminus B^+_{x_{i,l}}(A)} G(x,y)T_l(y)e^{3u_l(y)}\,dS_g(y)d\sigma_g(x)\\ &+\int_{\partial B^+_{x_{i,l}}(r)\cap\partial M}\int_{B^+_{x_{i,l}}(A)\cap\partial M} K(x,y)\overline{T}_l(y)e^{3u_l(y)}\,dS_g(y)d\sigma_g(x)\biggr).\end{aligned}$$

Since G is smooth outside of $\mathrm{diag}(M)$, it follows that

$$H_{i,l}\in C^1\left(0,\min\{\tfrac{1}{2}\mathrm{inj}_g(M), \tfrac{1}{2}\mathrm{inj}_{\hat{g}}(\partial M)\}\right) \quad \text{for all } i, l,$$

and

$$|H'_{i,l}(r)| \le C \quad \text{for all } r\in\left(0,\min\{\tfrac{1}{2}\mathrm{inj}_g(M), \tfrac{1}{2}\mathrm{inj}_{\hat{g}}(\partial M)\}\right). \tag{3-29}$$

Now using the change of variable $x = r\theta$ and $y = s\tilde{\theta}$, we obtain

$$\bar{u}_{i,l} = \bar{u}_{\partial M} - F_{i,l}(r) + H_{i,l}(r) + \frac{1}{\mathrm{vol}(S^2)}\int_{S^2}\int_{S^2}\int_0^A f(r,\theta)(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta}))\overline{T}(s\tilde{\theta})e^{3u_l(s\tilde{\theta})}s^2 f(s,\tilde{\theta})\,ds\,d\tilde{\theta}\,d\theta.$$

So differentiating with respect to r, we have

$$\bar{u}'_{i,l}(r) = (\mathrm{vol}(S^2))^{-1} \times \int_{S^2}\int_{S^2}\int_0^A \frac{\partial}{\partial r}\big(f(r,\theta)(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta}))\big)\overline{T}(s\tilde{\theta})e^{3u_l(s\tilde{\theta})}s^2 f(s,\tilde{\theta})\,ds\,d\tilde{\theta}\,d\theta - F'_{i,l}(r) + H'_{i,l}(r).$$

From the asymptotics of G (see Lemma 2.2) and the fact that f is bounded in C^2, it follows that

$$\frac{1}{\mathrm{vol}(S^2)}\int_{S^2}\int_{S^2}\big(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta})\big)d\tilde{\theta}\,d\theta = \hat{f}(r,s)\log\Big(\frac{1}{|r-s|}\Big) + H(r,s),$$

with H of class C^α and $\hat{f}$ of class C^2. Hence setting

$$\tilde{G}(r,s) = \frac{1}{\mathrm{vol}(S^2)}\int_{S^2}\int_{S^2}\frac{\partial}{\partial r}\big(f(r,\theta)(G(r\theta,s\tilde{\theta}) - K(r\theta,s\tilde{\theta}))\big)\overline{T}(s\tilde{\theta})f(s,\tilde{\theta})\,d\tilde{\theta}\,d\theta.$$

we obtain

$$\tilde{G}(r,s) = \hat{f}(r,s)\frac{1}{r-s} + \tilde{H}(r,s), \tag{3-30}$$

where $\tilde{H}(r,\cdot)$ is integrable for every fixed r.

On the other hand, using the Harnack-type inequality (see (3-20)), we have

$$u_l(s\tilde{\theta}) \le \bar{u}_{i,l}(s) + C \qquad \text{uniformly in } \tilde{\theta}.$$

Hence, we obtain

$$\bar{u}_{i,l}(r) \le C\int_0^A s^2\tilde{G}(r,s)e^{3\bar{u}_{i,l}(s)}ds - F'_{i,l}(r) + H'_{i,l}(r).$$

Now let us study $\int_0^A s^2\tilde{G}(r,s)e^{3\bar{u}_{i,l}(s)}ds$. Let R be so large that $r_{i,l} \le R_{i,l}/(4R)$ (this is possible because of the assumption of contradiction). Now let us split the integral as

$$\int_0^A s^2\tilde{G}(r,s)e^{3\bar{u}_{i,l}(s)}ds = \int_0^{r_{i,l}/R} s^2\tilde{G}(r,s)e^{3\bar{u}_{i,l}(s)}ds + \int_{r_{i,l}/R}^{r_{i,l}R} s^2\tilde{G}(r,s)e^{3\bar{u}_{i,l}(s)}ds + \int_{r_{i,l}R}^{R_{i,l}/C} s^2\tilde{G}(r,s)e^{3\bar{u}_{i,l}(s)}ds + \int_{R_{i,l}/C}^{A} s^2\tilde{G}(r,s)e^{3\bar{u}_{i,l}(s)}ds.$$

