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We show how to derive structure relations for general orthogonal polynomials, that is, we find operators whose action on p_n is a combination of p_n and p_{n+1} with variable coefficients. We also provide an analogue of the string equation for general orthogonal polynomials. We explore the connection with the Toda lattice and polynomials orthogonal with respect to general exponential weights.

1. Introduction

By a structure relation for a sequence of orthogonal polynomials $\{p_n(x)\}$, we mean a functional recurrence relation of the form

$$L\,p_n(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x), \tag{1-1}$$

where L is a linear operator whose domain contains all polynomials. Every orthonormal polynomial satisfies a three term recurrence relation of the form

$$xp_n(x) = a_{n+1}p_{n+1}(x) + \alpha_n p_n(x) + a_n p_{n-1}(x). \tag{1-2}$$

See [Chihara 1978; Szegő 1975; Ismail 2005].

Let $\{p_n(x)\}$ be orthonormal with respect to the weight function

$$w(x) = e^{-v(x)}, \tag{1-3}$$

that is,

$$\int_a^b p_m(x)p_n(x)e^{-v(x)}\,dx = \delta_{m,n}. \tag{1-4}$$

It is known that in this case the p_n have the structural relation

$$\frac{d}{dx}p_n(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x), \tag{1-5}$$

where A_n and B_n are given by (2-1) and (2-2).

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In view of (1-2), we can rewrite (1-1) in the form

$$L\, p_n(x) = C_n(x) p_{n+1}(x) + D_n(x) p_{n-1}(x). \tag{1-6}$$

Koornwinder [2007] considered the case when $C_n(x)$ and $D_n(x)$ are constants. He proved the following theorems.

Theorem 1.1. *Let L be linear operator acting on* $\mathbb{R}[x]$ *that is skew symmetric with respect to the inner product*

$$(f, g)_w := \int_a^b f(x)\overline{g(x)} w(x)\, dx. \tag{1-7}$$

If $L x^n = c_n x^{n+1} +$ *lower order terms, then*

$$L\, p_n(x) = c_n a_{n+1} p_{n+1}(x) - c_{n-1} a_n p_{n-1}(x). \tag{1-8}$$

Theorem 1.2. *Let D be a linear operator acting on* $\mathbb{R}[x]$ *that is symmetric with respect to the inner product (1-7). Assume further* $Dx^n = \lambda_n x^n +$lower order terms, *with* $\lambda_n \neq \lambda_{n-1}$ *for all* $n > 0$, *and that* $\{p_n(x)\}$ *are eigenfunctions of D with eigenvalues* $\{\lambda_n\}$, *that is,* $Dp_n(x) = \lambda_n p_n(x)$. *Then the operator* $L := [D, X]$ *is skew symmetric with respect to the inner product (1-7), where*

$$(Xf)(x) = x f(x). \tag{1-9}$$

Furthermore (1-8) holds with $c_n = \lambda_{n+1} - \lambda_n$.

Bangerezako [1999] proved that the Askey–Wilson polynomials have the structure relation when L is given by L_{AW}, defined by

$$(L_{AW} f)(z) := \frac{1}{z - 1/z}\left(\prod_{j=1}^{4}(1 - a_j z) z^{-2} f(qz) - \prod_{j=1}^{4}(1 - a_j/z) z^2 f(z/q)\right), \tag{1-10}$$

where a_1, a_2, a_3, a_4 are the parameters in the Askey–Wilson polynomials; see [Ismail 2005]. Here $x = (z + 1/z)/2$, and f is a Laurent polynomial assumed to be symmetric in z and $1/z$. Koornwinder also showed that L_{AW} can be computed.

The models for Koornwinder's results are the classical orthogonal polynomials, in which L and D are differential, difference, q-difference, or divided difference operators. Koornwinder applied his results to the Jacobi polynomials and the Askey–Wilson polynomials [Askey and Wilson 1985]. In all these models the operator D does not depend on the n in the equation $Dp_n = \lambda_n p_n$. There is also a connection to the work [Grünbaum and Haine 1996] on the bispectral problem.

