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In a previous paper, we classified all Lie bialgebras structures of Hamiltonian type. In this paper, we give an explicit formula for the quantization of Hamiltonian-type Lie algebras.

1. Introduction

The quantization of Lie algebras is closely related to the construction of Lie bialgebras. Drinfel'd [1983] (or see [Giaquinto and Zhang 1998]) proved that the equivalence classes of quantization of Lie algebras L are in bijection with unitary solutions in $L \otimes L$ of the classical Yang–Baxter equation. However, it is very difficult to find the Drinfel'd element (whose definition we give below), so it is not surprising that few explicit formulas of quantization of Lie algebras are known.

Constructing quantizations of Lie algebras is an important approach to producing new Hopf algebras. In the theory of quantum groups and Hopf algebras, there are two standard methods for obtaining new bialgebras from old ones. One method twists the product by a 2-cocycle but keeps the coproduct unchanged; the other twists the coproduct by a Drinfel'd twist element but keeps the product unchanged.

A number of recent papers have studied the structure of infinite-dimensional Lie bialgebras. In [1994], Michaelis defined a class of infinite-dimensional Lie bialgebras containing the Virasoro algebras, and then showed how to construct the Lie bialgebra structure based on the Lie algebras L that contain two elements $a, b \in L$ such that $[a, b] = kb$ for $k \neq 0$. This type of Lie bialgebra was classified by Ng and Taft [2000], and then quantized by Grunspan [2004]. Lie bialgebra structures of generalized Witt type were classified by Song and Su [2006], and then quantized by Hu and Wang [2007]. Wu, Song, and Su [2006] defined Lie bialgebras of generalized Virasoro-like type, and the authors quantized these algebras in [Song et al. 2008]. Here, we will present the quantization of Hamiltonian-type Lie algebras, whose Lie bialgebra structures were classified by Xin, Song and Su [2007].

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2. Preliminaries

2.1. ***Hamiltonian-type Lie bialgebras.*** Let $\mathbb{F}$ be a field with characteristic zero, and let Γ be any nondegenerate additive subgroup of $\mathbb{F}^{2n}$ (so that Γ contains an $\mathbb{F}$-basis of $\mathbb{F}^{2n}$). Choose an $\mathbb{F}$-basis $\{\epsilon_p \mid 1 \le p \le 2n\} \subset \Gamma$ of $\mathbb{F}^{2n}$. Any element $\alpha \in \Gamma$ can be written as

$$\alpha = (\alpha_1, \alpha_{\bar{1}}, \alpha_2, \alpha_{\bar{2}}, \ldots, \alpha_n, \alpha_{\bar{n}}), \quad \text{with } \alpha_1, \alpha_{\bar{1}}, \ldots, \alpha_n, \alpha_{\bar{n}} \in \mathbb{F} \text{ and } \bar{p} = p + n.$$

Set $\Sigma_p = \Sigma_{\bar{p}} = \epsilon_p + \epsilon_{\bar{p}}$ for $1 \le p \le n$.

Let $\overline{\mathcal{H}} = \overline{\mathcal{H}}(2n, \Gamma) = \mathrm{span}\{x^\alpha \mid \alpha \in \Gamma\}$ be the group algebra with product given by $x^\alpha \cdot x^\beta = x^{\alpha+\beta}$. Define $[\,\cdot\,,\cdot\,] : \overline{\mathcal{H}} \times \overline{\mathcal{H}} \to \overline{\mathcal{H}}$ by

$$[x^\alpha, x^\beta] = \sum_{i=1}^{n} (\alpha_i \beta_{\bar{i}} - \alpha_{\bar{i}} \beta_i) x^{\alpha+\beta-\Sigma_i} \quad \text{for } \alpha, \beta \in \Gamma.$$

Then it can be shown [Xu 2000; Su and Xu 2004] that $(\overline{\mathcal{H}}, [\,\cdot\,,\cdot\,])$ is a Poisson algebra satisfying the compatibility condition that

$$[u, vw] = [u, v]w + v[u, w] \quad \text{for all } u, v, w \in \overline{\mathcal{H}}.$$

Let $\mathcal{H} = \overline{\mathcal{H}}/\mathbb{F}\cdot 1$, where $1 = x^0$. Then $\mathcal{H}$ is simple Lie algebra; see again [Xu 2000; Su and Xu 2004].

