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BORING SPLIT LINKS

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Boring is an operation that converts a knot or two-component link in a 3-manifold into another knot or two-component link. It generalizes rational tangle replacement and can be described as a type of 2-handle attachment. Sutured manifold theory is used to study the existence of essential spheres and planar surfaces in the exteriors of knots and links obtained by boring a split link. It is shown, for example, that if the boring operation is complicated enough, a split link or unknot cannot be obtained by boring a split link. Particular attention is paid to rational tangle replacement. If a knot is obtained by rational tangle replacement on a split link, and a few minor conditions are satisfied, the number of boundary components of a meridional planar surface is bounded below by a number depending on the distance of the rational tangle replacement. This result is used to give new proofs of two results of Eudave-Muñoz and Scharlemann's band sum theorem.

1. Introduction

Refilling and boring. Given a genus 2 handlebody W embedded in a 3-manifold M , a knot or two component link can be created by choosing an essential disc $\alpha \subset W$ and boundary-reducing W along α . Then $W - \mathring{\eta}(\alpha)$ is the regular neighborhood of a knot or link L_α . We say that the exterior $M[\alpha]$ of this regular neighborhood is obtained by *refilling* the meridian disc α . Similarly, given a knot or link $L_\alpha \subset M$, we can obtain another knot or link L_β by the following process:

- (1) Attach an arc to L_α forming a graph.
- (2) Thicken the graph to form a genus 2 handlebody W .
- (3) Choose a meridian β for W and refill β .

Refilling the meridian α of the attached arc returns L_α . Any two knots in S^3 can be related by such a move if we allow α and β to be disjoint: Just let W be a neighborhood of the wedge of the two knots. Therefore we'll restrict attention to

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meridians of W that cannot be isotoped to be disjoint. If a knot or link L_β can be obtained from L_α by this operation, we say L_β is obtained by *boring* L_α . Since the relation is symmetric we may also say L_α and L_β are *related by boring*.

Boring generalizes several well-known operations in knot theory. Band sums, crossing changes, generalized crossing changes, and, more generally, rational tangle replacement can all be realized as boring. The band move from the Kirby calculus [Fenn and Rourke 1979; Kirby 1978] is also a type of boring. If W is the standard genus 2 handlebody in S^3 and L_α is the unlink of two components, then all tunnel number 1 knots can be obtained by boring L_α using W .

If L_α and L_β are related by boring, it is natural to ask under what circumstances both links can be split, both the unknot, both composite, and so on. Many of these questions have been effectively addressed for special types of boring, such as rational tangle replacement [Eudave-Muñoz 1988]. This paper, following [Scharlemann 2008], will focus on the exteriors $M[\alpha]$ and $M[\beta]$ of the knots L_α and L_β , respectively. There, Scharlemann conjectured that, with certain restrictions (discussed in Section 6), if $M[\alpha]$ and $M[\beta]$ are both reducible or boundary reducible, then either W is an unknotted handlebody in S^3 or α and β are positioned in a particularly nice way in W . He was able to prove his conjecture (with slightly varying hypotheses and conclusions) when $M - \mathring{W}$ is boundary reducible, when $|\alpha \cap \beta| \leq 2$, or when one of the discs is separating.

This paper looks again at these questions and completes, under stronger hypotheses, the proof of Scharlemann’s conjecture except when $M = S^3$ and $M[\alpha]$ and $M[\beta]$ are solid tori. With these stronger hypotheses, however, we reach conclusions that are stronger than those obtained in [Scharlemann 2008]. Even in the one situation that is not completed, we do gain significant insight. The remaining case is finally completed in [Taylor 2008]. Here is a simplified version of one of the main theorems:

Simplified Theorem 6.1. *Suppose that M is S^3 or the exterior of a link in S^3 and that $M - \mathring{W}$ is irreducible and boundary irreducible. If α and β cannot be isotoped to be disjoint, then at least one of $M[\alpha]$ or $M[\beta]$ is irreducible. Also, if one is boundary reducible (for example, a solid torus), then the other is not reducible.*

The conclusions of Theorem 6.1 are an “arc version” of the conclusions of the main theorem of [Scharlemann 1990], which considers surgeries on knots producing reducible 3-manifolds. The methods of this paper are similar in outline to those of [Scharlemann 1990] but differ in detail.

Perhaps the most interesting application of these techniques to rational tangle replacement is the following theorem, which generalizes some results of Eudave-Muñoz and Scharlemann:

Theorem 7.3. *Suppose that L_β is a knot or link in S^3 and that $B' \subset S^3$ is a 3-ball intersecting L_β so that $(B', B' \cap L_\beta)$ is a rational tangle. Let (B', r_α) be another rational tangle of distance $d \geq 1$ from $r_\beta = B' \cap L_\beta$, and let L_α be the knot obtained by replacing r_β with r_α . Let $(B, \tau) = (S^3 - \mathring{B}', L_\beta - \mathring{B}')$. Suppose that L_α is a split link and that (B, τ) is prime. Then L_β is not a split link or unknot. Furthermore, if L_β has an essential properly embedded meridional planar surface with m boundary components, it contains such a surface \bar{Q} with $|\partial \bar{Q}| \leq m$ such that either*

$$\bar{Q} \subset B \quad \text{or} \quad |\bar{Q} \cap \partial B|(d-1) \leq |\partial \bar{Q}| - 2.$$

One consequence of this is a new proof of Scharlemann's band sum theorem: If the unknot is obtained by attaching a band to a split link, then the band sum is the connected sum of unknots. This and other rational tangle replacement theorems are proved in [Section 7](#).

