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A REMARK ON INSTABILITY OF HARMONIC MAPS BETWEEN SPHERES

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We give a new characterization of the identity map between spheres.

1. Introduction

Let k be an integer greater that two, and $\mathbb{S}^k$ be a k-dimensional unit Euclidean sphere. For a smooth map $u : \mathbb{S}^k \to \mathbb{S}^k$, the energy density $e(u)$ is the function $e(u) : \mathbb{S}^k \to \mathbb{R}$ defined by $e(u)(x) = |du(x)|^2$ $(x \in \mathbb{S}^k)$, where $|du(x)|$ is the Hilbert–Schmidt norm of a linear map $du(x) \in T_x^*\mathbb{S}^k \otimes T_{u(x)}\mathbb{S}^k$. The Dirichlet energy $\mathbf{E}(u)$ of a smooth map $u : \mathbb{S}^k \to \mathbb{S}^k$ is defined by

$$\mathbf{E}(u) = \frac{1}{2}\int_{\mathbb{S}^k} e(u)\, d\mu,$$

where μ is the canonical measure induced by the Riemannian metric of $\mathbb{S}^k$. We often regard u as a map from $\mathbb{S}^k$ to $(k+1)$-dimensional Euclidean space $\mathbb{R}^{k+1}$ and denote

$$u(x) = (u_1(x), \dots, u_{k+1}(x)).$$

A map u is called a harmonic map if it is a critical point of the functional $\mathbf{E}$. They necessarily satisfy the Euler–Lagrange equation

$$\Delta_{\mathbb{S}^k} u + |du|^2 u = 0 \quad \text{in } \mathbb{S}^k,$$

where $\Delta_{\mathbb{S}^k}$ is the Laplacian on $\mathbb{S}^k$. All harmonic maps are assumed to be smooth in this paper.

Let $u^{-1}T\mathbb{S}^k$ be the pull-back bundle of $T\mathbb{S}^k$ by u and $C^\infty(u^{-1}T\mathbb{S}^k)$ be the vector space of smooth sections of $u^{-1}T\mathbb{S}^k$. For $\phi \in C^\infty(u^{-1}T\mathbb{S}^k)$, we consider a one-parameter variation

$$u_t \in C^\infty((-\epsilon, \epsilon), C^\infty(\mathbb{S}^k, \mathbb{S}^k)), \quad \epsilon > 0,$$

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such that $u_0 = u$ and

$$\frac{\partial u_t}{\partial t}(x)\bigg|_{t=0} = \phi(x).$$

The second variation $\delta_u^2\mathbf{E}$ of $\mathbf{E}$ at u is a functional on $C^\infty(u^{-1}T\mathbb{S}^k)$ defined by

$$\delta^2\mathbf{E}_u(\phi) = \frac{d^2}{dt^2}\mathbf{E}(u_t)\bigg|_{t=0}$$

and has an expression

$$\delta_u^2\mathbf{E}(\phi) = \int_{\mathbb{S}^k} \{|\nabla'\phi|^2 - \langle \mathrm{Tr} R(\phi, du)du, \phi\rangle\}\, d\mu.$$

where ∇' is the induced connection on $C^\infty(u^{-1}T\mathbb{S}^k)$, R is a Riemannian curvature of $\mathbb{S}^k$ and Tr is a trace. We define the number $\lambda_1(u)$ by

$$\lambda_1(u) = \inf_{\substack{\phi\in C^\infty(u^{-1}T\mathbb{S}^k)\\ \phi\neq 0}} \frac{\delta_u^2\mathbf{E}(\phi)}{\|\phi\|^2_{L^2(\mathbb{S}^k)}}$$

where

$$\|\phi\|_{L^2(\mathbb{S}^k)} = \left(\int_{\mathbb{S}^k} |\phi|^2\, d\mu\right)^{1/2}.$$

By the result of Xin [1980], if u is not a constant harmonic map, then $\lambda_1(u) \le 2-k$. (On instability of harmonic maps, see also [Eells and Lemaire 1983], [Urakawa 1993] and [Xin 1996].) The purpose of this paper is to prove the following.

Theorem 1.1. *Suppose $k \ge 3$ and $u : \mathbb{S}^k \to \mathbb{S}^k$ is a harmonic map. If $\lambda_1(u) = 2-k$, then there exists a $(k+1)\times(k+1)$ orthogonal matrix R satisfying*

$$u(x) = Rx, \quad x \in \mathbb{S}^k.$$

Remark. For any nonconstant harmonic map $u : \mathbb{S}^2 \to \mathbb{S}^2$, the number $\lambda_1(u)$ equals 0, since they minimize the Dirichlet energy in their homotopy classes and the Dirichlet energy is conformal invariant in this case.