The purpose of this paper is to extend Theorem 1.1 to general orthogonal polynomials, including the Freud polynomials when the weight function is e^{-v} with polynomial v. We have a different point of view, which we hope may shed more light on the subject. Theorem 1.2 requires that the orthogonal polynomials satisfy a Sturm–Liouville-type operator equation, a property only shared by the classical polynomial of the Askey scheme; see [Koekoek and Swarttouw 1999]. Our approach gives an alternative to the Sturm–Liouville property.

Section 2 gives the required preliminaries. Section 3 extends Koornwinder's results to polynomials orthogonal with respect to weight functions of the form $e^{-v(x)}$ with certain smoothness conditions on v. Section 4 extends the results of Bangerezako [1999] to the more general class of weight functions defined in (4-3). In Section 5 we treat the polynomials orthogonal with respect to a generalized Jacobi weight evolved under a Toda-type modification $\exp(-\sum_{j=1}^{m} t_j x^j)$.

We shall follow the notations and terminology of basic hypergeometric functions and orthogonal polynomials as in [Andrews et al. 1999; Gasper and Rahman 2004; Rainville 1960; Ismail 2005].

2. Preliminaries

When v' is continuous and $x^n e^{-v}[v'(x)-v'(y)]/(x-y)$ is integrable over $[a,b]$ for $n=0,1,2,\dots$, then $A_n(x)$ and $B_n(x)$ of (1-1) are given by

$$\frac{A_n(x)}{a_n} = \frac{w(y)p_n^2(y)}{y-x}\bigg|_a^b + \int_a^b \frac{v'(x)-v'(y)}{x-y} p_n^2(y)w(y)\,dy, \tag{2-1}$$

$$\frac{B_n(x)}{a_n} = \frac{w(y)p_n(y)p_{n-1}(y)}{y-x}\bigg|_a^b + \int_a^b \frac{v'(x)-v'(y)}{x-y} p_n(y)p_{n-1}(y)w(y)\,dy. \tag{2-2}$$

See [Bauldry 1990; Bonan and Clark 1990; Chen and Ismail 1997; Ismail 2005, Chapter 3]. It is assumed that the boundary terms in (2-1) and (2-2) exist.

Under the assumptions above on v, the orthonormal polynomials satisfy the differential equation

$$p_n''(x) + R_n(x)p_n'(x) + S_n(x)p_n(x) = 0, \tag{2-3}$$

where

$$R_n(x) := -\left(v'(x) + \frac{A_n'(x)}{A_n(x)}\right), \tag{2-4}$$

$$S_n(x) := A_n(x)\left(\frac{B_n(x)}{A_n(x)}\right)' - B_n(x)(v'(x)+B_n(x)) + A_n(x)A_{n-1}(x)\frac{a_n}{a_{n-1}}. \tag{2-5}$$

The differential equations (2-3) were used to determine the equilibrium position of the particles in a Coulomb gas model; see [Ismail 2000].

3. Differential operators and exponential weights

Let D_n denote the operator

$$D_n = \frac{1}{A_n(x)}\Big(\frac{d^2}{dx^2} + R_n(x)\frac{d}{dx} + S_n(x)\Big), \tag{3-1}$$

which acts on $\mathbb{C}[x]$.

Theorem 3.1. *Let*

$$H_n := \tfrac{1}{2}\,[D_n, X] = \frac{1}{A_n(x)}\frac{d}{dx} - \frac{v'(x)}{2A_n(x)} - \frac{A_n'(x)}{2A_n^2(x)}, \tag{3-2}$$

where X is the operator of multiplication by x as in (1-9). Then H_n is skew symmetric with respect to the inner product (1-7).