Define a linear map $\pi : \Gamma \to \mathbb{F}^n$ by

$$\pi(\alpha) = (\alpha_1 - \alpha_{\bar{1}}, \alpha_2 - \alpha_{\bar{2}}, \ldots, \alpha_n - \alpha_{\bar{n}}) \in \mathbb{F}^n \quad \text{for } \alpha \in \Gamma.$$

Let $G := \pi(\Gamma) = \{\pi(\alpha) \mid \alpha \in \Gamma\} \subset \mathbb{F}^n$. We will always denote an element μ of $\mathbb{F}^n$ by $\mu = (\mu_1, \mu_2 \cdots, \mu_n)$. Then $\mathcal{H}$ is a G-graded Lie algebra (but is not generally finitely-graded). That is, $\mathcal{H} = \oplus_\mu \mathcal{H}_\mu$, where $\mathcal{H}_\mu = \mathrm{span}\{t^\beta \mid \pi(\beta) = \mu\}$. It is easy to see that $[x^{\Sigma_p}, x^\alpha] = \pi(\alpha)_p x^\alpha$ for all $\alpha \in \Gamma$.

Lie bialgebras of Hamiltonian type are characterized as follows:

Theorem 2.2 [Xin et al. 2007]. *Every Lie bialgebra structure on $\mathcal{H}$ is coboundary triangular. That is, it is defined by the r-matrix $r - r^{12}$ with $r = H \otimes T$, where $H \in \mathrm{span}\{x^{\Sigma_p} \mid p = 1, 2, \ldots, gn\}$ and $T \in \mathcal{H}$.*

2.3. ***The quantization of*** $\mathcal{U}(\mathcal{H})$**.** The quantization of $\mathcal{U}(\mathcal{H})$ by the Drinfel'd twist is closely related to the construction of the Lie bialgebras. Before quantizing $\mathcal{U}(\mathcal{H})$, we present some notation.

Let A be a unital R-algebra, where R is a ring. For any elements $x \in A$, $a \in R$, and $n \in \mathbb{Z}$, we set

$$x_a^{\langle n \rangle} = (x+a)(x+a+1)\cdots(x+a+n-1),$$
$$x_a^{[n]} = (x+a)(x+a-1)\cdots(x+a-n+1).$$

We set $x^{\langle n\rangle}=x_0^{\langle n\rangle}$ and $x^{[n]}=x_0^{[n]}$.

Lemma 2.4 [Grunspan 2004; Giaquinto and Zhang 1998]. *Let* $\mathbb{F}$ *be a field with* char $\mathbb{F}=0$, *and let* x *be any element of a unital* $\mathbb{F}$*-algebra* A. *For* $a, d\in\mathbb{F}$ *and* $m, n, r\in\mathbb{Z}_+$, *one has*

$$x_a^{\langle m+n\rangle}=x_a^{\langle m\rangle}x_{m+a}^{\langle n\rangle},\quad x_a^{[m+n]}=x_a^{[m]}x_{a-m}^{[n]},\quad x_a^{[m]}=x_{a-m+1}^{\langle m\rangle},$$

$$\sum_{m+n=r}\frac{(-1)^n}{m!\,n!}x_a^{[m]}x_d^{\langle n\rangle}=\binom{a-d}{r}=\frac{(a-d)(a-d+1)\cdots(a-d-r+1)}{r!},\tag{2-1}$$

$$\sum_{m+n=r}\frac{(-1)^n}{m!\,n!}x_a^{[m]}x_{d-m}^{[n]}=\binom{a-d+r-1}{r}=\frac{(a-d)(a-d+1)\cdots(a-d+r-1)}{r!}.\tag{2-2}$$

Definition 2.5 [Drinfel'd 1983; 1987]. An element $\mathcal{F}\in H\otimes H$ is called a Drinfel'd twisting element if it is invertible and satisfies

$$(\mathcal{F}\otimes 1)(\Delta_0\otimes\mathrm{Id})(\mathcal{F})=(1\otimes\mathcal{F})(1\otimes\Delta_0)(\mathcal{F}),$$
$$(\epsilon_0\otimes\mathrm{Id})(\mathcal{F})=1\otimes 1=(\mathrm{Id}\otimes\epsilon_0)(\mathcal{F}).$$