Using the fact that we are at the scale $r_{i,l}/R$, Claim 1(b) implies the estimates

$$\int_0^{r_{i,l}/R} s^2\tilde{G}(r_{i,l},s)e^{3\bar{u}_{i,l}(s)}ds = -\frac{2}{r_{i,l}} + o_l(1)\frac{1}{r_{i,l}}$$

for the first term of the equality above, with $r = r_{i,l}$. On the other hand using Claim 1(c) we obtain the estimate

$$\int_{r_{i,l}R}^{R_{i,l}/C} s^2\tilde{G}(r_{i,l},s)e^{3\bar{u}_{i,l}(s)}ds = o_l(1)\frac{1}{r_{i,l}}.$$

for the third term, with $r = r_{i,l}$. Using Claim 1(c) and the fact that $R_{i,l}/r_{i,l} \to +\infty$, we have the estimate

$$\int_{R_{i,l}/C}^{A} s^2\tilde{G}(r_{i,l},s)e^{3\bar{u}_{i,l}(s)}ds = o_l(1)\frac{1}{r_{i,l}}.$$

for the fourth term, with $r = r_{i,l}$, Now let us estimate the second term, using Claim 3(c). First we recall that $r_{i,l}$ and R satisfy the its assumptions. Hence

$$\|r_{i,l}^3 e^{3\tilde{u}_l}\|_{C^\alpha(A_R^+)} = o_l(1). \tag{3-31}$$

For the definition of A_R^+ and $\tilde{u}_l$, see Claim 3(c), with $d_{i,l}$ replaced by $r_{i,l}$. Now, performing a change of variable, say $r_{i,l}y = s$, we obtain the equality

$$\int_{r_{i,l}/R}^{r_{i,l}R} s^2\tilde{G}(r,s)e^{3\bar{u}_{i,l}(s)}ds = \int_{1/R}^{R} y^2\hat{G}_{i,l}(y)r_{i,l}^3 e^{3\hat{u}_{i,l}(y)}dy, \tag{3-32}$$

where

$$\hat{u}_{i,l}(y) = \bar{u}_{i,l}(r_{i,l}y) \quad \text{and} \quad \hat{G}_{i,l}(y) = \tilde{G}(r_{i,l}, r_{i,l}y).$$

From the asymptotics of $\tilde{G}$ (see (3-30)), we deduce for $\hat{G}_{i,l}$ that

$$\hat{G}_{i,l}(y) = \hat{f}_{i,l}(y)\frac{1}{r_{i,l}(1-y)} + \hat{H}_{i,l}(y), \tag{3-33}$$

where $\hat{H}_{i,l}$ is integrable and $\hat{f}_{i,l}$ of class C^2.

Hence, using (3-32) and (3-33), we get the equation

$$\int_{r_{i,l}/R}^{r_{i,l}R} s^2\tilde{G}(r_{i,l},s)e^{3\bar{u}_{i,l}(s)}ds = \frac{1}{r_{i,l}}\int_{1/R}^{R} y^3\left(\frac{\hat{f}_{i,l}(y)}{(1-y)} + r_{i,l}\hat{H}_{i,l}(y)\right)r_{i,l}^3 e^{3\hat{u}_{i,l}(y)}dy. \tag{3-34}$$

Moreover, using the Harnack-type inequality for u_l (see (3-20)) and (3-31), we have

$$\|r_{i,l}^3 e^{3\hat{u}_{i,l}}\|_{C^\alpha(]1/R,R[)} = o_l(1). \tag{3-35}$$

So using techniques of the theory of singular integral operators as in [Gilbarg and Trudinger 1983, Lemma 4.4] to obtain Holder estimates, we find

$$\int_{1/R}^{R} y^3\left(\frac{\hat{f}_{i,l}(y)}{(1-y)} + r_{i,l}\hat{H}_{i,l}(y)\right)r_{i,l}^3 e^{3\hat{u}_{i,l}(y)}dy = o_l(1).$$

So, with (3-32), we deduce that

$$\int_{r_{i,l}/R}^{r_{i,l}R} s^3 \tilde{G}(r,s) e^{3\bar{u}_{i,l}(s)} ds = o_l(1/r_{i,l}).$$

Hence, we arrive to

$$\bar{u}'_{i,l}(r_{i,l}) \leq -2C\frac{1}{r_{i,l}} + o_l(1)\frac{1}{r_{i,l}} - F'_{i,l}(r_{i,l}) + H'_{i,l}(r). \tag{3-36}$$

Next let compute $\varphi'_{i,l}(r_{i,l})$. From straightforward computations, we have

$$\varphi'_{i,l}(r_{i,l}) = (r_{i,l})^{3\nu-1} \exp(\bar{u}_{i,l}(r_{i,l})) \left(3\nu + r_{i,l}\bar{u}'_{i,l}(r_{i,l})\right).$$

Thus, using (3-36), we get the inequality

$$\varphi'_{i,l}(r_{i,l}) \leq (r_{i,l})^{3\nu-1} \exp(\bar{u}_{i,l}(r_{i,l}) \left(3\nu - 2C + o_l(1) - r_{i,l}F'_{i,l}(r_{i,l}) + r_{i,l}H'_{i,l}(r_{i,l})\right).$$

So, since $\nu < 2/3$, we have $3\nu - 2C + o_l(1) < 0$ for l sufficiently large.