The main tool in this paper is Scharlemann's combinatorial version of Gabai's sutured manifold theory. The relationship of this paper to [\[Scharlemann 2008\]](#), where he states his conjecture about refilling meridians, is similar to the relationship between Gabai's and Scharlemann's proofs of the band sum theorem. Earlier, Scharlemann [\[1985\]](#) proved that the band sum of two knots is unknotted only if it is the connect sum of two unknots. Later Scharlemann and Gabai simultaneously and independently proved that

$$\text{genus}(K_1 \#_b K_2) \geq \text{genus}(K_1) + \text{genus}(K_2),$$

where $\#_b$ denotes a band sum. Gabai [\[1987\]](#) used sutured manifold theory to give a particularly simple proof. Scharlemann's proof [\[1989\]](#) uses a completely combinatorial version of sutured manifold theory. Since rational tangle replacement is a special type of boring, a similar relationship also holds between this paper and some of Eudave-Muñoz's extensions [\[1988\]](#) of the original band sum theorem. The techniques of this paper can be specialized to rational tangle replacement to recapture and extend some, but not all, of his results. In [\[2008\]](#), Scharlemann suggests that sutured manifold theory might contribute to a solution to his conjecture. This paper vindicates that idea.

The paper [\[Taylor 2008\]](#) uses sutured manifold theory in a different way. The two approaches are often useful in different circumstances. For example, the one used in this paper is more effective for studying the existence of certain reducing spheres in a manifold obtained by refilling meridians and for studying non-separating surfaces that are not homologous to a surface with interior disjoint from W . The approach of [\[Taylor 2008\]](#) is more effective for studying essential discs and separating surfaces.

Notation. We work in the PL or smooth categories. All manifolds and surfaces will be compact and orientable, except where indicated. $|A|$ denotes the number of components of A . If A and B are embedded curves on a surface, $|A \cap B|$ will generally be assumed to be minimal among all curves isotopic to A and B . For a subcomplex $B \subset A$, we denote by $\eta(B)$ a closed regular neighborhood of B in A . Both \mathring{B} and $\text{int } B$ denote the interior of B , and $\text{cl}(B)$ denotes the closure of B . ∂B denotes the boundary of B . All homology groups have \mathbb{Z} (integer) coefficients.

2. Sutured manifold theory

We begin by reviewing a few relevant concepts from combinatorial sutured manifold theory [Scharlemann 1989].

2.1. Definitions. A *sutured manifold* is a triple (N, γ, ψ) , where N is a compact, orientable 3-manifold, γ is a collection of oriented simple closed curves on ∂N , and ψ is a properly embedded 1-complex. $T(\gamma)$ denotes a collection of torus components of ∂N . The curves γ divide $\partial N - T(\gamma)$ into two surfaces that intersect along γ . Removing $\mathring{\eta}(\gamma)$ from these surfaces creates the surfaces $R_+(\gamma)$ and $R_-(\gamma)$. Let $A(\gamma) = \eta(\gamma)$.

For an orientable, connected surface $S \subset N$ in general position with respect to ψ , we define

$$\chi_\psi(S) = \max\{0, |S \cap \psi| - \chi(S)\}.$$

If S is disconnected, $\chi_\psi(S)$ is the sum of $\chi_\psi(S_i)$ for each component S_i . For a class $[S] \in H_2(N, X)$, we define $\chi_\psi([S])$ be the minimum of $\chi_\psi(S)$ over all embedded surfaces S representing $[S]$. If $\psi = \emptyset$, then $\chi_\psi(\cdot)$ is the Thurston norm.

Of utmost importance is the notion of ψ -tautness for both surfaces in a sutured manifold (N, γ, ψ) and for a sutured manifold itself. Let S be a properly embedded surface in N .

- S is ψ -minimizing in $H_2(N, \partial S)$ if $\chi_\psi(S) = \chi_\psi[S, \partial S]$.
- S is ψ -incompressible if $S - \psi$ is incompressible in $N - \psi$.
- S is ψ -taut if it is ψ -incompressible, ψ -minimizing in $H_2(N, \partial S)$, and each edge of ψ intersects S with the same sign. If $\psi = \emptyset$, we say either that S is \emptyset -taut or that S is taut in the Thurston norm.

