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JULIUS BORCEA AND RIKARD BØGVAD

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We give a local characterization of the class of functions having positive distributional derivative with respect to $\bar{z}$ that are almost everywhere equal to one of finitely many analytic functions and satisfy some mild nondegeneracy assumptions. As a consequence, we give conditions that guarantee that any subharmonic piecewise harmonic function coincides locally with the maximum of finitely many harmonic functions and we describe the topology of their level curves. These results are valid in a quite general setting as they assume no *à priori* conditions on the differentiable structure of the support of the associated Riesz measures. We also discuss applications to positive Cauchy transforms and we consider several examples and related problems.

1. Introduction

In a frequently used construction in complex analysis and geometry, one considers the maximum of a finite number of pairwise distinct harmonic functions. As is well known, the result is a subharmonic function which is also piecewise harmonic. It is quite natural to investigate the converse direction, namely to study the class of functions generated by this basic albeit fundamental procedure. Its classical flavor [Hayman and Kennedy 1976] and some important applications — some of which are listed below — further motivate a deeper study of this question, on which

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surprisingly little seems to be known. In this paper we answer by giving a local characterization of the aforementioned class of functions in generic cases, and in the process we establish several remarkable properties for this class. In particular, we show that any subharmonic piecewise harmonic function may essentially be realized as the maximum of finitely many harmonic functions.

1.1. *Piecewise harmonic and piecewise analytic functions.* Let us first define a fairly general notion.

Definition 1.2. Let X be a real or complex subspace of the space of smooth functions in a domain (open connected set) U in $\mathbb{R}^2$ or $\mathbb{C}$. We say that a function φ is *piecewise in* X if one can find finitely many pairwise disjoint open sets M_i for $1 \le i \le r$ in U and pairwise distinct functions $\varphi_i \in X$ for $1 \le i \le r$, such that

(i) $\varphi = \varphi_i$ in M_i for $1 \le i \le r$;

(ii) $U \setminus \bigcup_{i=1}^{r} M_i$ is of Lebesgue measure 0.

The set of all functions that are piecewise in X is denoted by PX.

Remark 1.3. It is not difficult to see that PX is actually a (real or complex) vector space. This as well as further properties of PX functions and related concepts are discussed in the appendix.

Note that since PX functions are locally integrable, they define distributions, and their derivatives are therefore defined in the distribution sense (and functions are identified if they define the same distributions). In particular, if $\varphi \in PX$, one can form $\Delta\varphi \in \mathcal{D}'(U)$ and also $\partial_z\varphi, \partial_{\bar{z}}\varphi \in \mathcal{D}'(U)$ if X is complex.

We now specialize Definition 1.2 to obtain the main objects of our study, namely the spaces of *piecewise harmonic* and *piecewise analytic functions*.

Notation 1.4. Fix a domain $U \subset \mathbb{C}$, let $H = H(U)$ be the real space of (real-valued) harmonic functions in U, and let $A = A(U)$ be the complex space of analytic functions in U. By Definition 1.2 the following holds:

(a) Given a piecewise harmonic function $\varphi \in PH$, there exists a finite family of pairwise disjoint open sets $\{M_i\}_{i=1}^{r}$ in U covering U up to a set of Lebesgue measure 0 and a corresponding family of pairwise distinct harmonic functions $\{H_i(z)\}_{i=1}^{r}$ in U such that

$$\varphi(z) = \sum_{i=1}^{r} H_i(z)\chi_i(z) \quad \text{almost everywhere in } U, \tag{1-1}$$

where χ_i is the characteristic function of the set M_i for $1 \le i \le r$.

(b) Similarly, any piecewise analytic function $\Phi \in PA$ may be represented as

$$\Phi(z) = \sum_{i=1}^{r} A_i(z)\chi_i(z) \quad \text{almost everywhere in } U, \tag{1-2}$$

where M_i and χ_i for $1 \le i \le r$ are as in (a) and $\{A_i(z)\}_{i=1}^r$ is a family of pairwise distinct analytic functions in U. Given this data and a point $p \in U$, we set

$$H_i(z) = \operatorname{Re}\Big(\int_p^z A_i(w)dw\Big) \quad \text{for } z \in U \text{ and } 1 \le i \le r. \tag{1-3}$$

These are well-defined harmonic functions in U provided U is simply connected, which we tacitly assume throughout unless otherwise stated.

We stress that in the above definitions no regularity (C^1) conditions are assumed on the negligible set $U \setminus \bigcup_{i=1}^r M_i$. Note also that Definition 1.2 and Notation 1.4 are merely a convenient way of saying that a *PH* function ϕ equals one of finitely many harmonic functions in certain prescribed sets. Therefore *PH* functions need not be continuous nor subharmonic, and one can hardly expect any interesting statements in this kind of generality. The same philosophy applies to *PA* functions: As defined above, a function Φ is *PA* if it is equal to one of finitely many analytic functions in certain open sets. Thus *PA* functions need not be continuous and this will not be case either in our situation.

1.5. *Canonical piecewise decompositions.* Conditions (i) and (ii) in Definition 1.2 remain valid if nonempty Lebesgue negligible sets are subtracted from the sets M_i, so it is in general impossible to say something about the boundaries of these sets. However, the inclusions $M_i \subseteq U \setminus \operatorname{supp}(\varphi - \varphi_i)$ for $1 \le i \le r$ always hold, where the supports are defined in the distribution sense (recall from Section 1.1 that *PX* functions are locally integrable and $L^1_{\text{loc}}(U)$ is viewed as a subspace of $\mathscr{D}'(U)$). Now both $X = H(U)$ and $X = A(U)$ are examples of function spaces satisfying the *unique continuation property*, that is, $f \equiv 0$ in U if $f \in X$ vanishes in some open nonempty subset of U. In view of the above inclusions, for spaces with this property one can reformulate Definition 1.2 in a more canonical way as follows.

Definition 1.6. Let X be a real or complex subspace of the space of smooth functions in a domain U in $\mathbb{R}^2$ or $\mathbb{C}$. Assume that X satisfies the unique continuation property, and let $\varphi \in L^1_{\text{loc}}(U)$. Then $\varphi \in PX$ (φ is piecewise in X) if one can find pairwise distinct elements $\varphi_i \in X$ for $1 \le i \le r$ such that the set $\Gamma := \bigcap_{1 \le i \le r} \operatorname{supp}(\varphi - \varphi_i)$ is of Lebesgue measure 0.

Setting $M_i = U \setminus \operatorname{supp}(\varphi - \varphi_i)$ for $1 \le i \le r$ in Definition 1.6, we see that M_i is the largest open set in which $\varphi - \varphi_i$ vanishes (as a distribution or almost everywhere). Further useful properties of the canonical piecewise decomposition of the *PX* function φ given in Definition 1.6 are gathered in the next lemma. Henceforth by a "continuous function" we mean a function in $L^1_{\text{loc}}(U)$ that agrees almost everywhere with a continuous function in U.

Lemma 1.7. *In the above notation the following holds*:

(i) $\bigcup_{1\le i\le r} M_i = U \setminus \Gamma$.

(ii) $\overline{M}_i \cap M_j = \varnothing$ *for* $1 \le i \ne j \le r$.

(iii) $M_i = \overline{M}_i^{\circ}$ (*that is*, M_i *is the interior of* $\overline{M}_i$) *for* $1 \le i \le r$.

(iv) $\Gamma = \bigcup_{1\le i<j\le r} \overline{M}_i \cap \overline{M}_j$.

(v) *If* φ *is continuous*, *then* $\Gamma \subseteq g^{-1}(0)$, *where* $g := \prod_{1\le i<j\le r}(\varphi_i - \varphi_j)$.

Proof. The first statement is obviously true by the (canonical) definition of the sets M_i for $1 \le i \le r$. To prove (ii) suppose that $i \ne j$ and $p \in \overline{M}_i \cap M_j$. Then one can find $q \in M_i$ arbitrarily close to p with $q \in M_i \cap M_j$. Since $q \notin \operatorname{supp}(\varphi - \varphi_i)$ and $q \notin \operatorname{supp}(\varphi - \varphi_j)$, one gets $q \notin \operatorname{supp}(\varphi_i - \varphi_j)$, and the unique continuation property implies that $\varphi_i = \varphi_j$, which contradicts the fact that $\varphi_i \ne \varphi_j$.

To show (iii), assume that $p \in \overline{M}_i^{\circ}$. Then there exists an (open) neighborhood N of p that is contained in $\overline{M}_i$. Since $\overline{M}_i \cap M_j = \varnothing$ if $j \ne i$ (see (ii)), it follows that $N \subset M_i \cup \Gamma$. Hence $\varphi = \varphi_i$ in N and $N \subset M_i$, so that in particular $p \in M_i$.

Clearly, $\bigcup_{1\le i\le r} \overline{M}_i = U$. Therefore, if $p \in \Gamma$, then $p \in \overline{M}_i$ for some i, and p must then be a boundary point of M_i. Assume that $p \notin \overline{M}_j$ whenever $j \ne i$. Then there is a neighborhood N of p such that $N \cap M_j = \varnothing$ for $j \ne i$. Hence $N \subset \overline{M}_i$ and it follows from (iii) that $p \in M_i^{\circ}$. This gives a contradiction (since p is a boundary point of M_i) and shows that $p \in \overline{M}_i \cap \overline{M}_j$ for some $j \ne i$, which proves (iv).

Finally, if φ is continuous, then $\varphi = \varphi_i$ in $\overline{M}_i$ and $\varphi = \varphi_j$ in $\overline{M}_j$; hence $\varphi_i = \varphi_j$ in $\overline{M}_i \cap \overline{M}_j$, and thus $g \equiv 0$ in $\overline{M}_i \cap \overline{M}_j$ for $i \ne j$, so that by (iv) $g \equiv 0$ in Γ. $\square$

The familiar "maximum construction" that we alluded to at the beginning of this introduction yields natural examples of *PH* and *PA* functions. We recall briefly the interplay between the classes of functions obtained in this case:

Example 1.8. Let $\{H_i(z)\}_{i=1}^r$ be a finite family of pairwise distinct harmonic functions in a domain $U \subset \mathbb{C}$. Then $\varphi(z) := \max_{1\le i\le r} H_i(z)$ is a (subharmonic) *PH* function. Indeed, set $\Omega := \{z \in U \mid H_k(z) \ne H_l(z), 1 \le k \ne l \le r\}$, let M_i be the (open) set consisting of those $z \in \Omega$ for which $\varphi(z) = H_i(z)$, and denote by χ_i the characteristic function of M_i for $1 \le i \le r$. It is clear that $U \setminus \Omega$ is Lebesgue negligible, so that $\{M_i\}_{i=1}^r$ forms a covering of U up to a set of Lebesgue measure 0 and

$$\varphi(z) = \sum_{i=1}^{r} H_i(z)\chi_i(z) \quad \text{almost everywhere in } U.$$

Moreover, the subharmonicity of φ implies that $\nu := \partial^2\varphi/\partial\bar{z}\partial z \geq 0$ in the sense of distributions. In fact, ν is a positive measure supported on the (finite) union of level curves $\{z \in U \mid H_i(z) - H_j(z) = 0\}$ for $1 \leq i \neq j \leq r$. One can show that in this case the support actually determines the measure (see Theorem 2.5).

Now the derivative of φ, again in the distribution sense, inherits a similar property, only this time with respect to analytic functions. Classical results yield namely

$$\partial\varphi(z)/\partial z = \sum_{i=1}^{r} A_i(z)\chi_i(z) \quad \text{almost everywhere in } U,$$

where $A_i := \partial H_i/\partial z$ for $1 \leq i \leq r$ are analytic functions in U (see Proposition 2.6). Hence $\partial\varphi(z)/\partial z$ is a *PA* function. Note that the above relation may be reformulated as saying that φ satisfies almost everywhere in U the differential equation $P(\partial\varphi(z)/\partial z, z) = 0$, where $P(y,z) := \prod_{i=1}^{r}(y - A_i(z))$ is a polynomial in y with coefficients that are holomorphic in U.

1.9. ***Main problem and results.*** *PA* functions occur naturally — and this was our original motivation — in various contexts, such as in the study of the asymptotic behavior of polynomial solutions to ordinary differential equations [Bergkvist and Rullgård 2002; Borcea et al. 2007; Fedoryuk 1993; Wasow 1965], the theory of Stokes lines [Kelly 1979; Wasow 1985] and orthogonal polynomials [Deift and Zhou 1993]. In these contexts, since *PA* functions are mostly constructed as limits, one has no control on the differentiable structure of the resulting sets M_i. Therefore it is important to describe the local and global structure of *PA* functions both with and without additional regularity assumptions — such as the piecewise C^1-boundary conditions on the sets M_i of Section 2 — and this is the primary objective of this paper. Another notation will help to state our main problem:

Notation 1.10. Given a domain $U \subset \mathbb{C}$, let $\Sigma(U) = \{f \in \mathcal{D}'(U) \mid \partial_{\bar{z}} f \geq 0\}$.

Clearly, $\partial_z\varphi \in \Sigma(U)$ if φ is subharmonic in U, which holds for example for the maximum of finitely many harmonic functions. For a (known) converse see the appendix.

The main problem. Let $\Phi \in \Sigma(U)$ be a *PA* function in a given domain $U \subset \mathbb{C}$. Find conditions that guarantee that Φ is locally (or globally) of the form $\partial_z\varphi$, where φ is the maximum of a finite number of harmonic functions in U.