In the case of Jacobi polynomials,

$$\frac{A_n(x)}{a_n} = \frac{(\alpha+\beta+1+2n)}{1-x^2}, \tag{3-3}$$

and the recursion coefficients $\{a_n\}$ are given by

$$a_n = \frac{2}{\alpha+\beta+2n}\sqrt{\frac{n(\alpha+n)(\beta+n)(\alpha+\beta+n)}{(\alpha+\beta-1+2n)(\alpha+\beta+1+2n)}}. \tag{3-4}$$

Thus H_n is a constant multiple of

$$(1-x^2)\frac{d}{dx} - \tfrac{1}{2}(\alpha-\beta+x(\alpha+\beta+2)), \tag{3-5}$$

which is the operator identified in [Koornwinder 2007].

If v is convex and the boundary terms in (2-1) vanish, then $A_n(x) > 0$ for real x. In general, we have

$$[H_n, X]f(x) = \frac{1}{A_n(x)}f(x), \tag{3-6}$$

that is, $[H_n, X] = A_n(X)^{-1}$. Equation (3-6), called the string equation, is also part of the relations for the Zhedanov algebra; see [Granovskiĭ et al. 1992, (3.2)].

Now we explore a different choice for the skew symmetric L when v' is a rational function. Let

$$v'(x) = \frac{\phi(x)}{\psi(x)} \quad\text{and}\quad L_1 := \psi(x)\frac{d}{dx} + \tfrac{1}{2}(\psi'(x) - \phi(x)). \tag{3-7}$$

In the case of Jacobi polynomials, $\phi(x) = \alpha-\beta+x(\alpha+\beta)$ and $\psi(x) = 1-x^2$, and L_1 reduces to the operator in (3-5). In this generality it is clear that

$$[L_1, X] = \psi(X). \tag{3-8}$$

We shall return to this topic in Section 5 and treat the generalized Jacobi weights under a Toda flow.

4. Askey–Wilson-type polynomials

We follow the notation in [Ismail 2005] and write

$$x = \tfrac{1}{2}(z + 1/z), \tag{4-1}$$

$$f(x) = \breve{f}(z). \tag{4-2}$$

Let $p_n(x)$ be orthonormal with respect to a weight function of the form

$$w(x) = \breve{w}(z) = \frac{(z^2, 1/z^2; q)_\infty}{\rho(z)\rho(1/z)} \frac{2i}{z - 1/z} \quad \text{for } -1 < x < 1. \tag{4-3}$$

The orthogonality is on $[-1, 1]$, so z is on the unit circle, so we think of z as $e^{i\theta}$ and x as $\cos\theta$. We assume that the function $\rho(z)$ has no zeros in $q \le |z| \le 1/q$. Define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle f, g \rangle = \frac{1}{2\pi i} \int_{|z|=1} \breve{f}(z)\overline{\breve{g}(z)} \frac{(z^2, 1/z^2; q)_\infty}{\rho(z)\rho(1/z)} \frac{dz}{z}. \tag{4-4}$$

The functions f and g in (4-4) are assumed to have the properties

$$\breve{h}(z) = \breve{h}(1/z) \quad \text{and} \quad \overline{h(z)} = h(\bar{z}), \tag{4-5}$$

and are such that $\breve{h}$ is analytic in $q \le |z| \le 1/q$. Since f and g are functions of $\cos\theta$, the inner product has the alternate form

$$\langle f, g \rangle = \frac{1}{\pi} \int_0^\pi f(\cos\theta)\overline{g(\cos\theta)} \left| \frac{(e^{2i\theta}; q)_\infty}{\rho(e^{i\theta})} \right|^2 d\theta. \tag{4-6}$$

This inner product is a weighted version of the inner product in [Brown et al. 1996].

Theorem 4.1. *The operator*

$$(Lf)(x) = \frac{1}{z - 1/z} \left(\frac{\rho(z)}{\rho(qz)} z^{-2}\, \breve{f}(qz) - \frac{\rho(1/z)}{\rho(q/z)} z^2 \breve{f}(z/q) \right) \tag{4-7}$$

is skew symmetric with respect to the inner product (4-4).