Let $\mathbb{F}[[t]]$ be the ring of formal power series over the field with characteristic zero, and $\mathcal{H}$ the coboundary triangular Lie bialgebra relative to the r-matrix $r-r^{21}$. Let $(\mathcal{U}(\mathcal{H}), \mu, \tau, \Delta_0, \epsilon_0, S_0)$ be the standard Hopf algebra structure on $\mathcal{U}(\mathcal{H})$. The topologically free $\mathbb{F}[[t]]$-algebra $\mathcal{U}(\mathcal{H})[[t]]$ can be viewed as an associative $\mathbb{F}$-algebra of formal powers with coefficients in $\mathcal{U}(\mathcal{H})$; see [Etingof and Schiffmann 2002]. Then $\mathcal{U}(\mathcal{H})[[t]]$ has a Hopf algebra structure induced from $(\mathcal{U}(\mathcal{H}), \mu, \tau, \Delta_0, \epsilon_0, S_0)$ naturally, and we still denote it by $(\mathcal{U}(\mathcal{H})[[t]], \mu, \tau, \Delta_0, \epsilon_0, S_0)$.

Definition 2.6. Let $\mathcal{H}$ be a coboundary triangular Lie bialgebra over a field $\mathbb{F}$ with characteristic zero. We call $\mathcal{U}(\mathcal{H})[[t]]$ a quantization of $\mathcal{U}(\mathcal{H})$ by a Drinfel'd twisting element $\mathcal{F}\in\mathcal{U}(\mathcal{H})[[t]]\otimes\mathcal{U}(\mathcal{H})[[t]]$ if $\mathcal{U}(\mathcal{H})[[t]]/t\mathcal{U}(\mathcal{H})\simeq\mathcal{U}(\mathcal{H})$, where $\mathcal{F}$ is defined by the r-matrix related to its Lie bialgebra structure.

The following theorem is well known (see [Drinfel'd 1983; 1987]) and can be found in any book of Hopf algebras.

Theorem 2.7. *Let* $(H, \mu, \tau, \Delta_0, \epsilon_0, S_0)$ *be a Hopf algebra over commutative ring, and let* $\mathcal{F}$ *be a Drinfel'd element of* $H\otimes H$. *Then*

(1) $u=\mu(\mathrm{Id}\otimes S_0)(\mathcal{F})$ *is an invertible element of* $H\otimes H$ *with* $u^{-1}=\mu(S_0\otimes\mathrm{Id})(\mathcal{F})$;

(2) *the algebra* $(H, \mu, \tau, \Delta, \epsilon, S)$ *is a new Hopf algebra if we keep the counit undeformed and define* $\Delta: H\to H\otimes H$ *and* $S: H\to H$ *by*

$$\Delta(h)=\mathcal{F}\Delta_0(h)\mathcal{F}^{-1}\quad\textit{and}\quad S(h)=uS_0(h)u^{-1}.$$

Let $(\mathcal{U}(\mathcal{H}), \mu, \tau, \Delta_0, \epsilon_0, S_0)$ be the standard Hopf algebra of $\mathcal{U}(\mathcal{H})$, that is, for $\alpha \in \Gamma$,

$$\Delta_0(x^\alpha) = x^\alpha \otimes 1 + 1 \otimes x^\alpha, \quad \epsilon_0(x^\alpha) = 0, \quad S_0(x^\alpha) = -x^\alpha.$$

The main result of this paper is this:

Theorem 2.8. *Let $\mathcal{H}$ be the Hamiltonian algebra over $\mathbb{F}$ with* char $\mathbb{F} = 0$*. Choose $h = \pi(\alpha)_p^{-1} x^{\Sigma_p}$ and $x^\alpha \in \mathcal{H}$, with $\pi(\alpha)_p \neq 0$ and $[h, x^\alpha] = x^\alpha$. Then the quantization of $\mathcal{U}(\mathcal{H})$ corresponding to the Drinfel'd twist element*

$$\mathcal{F} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h^{[i]} \otimes (x^\alpha)^i t^i \quad \textit{is} \quad (\mathcal{U}(\mathcal{H})[[t]], \mu, \tau, \Delta, \epsilon, S),$$

where μ, τ and $\epsilon = \epsilon_0$ are unchanged from those in $(\mathcal{U}(\mathcal{H}), \mu, \tau, \Delta_0, \epsilon_0, S_0)$, but the coproduct Δ and the antipode S are given by