The necessity of assuming $\partial_{\bar{z}}\Phi \geq 0$ in the main problem will soon become quite clear and is further illustrated in Example 1.14; see also Lemma A.3 in the appendix. We give four answers to the above problem, which may be summarized (in terms of the mutual implications among them) as follows:

$$\text{(1-4)} \qquad \text{Theorem 1.11} \Longrightarrow \text{Corollary 4.8} \Longrightarrow \text{Corollary 1.12} \Longrightarrow \text{Theorem 6.2}.$$

We formulate here just the first (Theorem 1.11) and third (Corollary 1.12) main results of this paper. The fourth one (Theorem 6.2) is an alternative approach to the main problem suggested by our referee, as were several ideas used in this paper.

Theorem 1.11. *Let $\Phi \in \Sigma(U)$ be a PA function as in (1-2) and assume that $p \in U$ satisfies the conditions that*

(i) *$p \in \overline{M}_i$ for $1 \leq i \leq r$;*

(ii) *$A_i(p) - A_k(p) \notin \mathbb{R}(A_j(p) - A_k(p))$ for any triple of distinct indices (i, j, k) in $\{1, \ldots, r\}$;*

(iii) *$A_i(p) \neq A_k(p)$ for any pair of distinct indices (i, k) in $\{1, \ldots, r\}$.*

There exists a neighborhood $\widetilde{N}(p)$ of p such that $\Phi = 2\partial\varphi/\partial z$ almost everywhere in $\widetilde{N}(p)$, where $\varphi(z) = \max_{1 \leq i \leq r} H_i(z)$ and the H_i are the harmonic functions defined in (1-3).

A word about each of the three conditions imposed in Theorem 1.11 is in order.

Condition (i) suggests defining the index set

$$I(p) = \{j \in \{1, \ldots, r\} \mid p \in \overline{M}_j\} \quad \text{for any } p \in U \tag{1-5}$$

and $i(p) = |I(p)|$. Condition (i) then requires that $i(p) = r$, that is, every set M_i is “active”. This will be tacitly assumed throughout.

Condition (ii) is the most important assumption and amounts to the requirement that for all distinct indices $i, j, k \in \{1, \ldots, r\}$, the level curves $H_i = H_k$ and $H_j = H_k$ should meet transversally at p (that is, the critical sets $\Gamma_{i,j,k}$ defined in (3-1) below do not contain p). For an illustration of the necessity of this assumption see Example 7.2 and Figure 1 in Section 7.

Condition (iii) means that locally the (0-)level curves of $H_i - H_j$ with $i \neq j$ form a foliation by 1-dimensional smooth curves of a small enough neighborhood of p. As will (ii) above, this assumption will be used in an essential way.

Let K be the convex hull of the points $A_i(p)$ for $i \in I(p)$, and denote by ∂K its boundary, which is clearly an $i(p)$-gon. From Theorem 1.11 and its proof sketched in Section 3 and completed in Sections 4 and 5 (see, in particular, Lemma 4.3 in Section 4.1 and Corollary 4.8 in Section 4.7) we deduce the following:

Corollary 1.12. *Assume all the hypotheses of Theorem 1.11 except conditions (i) and (ii), and set $S(p) = \{i \in I(p) \mid A_i(p)$ is an extreme point of $K\}$. If $A_k(p) \notin \partial K$ for $k \in I(p) \setminus S(p)$, then the conclusion of Theorem 1.11 holds.*

Remark 1.13. In particular, Corollary 1.12 holds if $S(p) = I(p)$, that is, all points $A_i(p)$ for $i \in I(p)$ are extreme in K.

We emphasize that results similar to those above cannot hold for arbitrary *PA* functions. Indeed, as we already noted, the requirement $\partial\Phi/\partial\bar{z} \geq 0$ is crucial. In particular, it implies that the open sets $\{M_i\}_{i=1}^r$ and the analytic functions $\{A_i(z)\}_{i=1}^r$ associated with Φ have to be intimately related to each other. The latter statement is illustrated (and further reinforced) in the next example.

Example 1.14. Let $r = 2$, $A_1(z) \equiv 1$ and $A_2(z) \equiv i$. Then the subharmonic function φ defined in Theorem 1.11 becomes $\varphi(x, y) = \max(x, -y)$, that is, $\varphi(x, y) = x$ if $x + y \geq 0$ and $\varphi(x, y) = -y$ if $x + y \leq 0$. Hence its derivative $2\partial\varphi/\partial z$ equals 1 if $x + y \geq 0$ and i if $x - y \leq 0$, respectively. Theorem 1.11 says (loosely) that among all *PA* functions Φ of the form $1 \cdot \chi_{M_1} + i \cdot \chi_{M_2}$ for varying sets M_1 and M_2 (covering some neighborhood of the origin up to a Lebesgue negligible set) $2\partial\varphi/\partial z$ is the only one that has a positive $\bar{z}$-derivative in the sense of distributions. To see why this is the case, consider the following simple example: Let l be a line through the origin with unit normal $n = n_1 + in_2$, so that $\mathbb{C} \setminus l$ consists of two half-planes. Let M_1 be the one with n as interior normal to its boundary and M_2 the other half-plane. Set $\Phi = 1 \cdot \chi_{M_1} + i \cdot \chi_{M_2}$. Then

$$\frac{\partial\Phi}{\partial\bar{z}} = \tfrac{1}{2}(1 - i)(n_1 + in_2)ds,$$

where ds is Euclidean length measure along the common boundary l to M_1 and M_2 (see Corollary 2.3). Clearly, $\partial\Phi/\partial\bar{z} \geq 0$ only if $n_1 + in_2 = (1+i)/\sqrt{2}$, that is, if the line l is given by $x + y = 0$. In other words one must indeed have $\Phi = 2\partial\varphi/\partial z$, where φ is the subharmonic function defined in Theorem 1.11. In this particular example we used the fact that the boundaries of the M_i are C^1 in order to explicitly calculate the derivative of Φ. Our theorems show that the corresponding result is true in a much more general situation with no assumptions on the boundaries.

The local characterization of subharmonic functions with *PA* derivatives is almost an immediate consequence of Theorem 1.11 and shows that at generic points such functions are indeed maxima of a finite set of harmonic functions:

Corollary 1.15. *Suppose Ψ is a subharmonic function such that $\partial\Psi/\partial z$ is a PA function with decomposition given by (1-2) and satisfying conditions (i)–(iii) of Theorem 1.11. Then there exist a neighborhood $\widetilde{N}(p)$ of p and harmonic functions H_i for $1 \leq i \leq r$ defined in $\widetilde{N}(p)$ such that $\Psi(z) = \max_{1 \leq i \leq r} H_i(z)$ almost everywhere in $\widetilde{N}(p)$.*

Let $\Phi \in \Sigma(U)$, so that by Notation 1.10 and [Hörmander 2003, Theorem 2.1.7] the measure $\nu := \partial\Phi/\partial\bar{z}$ is positive. Let further $p \in U$ and $N(p)$ be a neighborhood of p such that $\overline{N(p)} \subset U$. Then the (positive) measure $\tilde{\nu} := \chi_{\overline{N(p)}} \cdot \nu$ extends to $\mathbb{C}$, and there exists some analytic function A such that $\Phi = C_{\tilde{\nu}} + A$ (as distributions) in $N(p)$, where $C_{\tilde{\nu}}$ is the Cauchy transform of $\tilde{\nu}$ defined by $C_{\tilde{\nu}} := (1/(\pi z)) * \tilde{\nu}$. The above decomposition for Φ is a consequence of formula [op. cit., (4.4.2)], which asserts that Φ and $C_{\tilde{\nu}}$ have the same derivative with respect to $\partial/\partial\bar{z}$, so that by

[Hörmander 2003, Theorem 4.4.1] they must differ by an analytic function. Hence we also have a corollary to Theorem 1.11:

Corollary 1.16. *Let $\Phi \in \Sigma(U)$ be a PA function with decomposition given by (1-2), and set $\nu = \partial\Phi/\partial\bar{z}$. Assume that $p \in U$ satisfies conditions (i)–(iii) of Theorem 1.11, and let $N(p)$ and $\tilde{\nu}$ be as above. Then $\Phi = C_{\tilde{\nu}} + A$ in $N(p)$, where A is an analytic function and the positive measure $\tilde{\nu}$ is supported in a union of segments of level sets for the functions $H_i - H_j$, where $1 \leq i \neq j \leq r$. Moreover, ν may be locally described by means of its support in the sense of formula (2-2) (see Theorem 2.5(3)).*

The results above hold in a surprisingly great generality since they assume no *à priori* knowledge of the differentiable structure of $\operatorname{supp}\nu$. We will construct an example showing that the picture is even more complex in nongeneric cases and in particular that Corollary 1.15 is not true if p is special enough; see Example 7.2.

The special case when the A_i in Theorem 1.11 are constant functions was treated in [Bergkvist and Rullgård 2002]. Our crucial Lemma 3.3 is mutatis mutandis generalized from that paper. In the simpler situation of [loc. cit.], some additional global results were obtained. These show essentially that any (locally) *PH* subharmonic function is globally (in U) a maximum of finitely many harmonic functions. Example 7.2 again shows that this is not true in general. However, it is not difficult to get complete results in the case when only two functions are involved; see Section 2. It would be interesting to establish when a subharmonic function with a *PA* derivative is globally a maximum of finitely many harmonic functions (see Problem 7.8).

2. Derivatives of sums

Recall the canonical piecewise decomposition of a *PH* function from Section 1.5 (see Definition 1.6 with $X = H(U)$). If $\Psi(z)$ is a *PH* subharmonic function of the form (1-1), then the support of the associated Riesz measure $\Delta\Psi$ is equal to $\Gamma := U \setminus \bigcup_{i=1}^{r} M_i$. Indeed, it is clear that $\operatorname{supp}(\Delta\Psi) \subseteq \Gamma$. For the reverse inclusion, note that Ψ is harmonic in a neighborhood of any point $p \in \Gamma \setminus \operatorname{supp}(\Delta\Psi)$. If such a point exists, one can find $i \neq j$ so that any neighborhood of p intersects M_i and M_j, and then H_i and H_j both agree with Ψ in some neighborhood of p; hence $H_i = H_j$ (by the unique continuation property), which is a contradiction.

In this section we first discuss the case of a *PA* function Φ with canonical piecewise decomposition as in Definition 1.6 such that the corresponding set $\Gamma = U \setminus \bigcup_{i=1}^{r} M_i$ is a locally finite union of piecewise C^1-curves. We show that if the distribution derivative $\partial\Phi/\partial\bar{z}$ is positive, then this measure is determined in a simple way by its support, see Theorem 2.5(3) below. Note that in view of Lemma 1.7(v), a situation where Γ is piecewise smooth occurs if one considers a

PA function of the form $\Phi = \sum_{1\le i\le r} (\partial H_i/\partial z)\chi_i$, where $\Psi = \sum_{1\le i\le r} H_i\chi_i$ is a continuous *PH* function (for instance, Ψ could be the maximum of finitely many harmonic functions). In this case we show the continuity assumption implies that Φ is actually the distribution derivative of Ψ (without any C^1-assumptions on Γ).

We start with the case when only two functions are involved. Assume that $\Phi(z)$ is defined in a domain U and that there exists a smooth curve $\Gamma \subset U$ dividing U into two open connected components $U = M_1 \cup \Gamma \cup M_2$ such that $\Phi(z) = A_i(z)$ in M_i for $i = 1, 2$, where $A_i(z)$ is a function analytic in some neighborhood of M_i. In particular, $\Phi(z)$ is a *PA* function.

Lemma 2.1. *If $\nu := \partial\Phi(z)/\partial\bar{z} \ge 0$ in the sense of distribution theory (that is, ν is a positive measure) then at each point $\tilde{z}$ of Γ the tangent line $l(\tilde{z})$ to Γ is orthogonal to $\overline{A_1(\tilde{z})} - \overline{A_2(\tilde{z})}$ and the measure ν at $\tilde{z}$ equals*

$$\tfrac{1}{2}|A_1(\tilde{z}) - A_2(\tilde{z})|ds,$$

where ds denotes length measure along Γ.

Lemma 2.1 is an immediate consequence of the following well-known result, see for example [Hörmander 2003, Theorem 3.1.9].

Proposition 2.2. *Let $Y \subset X$ be open subsets of $\mathbb{R}^k$ such that Y has a C^1-boundary ∂Y in X, and let $u \in C^1(X)$. If χ_Y denotes the characteristic function of Y, dS the Euclidean surface measure on ∂Y, and n the interior unit normal to ∂Y, then*

$$\partial_j(u\chi_Y) = (\partial_j u)\chi_Y + un_j dS,$$

where ∂_j and n_j are the partial derivative with respect to the j-th coordinate and the j-th component of n, respectively.

Corollary 2.3. *In the notation of Proposition 2.2, one has*

$$\begin{aligned}\frac{\partial(u\chi_Y)}{\partial\bar{z}} &= \Big(\frac{\partial u}{\partial\bar{z}}\Big)\chi_Y + \tfrac{1}{2}u(n_1 + in_2)ds,\\ \frac{\partial(u\chi_Y)}{\partial z} &= \Big(\frac{\partial u}{\partial z}\Big)\chi_Y + \tfrac{1}{2}u(n_1 - in_2)ds.\end{aligned} \tag{2-1}$$

Proof of Lemma 2.1. Suppose that the function $\Phi(z) = A_1(z)\chi_1(z) + A_2(z)\chi_2(z)$ satisfies the conditions of Lemma 2.1, where χ_i is the characteristic function of M_i for $i = 1, 2$. Corollary 2.3 implies in particular that ν is supported on the smooth separation curve Γ and that with an appropriate choice of coorientation one has $\nu = \frac{1}{2}(A_1 - A_2)nds$, which proves the lemma. □

Proposition 2.2 remains true if the boundary of Y is assumed to be only piecewise C^1 or just Lipschitz continuous; see [Hörmander 2003]. We may therefore apply it to functions of the form $\max_{1\le i\le r} H_i(z) = \sum_{i=1}^{r} H_i(z)\chi_i(z)$ in U and get the description of their derivatives given in the introduction. In this case, the normal n is defined almost everywhere with respect to length measure on the boundary, and the equality in Corollary 1.15 is interpreted in this sense.