2. Proof of Theorem 1.1

We need the following lemma from [Ramanathan 1986]. (See also [Nakajima 2003].)

Theorem 2.1. *Let k be an integer greater than two. Suppose that $u \in C^\infty(\mathbb{S}^k, \mathbb{S}^k)$ is a harmonic map. If u is a conformal diffeomorphism, then u is an isometry. That is, there exits a $(k+1)\times(k+1)$ orthogonal matrix R satisfying*

$$u(x) = Rx, \quad x \in \mathbb{S}^k.$$

The next lemma is proved in [Xin 1996], but we include the proof here for the convenience of the reader.

Lemma 2.1. *Let $u : \mathbb{S}^k \to \mathbb{S}^k$ be a harmonic map. For any smooth function $f : \mathbb{S}^k \to \mathbb{R}$, we have*

$$\int_{\mathbb{S}^k} |\nabla f|^2 \, d\mu - \frac{k-2}{k} \int_{\mathbb{S}^k} |du|^2 |f(x)|^2 \, d\mu \geq \lambda_1(u) \int_{\mathbb{S}^k} |f|^2 \, d\mu$$

Proof. Let $\{\mathbf{e}_i\}_{i=1}^{k+1}$ be a canonical basis of $\mathbb{R}^{k+1}$. For any $1 \leq i \leq k+1$ we define $\mathbf{e}_i^\top \in C^\infty(u^{-1} T\mathbb{S}^k)$ by

$$\mathbf{e}_i^\top(x) = \mathbf{e}_i(x) - \langle \mathbf{e}_i, u(x) \rangle \, u(x),$$

where $\langle \ , \ \rangle$ is a Riemannian metric of $\mathbb{R}^{k+1}$. For $f \in C^\infty(\mathbb{S}^k)$, we put

$$\phi_i(x) = f(x)\mathbf{e}_i^\top(x).$$

By the definition of $\lambda_1(u)$, we have

$$\sum_{i=1}^{k+1} \delta_u^2 \mathbf{E}(\phi_i) \geq \lambda_1(u) \sum_{i=1}^{k+1} \|\phi_i\|_{L^2(\mathbb{S}^k)}^2. \tag{2-1}$$

Let $\{\sigma_\alpha\}_{\alpha=1}^k$ be an orthogonal basis of $T_x\mathbb{S}^k$. By a direct calculation, we have

$$\nabla'_{\sigma_\alpha} \mathbf{e}_i^\top = u_i \, du(\sigma_\alpha).$$

Therefore we have

$$\nabla'_{\sigma_\alpha} \phi_i = (\sigma_\alpha f)\mathbf{e}_i^\top + f u_i \, du(\sigma_\alpha).$$

Since $\sum_{i=1}^{k+1} \mathbf{e}_i^\top u_i = 0$, we have

$$\sum_{i=1}^{k+1} \sum_{\alpha=1}^{k} |\nabla'_{\sigma_\alpha} \phi_i|^2 = \sum_{i=1}^{k+1} \sum_{\alpha=1}^{k} \{ |(\sigma_\alpha f)\mathbf{e}_i^\top|^2 + |f u_i \, du(\sigma_\alpha)|^2 \} = k|\nabla f|^2 + |du|^2 |f|^2.$$

On the other hand

$$\sum_{i=1}^{k+1} |\phi_i|^2 = |f|^2 \sum_{i=1}^{k+1} (1 - u_i^2) = k|f|^2,$$

and we have

$$\begin{aligned}\sum_{i=1}^{k+1}\langle \mathrm{Tr}R(\phi_i, du)du, \phi_i\rangle &= \sum_{i=1}^{k+1}\sum_{\alpha=1}^{k}\left(|du(\sigma_\alpha)|^2|\phi_i|^2 - \langle du(\sigma_\alpha), \phi_i\rangle^2\right)\\ &= k|du|^2|f|^2 - \sum_{i=1}^{k+1}\sum_{\alpha=1}^{k}\langle du(\sigma_\alpha), f\mathbf{e}_i^{\top}\rangle^2\\ &= (k-1)|du|^2|f|^2.\end{aligned}$$

Thus the left hand side of (2-1) is

$$k\int_{\mathbb{S}^k}|\nabla f|^2\,d\mu - (k-2)\int_{\mathbb{S}^k}|du|^2|f|^2\,d\mu.$$