Proof. It is clear that

$$\begin{aligned}\langle Lf, g \rangle = -\frac{1}{2\pi i} \int_{|z|=1} \breve{f}(qz)\breve{g}(1/z) \frac{(qz^2, 1/z^2; q)_\infty}{z^2 \rho(qz)\rho(1/z)}\, dz \\ + \frac{1}{2\pi i} \int_{|z|=1} \breve{f}(z/q)\breve{g}(1/z) z^2 \frac{(qz^2, 1/z^2; q)_\infty}{\rho(z)\rho(q/z)}\, dz.\end{aligned}$$

In the first and second integrals, let $z = q^{\mp 1}\zeta$ so the integral now is on $|\zeta| = q^{\pm 1}$. Then use the analyticity of $\breve{f}$ and $\breve{g}$ to replace the contour by $|\zeta| = 1$. The result is that

$$\langle Lf, g\rangle = -\frac{1}{2\pi i}\int_{|z|=1} \breve{f}(z)\breve{g}(q/z)z^2\frac{(qz^2, 1/z^2; q)_\infty}{\rho(z)\rho(q/z)}\,dz \\ + \frac{1}{2\pi i}\int_{|z|=1} \breve{f}(z/q)\breve{g}(1/z)\frac{(qz^2, 1/z^2; q)_\infty}{z^2\rho(qz)\rho(1/z)}\,dz,$$

which is clearly equal to $-\langle f, Lg\rangle$. □

Our next result gives a structure relation when $\rho(z)/\rho(qz)$ is a polynomial. Let

$$\rho(z)/\rho(qz) = \prod_{j=1}^{m}(1 - t_j z). \tag{4-8}$$

It is easy to see that

$$L\,T_n(x) = \sum_{j=1}^{m}(-1)^j\sigma_j(\boldsymbol{t})\big(q^n U_{j+n-3} - q^{-n}U_{n-j+1}\big),$$

where $\sigma_j\boldsymbol{t}$ for $1 \le j \le m$ are the elementary symmetric functions of $t_1, t_2, \dots, t_m$ and $\sigma_j\boldsymbol{t} := 1$. The polynomials $\{T_k\}$ and $\{U_k\}$ are the Chebyshev polynomials of the first and second kinds, respectively. Note that $U_{-1}(x) = 0$ and that $U_{-k}(x)$ has degree $k-1$ for $k > 1$. Thus L maps a polynomial of degree n to a polynomial of degree $n+m-3$. Let $m \ge 3$. Then Lp_n must be of the form

$$Lp_n(x) = \sum_{j=0}^{n+m-3} c_{n,j}\,p_j(x).$$

Now (4-6) implies $c_{n,j} = \langle Lp_n, p_j\rangle = -\langle p_n, Lp_j\rangle = 0$ if $j + m - 3 < n$. This establishes the following theorem.

Theorem 4.2. *When $m \ge 3$, we have the structure relation*

$$Lp_n(x) = \sum_{j=0\wedge(n-m+3)}^{n+m-3} c_{n,j}\,p_j(x). \tag{4-9}$$

Al-Salam and Chihara [1972] solved the problem of classifying all systems of orthogonal polynomials satisfying

$$\phi(x)p_n'(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \tag{4-10}$$

for all n, $n = 1, 2, \dots,$ where $\phi(x)$ is a polynomial of degree at most 2 and $\alpha_n, \beta_n, \gamma_n$ are constants. They showed the polynomials $\{p_n(x)\}$ must be Jacobi, Hermite or Laguerre polynomials or special cases of them. Askey asked how the p_n are characterized when (4-10) is replaced by

$$\phi(x)p_n'(x) = \sum_{j=-r}^{s} c_{n,j}\, p_{n+j}(x) \tag{4-11}$$

and $\phi(x)$ has degree $k > 2$. Maroni [1985; 1987], and Bonan, Lubinsky and Nevai [Bonan et al. 1987] showed that only orthogonal polynomials in this class satisfy (1-4), with $v'(x)$ a rational function.

One can raise similar questions about the operator L. Given a real polynomial $\phi(z)$ with $\phi(0) \neq 0$, we associate an operator

$$(L_\phi f)(x) = \frac{1}{z-1/z}\big(\phi(z)z^{-2}\breve{f}(qz) - \phi(1/z)z^2\breve{f}(z/q)\big). \tag{4-12}$$

There is no loss of generality in assuming $\phi(0) = 1$. We let the degree of ϕ be m and write ϕ in the form

$$\phi(x) = \prod_{j=1}^{m}(1-t_j z)(1-t_j/z). \tag{4-13}$$

We shall assume that $|t_j| < q$ for all j.