Notation 2.4. Given a *PH* function $\Psi(z) = \sum_{i=1}^{r} H_i(z)\chi_i(z)$ as in (1-1), let $\Gamma_\Psi = U \setminus \bigcup_{i=1}^{r} M_i$ and denote by Γ_Ψ^d the set of points where the normal to Γ_Ψ is not defined. In similar fashion, for a *PA* function $\Phi(z) = \sum_{i=1}^{r} A_i(z)\chi_i(z)$ as in (1-2), we set $\Gamma_\Phi = U \setminus \bigcup_{i=1}^{r} M_i$, and let Γ_Φ^d be the set of points where the normal to Γ_Φ is not defined.

Essentially the same arguments yield this generalization of Lemma 2.1:

Theorem 2.5. *Let*

$$\Phi(z) = \sum_{i=1}^{r} A_i(z)\chi_i(z)$$

be a PA function in a simply connected domain $U \subset \mathbb{C}$ such that

(i) *Γ_Φ is a locally finite union of piecewise C^1-curves, and*

(ii) $\partial\Phi/\partial\bar{z} \geq 0$.

Let H_i for $1 \leq i \leq r$ be real-valued harmonic functions as in (1-3). Then for any $z \in \Gamma_\Phi \setminus \Gamma_\Phi^d$ there is a neighborhood $N(z)$ such that

(1) *$N(z) \setminus \Gamma_\Phi$ consists of two components $N(z)_i$, $N(z)_j$ such that $\Phi(z) = A_k(z)$ in $N(z)_k$ for $k = i, j$;*

(2) *$N(z) \cap \Gamma_\Phi$ is contained in a level curve of $H_i - H_j$ for some i, j;*

(3) *in $N(z)$ one has*

$$\partial\Phi(z)/\partial\bar{z} = \tfrac{1}{2}|A_i(z) - A_j(z)|ds. \tag{2-2}$$

The restriction of $\partial\Phi(z)/\partial\bar{z}$ to $U \setminus \Gamma_\Phi^d$, determined locally by (2-2), extends to a measure μ on U which is absolutely continuous with respect to length measure on Γ_Φ. Also $\partial\Phi(z)/\partial\bar{z} = \mu$ in U. If any two level curves Γ_{ij} and Γ_{kl} with $i < j$, $k < l$, and $(i, j) \neq (k, l)$ intersect in at most a finite number of points, then the measure μ, hence also $\partial\Phi(z)/\partial\bar{z}$, is determined by its support Γ_Φ.

Proof. Assertions (1), (2) and identity (2-2) are direct consequences of Lemma 2.1. Since by (i), Γ_Φ is a locally finite union of piecewise C^1-curves, the set Γ_Φ^d has measure 0 with respect to length measure ds on Γ_Φ, and thus the measure μ extending the right side of (2-2) to Γ_Φ exists. It remains to show that

$$\partial\Phi/\partial\bar{z} = \mu. \tag{2-3}$$

Note that $\partial\Phi/\partial\bar{z} = \mu + G$, where G is a sum of Dirac measures supported at (singular) points in Γ_Φ^d. Consider now a singular point $p \in \Gamma_\Phi^d$, a small neighborhood N of p, and the Cauchy transform $C_{\tilde{\mu}}$ of (the extension to $\mathbb{C}$ of) the measure $\tilde{\mu} := \chi_{\overline{N}} \cdot \mu$. Suppose that locally at p the measure G is given by $c\delta_p$ for some $c \geq 0$. Then the function $\Phi - C_{\tilde{\mu}} - c/(z-p)$ is analytic at p. On the other hand, Φ is bounded and by the classical Plemelj–Sokhotski formulas (see for example [Berenstein and Gay 1991, Section 3.6]) the Cauchy transform $C_{\tilde{\mu}}$ has at most a logarithmic singularity at p. It follows that $c = 0$, which proves (2-3). For the last statement in part (3) of the theorem, note that the assumption on the level curves made there guarantees that each regular point of Γ_Φ belongs to a unique Γ_{ij}; hence in view of (2-2) the measure $\partial\Phi/\partial\bar{z}$ is locally determined by Γ_{ij}. □

In the remainder of this paper we will see that results similar to Theorem 2.5 actually hold without its local regularity assumptions as in (i).

Obviously, a *PH* function Ψ has a *PA* derivative almost everywhere. However, this is not necessarily the same as the distribution derivative of Ψ. The next result shows that this is true for continuous *PH* functions.

Proposition 2.6. *If the canonically decomposed PH function*

$$\Psi(z) = \sum_{i=1}^{r} H_i(z)\chi_i(z)$$

is continuous in U (*see Section 1.5*) *then*

$$\partial\Psi(z)/\partial z = \sum_{i=1}^{r} A_i(z)\chi_i(z) \tag{2-4}$$

in the sense of distributions, *where $A_i := \partial H_i/\partial z$ for $1 \leq i \leq r$.*

Proof. Let Γ_Ψ be as in Notation 2.4. By Lemma 1.7(v), Γ_Ψ is contained in the zero set of the function $g = \prod_{1\leq i<j\leq r}(H_i - H_j)$. Let $p \in \Gamma_\Psi \setminus \Gamma_\Psi^d$ be a regular point of Γ_Ψ and N be a small (open) neighborhood of p. Let further $N_\pm$ be N intersected with the two sides of Γ_Ψ. It follows that $N_+ \subset M_i$ and $N_- \subset M_j$ for some $i \neq j$ if N is small enough, and the restriction of Ψ to N is a smooth function plus $f\chi_i$, where $f \equiv 0$ in Γ_Ψ. Then $\partial(f\chi_i)/\partial z$ is a function in N and we conclude that $\partial\Psi/\partial z = \sum_{i=1}^{r} A_i\chi_i + G$, where G is a distribution supported at the singular points $\Gamma_\Psi^d \subset \Gamma_\Psi$. Since Γ_Ψ^d is a discrete set, by choosing a continuous solution h to $\partial h/\partial z = \sum_{i=1}^{r} A_i\chi_i$, we get a continuous solution $\Psi - h$ to $\partial(\Psi - h)/\partial z = G$, and it follows that $G \equiv 0$, which proves the proposition. □

3. Local characterization in generic cases: Sketch of proof

In this section we give an equivalent formulation of Theorem 1.11 and sketch its proof. Under some mild nondegeneracy assumptions, this provides a local description of functions with positive (distributional) $\bar{z}$-derivative that is equal almost everywhere to one of a finite number of given analytic functions.

Let us first fix notations and assumptions.

Notation 3.1. Let $\{M_i\}_{i=1}^r$ for $r \geq 2$ be a finite family of disjoint open subsets of a simply connected domain $U \subset \mathbb{C}$ covering U up to a set of zero Lebesgue measure, and denote by χ_i the characteristic function of M_i. Given a family $\{A_i(z)\}_{i=1}^r$ of pairwise distinct analytic functions in U, define the (measurable) function

$$\Psi(z) = \sum_{i=1}^{r} A_i(z)\chi_i(z).$$

Fix a point $p \in U$. As in (1-3) we let

$$H_i(z) = \mathrm{Re}\left(\int_p^z A_i(w)\,dw\right) \quad \text{for } 1 \leq i \leq r.$$

Each H_i is a well-defined harmonic function in U satisfying $\partial H_i/\partial z = \frac{1}{2}A_i(z)$. If $r \geq 3$ we associate to each triple (i, j, k) of distinct indices in $\{1, \dots, r\}$ the "critical set"

$$\Gamma_{i,j,k} = \{z \in U \mid A_i(z),\, A_j(z),\, A_k(z) \text{ are collinear}\}. \tag{3-1}$$

Alternatively, $\Gamma_{i,j,k}$ consists of the set of $z \in U$ such that $A_i(z) - A_k(z)$ and $A_j(z) - A_k(z)$ are linearly dependent over the reals. This is the set where the gradients of $H_i - H_k$ and $H_j - H_k$ are parallel, or equivalently, the level curves through z to these functions are parallel. Clearly, $\Gamma_{i,j,k}$ is either a real analytic curve or there exists $c \in \mathbb{R}$ such that $A_i(z) - A_k(z) = c(A_j(z) - A_k(z))$ for all $z \in U$.

In this notation Theorem 1.11 may then be restated as follows. Suppose — using the labeling in the theorem — that $i(p) = r$ (see (1-5)), assume that $\partial\Psi/\partial\bar{z} \geq 0$ as a distribution supported in U, and let $p \in U$ be such that

(i) $p \in \overline{M}_i$ for $1 \leq i \leq r$;

(ii) there is no critical set $\Gamma_{i,j,k}$ that contains p;

(iii) $A_i(p) \neq A_j(p)$ for $1 \leq i \neq j \leq r$, that is, p is a nonsingular point of $H_i - H_j$.

Then there exists a neighborhood $\widetilde{N}(p)$ of p such that

$$\Psi = 2\partial\varphi/\partial z \quad \text{almost everywhere in } \widetilde{N}(p),$$

where φ is the subharmonic function defined by $\varphi(z) = \max_{1 \leq i \leq r} H_i(z)$.

Remark 3.2. Generically, the sets $\Gamma_{i,j,k}$ are curves and so conditions (ii) and (iii) above hold outside some real analytic set.

Strategy of the proof and two fundamental lemmas. The proof of Theorem 1.11 is rather technical and the main parts of the argument are contained in Lemma 3.3 and Lemma 3.5 below, which to some extent hold independently of condition (ii) in Theorem 1.11. We will now show that Theorem 1.11 follows in fact from these two lemmas. First, a convenient reformulation of the conclusion of Theorem 1.11 is that for $1 \le i \le r$ one has $\chi_i = 1$ almost everywhere in the set where $\varphi(z) = H_i(z)$, and this is what we will actually show. Clearly, it is enough to prove this statement for $i = 1$.

Assumption I. *By considering the function* $\Psi - A_1$ *and using the fact that* A_1 *is analytic in* U (*hence* $\partial A_1/\partial \bar{z} = 0$), *we may assume without loss of generality that*

$$A_1(z) = H_1(z) = 0 \quad \textit{for } z \in U, \tag{I}$$

which we do, *except when otherwise stated*, *throughout the remainder of this section as well as in Sections 4 and 5.*

Define now

$$\begin{aligned} W = W_1(p) &:= \{z \in U \mid \varphi(z) = 0\}, \\ W_i(p) &:= \{z \in U \mid \varphi(z) = H_i(z)\} \quad \text{if } 2 \le i \le r. \end{aligned} \tag{3-2}$$

We have to prove that $\Psi = 0$ almost everywhere in $N \cap W$, or equivalently $\Psi = 0$ almost everywhere in $N \cap W^\circ$ for some small enough neighborhood N of p, where W° denotes the interior of W.

The first lemma asserts that χ_1 is increasing along every path along which all functions H_i for $2 \le i \le r$ are decreasing.

Lemma 3.3. *Let* $p \in U$ *satisfy all the assumptions of Theorem 1.11 except condition (ii). If* γ *is a piecewise* C^1*-path from* $z_1 = \gamma(0)$ *to* $z_2 = \gamma(1)$ *such that each of the functions* $[0, 1] \ni t \mapsto H_i(\gamma(t))$ *for* $2 \le i \le r$ *is decreasing*, *then*

$$(\chi_1 * \phi)(z_1) \le (\chi_1 * \phi)(z_2) \tag{3-3}$$

for any positive test function ϕ *with* supp ϕ *small enough.*

The second lemma guarantees that enough many points may be reached by paths of the form given in Lemma 3.3. To make a precise statement we need the following definition: To each $z \in U$, we associate the set

$$\begin{aligned} V(z) = \{\zeta \in U \mid\ & \text{there exists a piecewise } C^1\text{-path} \\ & \text{from } z \text{ to } \zeta \text{ along which all } H_i \text{ decrease}\}. \end{aligned}$$

Definition 3.4. Given $p \in U$ and two subsets $M, X \subset U$ with $p \in \overline{M}$, we say that $V(z)$ *tends to* X *through* M *as* $z \to p$, which we denote by $\lim_{M \ni z \to p} V(z) = X$, if for each $\alpha \in X$ and any sequence $\{z_n\}_{n\in\mathbb{N}} \subset M$ converging to p, one has $\alpha \in V(z_n)$ for all but finitely many indices $n \in \mathbb{N}$.

Lemma 3.5. *Let $p \in U$ satisfy all the assumptions of Theorem 1.11, in particular $p \notin \Gamma_{i,j,1}$ for any i, j. Then there is a neighborhood N of p with*

$$\lim_{U \ni z \to p} V(z) = N \cap W^\circ.$$

Remark 3.6. There are actually no sets $\Gamma_{i,j,k}$ at all if $r = 2$ in Lemma 3.5.

Theorem 1.11: Outline of the proof. As noted in the paragraph before Lemma 3.3, we have to show that there exists a sufficiently small neighborhood N of p such that $\Psi = 0$ almost everywhere in $N \cap W^\circ$. This is trivially true if W has no interior points (that is, if W° has zero Lebesgue measure) and so we may assume that W° has positive Lebesgue measure.