The right hand side of (2-1) is

$$k\lambda_1(u)\int_{\mathbb{S}^k}|f|^2\,d\mu$$

and we obtain the desired inequality. □

Let $H^1(\mathbb{S}^k)$ be the completion of $C^\infty(\mathbb{S}^k)$ with respect to the norm

$$\|f\|_{H^1(\mathbb{S}^k)} = \left(\int_{\mathbb{S}^k}|f|^2\,d\mu + \int_{\mathbb{S}^k}|\nabla f|^2\,d\mu\right)^{1/2}.$$

We define a functional $J : H^1(\mathbb{S}^k) \to \mathbb{R}$ by

$$J(f) = \int_{\mathbb{S}^k}|\nabla f|^2\,d\mu - \frac{k-2}{k}\int_{\mathbb{S}^k}|f|^2\,d\mu.$$

By Lemma 2.1,

$$J(f) \geq \lambda_1(u)\|f\|^2_{L^2(\mathbb{S}^k)} \quad \text{for any } f \in H^1(\mathbb{S}^k).$$

The next lemma was proved in [Lin and Wang 2006] and [Okayasu 1994]. See also [Nakajima 1989].

Lemma 2.2. *Let $u : \mathbb{S}^k \to \mathbb{S}^k$ be a harmonic map.*

$$|\nabla du|^2 \geq \frac{k}{k-1}\big|\nabla|du|\big|^2$$

Finally, we recall the Bochner formula of harmonic maps between spheres. For the proof, see [Eells and Lemaire 1983; Urakawa 1993; Xin 1996].

Lemma 2.3. *Let $u : \mathbb{S}^k \to \mathbb{S}^k$ be a harmonic map. Then*

$$\begin{aligned}&\tfrac{1}{2}\Delta(|du|^2)\\ &\qquad = |\nabla du|^2 + (k-1)|du|^2 - \sum_{\alpha,\beta=1}^{k}\bigl(|du(\sigma_\alpha)|^2|du(\sigma_\beta)|^2 - \langle du(\sigma_\alpha), du(\sigma_\beta)\rangle^2\bigr),\end{aligned}$$

where $\{\sigma_\alpha\}_{\alpha=1}^k$ is an orthonormal local frame on $\mathbb{S}^k$.

Proof of Theorem 1.1. We use the method in [Nakajima 2006]. Since $\lambda_1(u) = 2-k < 0$, u is not a constant map. Therefore $e(u)$ is not identically zero. For small $\epsilon > 0$, we define $e_\epsilon(u)$ by

$$e_\epsilon(u) = e(u) + \epsilon,$$

and set

$$\theta = \frac{k-2}{k-1} > 0.$$

By direct calculation, we have

$$\Delta(e_\epsilon(u)^\theta) = \theta(\theta-1)e_\epsilon(u)^{\theta-2}|\nabla e_\epsilon(u)|^2 + \theta e_\epsilon(u)^{\theta-1}\Delta e_\epsilon(u). \tag{2-2}$$

From Lemma 2.2 and Lemma 2.3 we have

$$\begin{aligned}\frac{1}{2}\Delta e(u) &= |\nabla du|^2 + (k-1)|du|^2 - |du|^4 + \sum_{\alpha,\beta=1}^{k}\langle du(\sigma_\alpha), du(\sigma_\beta)\rangle^2\\ &\geq \frac{k}{k-1}\bigl|\nabla|du|\bigr|^2 + (k-1)|du|^2 - |du|^4 + \sum_{\alpha=1}^{k}|du(\sigma_\alpha)|^4\\ &\geq \frac{k}{k-1}\bigl|\nabla|du|\bigr|^2 + (k-1)|du|^2 - \frac{k-1}{k}|du|^4.\end{aligned}\tag{2-3}$$

Here we used the elementary inequality

$$\sum_{\alpha,\beta=1}^{k}\langle du(\sigma_\alpha), du(\sigma_\beta)\rangle^2 \geq \sum_{\alpha=1}^{k}|du(\sigma_\alpha)|^4 \geq \frac{1}{k}\biggl(\sum_{\alpha=1}^{k}|du(\sigma_\alpha)|^2\biggr)^2 \tag{2-4}$$