$$\Delta(x^\beta) = x^\beta \otimes (1 - x^\alpha t)^b + \sum_{l=0}^{\infty} (-1)^l h^{\langle l \rangle} \otimes (1 - x^\beta t)^{-l} \frac{1}{l!} \operatorname{ad}(x^\alpha)^l (x^\beta),$$

$$S(x^\beta) = -(1 - x^\alpha t)^{-b} \Big(\sum_{l=0}^{\infty} h_{-b}^{[l]} (\operatorname{ad}(x^\alpha))^l (x^\beta) t^l \Big),$$

where $b = \pi(\alpha)_p^{-1} \pi(\beta)_p$.

3. Proof of the main result

We divide the proof of Theorem 2.8 into several lemmas.

Lemma 3.1. *Let $\mathcal{H}$ be Hamiltonian Lie algebra defined as Section 2.1, and assume $x^{\Sigma_p}, x^\alpha \in \mathcal{H}$ and $[x^{\Sigma_p}, x^\alpha] = \pi(\alpha)_p x^\alpha$, with $\mu_p \neq 0$. Take $h = \pi(\alpha)_p^{-1} x^{\Sigma_p}$ for $a \in \mathbb{F}$. Let m and n be nonnegative integers. Then we have these equations in $\mathcal{U}(\mathcal{H})$:*

(3-1) $$x^\beta h_a^{[m]} = h^{[m]}_{a - \pi(\alpha)_p^{-1} \pi(\beta)_p} x^\beta,$$

(3-2) $$x^\beta h_a^{\langle m \rangle} = h^{\langle m \rangle}_{a - \pi(\alpha)_p^{-1} \pi(\beta)_p} x^\beta,$$

(3-3) $$(x^\alpha)^k h_a^{[m]} = h_{a-k}^{[m]} (x^\alpha)^k,$$

(3-4) $$(x^\alpha)^k h_a^{\langle m \rangle} = h_{a-k}^{\langle m \rangle} (x^\alpha)^k,$$

(3-5) $$x^\beta (x^\gamma)^m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x^\gamma)^{(m-i)} (\operatorname{ad}(x^\gamma)^i (x^\beta)).$$

Proof. Since

$$[x^{\Sigma_p}, x^\beta] = \pi(\beta)_p x^\beta = x^{\Sigma_p} x^\beta - x^\beta x^{\Sigma_p},$$

we have $x^\beta h = (h - \pi(\alpha)_p^{-1}\pi(\beta)_p)x^\beta$. This is (3-1) for $m = 1$. Now suppose that (3-1) is true for m. Then for $m+1$ we have

$$\begin{aligned}x^\beta h_a^{[m+1]} &= x^\beta h_a^{[m]}(h+a-m)\\ &= h^{[m]}_{a-\pi(\alpha)_p^{-1}\pi(\beta)_p} x^\beta (h+a-m)\\ &= h^{[m]}_{a-\pi(\alpha)_p^{-1}\pi(\beta)_p}(h+a-\pi(\alpha)_p^{-1}\pi(\beta)_p - m) = h^{[m+1]}_{a-\pi(\alpha)_p^{-1}\pi(\beta)_p}.\end{aligned}$$

By induction on m, we see that (3-1) holds. Similarly, we can obtain (3-2)–(3-4). Equation (3-5) is a general result of associative algebra. □

Now for $a \in \mathbb{F}$, set

$$\mathcal{F}_a = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h_a^{[i]} \otimes (x^\alpha)^i t^i, \qquad F_a = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h_a^{\langle i\rangle} \otimes (x^\alpha)^i t^i,$$

$$u_a = \mu \cdot (S_0 \otimes \mathrm{Id})(F_a), \qquad v_a = \mu \cdot (\mathrm{Id} \otimes S_0)(\mathcal{F}_a).$$

Write $\mathcal{F} = \mathcal{F}_0$, $F = F_0$, $u = u_0$ and $v = v_0$. Since $S_0(h_a^{\langle i\rangle}) = (-1)^i h_{-a}^{[i]}$ and $S_0((x^\alpha)^i) = (-1)^i (x^\alpha)^i$, we have

$$u_a = \mu(S_0 \otimes \mathrm{Id})\Big(\sum_{i=0}^{\infty} \frac{1}{i!} h_a^{\langle i\rangle} \otimes (x^\alpha)^i t^i\Big) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h_{-a}^{[i]} (x^\alpha)^i t^i,$$

$$v_a = \mu(\mathrm{Id} \otimes S_0)\Big(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h_a^{[i]} \otimes (x^\alpha)^i t^i\Big) = \sum_{i=0}^{\infty} \frac{1}{i!} h_a^{[i]} (x^\alpha)^i t^i.$$