Let now $\{\phi_s\}_{s\in\mathbb{N}}$ be a sequence of test functions satisfying $\operatorname{supp}\phi_s \to \{0\}$ as $s \to \infty$ and $\int \phi_s d\lambda = 1$ for $s \in \mathbb{N}$, where λ denotes Lebesgue measure. Note that $\{\phi_s * \chi_1\}_{s\in\mathbb{N}}$ converges in L^1_{loc} to χ_1. In particular, this implies that for all $\epsilon > 0$ and $\delta > 0$, there exists a sufficiently large $s(\epsilon, \delta) \in \mathbb{N}$ such that if $s \in \mathbb{N}$ satisfies $s \geq s(\epsilon, \delta)$, there is a point $z_1 = z_1(\epsilon, \delta, s) \in U$ satisfying

$$|z_1 - p| < \delta \quad \text{and} \quad (\phi_s * \chi_1)(z_1) > 1 - \epsilon. \tag{3-4}$$

To see this let $N_\delta = \{z \in U \mid |z - p| < \delta\}$, and suppose that $(\phi_{s_k} * \chi_1)(z) \leq 1 - \epsilon$ for some infinite sequence $\{s_k\}_{k\in\mathbb{N}}$ and almost all $z \in N_\delta$. Then

$$\int_{N_\delta} \left|(\phi_{s_k} * \chi_1)(z) - \chi_1(z)\right| d\lambda(z) > \epsilon\lambda(M_1 \cap N_\delta),$$

and since by assumption $\lambda(M_1 \cap N_\delta) > 0$, this contradicts the fact that $\{\phi_{s_k} * \chi_1\}_{s\in\mathbb{N}}$ converges to χ_1 in L^1_{loc} as $k \to \infty$, so that (3-4) must hold.

From (3-3) and (3-4) it follows that $(\phi_s * \chi_1)(z) > 1 - \epsilon$ for $z \in V(z_1)$, which together with the identity $\phi_s * 1 = 1$ yields $\left(\phi_s * \sum_{i=2}^r \chi_i\right)(z) < \epsilon$ and therefore

$$\begin{aligned} |(\phi_s * \Psi)(z)| &= \left|\int \phi_s(z - \zeta)\Psi(\zeta) d\lambda(\zeta)\right| \\ &\leq \epsilon \max_{2\leq d\leq r} \sup_{\zeta \in z - \operatorname{supp}\phi_s} |A_d(\zeta)| =: \epsilon C_s(z) \quad \text{for } z \in V(z_1). \end{aligned} \tag{3-5}$$

Now we also assume that all the conditions of Theorem 1.11 and Lemma 3.5 are true. Fix $\epsilon > 0$. The arguments above show that one can construct a sequence $\{z_n\}_{n\in\mathbb{N}} \subset U$ such that

$$|z_n - p| < 1/n \quad \text{and} \quad (\phi_{s_n} * \chi_1)(z_n) > 1 - \epsilon \tag{3-6}$$

for some strictly increasing sequence of positive integers $\{s_n\}_{n\in\mathbb{N}}$. By Lemma 3.5 there exists a neighborhood N of p such that each $z \in N \cap W^\circ$ belongs to all but finitely many sets $V(z_n)$ for $n \in \mathbb{N}$. Combined with (3-5) this shows that for every $z \in N \cap W^\circ$ there exists $n_z \in \mathbb{N}$ such that

$$\left|(\phi_{s_n} * \Psi)(z)\right| \le C_{s_n}(z)\epsilon \quad \text{for } n \ge n_z. \tag{3-7}$$

Since A_d for $2 \le d \le r$ are analytic functions and $\operatorname{supp}\phi_{s_n} \to \{0\}$ as $n \to \infty$, it follows from (3-5) that by shrinking the neighborhood N (if necessary) one can find $C > 0$ such that $C_{s_n}(z) \le C$ for $n \in \mathbb{N}$ and $z \in N \cap W^\circ$. Together with (3-7) and the fact that $\lim_{n\to\infty} \phi_{s_n} * \Psi = \Psi$ in L^1_{loc}, this clearly implies that $\Psi = 0$ almost everywhere in $N \cap W^\circ$, which proves Theorem 1.11. □

4. Proof of Lemma 3.5

To complete the proof of Theorem 1.11, it remains to show Lemmas 3.3 and 3.5. We start with the latter, which we prove in this section.

4.1. ***Preliminaries.*** Let $A(z)$ be an analytic function defined in a neighborhood of some point $z_0 \in \mathbb{C}$ and set $H(z) := \operatorname{Re}(\int_{z_0}^{z} A(w)\,dw)$, so that $\partial H(z)/\partial z = \frac{1}{2}A(z)$. The directional derivative of H with respect to a complex number $v = \alpha + \beta i$ is given by

$$D_v H(z) = \alpha\partial H(z)/\partial x + \beta\partial H(z)/\partial y = \operatorname{Re}(vA(z)), \tag{4-1}$$

and the gradient of $H(x, y)$ considered as a vector in $\mathbb{C}$ is just

$$\nabla H(x, y) = 2\partial H(z)/\partial\bar{z} = \overline{A(z)}. \tag{4-2}$$

If $A(z_0) \ne 0$, then z_0 is a noncritical point for $H(z)$ and locally the 0-level curves of H form a foliation by 1-dimensional smooth curves of a small enough neighborhood N of z_0 [Spivak 1970, Theorem 5.7]. In particular, the (0-)level curve C_H of H through z_0 divides N into two components

$$N_H^+ = \{z \in N \mid H(z) > 0\} \quad \text{and} \quad N_H = \{z \in N \mid H(z) < 0\}.$$

Correspondingly, the tangent to C_H at z_0 divides the plane into two opposite half-planes

$$\tau(z_0)^+ = \{v + z_0 \mid v \cdot \nabla H(z_0) \ge 0\} = \{v + z_0 \mid \operatorname{Re}(vA(z_0)) \ge 0\},$$
$$\tau(z_0) = \{v + z_0 \mid v \cdot \nabla H(z_0) \le 0\} = \{v + z_0 \mid \operatorname{Re}(vA(z_0)) \le 0\}.$$

We now return to the functions A_i for $1 \le i \le r$, suspending for the moment Assumption I stating that $A_1 = 0$. As before, we suppose that $A_i(p) \ne A_j(p)$ if $i \ne j$. Consider the convex hull K of the points $A_i(p)$ for $1 \le i \le r$. For each i

define the dual cone (with vertex at p) to the sector consisting of all rays from $\nabla H_i(p) = \overline{A_i(p)}$ to points in the complex dual $\overline{K}$ by

$$\begin{aligned}\sigma_i(p) :&= \bigcap_{k\in K} \{v + p \mid v \cdot (\bar{k} - \nabla H_i(p)) \le 0\} \\ &= \bigcap_{j\neq i}^{r} \{v + p \mid v \cdot (\nabla H_j(p) - \nabla H_i(p)) \le 0\} \\ &= \bigcap_{j\neq i}^{r} \{v + p \mid \mathrm{Re}(v(A_j(p) - A_i(p))) \le 0\}.\end{aligned} \tag{4-3}$$

Clearly, this cone is the infinitesimal analogue of the set $W_i(p)$ defined in (3-2). The interior of $\sigma_i(p)$ contains the directions in which H_i grows faster (up to the first order) than any other H_k with $k \neq i$.

There are several possibilities for the cone $\sigma_i(p)$: it may have a top angle strictly between 0 and π, in which case we say that it is a *pointed cone*; it consists just of the point p; or it is either a line, a half-line or a half-plane.

The next lemma is a direct consequence of basic convex geometry.

Lemma 4.2. *With the above notations and assumptions the following holds*:

(i) *If $A_i(p)$ lies in the interior of K, then $\sigma_i(p) = \{p\}$*;

(ii) *If K is not a segment, then $A_i(p)$ is an extreme point of K if and only if $\sigma_i(p)$ is a pointed cone.*

Now consider condition (ii) in Theorem 1.11, which is also part of the assumptions of Lemma 3.5. By Lemma 4.2(ii), this condition is strictly stronger than the hypothesis in the following lemma.

Lemma 4.3. *Assume that the only points $A_i(p)$ contained in the boundary ∂K of K are extreme points. If $S(p) = \{i \in \{1, \dots, r\} \mid A_i(p)$ is an extreme point of $K\}$, then*

(i) $\max_{1\le i\le r} H_i(z) = \max_{i\in S(p)} H_i(z)$ *in a neighborhood of p*;

(ii) *there is a neighborhood N of p such that $\bigcup_{i\in S(p)} N \cap W_i = N$.*

Proof. Clearly, (ii) follows from (i). Let now $j \notin S(p)$, so that by Lemma 4.2 and the assumption of Lemma 4.3 one has $\sigma_j(p) = \{p\}$. This means that for each ray from p in the unit vector direction $v \in S^1$, there is at least one H_i with $i \in S(p)$ such that

$$v \notin \{u + p \mid u \cdot (\nabla H_i(p) - \nabla H_j(p)) \le 0\}.$$

Thus, for each $v \in S^1$ there is a product neighborhood $I(v) \times J(v, p) \subset S^1 \times U$ of $\{v\} \times \{p\}$ such that there exists $i = i(v) \in S(p)$ so that the continuous function $u \cdot (\nabla H_i(z) - \nabla H_j(z))$ is positive if $(u, z) \in I(v) \times J(v, p)$. By the compactness of

$S^1 \times \{p\} \subset S^1 \times U$, a finite number of neighborhoods $I(v_l) \times J(v_l, p)$ for $1 \le l \le s$ cover $S^1 \times \{p\}$. Hence the neighborhood $J(p) := \bigcap_{1 \le l \le s} J(v_l, p)$ of p has the property that along each ray from p with direction $v \in S^1$ there is some $i \in S(p)$ such that $H_i(z) > H_j(z)$ if $z \in J(p) \setminus \{p\}$, which proves (i). □

For the rest of this section we will again assume that $W = W_1$ and $A_1 = H_1 = 0$ (see Assumption I), and furthermore that $p = 0$. By condition (ii) in Theorem 1.11 (which, as we already pointed out, is also assumed in Lemma 3.5) and Lemma 4.3, it is then enough to prove Lemma 3.5 in the case when the index 1 belongs to the set $S(p)$ defined above, which we now proceed to do.

4.4. ***Changing coordinates.*** To prove Lemma 3.5 in the above situation we will further simplify the picture by making suitable coordinate changes as follows. Let G be a C^1-homeomorphism from a domain U' to U that takes a neighborhood $N' \subset U'$ of $p' = G^{-1}(p)$ one-to-one onto N. Then $W(p) \cap N$ is the homeomorphic image under G of the set

$$W'(p) = \{w \in N' \mid H_i(G(w)) \le 0,\ 2 \le i \le r\}$$

(note that we do not need to assume that G is analytic since we are not concerned with preserving subharmonicity in the present situation). Furthermore, if $z \in U$ and $z' = G^{-1}(z)$, then $V(z)$ is the homeomorphic image under G of the set

$$V'(z') = \{\zeta' \in U' \mid \text{there exists a piecewise } C^1\text{-path from } z' \text{ to } \zeta' \text{ along which all } H_i \circ G \text{ decrease}\}.$$

Clearly, since G is one-to-one it suffices for the proof of Lemma 3.5 to show that there exists a neighborhood N' of p' such that $V'(z')$ tends to W'° through an appropriate set as $z' \to p'$ (see Definition 3.4).

As an immediate application of this observation we may prove Lemma 3.5 in the case when K is a line segment. Indeed, suppose that $A_1(0) = 0$ and $A_2(0)$ are the (only) two extreme points of K. By Lemma 4.3, the functions $A_1(z) \equiv 0$ and $A_2(z)$ are the only active ones at p, and it suffices to show that $V(z)$ tends to W through W as $z \to p$ in a suitable neighborhood. We may change coordinates as above in order to reduce this case to the situation when $H_2(x, y) = y$. Then just consider the harmonic conjugate Q of H_2, and note that $N \ni z \mapsto (Q(z), H_2(z))$ is a local homeomorphism for a sufficiently small neighborhood N of $p = 0$. It follows that

$$V(z) \cap N = \{w \in N \mid \operatorname{Re} w \le \operatorname{Re} z\} \quad \text{and} \quad W^{\circ} \cap N = \{w \in N \mid \operatorname{Re} w < \operatorname{Re} p = 0\},$$

so the conclusion of Lemma 3.5 is immediate in this case.

4.5. ***The general case r ≥ 3.*** From the discussion at the beginning of this section it follows that if W is as in (3-2) and as before W° is its interior, we get that the open set

$$\Omega(p) := \bigcap_{i=2}^{r} N_{H_i} = W^\circ \cap N$$

is bounded by parts of some of the (0-)level curves through $p=0$ of H_i for $2 \le i \le r$, and part of the boundary of N. Furthermore, $\sigma_1(p)$ is a pointed cone subtending an angle $\alpha \in (0, \pi)$ at its vertex (which is the origin), and it is bounded in a small neighborhood of p by tangents to some level curves, say $H_2 = 0$ and $H_3 = 0$, that meet transversally at p. Since two nonidentical real analytic curves can intersect each other only in a discrete set, it follows that for a small enough neighborhood N of p the boundary of $\Omega(p)$ will consist of at most part of two level curves (and part of the boundary of N).

By the inverse function theorem, the map

$$(x, y) \mapsto R(x, y) := (H_2(x, y), H_3(x, y))$$

is a homeomorphism from a neighborhood (also called N) of p to a neighborhood of p. This map takes $W \cap N$ to an open subset of the third quadrant, and p is an interior point in the induced topology of the third quadrant. Clearly, the homeomorphism $G(x, y) = R^{-1}(x, y)$ satisfies $H_3(G(x, y)) = x$ and $H_2(G(x, y)) = y$, so that by Section 4.4 we may assume without loss of generality throughout the rest of this section that

> $H_2(x, y) = y$, $H_3(x, y) = x$, $\sigma_1(p)$ *is the third quadrant, and* $W \cap N$ *is the corresponding quadrant of a disk.*

The assumption on the boundary of the convex hull of the $A_i(p)$ (see Lemma 4.3 and the discussion following it) implies that there are no other level curves through p that are parallel to either of the level curves of H_2 or H_3 through p except the latter curves themselves.