Conjecture 4.3. *Under the above assumptions on ϕ, if $\{p_n(x)\}$ is a system of orthogonal polynomials that satisfies*

$$(L_\phi f)(x) = \sum_{j=-r}^{s} c_{n,j}\, p_{n+j}(x) \quad \textit{for all } n, \tag{4-14}$$

then $p_n(x)$ is orthogonal with respect to the weight function w in (4-3), with

$$\rho(z) = \prod_{j=1}^{m}(t_j z; q)_\infty. \tag{4-15}$$

5. The Jacobi–Toda lattice

Adler and Van Moerbeke [1995], and Van Moerbeke [1994] studied the orthogonal polynomials arising from applying a Toda flow to Jacobi polynomials. The weight function is

$$w(x) = C(1-x)^\alpha(1+x)^\beta \exp\big(-\textstyle\sum_{j=1}^{m} t_j x^j\big), \tag{5-1}$$

where C is a normalization constant that makes $\int_{\mathbb{R}} w(x)dx = 1$. By writing $w(x) = e^{-v(x)}$, we now compute $A_n(x)$ and $B_n(x)$ of (2-1) and (2-2). In this case

$$\frac{v'(x)-v'(y)}{x-y} = \frac{\alpha}{(1-x)(1-y)} + \frac{\beta}{(1+x)(1+y)} + \sum_{j=2}^{m}\sum_{k=0}^{j-2} j t_j x^{j-2-k} y^k.$$

We set

$$\begin{aligned} &\xi_n := \int_{\mathbb{R}} \frac{p_n(y)}{1-y} w(y)dy, \qquad \eta_n := \int_{\mathbb{R}} \frac{p_n(y)}{1+y} w(y)dy \\ &x^k p_n(x) = \sum_{j=0}^{n+k} C_{k,n,j}\, p_j(x). \end{aligned} \tag{5-2}$$

Note that $C_{k,n,j} = 0$ if $k + j < n$ and that ξ_n and η_n are the values of the functions of the second kind at $x = \pm 1$. It is straightforward to see that

$$\int_{\mathbb{R}} \frac{p_n^2(y)}{1-y} w(y)dy = p_n(1)\xi_n \quad \text{and} \quad \int_{\mathbb{R}} \frac{p_n^2(y)}{1+y} w(y)dy = p_n(-1)\eta_n. \tag{5-3}$$

Therefore

$$\begin{aligned} \frac{A_n(x)}{a_n} &= \xi_n \frac{\alpha p_n(1)}{1-x} + \eta_n \frac{\beta p_n(-1)}{1+x} + \sum_{j=2}^{m}\sum_{k=0}^{j-2} j t_j x^{j-2-k} C_{k,n,n}, \\ \frac{B_n(x)}{a_n} &= \xi_n \frac{\alpha p_{n-1}(1)}{1-x} + \eta_n \frac{\beta p_{n-1}(-1)}{1+x} + \sum_{j=2}^{m}\sum_{k=0}^{j-2} j t_j x^{j-2-k} C_{k,n,n-1}. \end{aligned} \tag{5-4}$$

With these values for A_n and B_n, the structure relation (1-5) holds.