Lemma 3.2. *For $a, d \in \mathbb{F}$, one has*

$$\mathcal{F}_a F_d = 1 \otimes (1 - x^\alpha t)^{(a-d)} \quad \textit{and} \quad v_a u_d = (1 - x^\alpha t)^{-(a+d)}.$$

Thus the elements $\mathcal{F}_a$, F_a and u_a, v_a are invertible with $\mathcal{F}_a^{-1} = F_a$ and $u_a^{-1} = v_{-a}$.

Proof. Using the formula (2-1) we have

$$\begin{aligned}\mathcal{F}_a F_d &= \Big(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h_a^{[i]} \otimes (x^\alpha)^i t^i\Big) \cdot \Big(\sum_{j=0}^{\infty} \frac{1}{j!} h_d^{\langle j\rangle} \otimes (x^\alpha)^j t^j\Big)\\ &= \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!\,j!} h_a^{[i]} h_d^{\langle j\rangle} \otimes (x^\alpha)^i (x^\alpha)^j t^{i+j}\\ &= \sum_{m=0}^{\infty} \Big(\sum_{i+j=m} (-1)^m \frac{(-1)^j}{i!\,j!} h_a^{[i]} h_d^{\langle j\rangle}\Big) \otimes (x^\alpha)^m t^m\\ &= \sum_{m=0}^{\infty} (-1)^m \binom{a-d}{m} (x^\alpha)^m t^m = 1 \otimes (1 - x^\alpha t)^{a-d}.\end{aligned}$$

Using (2-2) and (3-3) we have

$$\begin{aligned}
v_a u_d &= \Big(\sum_{i=0}^{\infty} \frac{1}{i!} h_a^{[i]} (x^\alpha)^i t^i\Big) \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} h_{-d}^{[j]} (x^\alpha)^j t^j \\
&= \sum_{i,j=0}^{\infty} \frac{1}{i!} h_a^{[i]} (x^\alpha)^i t^i \frac{(-1)^j}{j!} h_{-d}^{[j]} (x^\alpha)^j t^j \\
&= \sum_{m=0}^{\infty} \sum_{i+j=m} \frac{(-1)^j}{i!\,j!} h_a^{[i]} h_{-d-i}^{[j]} (x^\alpha)^{i+j} t^{i+j} \\
&= \sum_{m=0}^{\infty} \binom{a+d+m-1}{m} (x^\alpha)^m t^m = (1-x^\alpha t)^{-(a+d)}. \qquad \square
\end{aligned}$$

Lemma 3.3. *For any nonnegative integer m and any $a \in \mathbb{F}$, we have*

$$\Delta_0 h^{[m]} = \sum_{i=0}^{m} \binom{m}{i} h_{-a}^{[i]} \otimes h_a^{[m-i]}.$$

In particular, $\Delta_0 h^{[m]} = \sum_{i=0}^{m} \binom{m}{i} h^{[i]} \otimes h^{[m-i]}$.

Proof. Since $\Delta_0(h) = h \otimes 1 + 1 \otimes h$, it is easy to see that the result is true for $m = 1$. Suppose it is true for m. Then for $m+1$, we have