Now by viewing gradients as complex numbers for each $z \in N$, we may write

$$\nabla H_k(z) = |\nabla H_k(z)| e^{\sqrt{-1}\theta_k(z)}, \quad \text{where } \theta_k(z) \in [0, 2\pi) \text{ for } 2 \le k \le r. \tag{4-4}$$

Our assumptions imply that $0 < \theta_k(p) < \pi/2$ for $2 \le k \le r$. Let us further shrink N, if necessary, so that

$$0 < \theta_k(z) < \pi/2 \quad \text{for } k \in \{2, \dots, r\} \setminus \{2, 3\} \quad \text{if } z \in N. \tag{4-5}$$

Claim 4.6. *For any* $z \in W^\circ \cap N$ *there exists a neighborhood* $\tilde{N}_z$ *of* 0 *such that every point in* $\tilde{N}_z$ *may be reached by a path from* z *along which each of the functions* H_k *for* $2 \le k \le r$ *increases.*

Proof. Let $z \in W^\circ \cap N$. Then clearly both coordinates x and y are increasing along the straight segment from z to $p = 0$ given by $\{(1-t)z \mid t \in [0, 1]\}$. Moreover, there is a disk N_z centered at p such that $w \in N_z$ implies that both x and y increase along the path $\gamma_w(t) = (1-t)z + tw$, with $t \in [0, 1]$, from z to w. (Note that N_z is the largest disk contained in $N \cap \{w \in \mathbb{C} \mid \operatorname{Re} w \geq \operatorname{Re} z,\ \operatorname{Im} w \geq \operatorname{Im} z\}$.) Thus the functions $[0, 1] \ni t \mapsto H_k(\gamma_w(t))$ with $k \in \{2, 3\}$ are both increasing. Let us show that this is true as well for each of the remaining functions with $k \in \{2, \dots, r\} \setminus \{2, 3\}$. By (4-5) one has $\nabla H_k(z) = (\alpha(z), \beta(z))$, where $\alpha(z), \beta(z) > 0$ if $k \notin \{2, 3\}$ and $z \in N$, so that the derivative

$$\frac{d}{dt} H_k(\gamma_w(t)) = \alpha(\gamma_w(t)) \operatorname{Re}(w - z) + \beta(\gamma_w(t)) \operatorname{Im}(w - z) \tag{4-6}$$

is positive for $w = 0$, $2 \leq k \leq r$, and $t \in [0, 1]$. Hence there is a neighborhood $\tilde{N}_z$ of 0 such that the expression in (4-6) is positive for all $w \in \tilde{N}_z$ and $t \in [0, 1]$. This means that each point in $\tilde{N}_z$ may be reached by a path from z along which each of the functions H_k for $2 \leq k \leq r$ increases. □

The proof of Lemma 3.5 is now immediate: If $\{z_n\}_{n \in \mathbb{N}}$ is a sequence converging to p, there is an $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $z_n \in \tilde{N}_z$ and by Claim 4.6 there is a path from z to z_n along which all H_k for $2 \leq k \leq r$ increase. Going in the other direction, there is a path from z_n to z along which all H_k for $2 \leq k \leq r$ decrease and hence $z \in V(z_n)$ for $n \geq n_0$. By the above remarks this completes the proof of Lemma 3.5.

4.7. *A more precise version of Theorem 1.11.* Revisiting the proof sketched in Section 3, we see that we can actually formulate a more precise result by using the terminology and arguments given in Sections 4.1–4.5 above.

Corollary 4.8. *Assume that all hypotheses of Theorem 1.11 are satisfied* except condition (ii). *Let $A_i(p)$ be an extremal point in K and consider the part ∂K_i of its boundary (that is, the union of the two edges of K) connecting $A_i(p)$ to its two neighboring extremal points. If $A_k(p) \notin \partial K_i$ for $k \neq i$, there exists a neighborhood N of p such that $\Psi = 2\partial\varphi/\partial z$ almost everywhere in $W_i(p) \cap N$.*

5. Proof of Lemma 3.3

In this section we prove the remaining lemma, which generalizes a corresponding result obtained in [Bergkvist and Rullgård 2002] in the (simpler) case when the A_i are constant functions. Recall Notation 3.1, the renormalization argument in Assumption I allowing $A_1 \equiv 0$, and also the assumptions of Lemma 3.3 and

Theorem 1.11 for our given *PA* function

$$\Psi(z)=\sum_{i=1}^{r}A_i(z)\chi_i(z)=0\cdot\chi_1(z)+\sum_{i=2}^{r}A_i(z)\chi_i(z) \tag{5-1}$$

and for the path γ. In particular, we assume that condition (iii) in Theorem 1.11 is fulfilled at all points on γ, that is, γ does not pass through singular points for the differences $H_i - H_j$ with $i \neq j$. We may reparametrize γ by arc-length using the parameter interval $[0, L]$,m and so we may assume that $|\dot\gamma(t)| = 1$ for $t \in [0, L]$. Note first that it is enough to prove the following modified form of Lemma 3.3: For each $t_1 \in [0, L]$ there exists $\eta > 0$ such that for any positive test function ϕ with supp ϕ small enough, one has

$$(\chi_1 * \phi)(z_1) \le (\chi_1 * \phi)(z_2), \quad \text{where } z_1 = \gamma(t_1) \text{ and } z_2 = \gamma(t_2), \text{ with } 0 < t_2 - t_1 < \eta. \tag{5-2}$$

Indeed, the fact that (5-2) implies Lemma 3.3 follows easily by a compactness argument: Fix t_1 and let s_2 be maximal such that (3-3) holds for $t_2 < s_2$. If $s_2 \neq L$, then (5-2) gives a contradiction to the maximality of s_2. For simplicity we make a translation so that $z_1 = 0$. Clearly, we may also assume that γ is C^1.

The idea of the proof of inequality (5-2) is to use the asymptotic properties of the logarithm of Ψ. For this we need to take the logarithm of the A_i, and we must therefore make sure that it is possible to choose a suitable branch. To this end we first prove the following assertion.

Claim 5.1. *There exists a neighborhood M of $z_1 = 0$ such that*

$$A_i(z) \in \mathbb{C} \setminus \{t\bar{\upsilon} \mid t \in (0,\infty)\} \quad \textit{for } z \in M \textit{ and } 1 \le i \le r$$

whenever υ is a unimodular complex number satisfying $\upsilon \in \sigma(z_1)$, where (see (4-3))

$$\sigma(z_1) = \bigcap_{i=2}^{r}\{u \mid \operatorname{Re}(uA_i(z_1)) \le 0\}.$$

Proof. Since $A_1 \equiv 0$ this is immediate for $i = 1$. By condition (iii) in Theorem 1.11, there exists $c' > 0$ such that $|A_i(z_1)| \ge c'$ for $i \in \{2,\dots,r\}$, so that there is a $c \in (0, c']$ and a neighborhood M of z_1 such that $|A_i(z)| \ge c$ for $i \in \{2,\dots,r\}$ and $z \in M$. It follows that for all unit vectors $\upsilon \in \sigma(z_1)$, we may assume up to shrinking M that $\operatorname{Re}(\upsilon A_i(z)) \le c/2$ for $z \in M$. Thus the angle ρ between $A_i(z)$ and $\bar\upsilon$ satisfies $\rho \in (\pi/3, 5\pi/3)$ since $\cos\rho = |A_i(z)|^{-1}\operatorname{Re}(\upsilon A_i(z)) < 1/2$, which proves the claim. □

We use this result to simplify the situation. We choose $\eta > 0$ such that $\gamma(t) \in M$ for $t \in [0, \eta]$, where the neighborhood M of $z_1 = 0$ is as in Claim 5.1, and we let $\upsilon = \dot\gamma(0)$. Note that since by the assumption in Lemma 3.3 all functions $[0, \eta] \ni t \mapsto$

$H_i(\gamma(t))$ for $2 \le i \le r$ are decreasing we have $\upsilon \in \sigma(z_1)$ by (4-1). Up to replacing Ψ by the function $e^{i\theta}\Psi(e^{i\theta}z)$, where $\upsilon = e^{i\theta}$, we may also assume that $\upsilon = 1$. In particular, we deduce that $\mathrm{Re}(\dot{\gamma}(0)) = 1 > 0$ so that by further shrinking M and the corresponding $\eta > 0$, we get the key property

$$\mathrm{Re}(\dot{\gamma}(t)) > 0 \quad \text{for } t \in [0, \eta]. \tag{5-3}$$

Let $\widetilde{\Psi}_\epsilon = \log(\Psi - \epsilon)$, where $\epsilon > 0$ is arbitrary and we have chosen a branch of the logarithm that is defined in the complex plane cut along the positive real axis. The composite distribution $\tilde{\Psi}_\epsilon$ is then defined by the above rotation of the complex plane, since $\upsilon = 1 \in \sigma(z_1)$. We now study its derivative along the path γ.

Given $\zeta \in M$, define as above (see (4-3))

$$\sigma(\zeta) = \bigcap_{i=2}^{r} \{u \mid \mathrm{Re}(uA_i(\zeta)) \le 0\}.$$

Then for any fixed $\epsilon > 0$ and $u \in \sigma(\zeta)$ with $\mathrm{Re}\, u > 0$, one has

$$\mathrm{Re}(u(A_i(w) - \epsilon)) < 0 \quad \text{for } 1 \le i \le r, \tag{5-4}$$

for all w in a (sufficiently small) neighborhood of ζ. In particular, inequality (5-4) holds for all vectors of the form $u = \dot{\gamma}(t)$ in view of (5-3) and the fact that all functions $[0, \eta] \ni t \mapsto H_i(\gamma(t))$ for $2 \le i \le r$ are decreasing (and thus $u \in \sigma(\zeta)$ by (4-1)). It follows that if ϕ is a positive test function with $\int \phi d\lambda = 1$ and $\mathrm{supp}\, \phi$ is small enough, then

$$\mathrm{Re}(u(\phi * \Psi - \epsilon)) < 0 \tag{5-5}$$

and therefore $\mathrm{Re}(\bar{u}/(\phi * \Psi - \epsilon)) \le 0$ in a neighborhood of ζ. Since $\partial(\phi * \Psi)/\partial\bar{z} \ge 0$ we get

$$\mathrm{Re}\Big(\bar{u}\frac{\partial}{\partial\bar{z}}\log(\phi * \Psi - \epsilon)\Big) = \mathrm{Re}\Big(\frac{\bar{u}}{\phi * \Psi - \epsilon} \cdot \frac{\partial(\phi * \Psi)}{\partial\bar{z}}\Big) \le 0.$$

Letting $\mathrm{supp}\, \phi \to 0$ with $\int \phi d\lambda = 1$, we see that $\log(\phi * \Psi - \epsilon) \to \widetilde{\Psi}_\epsilon$ in L^1_{loc} (hence as a distribution), and by passing to the limit we get

$$\mathrm{Re}\Big(\bar{u}\frac{\partial\widetilde{\Psi}_\epsilon}{\partial\bar{z}}\Big) \le 0.$$

Write now $\widetilde{\Psi}_\epsilon = \sigma_\epsilon + i\tau_\epsilon$, where σ_ϵ and τ_ϵ are real-valued distributions. Then the latter inequality yields

$$\mathrm{Re}\Big(\bar{u}\frac{\partial\sigma_\epsilon}{\partial\bar{z}}\Big) \le \mathrm{Im}\Big(\bar{u}\frac{\partial\tau_\epsilon}{\partial\bar{z}}\Big), \tag{5-6}$$

where (5-6) is interpreted as being valid for the restrictions of the corresponding distributions to a neighborhood of ζ. Note that up to further shrinking M (and the corresponding $\eta > 0$) by our choice of the branch of the logarithm used in the definition of $\widetilde{\Psi}_\epsilon$, we have

$$\tau_\epsilon(z) \in \left(\tfrac{1}{2}\pi, \tfrac{3}{2}\pi\right) \quad \text{for } z \in 2M = \{a+b \mid a, b \in M\}. \tag{5-7}$$

Let us show that relations (5-6) and (5-7) produce the desired result. Recall that for a real-valued function $\omega(z)$ one has

$$\frac{\partial \omega(z)}{\partial \bar{z}} = \overline{\frac{\partial \omega(z)}{\partial z}} \tag{5-8}$$

in the sense of distributions. We consider the derivative of $\widetilde{\Psi}_\epsilon$ along the path γ: If ϕ is a positive test function, then since σ_ϵ is a real-valued distribution we deduce from (5-8) and (5-6) that the following holds in the interval $(0, \eta)$:

$$\begin{aligned} \frac{d}{dt}\big((\phi * \sigma_\epsilon)(\gamma(t))\big) &= 2\operatorname{Re}\Big(\dot{\gamma}(t)\frac{\partial \phi * \sigma_\epsilon}{\partial z}(\gamma(t))\Big) \\ &= 2\operatorname{Re}\Big(\overline{\dot{\gamma}(t)}\frac{\partial \phi * \sigma_\epsilon}{\partial \bar{z}}(\gamma(t))\Big) \\ &= 2\int \operatorname{Re}\Big(\overline{\dot{\gamma}(t)}\frac{\partial \phi}{\partial \bar{z}}(\gamma(t)-w)\sigma_\epsilon(w)\Big)d\lambda(w) \\ &\le 2\int \operatorname{Im}\Big(\overline{\dot{\gamma}(t)}\frac{\partial \phi}{\partial \bar{z}}(\gamma(t)-w)\tau_\epsilon(w)\Big)d\lambda(w). \end{aligned} \tag{5-9}$$