Now, because $F'_{i,l}$ and $H'_{i,l}$ are bounded in $(0, \min\{\frac{1}{4}\mathrm{inj}_g(M), \frac{1}{4}\mathrm{inj}_{\hat{g}}(\partial M)\})$ uniformly in l, and because $r_{i,l} \to 0$, we have $\varphi'_{i,l}(r_{i,l}) < 0$ for l big enough, This is the contradiction that proves Claim 4. □

Conclusion of the proof of Theorem 1.8. Following the arguments of [Ndiaye 2007b, Step 5], we have

$$\int_{\partial M \setminus (\bigcup_{i=1}^{i=N} B^+_{x_{i,l}}(R_{i,l}/C) \cap \partial M)} e^{3u_l(y)} dS_g(y) = o_l(1).$$

So, since $B^+_{x_{i,l}}(R_{i,l}/C) \cap \partial M$ are disjoint, Claim 4 implies

$$\int_{\partial M} \overline{T}_l(y) e^{3u_l(y)} dS_g(y) = 4N\pi^2 + o_l(1),$$

Thus from (1-12) we derive

$$\int_M Q_0(y) dV_g(y) + \int_{\partial M} T_0(y') dS_g(y') = 4N\pi^2. \qquad \square$$

4. Proof of Theorem 1.2

This section has four subsections. The first concerns an improvement of the trace Moser–Trudinger-type inequality (see Proposition 2.4) and its corollaries. The second is about the existence of a nontrivial global projection from some negative sublevels of II onto ∂M_k (for the definition see (2-1)). The third deals with the construction of a map from ∂M_k into suitable negative sublevels of II. The last describes the min-max scheme.

4.1. *Improved trace Moser–Trudinger-type inequality.* Here we give an improvement of the trace Moser–Trudinger-type inequality; see Proposition 2.4. Then we state a lemma that gives some sufficient conditions for the improvement to hold; see (4-1). By these results, we derive that, if $u \in H_{\partial/\partial n}$ with $\int_{\partial M} e^{3u}\, dS_g = 1$, then that $\mathrm{II}(u)$ attains large negative values implies e^{3u} can concentrate at most at k points of ∂M; see Lemma 4.3. Finally from these results, we derive a corollary that gives the distance of e^{3u} (for some functions $u \in H_{\partial/\partial n}$ with $\int_{\partial M} e^{3u}\, dS_g = 1$) from ∂M_k.

The aforementioned improvement of the trace Moser–Trudinger-type inequality (Proposition 2.4) is proved by a trivial adaptation of the arguments of [Djadli and Malchiodi 2006, Lemma 2.2].

Lemma 4.1. *For a fixed $l \in \mathbb{N}$, suppose $S_1, \dots, S_{l+1}$ are subsets of ∂M satisfying* $\mathrm{dist}(S_i, S_j) \geq \delta_0$ *for $i \neq j$. Let $\gamma_0 \in (0, 1/(l+l))$.*

Then, for any $\bar{\epsilon} > 0$, there exists a constant $C = C(\bar{\epsilon}, \delta_0, \gamma_0, l, M)$ such that

$$\log \int_{\partial M} e^{3(u-\bar{u}_{\partial M})} \leq C + \frac{3}{16\pi^2}\left(\frac{1}{l+1-\bar{\epsilon}}\right)\langle P_g^{4,3} u, u\rangle_{L^2(M)}$$

for all the functions $u \in H_{\partial/\partial n}$ satisfying

$$\frac{\int_{S_i} e^{3u}\, dSg}{\int_{\partial M} e^{3u}\, dSg} \geq \gamma_0 \quad \text{for } i \in \{1, \dots, l+1\}. \tag{4-1}$$

In the next lemma we show a criterion which implies that condition (4-1) holds. Its proof is the same as that of [Djadli and Malchiodi 2006, Lemma 2.3].

Lemma 4.2. *Suppose l is a given positive integer, and ϵ and r are positive numbers. Suppose for a nonnegative function $f \in L^1(\partial M)$ with $\|f\|_{L^1(\partial M)} = 1$ that*

$$\int_{\bigcup_{i=1}^{\ell} B_r^{\partial M}(p_i)} f\, dS_g < 1 - \varepsilon \quad \text{for all } \ell\text{-tuples } p_1, \dots, p_\ell \in \partial M.$$

Then there exist positive numbers $\bar{\varepsilon}$ and $\bar{r}$ depending only on ε, r, ℓ and ∂M (but not on f), and $\ell + 1$ points $\bar{p}_1, \dots, \bar{p}_{\ell+1} \in \partial M$ (which do depend on f) satisfying

$$\int_{B_{\bar{r}}^{\partial M}(\bar{p}_i)} f\, dS_g > \bar{\varepsilon} \quad \text{for } i = 1, \dots, \ell+1,$$

$$B_{2\bar{r}}^{\partial M}(\bar{p}_i) \cap B_{2\bar{r}}^{\partial M}(\bar{p}_j) = \varnothing \quad \text{for } i \neq j.$$

The following interesting consequence of Lemma 4.1 characterizes some functions in $H_{\partial/\partial n}$ for which the value of II is large and negative.