$$\begin{aligned}
\Delta_0(h^{[m+1]}) &= \Delta_0(h^{[m]}) \Delta_0(h-m) \\
&= \Big(\sum_{i=0}^{m} \binom{m}{i} h_{-a}^{[i]} \otimes h_a^{[m-i]}\Big) \big((h-a-m) \otimes 1 + 1 \otimes (h+a-m) + m(1 \otimes 1)\big) \\
&= \Big(\sum_{i=1}^{m-1} \binom{m}{i} h_{-a}^{[i]} \otimes h_a^{[m-i]}\Big) \big((h-a-m) \otimes 1 + 1 \otimes (h+a-m)\big) \\
&\qquad + m\Big(\sum_{i=0}^{m} \binom{m}{i} h_{-a}^{[i]} \otimes h_a^{[m-i]}\Big) + (1 \otimes h_a^{[m+1]} + h_{-a}^{[m+1]} \otimes 1) \\
&\qquad\qquad + (h-a-m) \otimes h_a^{[m]} + h_{-a}^{[m]} \otimes (h+a-m) \\
&= 1 \otimes h_a^{[m+1]} + h_{-a}^{[m+1]} \otimes 1 + m\Big(\sum_{i=1}^{m-1} \binom{m}{i} h_{-a}^{[i]} \otimes h_a^{[m-i]}\Big) \\
&\qquad\qquad + (h-a) \otimes h_a^{[m]} + h_{-a}^{[m]} \otimes (h+a) \\
&\qquad + \sum_{i=1}^{m-1} \binom{m}{i} h_{-a}^{[i+1]} \otimes h_a^{[m-i]} + \sum_{i=1}^{m-1} (i-m) \binom{m}{i} h_{-a}^{[i]} \otimes h_a^{[m-i]} \\
&\qquad + \sum_{i=1}^{m-1} \binom{m}{i} h_{-a}^{[i]} \otimes h_a^{[m-i+1]} + \sum_{i=1}^{m-1} (-i) \binom{m}{i} h_{-a}^{[i]} \otimes h_a^{[m-i]}
\end{aligned}$$

$$= 1 \otimes h_a^{[m+1]} + h_{-a}^{[m+1]} \otimes 1 + \sum_{i=1}^{m} \left(\binom{m}{i-1} + \binom{m}{i} \right) h_{-a}^{[i]} \otimes h_a^{[m+1-i]}$$
$$= \sum_{i=0}^{m+1} \binom{m+1}{i} h_{-a}^{[i]} \otimes h_a^{[m+1-i]}.$$

By induction, the result holds for arbitrary m. □

Lemma 3.4. *The element*

$$\mathcal{F} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h^{[i]} \otimes (x^\alpha)^i t^i$$

is a Drinfel'd twist element of $\mathcal{U}(\mathcal{L}(\Gamma))[[t]]$, *that is,*

$$(\mathcal{F} \otimes 1)(\Delta_0 \otimes \mathrm{Id})(\mathcal{F}) = (1 \otimes \mathcal{F})(1 \otimes \Delta_0)(\mathcal{F}),$$
$$(\epsilon_0 \otimes \mathrm{Id})(\mathcal{F}) = 1 \otimes 1 = (\mathrm{Id} \otimes \epsilon_0)(\mathcal{F}).$$

Proof. The second equality obviously holds, so we just need to prove the first. First,

$$(\mathcal{F} \otimes 1)(\Delta_0 \otimes \mathrm{Id})(\mathcal{F})$$
$$= \Big(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h^{[i]} \otimes (x^\alpha)^i t^i \otimes 1 \Big) (\Delta_0 \otimes \mathrm{Id}) \Big(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} h^{[j]} \otimes (x^\alpha)^j t^j \Big)$$
$$= \Big(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} h^{[i]} \otimes (x^\alpha)^i t^i \otimes 1 \Big) \Big(\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=0}^{j} \binom{j}{k} h_{-i}^{[k]} \otimes h_i^{[j-k]} \otimes (x^\alpha)^j t^j \Big)$$
$$= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!\, j!} t^{i+j} \sum_{k=0}^{j} \binom{j}{k} h^{[i]} h_{-i}^{[k]} \otimes (x^\alpha)^i h_i^{[j-k]} \otimes (x^\alpha)^j$$
$$= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!\, j!} t^{i+j} \sum_{k=0}^{j} \binom{j}{k} h^{[i+k]} \otimes h^{[j-k]} (x^\alpha)^i \otimes (x^\alpha)^j.$$

On the other hand,

$$(1 \otimes \mathcal{F})(\mathrm{Id} \otimes \Delta_0)(\mathcal{F})$$
$$= \Big(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} t^r 1 \otimes h^{[r]} \otimes (x^\alpha)^r \Big) \Big(\sum_{s=0}^{\infty} \frac{(-1)^s}{s!} t^s h^{[s]} \otimes \sum_{q=0}^{s} \binom{s}{q} (x^\alpha)^q \otimes (x^\alpha)^{s-q} \Big)$$
$$= \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s}}{r!\, s!} t^{r+s} \sum_{q=0}^{s} \binom{s}{q} h^{[s]} \otimes h^{[r]} (x^\alpha)^q \otimes (x^\alpha)^{r+s-q}.$$