Now if supp ϕ is small enough, say supp $\phi \subset M$, then from (5-7) and the fact that $|\dot{\gamma}(t)| = 1$ for $t \in [0, \eta]$ (see the reparametrization argument at the beginning of this section), we get

$$\begin{aligned} 2\Big|\int \operatorname{Im}\Big(\overline{\dot{\gamma}(t)}\frac{\partial \phi}{\partial \bar{z}}(\gamma(t)-w)\tau_\epsilon(w)\Big)d\lambda(w)\Big| \\ \le 2\cdot\frac{3\pi}{2}\cdot\frac{1}{2}\Big(\Big\|\frac{\partial \phi}{\partial x}\Big\|_1 + \Big\|\frac{\partial \phi}{\partial y}\Big\|_1\Big) =: \kappa(\phi), \end{aligned} \tag{5-10}$$

where $\|\cdot\|_1$ denotes the L^1-norm. Note that the (positive) constant $\kappa(\phi)$ defined above does not depend on ϵ. Combining (5-9) and (5-10), we obtain

$$(\phi * \sigma_\epsilon)(z_2) - (\phi * \sigma_\epsilon)(z_1) \le \kappa(\phi)\eta. \tag{5-11}$$

On the other hand, by (5-1) we have

$$\tilde{\Psi}_\epsilon(z) = \log\Big(-\epsilon\chi_1(z) + \sum_{i=2}^{r}(A_i(z)-\epsilon)\chi_i(z)\Big);$$

hence

$$\sigma_\epsilon(z) = (\log \epsilon)\cdot\chi_1(z) + f_\epsilon(z), \quad \text{where } f_\epsilon(z) = \sum_{i=2}^{r}\log|A_i(z)-\epsilon|\cdot\chi_i(z),$$

and therefore $(\phi * \sigma_\epsilon)(z) = (\log \epsilon) \cdot (\phi * \chi_1)(z) + (\phi * f_\epsilon)(z)$. By condition (iii) in Theorem 1.11, there exists a $c > 0$ such that $|A_i(z)| \geq c$ for $i \in \{2, \ldots, r\}$ and $z \in M$ (see the proof of Claim 5.1). We deduce that there exists a $c' > 0$ (independent of ϵ and ϕ) such that $|(\phi * f_\epsilon)(z)| \leq c' \|\phi\|_\infty$ for $z \in M$, where $\|\cdot\|_\infty$ denotes the L^∞-norm. It follows that

$$(\phi * \sigma_\epsilon)(z) = (\log \epsilon) \cdot (\phi * \chi_1)(z) + O(1). \tag{5-12}$$

Substituting (5-12) in (5-11) and letting $\epsilon \to 0$, we conclude that (5-2) holds, which by the preliminary remarks at the beginning of this section completely settles Lemma 3.3.

6. An alternative approach under extra conditions

In the previous sections we formulated and proved three results answering the main problem stated in Section 1 under fairly mild assumptions, namely Theorem 1.11 and its consequences Corollary 4.8 and Corollary 1.12 (see (1-4)). We will now prove Theorem 6.2 below, which provides a fourth answer to the main problem under some extra (yet still mild) conditions. Although this result may be obtained directly from Corollary 1.12, the point in what follows is to present an approach[1] different from the one used in Sections 3–5, and one that does not rely on Lemmas 3.3 and 3.5.

Notation 6.1. Let $\Phi \in PA$ be as in (1-2), which we assume to be the canonical piecewise decomposition of Φ in the sense of Definition 1.6. We may write

$$U \setminus \bigcup_{i=1}^{r} M_i = Z,$$

where M_i for $1 \leq i \leq r$ are pairwise disjoint open sets and Z is Lebesgue negligible. Note that each ∂M_i is also Lebesgue negligible since $\partial M_i \subset Z$ for $1 \leq i \leq r$. As before we let χ_i be the characteristic function of M_i. Recall from (1-5) the set $I(p)$ and its cardinality $i(p)$ defined for any $p \in U$. To simplify some discussions, assume that U is simply connected and choose $f_i \in A(U)$ such that $f_i'(z) = A_i(z)$ for $1 \leq i \leq r$, where the A_i are the given (analytic) functions appearing in the decomposition (1-2) of Φ. Hence

$$\Phi(z) = \sum_{i=1}^{r} f_i'(z) \chi_i(z).$$

[1] This approach and the subsequent proofs were suggested by the referee, whom we would like to thank for generously sharing his ideas with us.

For arbitrarily fixed $p \in U$ we let

$$\phi(z) \, (= \phi_p(z)) = \max_{j \in I(p)} \operatorname{Re}(f_j(z) - f_j(p)) = \max_{j \in I(p)} H_j(z),$$

where the H_i are the harmonic functions defined in (1-3) (see Section 1 in the case when $i(p) = r$). Clearly, ϕ is a continuous subharmonic function in U that vanishes at p. Finally, if $k \in I(p)$ and $i(p) > 1$, set

$$V_k(p) = \Big\{ \sum_{j \in I(p) \setminus \{k\}} \theta_j (f_k'(p) - f_j'(p)) \,\Big|\, \theta_j \geq 0, \; j \in I(p) \setminus \{k\}, \sum_{j \in I(p) \setminus \{k\}} \theta_j > 0 \Big\}.$$

Recall the definition of $\Sigma(U)$ from Notation 1.10.

Theorem 6.2. *In the above notations assume that $\Phi \in \Sigma(U)$ and that the following conditions hold*:

(i) *The one-dimensional Hausdorff measure of $\partial M_j \cap \partial M_k \cap \partial M_l$ is 0 whenever $j < k < l$.*

(ii) *If $i(p) > 1$ and $k \in I(p)$, then $0 \notin V_k(p)$.*

Then one has $\Phi = 2\partial_z \phi \, (= 2\partial_z \phi_p)$ almost everywhere in a neighborhood of every $p \in U$.

Remark 6.3. Recall the assumptions used in Corollaries 1.12 and 4.8 involving the (extremal points of the) convex hull K of the points $A_i(p) = f_i'(p)$ for $1 \leq i \leq i(p)$. Although still mild (since it is generically true), requirement (ii) in Theorem 6.2 is actually stronger than these assumptions.

The remainder of this section is devoted to the proof of Theorem 6.2, which uses induction on $i(p)$.

Consider first the case $i(p) = 1$. By relabeling the indices we may assume that $I(p) = \{1\}$, that is, $p \notin \overline{M}_j$ for $j > 1$. Hence p is either an interior point of M_1 or $p \in Z$, and every neighborhood of p intersects M_1. If the former occurs, then $\Phi(z) = 2\partial_z \operatorname{Re} f_1(z)$ in an open neighborhood of p and thus $\Phi = 2\partial\phi/\partial z$ in that neighborhood. If $p \in Z$, then there is a small open neighborhood Ω of p contained in $M_1 \cup Z$, and we conclude that $\Phi(z) = 2\partial_z \operatorname{Re} f_1(z)$ almost everywhere in Ω; hence equality holds in Ω in the distribution sense. This settles the case $i(p) = 1$.

Assume next that $i(p) = 2$ and (without loss of generality) $I(p) = \{1, 2\}$. Since the M_i are pairwise disjoint it follows that $p \in Z$ and $p \notin \overline{M}_k$ for $k > 2$. Therefore, there is an open neighborhood Ω of p such that

$$\Phi(z) = f_1'(z)\chi_1(z) + f_2'(z)\chi_2(z) \quad \text{for } z \in \Omega.$$

Let $\chi = \chi_2|_\Omega$, $f = f_2 - f_1|_\Omega$, and define

$$\Psi(z) = f'(z)\chi(z) = \Phi(z) - f_1'(z).$$

Note that $\partial_{\bar z}\Psi(z) \geq 0$ in Ω. Condition (ii) in Theorem 6.2 implies that $f'(p) \neq 0$, and we may assume (after shrinking Ω, if necessary) that f is a diffeomorphism from Ω onto some open disk $D \subset \mathbb{C}$. We may then write $\chi(z) = \eta(f(z))$, where $\eta = \eta(w) = \eta(u + iv)$ is the characteristic function of some open subset ω of D, and we get

$$0 \leq \partial_{\bar z}\Phi(z) = \partial_{\bar z} f'(z)\eta(f(z)) = |f'(z)|^2(\partial_{\bar w}\eta)(f(z)),$$

so that $\partial_{\bar w}\eta \geq 0$ in D. Since η is real-valued, this means that η is an increasing function of u. Hence the open set ω is defined by an inequality of the form $\operatorname{Re} w > a$, and then $M_2 \cap \Omega$ is defined by $\operatorname{Re}(f_2(z) - f_1(z)) > a$. Also since p is in the closure of the set where $\chi = 1$, we must have $a = \operatorname{Re}(f_2(p) - f_1(p))$. Clearly, we may assume that $f_1(p) = f_2(p) = 0$. Then $\Phi(z) = f_1'(z)$ when $z \in \Omega$ and $\operatorname{Re} f_1(z) > \operatorname{Re} f_2(z)$, while $\Phi(z) = f_2'(z)$ when $z \in \Omega$ and $\operatorname{Re} f_1(z) < \operatorname{Re} f_2(z)$. This shows that $\Phi = 2\partial\phi/\partial z$ in a neighborhood of p, which completes the proof in the case when $i(p) = 2$.

The above observations also give us a result that will be used later on:

Lemma 6.4. *Assume that $I(p) = \{j, k\}$, where $j < k$, and that $\gamma(t)$ is a C^1-curve escaping from M_j into M_k when $t = \tau$ in the sense that $\gamma(t) \in M_j$ for $t < \tau$ and that there is a sequence $\{\tau_\nu\}_1^\infty$ with $\tau_\nu > \tau$ and $\tau_\nu \to \tau$ as $\nu \to \infty$ such that $\gamma(\tau_\nu) \in M_k$. Then $\partial_t \operatorname{Re}(f_j(\gamma(t)) - f_k(\gamma(t)))\big|_{t=\tau} \leq 0$.*

Let us now pass to the case when $i(p) \geq 3$. Then $p \in Z$ and there is an open neighborhood of p that does not intersect $r - i(p)$ of the $\overline{M}_j$. By deleting these sets from U we may assume that $i(p) = r \geq 3$ (see the comments after (1-5)). We then know that $p \in \bigcap_{i=1}^r \partial M_j$. It is no restriction to further assume that the f_j are normalized so that $f_j(p) = 0$ for every j. Then $\phi(z)\ (= \phi_p(z)) = \max_j \operatorname{Re} f_j(z)$, and we have to prove that

$$\operatorname{Re} f_k = \phi \quad \text{in } M_k \cap N, \tag{6-1}$$

where $N \subset U$ is a sufficiently small open neighborhood of p. Let

$$N_k = \{z \in N \mid \operatorname{Re} f_k(z) > \operatorname{Re} f_j(z) \text{ when } j \neq k\}.$$

Suppose now that we can show that

$$N_k \subset \overline{M}_k \text{ for every } k \text{ if } N \text{ is sufficiently small.} \tag{6-2}$$

Since the $\operatorname{Re} f_j$ must be pairwise distinct harmonic functions in U (as a consequence of condition (ii) in Theorem 6.2), the set where $\operatorname{Re} f_j = \operatorname{Re} f_k$ for some j, k with $j \neq k$ is of Lebesgue measure 0. It follows that N is the disjoint union of the sets N_k together with a set of measure 0. Since the M_j are pairwise disjoint and ∂M_j is of Lebesgue measure 0 for every j (since $\partial M_j \subset Z$ — see Notation 6.1) we deduce that $(M_k \cap N) \setminus N_k$ is Lebesgue negligible. From this we conclude that $\operatorname{Re} f_k = \phi$ in $M_k \cap N$ and hence $\Phi = 2\partial_z \phi$ in N, which proves Theorem 6.2.