If we wish to write (1-5) in the form (4-10), then we rationalize A_n and B_n. The result is

$$\begin{aligned} (1-x^2)p_n'(x) = {} & a_n(\xi_n \alpha p_n(1)(1+x) + \eta_n \beta p_n(-1)(1-x))p_{n-1}(x) \\ & + a_n(\xi_n \alpha p_{n-1}(1)(1+x) + \eta_n \beta p_{n-1}(-1)(1-x))p_n(x) \\ & + a_n(1-x^2)\left(\sum_{j=2}^{m}\sum_{k=0}^{j-2} j t_j x^{j-2-k} C_{k,n,n}\right) p_{n-1}(x) \\ & + a_n(1-x^2)\left(\sum_{j=2}^{m}\sum_{k=0}^{j-2} j t_j x^{j-2-k} C_{k,n,n-1}\right) p_n(x). \end{aligned} \tag{5-5}$$

To reduce (5-5) to the form (4-10), we repeatedly use the three-term recurrence relation (1-2). The relevant skew symmetric operator in this case is

$$L := (1-x^2)\frac{d}{dx} + x(\alpha+\beta+1) + \alpha - \beta. \tag{5-6}$$

It is clear that $[L, X] = 1 - X^2$.

We next consider the generalized Jacobi weights

$$w(x; \boldsymbol{c}) = C \prod_{j=0}^{m} |x - c_j|^{\delta_j} \quad \text{for } x \in (-1, 1), \tag{5-7}$$

where $\boldsymbol{c}$ stands for $(c_0, c_1, \dots, c_m)$, $c_0 = 1$, $c_1 = -1$, and $c_j \in \mathbb{R} \setminus [-1, 1]$ for $1 < j \le m$. Usually we choose $\delta_0 = \alpha$ and $\delta_1 = \beta$. In this case,

$$\begin{aligned} \frac{A_n(x)}{a_n} &= \sum_{j=0}^{m} \frac{\delta_j p_n(c_j)}{x - c_j} \int_{-1}^{1} \frac{p_n(y)}{y - c_j}\, w(y) dy, \\ \frac{B_n(x)}{a_n} &= \sum_{j=0}^{m} \frac{\delta_j p_{n-1}(c_j)}{x - c_j} \int_{-1}^{1} \frac{p_n(y)}{y - c_j}\, w(y) dy. \end{aligned} \tag{5-8}$$

In the notation of Section 3,

$$v'(x) = \sum_{j=0}^{m} \frac{\delta_j}{x - c_j} = \frac{\phi(x)}{\psi(x)}, \tag{5-9}$$

$$\psi(x) = \prod_{j=0}^{m} (x - c_j) = (1 - x^2) \prod_{j=2}^{m} (x - c_j). \tag{5-10}$$

Moreover $[L, X] = \psi(X)$.

Finally we consider the Toda-evolved generalized Jacobi weights

$$w(x; \boldsymbol{c}, \boldsymbol{t}) = C \prod_{j=0}^{m} |x - c_j|^{\delta_j} \exp\bigl(-\textstyle\sum_{k=1}^{s} t_k x^k\bigr) \quad \text{for } x \in (-1, 1), \tag{5-11}$$

where $\boldsymbol{c}$ and $\boldsymbol{t}$ stand for $(c_0, c_1, \dots, c_m)$ and $(t_{1,2}, \dots, t_s)$. As before $c_0 = 1$, $c_1 = -1$, and $c_j \in \mathbb{R} \setminus [-1, 1]$ for $1 < j \le m$ and $t_k \in \mathbb{R}$ for $1 \le k \le s$. Set

$$c_{k,j,n} = \int_{-1}^{1} x^k p_n(x) p_j(x) w(x; \boldsymbol{c}, \boldsymbol{t})\, dx. \tag{5-12}$$

In this case, we have

$$\begin{aligned} \frac{A_n(x)}{a_n} &= \sum_{j=0}^{m} \frac{\delta_j p_n(c_j)}{x - c_j} \int_{-1}^{1} \frac{p_n(y)}{y - c_j}\, w(y) dy + \sum_{j=2}^{s} \sum_{k=0}^{j-2} j t_j x^{j-2-k} C_{k,n,n}, \\ \frac{B_n(x)}{a_n} &= \sum_{j=0}^{m} \frac{\delta_j p_{n-1}(c_j)}{x - c_j} \int_{-1}^{1} \frac{p_n(y)}{y - c_j}\, w(y) dy + \sum_{j=2}^{s} \sum_{k=0}^{j-2} j t_j x^{j-2-k} C_{k,n,n-1}. \end{aligned}$$

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