Lemma 4.3. *Under the assumptions of Theorem 1.2, and for $k \geq 1$ as in Case 1.5, the following property holds. For any $\epsilon > 0$ and any $r > 0$ there exists a large* *positive real number $L = L(\epsilon, r)$ such that for any $u \in H_{\partial/\partial n}$ with $\mathrm{II}(u) \le -L$ and $\int_{\partial M} e^{3u}\, dS_g = 1$, there exist k points $p_{1,u}, \ldots, p_{k,u} \in \partial M$ such that*

$$\int_{\partial M \setminus \bigcup_{i=1}^{k} B^{\partial M}_{p_{i,u}}(r)} e^{3u}\, dS_g < \epsilon. \tag{4-2}$$

Proof. Suppose the statement is not true. Then there exist $\epsilon > 0$, $r > 0$, and a sequence $(u_n) \in H_{\partial n}$ such that $\int_{\partial M} e^{3u_n}\, dS_g = 1$ and $\mathrm{II}(u_n) \to -\infty$ as $n \to +\infty$ and such that

$$\int_{\bigcup_{i=1}^{k} B^{\partial M}_{p_{i,u}}(r)} e^{3u}\, dS_g < 1 - \epsilon \tag{4-3}$$

for any k tuples of points $p_1, \ldots, p_k \in \partial M$. Now by applying Lemma 4.2 with $f = e^{3u_n}$, and by using Lemma 4.1 with $\delta_0 = 2\bar{r}$, $S_i = B^{\partial M}_{\bar{p}_i}(\bar{r})$, and $\gamma_0 = \bar{\epsilon}$, where $\bar{\epsilon}$, $\bar{r}$ and $\bar{p}_i$ are given as in Lemma 4.2, we find that for every $\tilde{\epsilon} > 0$ there exists a postive real number C depending on ϵ, r and $\tilde{\epsilon}$ (and not on n) such that

$$\begin{aligned}\mathrm{II}(u_n) \ge \langle P^{4,3}_g u_n, u_n\rangle &+ 4\int_M Q_g u_n\, dV_g + 4\int_{\partial M} T_g u_n\, dS_g \\ &- \tfrac{4}{3}\kappa_{(P^4,P^3)} \frac{3}{16\pi^2(k+1-\tilde{\epsilon})} \langle P^{4,3}_g u_n, u_n\rangle - C\kappa_{(P^4,P^3)} - 4\kappa_{(P^4,P^3)} \bar{u}_{n\partial M}.\end{aligned}$$

Using elementary simplifications, the above inequality becomes

$$\begin{aligned}\mathrm{II}(u_n) \ge \langle P^{4,3}_g u_n, u_n\rangle &+ 4\int_M Q_g u_n\, dV_g + 4\int_{\partial M} T_g u_n\, dS_g \\ &- \frac{\kappa_{(P^4,P^3)}}{4\pi^2(k+1-\tilde{\epsilon})} \langle P^{4,3}_g u_n, u_n\rangle - C\kappa_{(P^4,P^3)} - 4\kappa_{(P^4,P^3)} \bar{u}_{n\partial M}.\end{aligned}$$

So, since $\kappa_{P^4,P^3} < (k+1)4\pi^2$, we get by choosing $\tilde{\epsilon}$ small enough that

$$\mathrm{II}(u_n) \ge \beta \langle P^{4,3}_g u_n, u_n\rangle - 4C\langle P^{4,3}_g u_n, u_n\rangle^{1/2} - C\kappa_{P^4,P^3},$$

where have used the Hölder inequality, Sobolev embedding, trace Sobolev embedding and the fact that $\ker P^{4,3}_g \simeq \mathbb{R}$ (where $\beta = 1 - \kappa_{(P^4,P^3)}/(4\pi^2(k+1-\tilde{\epsilon})) > 0$). Thus we arrive at $\mathrm{II}(u_n) \ge -C$, a contradiction. □

The next lemma, a direct consequence of the previous one, gives the distance of e^{3u} from ∂M_k for some functions u belonging to very negative sublevels of II such that $\int_{\partial M} e^{3u}\, dS_g = 1$. Its proof is the same as that of the corollary in [Djadli and Malchiodi 2006].