So it is sufficient to show for a fixed m that

$$\sum_{i+j=m}\frac{1}{i!\,j!}t^{i+j}\sum_{k=0}^{j}\binom{j}{k}h^{[i+k]}\otimes h^{[j-k]}(x^\alpha)^i\otimes(x^\alpha)^j = \sum_{r+s=m}\frac{1}{r!\,s!}t^{r+s}\sum_{q=0}^{s}\binom{s}{q}h^{[s]}\otimes h^{[r]}(x^\alpha)^q\otimes(x^\alpha)^{r+s-q}.$$

Now, fix r and suppose $0\le i\le s$. Set $i=q$ and $i+k=s$. Then we have $j-k=j-(s-i)=i+j-s=m-s=r$. We also see that the coefficients of $h^{[s]}\otimes h^{[r]}(x^\alpha)^q\otimes(x^\alpha)^{m-q}$ in both are equal. So the result holds. □

Lemma 3.5. *For $a\in\mathbb{F}$, $\beta\in\Gamma$, we have*

$$(x^\beta\otimes 1)F_a = F_{a-b}(x^\beta\otimes 1), \tag{3-6}$$

$$(1\otimes x^\beta)F_a = \sum_{l=0}^{\infty}(-1)^l F_{a+l}(h_a^{\langle l\rangle}\otimes(1/l!)\,\mathrm{ad}(x^\alpha)^l(x^\beta)t^l), \tag{3-7}$$

$$x^\beta u_a = u_{a+b}\sum_{l=0}^{\infty}h_{-a-b}^{[l]}(\mathrm{ad}(x^\alpha))^l t^l,\quad \textit{where } b=\pi_p(\alpha)^{-1}\pi_p(\beta). \tag{3-8}$$

Proof. From (3-2) we have

$$(x^\beta\otimes 1)F_a = (x^\beta\otimes 1)\sum_{i=0}^{\infty}\frac{1}{i!}h_a^{\langle i\rangle}\otimes(x^\alpha)^i t^i = \sum_{i=0}^{\infty}\frac{1}{i!}x^\beta h_a^{\langle i\rangle}\otimes(x^\alpha)^i t^i = \sum_{i=0}^{\infty}\frac{1}{i!}h_{a-b}^{\langle i\rangle}x^\beta\otimes(x^\alpha)^i t^i = F_{a-b}(x^\beta\otimes 1),$$

where $b=\pi(\alpha)_p^{-1}\pi(\beta)_p$. This proves (3-6).

For (3-7), using (3-5) we have

$$\begin{aligned}(1\otimes x^\beta)F_a &= (1\otimes x^\beta)\sum_{i=0}^{\infty}\frac{1}{i!}h_a^{\langle i\rangle}\otimes(x^\alpha)^i t^i = \sum_{i=0}^{\infty}\frac{1}{i!}h_a^{\langle i\rangle}\otimes x^\beta(x^\alpha)^i t^i\\
&= \sum_{i=0}^{\infty}\frac{1}{i!}h_a^{\langle i\rangle}\otimes\sum_{l=0}^{i}(-1)^l\binom{i}{l}(x^\alpha)^{i-l}\,\mathrm{ad}(x^\alpha)^l(x^\beta)t^i\\
&= \sum_{i=0}^{\infty}\Big(\sum_{l=0}^{i}(-1)^l\frac{1}{(i-l)!\,l!}h_a^{\langle i\rangle}\otimes(x^\alpha)^{i-l}\,\mathrm{ad}(x^\alpha)^l(x^\beta)t^i\\
&= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}(-1)^l\frac{1}{i!\,l!}h_a^{\langle i+l\rangle}\otimes(x^\alpha)^i\,\mathrm{ad}(x^\alpha)^l(x^\beta)t^{i+l}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty}(-1)^l \sum_{i=0}^{\infty}\Big(\frac{1}{i!}h_{a+l}^{\langle i\rangle}\otimes (x^\alpha)^i t^i\Big)\frac{1}{l!}(h_a^{\langle l\rangle}\otimes \operatorname{ad}(x^\alpha)^l(x^\beta))t^l \\
&= \sum_{l=0}^{\infty}(-1)^l F_{a+l}\Big(h_a^{\langle l\rangle}\otimes \frac{1}{l!}\operatorname{ad}(x^\alpha)^l(x^\beta)t^l\Big).
\end{aligned}
$$