Thus the main issue is to show that (6-2) holds. When doing this we may assume that $k = r$ and consider the harmonic functions $h_j = \operatorname{Re}(f_r - f_j)$ for $1 \le j \le r-1$. We know that $h_j(p) = 0$. Let $q \in N_r$, that is, $q \in N$ and $h_j(q) > 0$ for $j < r$. We want to show that $q \in \overline{M}_r$. For this we define

$$\Lambda = \bigcup_{j<k<l} \left(\partial M_j \cap \partial M_k \cap \partial M_l\right).$$

By assumption (i) in Theorem 6.2, Λ has vanishing one-dimensional Hausdorff measure. We need this lemma:

Lemma 6.5. *There is an open set $N \subset U$ containing p such that the following holds: If $w \in N$ and $h_k(w) := \operatorname{Re}(f_r(w) - f_k(w)) > 0$ when $k < r$, then there exist an open neighborhood $\mathcal{M} = \mathcal{M}_w \subset U$ of p and for every $z \in \mathcal{M}$ a real analytic mapping $\gamma = \gamma(s,t)$ from a neighborhood of* $[0,1] \times [0,1]$ *into U such that*

(a) *the restriction of γ to any set where $t < t_0 < 1$ is a diffeomorphism onto its image*;

(b) *$\gamma(1/2, 0) = z$ and $\gamma(s,1) = w$ for all s*;

(c) *$\partial_t h_k(\gamma(s,t)) > 0$ for all (s,t) when $k < r$.*

Assertion (6-2) — and thus, as explained above, Theorem 6.2 as well — is now a consequence of Lemma 6.5. Indeed, let N be a small neighborhood of p satisfying its assumptions, and let $w \in N$ be such that $h_k(w) > 0$ for $k < r$. We need to prove that $p \in \overline{M}_r$. For this let $\mathcal{M} = \mathcal{M}_w$ be as in the conclusion of Lemma 6.5. Since $p \in \overline{M}_r$ we know that $\mathcal{M}$ contains a point $z \in M_r$. Let γ be the mapping corresponding to z and w. By shrinking the domain in which the variable s ranges, we may assume that $\gamma(s,0) \in M_r$ when $s \in [0,1]$. Set

$$\mathcal{A}_\nu = \{(s,t) \mid 0 \le s \le 1,\ 0 \le t \le 1 - \nu^{-1}\}$$

for each integer $\nu \ge 2$. Since the one-dimensional Hausdorff measure of Λ vanishes, this is also true for the one-dimensional Hausdorff measure of

$$K_\nu := \{(s,t) \in \mathcal{A}_\nu \mid \gamma(s,t) \in \Lambda\}.$$

It follows that

$$J_\nu := \{s \in [0,1] \mid (s,t) \in K_\nu \text{ for some } t\}$$

is a closed set of Lebesgue measure 0. In fact, J_ν is the projection of a set with vanishing one-dimensional Hausdorff measure; see for example [Mattila 1995, Theorem 7.5]. Therefore the set J_ν is of the first category, which implies that $\bigcup_\nu J_\nu$ is also of the first category. This gives us an $s \in [0,1]$ such that $\gamma(s,t) \notin \Lambda$

when $0 \le t < 1$. From condition (c) in Lemma 6.5 and from Lemma 6.4, it follows that the curve $t \mapsto \gamma(s, t)$, which starts at $\gamma(s, 0) \in M_r$, cannot leave $\overline{M}_r$ until $t = 1$. Hence $w \in \overline{M}_r$, which proves (6-2) and we are done.

It remains to prove Lemma 6.5. In doing so we will use the fact that the functions $h_j = \operatorname{Re}(f_r - f_j)$ for $1 \le j \le r-1$ introduced above are real-valued and real analytic, but we will make no use of their harmonicity. Condition (ii) in Theorem 6.2 implies that the set of all linear combinations $\sum_{j=1}^{r-1} \theta_j dh_j(p)$, where $\theta_j \ge 0$ for all j and dh denotes differential, is contained in a convex cone Γ with positive opening angle less than π. We make an affine change of coordinates, keeping only the affine space structure of $\mathbb{C}$. This change of coordinates will allow us to replace Γ with any other cone with positive opening angle, and without loss of generality we may further assume that p is the origin. Then we are in the situation where a set of $m = r - 1$ real analytic and real-valued functions $h_1, \dots, h_m$ are defined in a neighborhood V of the origin in $\mathbb{R}^2$ and satisfy the conditions

(I) $h_j(0) = 0$ and $dh_j(0) \ne 0$ when $1 \le j \le m$;

(II) the closed convex cone generated by the gradients $\nabla h_j(0)$ for $1 \le j \le m$ is contained in the cone $\Gamma := \{(x, y) \in \mathbb{R}^2 \mid |x| \le y\}$.

To complete the proof of Lemma 6.5 we need only establish this result:

Lemma 6.6. *Assume conditions* (I) *and* (II) *above. Then there is an open set* $0 \in N \subset V$ *such that the following holds*: *If*

$$w \in \Omega_N := \{z = (x, y) \in N \mid h_j(z) > 0,\ 1 \le j \le m\}$$

one can find an open neighborhood $\mathcal{M} = \mathcal{M}_w$ *of the origin and for each* $z \in \mathcal{M}$ *a* C^1*-mapping* $\gamma(s, t)$ *from a neighborhood of* $[0, 1] \times [0, 1]$ *into* V *such that*

(a) *the restriction of* γ *to any set where* $t < t_0 < 1$ *is a diffeomorphism onto its image*;

(b) $\gamma(1/2, 0) = z$ *and* $\gamma(s, 1) = w$ *for all* s;

(c) $\partial_t h_k(\gamma(s, t)) > 0$ *for all* (s, t) *when* $k \le m$.

Proof. Define $\Omega_N^{\pm} = \Omega_N \cap \{(x, y) \in \mathbb{R}^2 \mid \pm x \ge 0\}$ whenever $N \subset V$. It suffices to prove that there exist an open set $0 \in N = N_+ \subset V$ such that the conclusion of the lemma holds when $w \in \Omega_N^+$. Indeed, by replacing $h_k(x, y)$ with $h_k(-x, y)$, we would obtain $N = N_-$, for which the lemma's conclusion would then be true when $w \in \Omega_N^-$, and thus the lemma's claims would follow for the open set $N = N_+ \cap N_-$.

It is no restriction to assume that $dh_j(0)$ is proportional to $-dx + dy$ for some j. By shrinking V if necessary and applying the implicit function theorem, we may also assume that every h_j is of the form

$$h_j(x, y) = \beta_j(x, y)(y - g_j(x)),$$

where β_j, g_j are real analytic functions and $\beta_j > 0$. Then by using the real analyticity of the functions g_j we may further assume — after shrinking V and relabeling the indices, if necessary — that $V = (-b, b) \times (-b, b)$ for some positive real number b and that $g_1(x) \le g_2(x) \le \cdots \le g_m(x)$ when $0 < x < b$. With these normalizations it follows that

$$-1 \le g_1'(0) \le g_2'(0) \le \cdots \le g_m'(0) = 1,$$

and finally, after making a nonlinear change of the x-coordinate, we may also assume that $g_m(x) = x$.

Below we let $a < b$ and δ be small positive numbers and we make generic use of the letter C to denote constants that are independent of a and δ when these stay small. Define

$$N(a) = \{z \in \mathbb{C} \mid |z| < a\},$$
$$\Omega^+(a) = \{z = (x, y) \in N(a) \mid x \ge 0 \text{ and } h_k(z) > 0 \text{ for all } k\},$$

so that $\Omega^+(a) = \{z = (x, y) \mid 0 \le x < y,\ |z| < a\}$.

Now, we clearly have the estimates

$$C^{-1} \le \beta_j(z) \quad \text{and} \quad |\nabla \beta_j(z)| \le C \quad \text{for } z \in N(a). \tag{6-3}$$

Let $w = (u, v) \in \Omega^+(a)$ and set $\rho = v - u$. Then ρ is a positive real number that depends on w, and we define

$$\mathcal{M} = \mathcal{M}_w = \{z \in \mathbb{C} \mid |z| < \delta\rho\}.$$

Take $z \in \mathcal{M}$ and let $\alpha \in \mathbb{R}^2$ be linearly independent from $w - z$ and such that $|\alpha| \le \delta\rho$. Introduce the mapping

$$\gamma(s, t) = (x(s, t), y(s, t)) = z + (s - 1/2)(1 - t)\alpha + t(w - z) \tag{6-4}$$

defined for all (s, t) in a small open neighborhood of $[0, 1] \times [0, 1]$. It is then immediate that assertions (a) and (b) in the lemma are satisfied.

To verify (c) we compute the t-derivative of $h_j(\gamma(s, t))$ as

$$\begin{aligned} \partial_t(h_j(\gamma(s, t))) = (y(s, t) - g_j(x(s, t)))\partial_t(\beta_j(\gamma(s, t))) \\ + \beta_j(\gamma(s, t))(\partial_t y(s, t) - g_j'(x(s, t))\partial_t x(s, t)). \end{aligned} \tag{6-5}$$

We see that

$$|\partial_t(\beta_j(\gamma(s, t)))| \le Ca. \tag{6-6}$$

Since $g_j(x) \le g_m(x) = x$ when $0 < x < a$ we may write $g_j(x) = x - p_j(x)$, where $p_j(x) \ge 0$. If $p_j(x) \not\equiv 0$, then $p_j(x) = x^{\mu_j} q_j(x)$, where μ_j is a positive integer and $q_j(0) > 0$. By taking a sufficiently small, we may then assume that

$$p_j'(x) = \mu_j x^{\mu_j - 1} q_j(x) + x^{\mu_j} q_j'(x) \geq C^{-1} p_j(x)/x \quad \text{for } 0 < x < a. \tag{6-7}$$

Moreover, since $x(s,t) = (1-t)x(s,0) + tx(s,1) \geq (1-t)x(s,0)$ it follows that $|x(s,t)| \leq C\delta\rho$ if $x(s,t) \leq 0$. Hence there is a constant C such that

$$\left| p_j'(x(s,t)) - p_j'(|x(s,t)|) \right| \leq C\delta\rho \quad \text{for } 0 \leq s, t \leq 1. \tag{6-8}$$

Next, one has

$$\begin{aligned} y(s,t) - g_j(x(s,t)) &= (1-t)y(s,0) + ty(s,1) - x(s,t) + p_j(x(s,t)) \\ &= (1-t)y(s,0) + ty(s,1) - (1-t)x(s,0) - tx(s,1) + p_j(x(s,t)) \\ &= (1-t)(y(s,0) - x(s,0)) + t(y(s,1) - x(s,1)) + p_j(x(s,t)) \\ &= (1-t)(y(s,0) - x(s,0)) + t\rho + p_j(x(s,t)). \end{aligned} \tag{6-9}$$

Recall that $w \in \Omega^+(a)$, so that in particular $|w| < a$. Since $|z| < \delta\rho$ and $|\alpha| \leq \delta\rho$, it follows from (6-4) that $|x(s,t)| < a$ if δ is small enough. We then deduce from (6-8) and (6-9) that

$$|y(s,t) - g_j(x(s,t))| \leq C\rho + p_j(|x(s,t)|). \tag{6-10}$$

Using (6-7) and (6-8) we find that

$$\begin{aligned} \partial_t y(s,t) - (\partial_t x(s,t)) g_j'(x(s,t)) &= \rho - (y(s,0) - x(s,0)) + (\partial_t x(s,t)) p_j'(x(s,t)) \\ &= \rho - (y(s,0) - x(s,0)) + (x(s,1) - x(s,0)) p_j'(x(s,t)) \\ &= \rho - (y(s,0) - x(s,0)) - x(s,0) p_j'(x(s,t)) + x(s,1) p_j'(x(s,t)) \\ &\geq (1 - C\delta)\rho + x(s,1) p_j'(x(s,t)) \geq (1 - 2C\delta)\rho + x(s,1) p_j'(|x(s,t)|) \\ &\geq (1 - 2C\delta)\rho + C^{-1} p_j(|x(s,t)|). \end{aligned}$$

We now choose δ small enough so that, say, $2C\delta < 1/2$. This gives the inequality

$$\partial_t y(s,t) - (\partial_t x(s,t)) g_j'(x(s,t)) \geq C^{-1}(\rho + p_j(|x(s,t)|)). \tag{6-11}$$

Combining (6-11) with (6-3), (6-5), (6-6) and (6-10), we get

$$\begin{aligned} \partial_t h_j(\gamma(s,t)) &\geq \beta_j(\gamma(s,t))(\partial_t y(s,t) - (\partial_t x(s,t)) g_j'(x(s,t))) \\ &\qquad - |(y(s,t) - g_j(x(s,t)))\partial_t \beta_j(\gamma(s,t))| \\ &\geq C^{-2}(\rho + p_j(|x(s,t)|)) - C^2 a(\rho + p_j(|x(s,t)|)) \\ &= (C^{-2} - C^2 a)(\rho + p_j(|x(s,t)|)). \end{aligned}$$

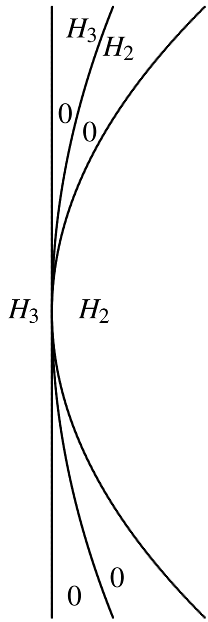

Figure 1. A nonmaximal subharmonic *PH* function.

Taking $a < C^{-4}/2$ we obtain a positive bound from below for the right side in the last expression, which completes the proof of the lemma. □

7. Examples and further problems

7.1. ***The necessity of nondegeneracy assumptions.*** If one of the cones $\sigma_i(p)$ in (4-3) is a line, it may happen that $W(p)\backslash\{p\}$ is the union of two components $W(p)_l$ and $W(p)_r$, each bounded by level curves as above. In this case there might be several different subharmonic *PH* functions that satisfy condition (i) in Theorem 1.11, as shown by Example 7.2 below. Hence something like condition (ii) is indeed necessary to obtain the conclusion of Theorem 1.11.

Example 7.2. Set $H_1(x, y) = 0$, $H_2(x, y) = 4x + x^2 - y^2$, and $H_3(x, y) = -x$. There are three level curves through $(0, 0)$ to functions of the form $H_i - H_j$ with $i \neq j$. These are depicted in Figure 1. Let $\varphi = \max\{H_1 \equiv 0, H_2, H_3\}$. The functions in the figure closest to the origin in each sector are the restriction of φ to that sector.

If one instead defines $\Psi(x, y)$ by changing the value in the two upper sectors from 0 to H_3 and H_2, respectively, then one obtains a different continuous *PH* function that is again subharmonic. Clearly, every neighborhood of the origin still has the property that Ψ is equal to each of the three harmonic functions in some subset of positive Lebesgue measure. So Ψ is a maximum of harmonic functions along the curves and hence is trivially subharmonic away from the origin. Letting $0 \le \chi \in C_0^\infty(\mathbb{R})$ be equal to 1 near the origin and $\chi_\epsilon(z) := \chi(z/\epsilon)$, this implies that $(1-\chi_\epsilon)\Delta\Psi \ge 0$ in $\mathcal{D}'$. But clearly $\chi_\epsilon \Delta\Psi \to 0$ in $\mathcal{D}'$ as $\epsilon \to 0$ since $\Psi = O(|z|)$. Hence Ψ is subharmonic.