Corollary 4.4. *Let $\bar{\varepsilon}$ be a (small) arbitrary positive number, and let k be as in Case 1.5. Then there exists an $L > 0$ such that, if $\mathrm{II}(u) \le -L$ and $\int_{\partial M} e^{3u}\, dS_g = 1$, then $d(e^{3u}, \partial M_k) \le \bar{\varepsilon}$.*

4.2. ***Mapping very negative sublevels of* II *into* ∂M_k.** In this short subsection, we show that one can nontrivially map some appropriate low energy sublevels of the Euler–Lagrange functional II into ∂M_k.

Arguing as in [Djadli and Malchiodi 2006, Proposition 3.1], we have this lemma:

Lemma 4.5. *Let m be a positive integer, and for $\varepsilon > 0$ let $\mathfrak{D}_{\varepsilon,m}$ be as in (2-3). Then there exists an $\varepsilon_m > 0$, depending on m and ∂M, such that for all $\varepsilon \le \varepsilon_m$ there exists a continuous map $\Pi_m : \mathfrak{D}_{\varepsilon,m} \to \partial M_m$.*

Using this lemma, we have the following nontrivial continuous global projection from low energy sublevels of II into ∂M_k.

Proposition 4.6. *For $k \ge 1$ as in Case 1.5, there is a large positive real number L and a continuous and topologically nontrivial map Ψ from the sublevel $\{u : \mathrm{II}(u) < -L,\ \int_{\partial M} e^{3u}\, dS_g = 1\}$ into ∂M_k .*

By the noncontractibility of ∂M_k, the nontriviality of Ψ will be apparent from Proposition 4.10(a) below.

Proof. We fix ε_k so small that Lemma 4.5 applies with $m = k$. Then we apply Corollary 4.4 with $\bar{\varepsilon} = \varepsilon_k$. We let L be the corresponding large positive real number, so that if $\mathrm{II}(u) \le -L$ and $\int_{\partial M} e^{3u}\, dS_g = 1$, then $d(e^{3u}, \partial M_k) < \varepsilon_k$. Thus for these ranges of u, that the map $u \mapsto e^{3u}$ is continuous from $H_{\partial/\partial n}$ into $L^1(\partial M)$ implies that the projection Π_k from $H_{\partial/\partial n}$ onto ∂M_k is well defined and continuous. Hence setting $\Psi(u) = \Pi_k(e^{3u})$ finishes the proof. □

4.3. ***Mapping* ∂M_k *into very negative sublevels of* II.** In this subsection, we define some test functions depending on a real parameter λ and estimate the quadratic part of the functional II on those functions as λ tends to infinity. As a corollary, we define a continuous map from ∂M_k into large negative sublevels of II.

For $\delta > 0$ small, consider a smooth nondecreasing cutoff function $\chi_\delta : \mathbb{R}_+ \to \mathbb{R}$ with the properties that (see [Djadli and Malchiodi 2006])

$$\begin{aligned} &\chi_\delta(t) = t && \text{if } t \in [0, \delta], \\ &\chi_\delta(t) = 2\delta && \text{if } t \ge 2\delta, \\ &\chi_\delta(t) \in [\delta, 2\delta] && \text{if } t \in [\delta, 2\delta]. \end{aligned}$$

Then, given $\partial M_k \ni \sigma = \sum_{i=1}^k t_i \delta_{x_i}$ and $\lambda > 0$, we define $\varphi_{\lambda,\sigma} : M \to \mathbb{R}$ by

$$\varphi_{\lambda,\sigma}(y) = \tfrac{1}{3} \log\left(\sum_{i=1}^k t_i \left(\frac{2\lambda}{1+\lambda^2 \chi_\delta^2(d_i(y))} \right)^3 \right); \tag{4-4}$$

where we have set $d_i(y) = d_g(y, x_i)$ for $x_i \in \partial M$ and $y \in M$, with d_g denoting the Riemannian distance on M.

Now, we state a lemma giving an estimate (uniform in $\sigma \in \partial M_k$) of the quadratic part $\langle P_g^{4,3}\varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma}\rangle$ of the Euler–Lagrange functional II as $\lambda \to +\infty$. Its proof is a straightforward adaptation of the arguments in [Ndiaye 2007b, Lemma 4.5].

Lemma 4.7. *Suppose $\varphi_{\lambda,\sigma}$ as in (4-4), and let ϵ be sufficiently small and positive. Then*

$$\langle P_g^{4,3}\varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma}\rangle \le (16\pi^2 k + \epsilon + o_\delta(1))\log\lambda + C_{\epsilon,\delta} \quad \text{as } \lambda \to +\infty. \tag{4-5}$$

The next lemma estimates the remainder of the functional II along $\varphi_{\sigma,\lambda}$. The proof is the same as that of [Djadli and Malchiodi 2006, Lemma 4.3, formulas (40) and (41)].