So (3-7) is right. Now we prove (3-8) by the calculation

$$
\begin{aligned}
x^\beta u_a &= x^\beta \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h_{-a}^{[r]}(x^\alpha)^r t^r = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}x^\beta h_{-a}^{[r]}(x^\alpha)^r t^r \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h_{-a-b}^{[r]}x^\beta (x^\alpha)^r t^r \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h_{-a-b}^{[r]}\sum_{l=0}^{r}(-1)^l\binom{r}{l}(x^\alpha)^{r-l}(\operatorname{ad}(x^\alpha))^l(x^\beta)t^r \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}h_{-a-b}^{[r]}\sum_{l=0}^{r}(-1)^l\frac{r!}{(r-l)!\,l!}(x^\alpha)^{r-l}(\operatorname{ad}(x^\alpha))^l(x^\beta)t^r \\
&= \sum_{r,l=0}^{\infty}\frac{(-1)^r}{r!\,l!}h_{-a-b}^{[r+l]}(x^\alpha)^r(\operatorname{ad}(x^\alpha))^l(x^\beta)t^{r+l} \\
&= \sum_{r,l=0}^{\infty}\frac{(-1)^r}{r!\,l!}h_{-a-b}^{[r]}h_{-a-b-r}^{[l]}(x^\alpha)^r(\operatorname{ad}(x^\alpha))^l(x^\beta)t^{r+l} \\
&= \sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\Big(\frac{(-1)^r}{r!}h_{-a-b}^{[r]}(x^\alpha)^r t^r\Big)h_{-a-b}^{[l]}(\operatorname{ad}(x^\alpha))^l(x^\beta)t^l \\
&= u_{a+b}\sum_{l=0}^{\infty}h_{-a-b}^{[l]}(\operatorname{ad}(x^\alpha))^l t^l. \qquad \square
\end{aligned}
$$

Now we complete the proof of Theorem 2.8. For arbitrary elements $x^\beta \in \mathcal{H}(\Gamma)$, we have

$$
\begin{aligned}
\Delta(x^\beta) &= \mathcal{F}\Delta_0(x^\beta)\mathcal{F}^{-1} = \mathcal{F}(x^\beta\otimes 1)\mathcal{F}^{-1} + \mathcal{F}(1\otimes x^\beta)\mathcal{F}^{-1} \\
&= \mathcal{F}(x^\beta\otimes 1)F + \mathcal{F}(1\otimes x^\beta)F \\
&= \mathcal{F}F_{-b}(x^\beta\otimes 1) + \mathcal{F}\sum_{l=0}^{\infty}(-1)^l F_l(h^{\langle l\rangle}\otimes (1/l!)\operatorname{ad}(x^\alpha)^l(x^\beta)
\end{aligned}
$$

$$
\begin{aligned}
&= (1\otimes(1-x^\alpha t)^b)(x^\beta\otimes 1)\\
&\qquad\qquad + \sum_{l=0}^{\infty}(-1)^l(1\otimes(1-x^\alpha t)^{-l})\otimes(h^{\langle l\rangle}\otimes(1/l!)\,\mathrm{ad}(x^\alpha)^l(x^\beta))\\
&= x^\beta\otimes(1-x^\alpha t)^b + \sum_{l=0}^{\infty}(-1)^l h^{\langle l\rangle}\otimes(1-x^\beta t)^{-l}(1/l!)\,\mathrm{ad}(x^\alpha)^l(x^\beta)
\end{aligned}
$$

and

$$
\begin{aligned}
S(x^\beta) &= u^{-1}S_0(x^\beta)u = -\upsilon x^\beta u\\
&= -\upsilon u_b\Big(\sum_{l=0}^{\infty} h_{-b}^{[l]}(\mathrm{ad}(x^\alpha))^l t^l\Big)\\
&= -(1-x^\alpha t)^{-b}\Big(\sum_{l=0}^{\infty} h_{-b}^{[l]}(\mathrm{ad}(x^\alpha))^l t^l\Big), \quad \text{where } b=\pi(\alpha)_p^{-1}\pi(\beta)_p. \qquad \square
\end{aligned}
$$

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