7.3. ***On global descriptions.*** In this paper we have only considered the problem of locally characterizing the maximum of a finite number of harmonic functions. A natural question is to study various situations when a subharmonic *PH* function is globally the maximum of a finite number of harmonic functions. Such a situation occurs for instance in [Bergkvist and Rullgård 2002], where the given harmonic functions are linear. The same conclusion holds when the number of given harmonic functions is two as well as in certain other cases. We discuss some of these cases in the following examples, which were inspired by personal communication with A. Melin in 2005 and 2006.

Example 7.4. Let A_1 and A_2 be entire functions such that $A_1(z) \ne A_2(z)$ for $z \in \mathbb{C}$, and assume that $\Phi := \chi_1 A_1 + \chi_2 A_2$ satisfies $\partial\Phi/\partial\bar{z} \ge 0$, where χ_1 and χ_2 are the characteristic functions of the sets M_1 and M_2, respectively (see Notation 1.4). The first assumption implies that $H_i(z) = \operatorname{Re}\int_0^z A_i(w)dw$ for $i = 1, 2$ are well-defined functions in $\mathbb{C}$ and that there are no singular points for $H_1 - H_2$. For simplicity assume further that level curves to $H_1 - H_2$ as well as the support $\partial\Phi/\partial\bar{z}$ are connected. If $p \in \overline{M}_1 \cap \overline{M}_2$, it follows from Theorem 1.11 (condition (ii) there being vacuous in this case) that there exists a neighborhood N of p and constants $c_1(p)$ and $c_2(p)$ such that

$$\Phi = 2\frac{\partial}{\partial z}\max(H_1 + c_1(p),\, H_2 + c_2(p)) = 2\frac{\partial}{\partial z}\max(H_1,\, H_2 + c_2(p) - c_1(p))$$

In particular, the common boundary of M_1 and M_2 in N is the level curve $H_1 - H_2 = c_2(p) - c_1(p)$, and this is also the support of $\partial\Phi/\partial\bar{z}$ in N. The local information implies, by the connectedness assumptions, that globally $c_2(p) - c_1(p)$ is a constant c independent of p, and that the support actually consists of the level curve $H_1 - H_2 = c$, and finally that $\Phi = 2\frac{\partial}{\partial z}\max(H_1,\, H_2 + c)$.

Example 7.5. This is an essentially one-dimensional example. We assume that $\mathbb{R} = \bigcup_{j=1}^r \bar{I}_j$, where the I_j are open pairwise disjoint intervals. Set $M_j = I_j \times \mathbb{R}$ for $1 \le j \le r$, and let $\chi_j(x)$ be the characteristic function of I_j, which we also view as the characteristic function of M_j. Let $h_j(x + \sqrt{-1}y) = a_j x + b_j$ for $1 \le j \le r$ be linear functions on $\mathbb{C}$, and assume as usual that

$$\chi := \frac{\partial}{\partial\bar{z}}\Big(\sum_{j=1}^r \frac{\partial h_j(z)}{\partial z}\chi_j\Big) = \sum_{j=1}^r \frac{\partial h_j(z)}{\partial z}\frac{\partial\chi_j}{\partial\bar{z}} = \sum_{j=1}^r \frac{a_j}{2}\frac{\partial\chi_j}{\partial\bar{z}} \ge 0.$$

Since $\partial\chi_j/\partial\bar{z} = \frac{1}{2}\partial\chi_j/\partial x$ we deduce that $\sum_{j=1}^{r} a_j\chi_j$ is an increasing function of x and thus $h(x) = \int_0^x \sum_{j=1}^{r} a_j\chi_j$ is a convex function. Set

$$H(x, y) = h(y) + h'(y+0)(x-y).$$

By convexity we have

$$h(x) \geq H(x, y) \quad \text{for } x, y \in \mathbb{R}, \tag{7-1}$$

with equality when $y = x$. The functions $H(x, y)$, viewed as linear functions of $x \in \mathbb{R}$, are independent of y when $y \in I_j$. We denote their common value for $y \in I_j$ by $\tilde{h}_j(x)$ and notice that $\tilde{h}_j - h_j = C_j$, where C_j is a constant. It follows from (7-1) that $h(x) = \max_{1\leq k\leq r} \tilde{h}_k(x)$ in M_j, and then differentiation implies that

$$h'(x) = \frac{\partial}{\partial x} \max_{1\leq k\leq r} \tilde{h}_k(x) = \frac{\partial}{\partial x} \max_{1\leq k\leq r} (h_k(x) + C_k).$$

This means that the *PA* function χ satisfies $\chi = 2\frac{\partial}{\partial z} \max_{1\leq j\leq r}(h_j(z) + C_j)$ and is therefore globally the maximum of a finite number of harmonic functions.

7.6. *Related questions.* Let us finally discuss some interesting related problems.

Problem 7.7. At the moment we do not know, although we strongly suspect, that locally there are in fact only a finite number of possibilities for Ψ even when conditions (i)–(iii) are weakened in Theorem 1.11. This holds for example for the function constructed in Example 7.2. In particular, it seems likely that there always exists a sufficiently small neighborhood of p that can be dissected into sectors bounded by level curves to $H_i - H_j$ such that Ψ is constant in each such sector. Example 7.2 suggests that the local behavior of a *PH* subharmonic function is determined by the geometry of the level curves $\Gamma_{i,j,k}$, whose study is essentially a problem of a combinatorial and topological nature. It would be interesting to give a description of this local behavior in terms of Morse theory (the study of level curves was Morse's original motivation for his theory; see [Kelly 1979]).

Problem 7.8. Another problem is to understand the global behavior of a *PH* subharmonic function and in particular to give criteria saying precisely when $\partial\Psi/\partial z$ is the derivative of the maximum of a finite number of harmonic functions as in the last two examples. This would have interesting applications to uniqueness theorems for Cauchy transforms that are algebraic functions, as in [Bergkvist and Rullgård 2002; Borcea et al. 2007].

Problem 7.9. There are also several connections between the questions studied in the present paper and the theory of asymptotic solutions to differential equations. For instance, sets like those that occur as the support of the measures in Theorem 2.5 play a remarkable role in the theory [Fedoryuk 1993; Kelly 1979;

Wasow 1985; 1965; Sibuya 1975]. Moreover, many similar techniques are used; for example the admissible sets in [Fedoryuk 1993; Kelly 1979] are closely related to (though not exactly the same as) the sets $V(z)$ in Lemma 3.3 above. These connections are quite close in the cases studied in [Bergkvist and Rullgård 2002; Borcea et al. 2007] (as well as other cases) and certainly deserve further investigation in view of their important applications.

Problem 7.10. Suppose U is a domain in C^n, where $n \geq 1$. By analogy with Definition 1.2 and Notation 1.4 one can define the notions of PH_n and PA_n functions in U as natural higher-dimensional generalizations of the concepts of PH and PA functions, respectively. It seems reasonable to conjecture that appropriate higher-dimensional analogues of Theorem 1.11 hold for the class PA_n and that as a consequence one would get a natural extension of (say) Corollary 1.15 to the class PH_n.

Appendix. Comments on some properties and definitions

As before, χ_Ω denotes the characteristic function of a set $\Omega \subset \mathbb{C}$ (or $\mathbb{R}^2$). Let us introduce the additional condition: An open set $\Omega \subset \mathbb{R}^2$ is said to have property (*) if $\partial\Omega$ is of Lebesgue measure 0 and $\partial_z \chi_\Omega$ and $\partial_y \chi_\Omega$ are measures.

Lemma A.1. *If* $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ *have property* (*), *then so does* $\Omega_1 \cap \Omega_2$.

Proof. It is clear that $\partial(\Omega_1 \cap \Omega_2)$ is Lebesgue negligible. Let $K \subset \mathbb{R}^2$ be any compact set, choose $\eta \in C_0^\infty(\mathbb{R}^2)$ with $\iint \eta(x, y)\,dx\,dy = 1$, define $\eta_\epsilon = \epsilon^{-2}\eta(x/\epsilon, y/\epsilon)$ for $\epsilon \in (0, 1)$ and set $\chi_{j,\epsilon} = \chi_j * \eta_\epsilon$, where $\chi_j = \chi_{\Omega_j}$ for $j = 1, 2$. Then $0 \leq \chi_{j,\epsilon} \leq 1$, $\chi_{j,\epsilon} \to \chi_j$ almost everywhere as $\epsilon \to 0$ and $\|\partial_x \chi_{j,\epsilon}\|_{L^1(K)} = \|\eta_\epsilon * \partial_x \chi_j\|_{L^1(K)} \leq C_K$, where C_K is independent of ϵ. Since $\partial_x(\chi_{1,\epsilon}\chi_{2,\epsilon}) = \chi_{1,\epsilon}\partial_x\chi_{2,\epsilon} + \chi_{2,\epsilon}\partial_x\chi_{1,\epsilon}$, it follows that if $\phi \in C_0^\infty(\mathbb{R}^2)$ then

$$\left|\iint \chi_{1,\epsilon}(x, y)\chi_{2,\epsilon}(x, y)\partial_x\phi(x, y)\,dx\,dy\right| \leq \iint |\phi(x, y)|(|\partial_x\chi_{1,\epsilon}(x, y)| + |\partial_x\chi_{2,\epsilon}(x, y)|)\,dx\,dy \leq 2C_K\|\phi\|_{L^\infty}.$$

When $\epsilon \to 0$ this shows that

$$\left|\iint \chi_1(x, y)\chi_2(x, y)\partial_x\phi(x, y)\,dx\,dy\right| \leq 2C_K\|\phi\|_{L^\infty},$$

and thus $\partial_x(\chi_1\chi_2)$ is a distribution of order 0 (which extends to a measure). This finishes the proof since $\partial_y(\chi_1\chi_2)$ can be dealt with in the same way. □

Lemma A.1 shows that if we define sets P^*X of functions "piecewise* in X" as in Definition 1.2 by demanding in addition that all sets M_i have property (*), then P^*X are again vector spaces.

Lemma A.2. *If $u \in P^*X$ is continuous then $\partial_x u, \partial_y u \in P^*X$, where derivatives are taken in the distribution sense.*

Proof. Let us write $u = \sum_{i=1}^r u_i \chi_i$, where χ_i is the characteristic function of the (open) set M_i, $\sum_{i=1}^r \chi_i = 1$ almost everywhere and $\partial_x \chi_i$ and $\partial_y \chi_i$ are measures for $1 \le i \le r$. Since u is continuous we can find $u_\epsilon \in C^\infty(U)$ tending uniformly to u on every compact set as $\epsilon \to 0$. Now

$$\begin{aligned}\partial_x u_\epsilon &= \sum_{i=1}^r (\partial_x u_i)\chi_i + \sum_{i=1}^r \chi_i \partial_x (u_\epsilon - u_i) \\ &= \sum_{i=1}^r (\partial_x u_i)\chi_i + \partial_x\Big(\sum_{i=1}^r (u_\epsilon - u_i)\chi_i\Big) - \sum_{i=1}^r (u_\epsilon - u_i)\partial_x \chi_i.\end{aligned} \tag{A-1}$$

For every i one has $u_\epsilon - u_i = u_\epsilon - u$ in a dense subset of M_i. It follows that $u_\epsilon \to u_i$ uniformly on every compact subset of $\overline{M}_i$ and hence also on every compact subset of the support of the measure $\partial_x \chi_i$. Therefore $(u_\epsilon - u_i)\partial_x \chi_i \to 0$ in $\mathcal{D}'(\mathbb{R}^2)$ as $\epsilon \to 0$. This is true for $(u_\epsilon - u_i)\chi_i$ as well and so by letting $\epsilon \to 0$ in (A-1) we conclude that $\partial_x u_\epsilon = \sum_{i=1}^r (\partial_x u_i)\chi_i$. The same argument applies to $\partial_y u$. □

Given a domain $U \subset \mathbb{C}$, let $S(U)$ be the class of subharmonic functions in U. Recall Notation 1.10, where we already noted the (well-known) fact that $\partial_z \phi \in \Sigma(U)$ whenever $\phi \in S(U)$. For completeness we give here a proof of an (also well-known) partial converse to this statement.

Lemma A.3. *If U is simply connected and $f \in \Sigma(U)$, then $f = \partial_z \phi$ for some $\phi \in S(U)$ that is uniquely determined modulo an additive constant.*

Proof. Since the operator ∂_z is elliptic, we may write f as $f = \partial_z w$, where $w = u + iv \in \mathcal{D}'(U)$; see for example [Hörmander 2003]. We get $\Delta u + i\Delta v = \Delta w = 4\partial_{\bar z}\partial_z w = 4\partial_{\bar z} f \ge 0$, which implies that $u \in S(U)$ for $v \in H(U)$, and thus $f = \partial_z u + g$, where $g = i\partial_z v \in A(U)$. Let $G \in A(U)$ be such that $G'(z) = g(z)$, and define $\phi = u + G + \bar{G}$. Then $\phi \in S(U)$ and $\partial_z \phi = \partial_z u + \partial_z G = \partial_z u + g = f$. The last assertion in the lemma follows from the fact that a function h in U is constant whenever $h = \bar{h}$ and $\partial_z h = 0$. □

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JULIUS BORCEA
DEPARTMENT OF MATHEMATICS
STOCKHOLM UNIVERSITY
106 91 STOCKHOLM
SWEDEN
julius@math.su.se

RIKARD BØGVAD
DEPARTMENT OF MATHEMATICS
STOCKHOLM UNIVERSITY
106 91 STOCKHOLM
SWEDEN
rikard@math.su.se