Lemma 4.8. *Suppose $\varphi_{\sigma,\lambda}$ is as in (4-4). Then as $\lambda \to +\infty$, we have*

$$\int_M Q_g\varphi_{\sigma,\lambda}\,dV_g = -\kappa_{P_g^4}\log\lambda + O(\delta^4\log\lambda) + O(\log\delta) + O(1),$$
$$\int_{\partial M} T_g\varphi_{\sigma,\lambda}\,dV_g = -\kappa_{P_g^3}\log\lambda + O(\delta^3\log\lambda) + O(\log\delta) + O(1),$$
$$\log\int_{\partial M} e^{3\varphi_{\sigma,\lambda}} = O(1).$$

Now for $\lambda > 0$ we define the map $\Phi_\lambda : \partial M_k \to H_{\partial/\partial n}$, $\sigma \mapsto \varphi_{\sigma,\lambda}$. This map appears in the following lemma, a trivial application of Lemmas 4.7 and 4.8.

Lemma 4.9. *For $k \ge 1$ as in Case 1.5 and for any positive L large enough, there exists a small δ and a large positive real number $\bar\lambda$ such that $\mathrm{II}(\Phi_{\bar\lambda}(\sigma)) \le -L$ for every $\sigma \in \partial M_k$.*

The next proposition shows the existence of a projection from ∂M_k onto large negative sublevels of II, and the nontriviality of the map Ψ of Proposition 4.6.

Proposition 4.10. *Let Ψ be the map defined in Proposition 4.6. For $k \ge 1$ as in Case 1.5 and for every positive L large enough that Proposition 4.6 applies, there exists a map $\Phi_{\bar\lambda} : \partial M_k \to H_{\partial/\partial n}$ such that*

(a) $\mathrm{II}(\Phi_{\bar\lambda}(z)) \le -L$ *for any $z \in \partial M_k$ and*

(b) $\Psi \circ \Phi_{\bar\lambda}$ *is homotopic to the identity on ∂M_k.*

Proof. The statement (a) follows from Lemma 4.9. To prove (b) it is sufficient to consider the family of maps $T_\lambda : \partial M_k \to \partial M_k$, $\sigma \mapsto \Psi(\Phi_\lambda(\sigma))$. We recall that this composition is well defined if λ is sufficiently large. On the other hand, one can check easily that $e^{3\varphi_{\sigma,\lambda}}/\int_{\partial M} e^{3\varphi_{\sigma,\lambda}}\,dS_g \rightharpoonup \sigma$ in the weak distributional sense. Thus, letting $\lambda \to +\infty$, we obtain a homotopy between $\Psi \circ \Phi$ and $\mathrm{Id}_{\partial M_k}$. □

4.4. ***Min-max scheme for existence of solutions.*** Here we describe the min-max scheme based on the set ∂M_k, which we will need to prove Theorem 1.2. As anticipated in the introduction, we define a modified functional II_ρ for which we can prove existence of solutions for almost every ρ in a neighborhood of 1. Following an idea of Struwe [1988], this is done by proving the almost everywhere differentiability of the map $\rho \to \overline{\mathrm{II}}_{\rho,\bar\lambda}$, where $\overline{\mathrm{II}}_{\rho,\bar\lambda}$ is the min-max value for the perturbed functional II_ρ.

We now introduce the min-max scheme that supplies the existence of solutions of (1-8) . Let $\widehat{\partial M_k}$ denote the (contractible) cone over ∂M_k, which can be represented as $\widehat{\partial M_k} = (\partial M_k \times [0, 1])$ with $\partial M_k \times 0$ collapsed to a single point. Now, let L be so large that Proposition 4.6 applies with $L/4$, and then let $\bar\lambda$ be so large that Proposition 4.10 applies for this value of L. Fixing $\bar\lambda$, we define the class

$$\mathrm{II}_{\bar\lambda} = \bigl\{\pi : \widehat{\partial M_k} \to H_{\partial/\partial n} : \pi \text{ is continuous and } \pi(\,\cdot\, \times 1) = \Phi_{\bar\lambda}(\,\cdot\,)\bigr\}. \tag{4-6}$$

Lemma. The set $\mathrm{II}_{\bar\lambda}$ is nonempty, and

$$\overline{\mathrm{II}}_{\bar\lambda} = \inf_{\pi\in \mathrm{II}_{\bar\lambda}} \sup_{m\in\widehat{\partial M_k}} \mathrm{II}(\pi(m)) \quad \text{satisfies } \overline{\mathrm{II}}_{\bar\lambda} > -L/2.$$

Proof. The proof is the same as that of [Djadli and Malchiodi 2006, Lemma 5.1], but we repeat it for convenience.