Combining (2-2) and (2-3), we have

$$\begin{aligned}\Delta(e_\epsilon(u)^\theta) &\geq -\frac{4(k-2)}{(k-1)^2}e_\epsilon(u)^{\theta-2}|du|^2|\nabla|du||^2\\ &\qquad + \frac{2(k-2)}{k-1}e_\epsilon(u)^{\theta-1}\left(\frac{k}{k-1}|\nabla|du||^2 + (k-1)|du|^2 - \frac{k-1}{k}|du|^4\right)\\ &\geq 2\theta^2 e_\epsilon(u)^{\theta-2}|du|^2|\nabla|du||^2 + 2(k-2)e(u)^\theta - \frac{2(k-2)}{k}e_\epsilon(u)^\theta|du|^2\\ &= 2\Bigl(|\nabla(e_\epsilon(u)^{\theta/2})|^2 + (k-2)e(u)^\theta - \frac{k-2}{k}|du|^2 e_\epsilon(u)^\theta\Bigr).\end{aligned}$$

Integrating this on $\mathbb{S}^k$, we obtain

$$\int_{\mathbb{S}^k} |\nabla(e_\epsilon(u)^{\theta/2})|^2 \, d\mu - \frac{k-2}{k} \int_{\mathbb{S}^k} |du|^2 e_\epsilon(u)^\theta \, d\mu \leq (2-k) \int_{\mathbb{S}^k} e(u)^\theta \, d\mu.$$

In particular ,

$$\int_{\mathbb{S}^k} |\nabla(e_\epsilon(u)^{\theta/2})|^2 \, d\mu \leq \frac{k-2}{k} \int_{\mathbb{S}^k} |du|^2 e_\epsilon(u)^\theta \, d\mu.$$

From this inequality, we have

$$\int_{\mathbb{S}^k} |\nabla(e_\epsilon(u)^{\theta/2})|^2 \, d\mu \leq C,$$

where $C > 0$ is a constant independent of small $\epsilon > 0$. Since $e_\epsilon(u)$ converges to $e(u)$ at every point of $\mathbb{S}^k$, $e(u)$ belongs to $H^1(\mathbb{S}^k)$. By Fatou's lemma, we have

$$\int_{\mathbb{S}^k} |\nabla(e(u)^{\theta/2})|^2 \, d\mu \leq \liminf_{\epsilon \searrow 0} \int_{\mathbb{S}^k} |\nabla(e_\epsilon(u)^{\theta/2})|^2 \, d\mu.$$

Thus

$$\int_{\mathbb{S}^k} |\nabla(e(u)^{\theta/2})|^2 \, d\mu - \frac{k-2}{k} \int_{\mathbb{S}^k} |du|^2 e(u)^\theta \, d\mu \leq (2-k) \int_{\mathbb{S}^k} e(u)^\theta \, d\mu.$$

Since $\lambda_1(u) = 2-k$, this inequality must be an equality. Thus $e(u)^{\theta/2}$ is a minimizer of the functional J. This implies that $e(u)^{\theta/2}$ is a solution of the Euler–Lagrange equation of J, that is,

$$-\Delta e(u)^{\theta/2} - \frac{k-2}{k} |du|^2 e(u)^{\theta/2} = (2-k) e(u)^{\theta/2} \quad \text{in } \mathbb{S}^k.$$

If $e(u)^{\theta/2}$ has a zero point $p \in \mathbb{S}^k$, then for small $r > 0$, we have

$$\sup_{\mathbb{B}_r(p)} e(u)^{\theta/2} \leq C \inf_{\mathbb{B}_r(p)} e(u)^{\theta/2} = 0$$

by the Harnack inequality, where $\mathbb{B}_r(p)$ is an open geodesic ball in $\mathbb{S}^k$ of radius r centered at p, and $C > 0$ is a constant. Thus u is constant in $\mathbb{B}_r(p)$ and this implies that u is a constant map by a unique continuation theorem for harmonic maps [Sampson 1978]. This contradicts the fact that u is not a constant map, therefore $e(u)^{\theta/2}$ has no zero points. On the other hand, considering (2-4), we have

$$\begin{aligned} |du(\sigma_\alpha)| &= |du(\sigma_\beta)| \qquad && (1 \leq \alpha, \beta \leq k), \\ \langle du(\sigma_\alpha), du(\sigma_\beta) \rangle &= 0 && (\alpha \neq \beta). \end{aligned}$$

This implies that $du(x)$ is a conformal isomorphism between $T_x\mathbb{S}^k$ and $T_{u(x)}\mathbb{S}^k$ for any $x \in \mathbb{S}^k$, so u is a covering map by the inverse function theorem. Since $\mathbb{S}^k$ is simply connected, u is a diffeomorphism of $\mathbb{S}^k$. Therefore u must be a conformal diffeomorphism of $\mathbb{S}^k$.

We complete the proof using Theorem 2.1. □

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