To prove that $\overline{\mathrm{II}}_{\bar\lambda}$ is nonempty, we just notice that the map $\bar\pi(\,\cdot\,, t) = t\Phi_{\bar\lambda}(\,\cdot\,)$ belongs to $\mathrm{II}_{\bar\lambda}$. Now suppose by contradiction that $\overline{\mathrm{II}}_{\bar\lambda} \le -L/2$. Then there exists a map $\pi \in \mathrm{II}_{\bar\lambda}$ such that $\sup_{m\in\widehat{\partial M_k}} \mathrm{II}(\pi(m)) \le -3L/8$. Writing $m = (z, t)$ with $z \in \partial M_k$. Then, since Proposition 4.6 applies with $L/4$, the map $t \mapsto \Psi\circ\pi(\,\cdot\,, t)$ is a homotopy in ∂M_k between $\Psi\circ\Phi_{\bar\lambda}$ and a constant map. But this is impossible since ∂M_k is noncontractible and $\Psi\circ\Phi_{\bar\lambda}$ is homotopic to the identity map on ∂M_k by Proposition 4.10. □

Next, we introduce a variant of the above min-max scheme following [Djadli and Malchiodi 2006; Struwe 1988; Ndiaye 2007b]. For ρ in a small neighborhood $[1-\rho_0, 1+\rho_0]$, we define the modified functional $\mathrm{II}_\rho : H_{\partial/\partial n} \to \mathbb{R}$ by

$$u \mapsto \langle P_g^{4,3}u, u\rangle + 4\rho \int_M Q_g u\, dV_g + 4\rho \int_{\partial M} T_g u\, dS_g - \tfrac{4}{3}\rho\kappa_{(P^4,P^3)} \log \int_{\partial M} e^{3u}\, dS_g. \tag{4-7}$$

Following the estimates of Section 3, one easily checks that the above min-max scheme applies uniformly for $\rho \in [1-\rho_0, 1+\rho_0]$ and for $\bar\lambda$ sufficiently large. More precisely, given any large number $L > 0$, there exists $\bar\lambda$ sufficiently large and ρ_0

sufficiently small so that

$$
\begin{aligned}
&\sup_{\pi \in \mathrm{II}_{\bar\lambda}} \sup_{m \in \widehat{\partial M_k}} \mathrm{II}(\pi(m)) < -2L, \\
&\overline{\mathrm{II}}_{\rho,\bar\lambda} = \inf_{\pi \in \mathrm{II}_{\bar\lambda}} \sup_{m \in \widehat{\partial M_k}} \mathrm{II}_\rho(\pi(m)) > -L/2 \quad \text{for } \rho \in [1-\rho_0, 1+\rho_0],
\end{aligned}
\tag{4-8}
$$

where $\mathrm{II}_{\bar\lambda}$ is defined as in (4-6). Moreover, using for example the test maps obtained by modifying the standard bubbles, one shows that for ρ_0 sufficiently small, there exists a large positive constant $\bar L$ such that

$$
\overline{\mathrm{II}}_{\rho,\bar\lambda} \le \bar L \quad \text{for every } \rho \in [1-\rho_0, 1+\rho_0]. \tag{4-9}
$$

We have the following result regarding the dependence in ρ of the min-max value $\overline{\mathrm{II}}_{\rho,\bar\lambda}$.

Lemma 4.11. *Let $\bar\lambda$ and ρ_0 be such that (4-8) holds. Then the function*

$$
\rho \mapsto \overline{\mathrm{II}}_{\rho,\bar\lambda}/\rho \quad \textit{is nonincreasing in } [1-\rho_0, 1+1-\rho_0].
$$

Proof. For $\rho \ge \rho'$, we have

$$
\frac{\mathrm{II}_\rho(u)}{\rho} - \frac{\mathrm{II}_{\rho'}(u)}{\rho'} = \Big(\frac{1}{\rho} - \frac{1}{\rho'}\Big)\langle P_g^{4,3}u, u\rangle.
$$

Therefore it follows easily that $\overline{\mathrm{II}}_{\rho,\bar\lambda}/\rho - \overline{\mathrm{II}}_{\rho',\bar\lambda}/\rho' \le 0$. □

From this lemma, it follows that the function $\rho \to \overline{\mathrm{II}}_{\rho,\bar\lambda}/\rho$ is almost everywhere differentiable in $[1-\rho_0, 1+\rho_0]$.

Corollary 4.12. *Let $\bar\lambda$ and ρ_0 be as in Lemma 4.11, and let $\Lambda \subset [1-\rho_0, 1+\rho_0]$ be the (dense) set of ρ for which the function $\overline{\mathrm{II}}_{\rho,\bar\lambda}/\rho$ is differentiable. Then for $\rho \in \Lambda$ the functional II_ρ has a bounded Palais–Smale sequence $(u_l)_l$ at level $\overline{\mathrm{II}}_{\rho,\bar\lambda}$.*

Proof. The existence of Palais–Smale sequence $(u_l)_l$ at level $\overline{\mathrm{II}}_{\rho,\bar\lambda}$ follows from (4-8); the boundedness is proved exactly as in [Ding et al. 1999, Lemma 3.2]. □

Next, we state a proposition saying that bounded Palais–Smale sequence of II_ρ converges weakly (up to a subsequence) to a solution of the perturbed problem. The proof is the same as that of [Djadli and Malchiodi 2006, Proposition 5.5].

Proposition 4.13. *Suppose $(u_l)_l \subset H_{\partial/\partial n}$ is a sequence for which*

$$
\mathrm{II}_\rho(u_l) \to c \in \mathbb{R}, \quad \mathrm{II}'_\rho[u_l] \to 0, \quad \int_{\partial M} e^{3u_l}\, dS_g = 1, \quad \|u_l\|_{H^2(M)} \le C.
$$

Then (u_l) *has a weak limit* u (*up to a subsequence*) *satisfying the BVP*

$$\begin{aligned} P_g^4 u + 2\rho Q_g &= 0 && \text{in } M, \\ P_g^3 u + \rho T_g &= \rho\kappa_{(P4,P^3)} e^{3u} && \text{on } \partial M, \\ \partial u/\partial n_g &= 0 && \text{on } \partial M. \end{aligned}$$

Proof of Theorem 1.2. By Corollary 4.12 and Proposition 4.13, there exists a sequence $\rho_l \to 1$ and u_l such that

$$\begin{aligned} P_g^4 u_l + 2\rho_l Q_g &= 0 && \text{in } M, \\ P_g^3 u_l + \rho_l T_g &= \rho\kappa_{(P4,P^3)} e^{3u_l} && \text{on } \partial M, \\ \partial u_l/\partial n_g &= 0 && \text{on } \partial M. \end{aligned}$$

Now since $\kappa_{(P^4,P^3)} = \int_M Q_g \, dV_g + \int_{\partial M} T_g \, dS_g \neq 4\pi^2 k$ for $k = 1, 2, 3, \ldots$ and by applying Corollary 1.9(c), we have that u_l is bounded in $C^{4+\alpha}$ for every $\alpha \in (0, 1)$. Hence up to a subsequence it converges in $C^1(M)$ to a solution of (1-8). □

Remark 4.1. As said in the introduction, we now discuss how to settle the general case. For clarity of exposition, we divide the discussion in three parts, each corresponding to a remaining case.

Case ($\bar{k} = 0$ and $\kappa_{(P^4,P^3)} < 4\pi^2$). This case was proved in [Ndiaye 2007c] using geometric flows. However, using direct methods in the calculus of variations, it can be obtained thanks to the trace Moser–Trudinger-type inequality (see Proposition 2.4). Indeed by the latter inequality, the functional is coercive and weakly lower-semicontinuous. Hence, from the Weierstrass theorem in the calculus of variations, one infers that the functional admits a minimizer.

Case ($\bar{k} \neq 0$ and $\kappa_{(P^4,P^3)} < 4\pi^2$). In this case, $P_g^{4,3}$ has some negative eigenvalues. We proceed as in [Djadli and Malchiodi 2006], but change the arguments as follows. To obtain the trace Moser–Trudinger-type inequality, we impose the additional condition $\|\hat{u}\| \leq C$, where $\hat{u}$ is the component of u in the direct sum of the negative eigenspaces. Thus the functional goes negative infinity only if $\|\hat{u}\|$ tends to infinity. Hence to run the min-max scheme we substitute ∂M_k with $S^{\bar{k}-1}$, the boundary of the unit ball in the $\bar{k}$-dimensional Euclidean space. Another modification for the min-max scheme is in the monotonicity formula, which now says that $\rho \mapsto \overline{\mathrm{II}}_\rho/\rho - C\rho$ is nonincreasing in $[1-\rho_0, 1+\rho_0]$ for a fixed constant $C > 0$.

Case ($\bar{k} \neq 0$ and $\kappa_{(P^4,P^3)} \in (4\pi^2 k,\ 4(k+1)\pi^2)$ for $k \geq 1$). In this case we mix ideas from the case that $\bar{k} = 0$ and $\kappa_{(P^4,P^3)} \in (4\pi^2 k,\ 4(k+1)\pi^2)$ and from the case that $\bar{k} \neq 0$ and $\kappa_{(P^4,P^3)} < 4\pi^2$. Precisely, to obtain the trace Moser–Trudinger inequality and its improvement, we impose the additional condition $\|\hat{u}\| \leq C$, where $\hat{u}$ is the component of u in the direct sum of the negative eigenspaces. Another issue that must be considered is that not only e^{3u} can concentrate but also $\|\hat{u}\|$ can also tend to infinity. To deal with this, we must substitute the set ∂M_k with an other one, $A_{k,\bar{k}}$, which is defined in terms of the integer k and the number $\bar{k}$ of negative eigenvalues of $P_g^{4,3}$. This was done in [Djadli and Malchiodi 2006]. Also required is suitable adaptation of the min-max scheme and of the monotonicity formula, which in general says that $\rho \mapsto \overline{\mathrm{II}}_\rho/\rho - C\rho$ is nonincreasing in $[1-\rho_0, 1+\rho_0]$ for a fixed constant $C > 0$.

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