A sutured manifold (N, γ, ψ) is ψ -taut if

- $\partial \psi$ (that is, valence one vertices) is disjoint from $A(\gamma) \cup T(\gamma)$;
- $T(\gamma)$, $R_+(\gamma)$, and $R_-(\gamma)$ are all ψ -taut; and
- $N - \psi$ is irreducible.

The final important notion is the concept of a conditioned surface. A *conditioned surface* $S \subset N$ is an oriented properly embedded surface such that

- if T is a component of $T(\gamma)$, then $\partial S \cap T$ consists of coherently oriented parallel circles;
- if A is a component of $A(\gamma)$, then $S \cap A$ consists of either circles parallel to γ and oriented the same direction as γ or arcs all oriented in the same direction;
- no set of simple closed curves of $\partial S \cap R(\gamma)$ is trivial in $H_1(R(\gamma), \partial R(\gamma))$;
- each edge of ψ that intersects $S \cup R(\gamma)$ does so always with the same sign.

Conditioned surfaces, along with product discs and annuli, are the surfaces along which a taut sutured manifold is decomposed to form a taut sutured manifold hierarchy. A hierarchy can be taken to be “adapted” to a parameterizing surface, that is, a surface $Q \subset N - \mathring{\eta}(\psi)$ no component of which is a disc disjoint from $\gamma \cup \eta(\psi)$. The index of a parameterizing surface is a certain number associated to Q that does not decrease as Q is modified during the hierarchy.

2.2. Satellite knots have property P. It will be helpful to review the essentials of the proof of [Scharlemann 1989, Theorem 9.1], where it is shown that satellite knots have property P.

In that theorem, which considers a 3-manifold N with ∂N a torus, it is assumed that $H_1(N)$ is torsion-free and that $k \subset N$ is a knot in N such that (N, \emptyset) is a k -taut sutured manifold. Suppose that some nontrivial surgery on k creates a manifold that has a boundary-reducing disc \bar{Q} and still has torsion-free first homology. The main goal is to show that (N, \emptyset) is \emptyset -taut. The surface $Q = \bar{Q} - \mathring{\eta}(k)$ acts as a parameterizing surface for a k -taut sutured manifold hierarchy of N . At the end of the hierarchy, there is at least one component containing pieces of k . A combinatorial argument using the assumption that $H_1(N)$ is torsion-free shows that, in fact, the last stage of the hierarchy is \emptyset -taut. Sutured manifold theory then shows that the original manifold N is \emptyset -taut, as desired. This argument is extended in [Scharlemann 1990] to study surgeries on knots in 3-manifolds that produce reducible 3-manifolds. In that paper, the surface \bar{Q} can be either a ∂ -reducing disc or a reducing sphere.

This paper extends these techniques in two other directions. First, we use an arc $\bar{\alpha} \subset M[\alpha]$ in place of the knot $k \subset N$. Second, we develop criteria that allow the surface $\bar{Q} \subset M[\beta]$ to be any of a variety of surfaces, including essential spheres and discs. Section 5 shows how to construct a useful surface \bar{Q} . Section 4 discusses the placement of sutures on $\partial M[\alpha]$. This allows theorems about sutured manifolds to be phrased without reference to sutured manifold terminology. Section 6 applies the sutured manifold results in order to (partially) answer Scharlemann’s conjecture about refilling meridians of genus 2 handlebodies. Section 7 uses the technology to reprove three classical theorems about rational tangle replacement and prove a new theorem about essential meridional surfaces in the exterior of a knot or link

obtained by boring a split link. Finally, [Section 8](#) shows how the sutured manifold theory results of this paper can significantly simplify certain combinatorial arguments.

3. Attaching a 2-handle

Let N be a compact orientable 3-manifold containing a component $F \subset \partial N$ of genus at least two. Let $a \subset F$ be an essential closed curve and let $\mathcal{B} = \{b_1, \dots, b_{|\mathcal{B}|}\}$ be a collection of disjoint, pairwise nonparallel essential closed curves in F isotoped so as to intersect a minimally. Suppose that $\gamma \subset \partial N$ is a collection of simple closed curves, disjoint from a , such that $(N, \gamma \cup a)$ is a taut sutured manifold and γ intersects the curves of \mathcal{B} minimally. Let $\Delta_i = |b_i \cap a|$ and $\nu_i = |b_i \cap \gamma|$.

Suppose that $Q \subset N$ is a surface with q_i boundary components parallel to the curve b_i , for each $1 \leq i \leq |\mathcal{B}|$. Let $\partial_0 Q$ be the components of ∂Q that are not parallel to any b_i . Assume that ∂Q intersects $\gamma \cup a$ minimally. Define $\Delta_\partial = |\partial_0 Q \cap a|$ and $\nu_\partial = |\partial_0 Q \cap \gamma|$. We need two definitions. The first defines a specific type of boundary compression and the second (as we shall see) is related to the notion of ‘‘Scharlemann cycle’’.

Definition. An *a*-boundary compressing disc for Q is a boundary compressing disc D for which $\partial D \cap F$ is a subarc of some essential circle in $\eta(a)$.

Definition. An *a*-torsion $2g$ -gon is a disc $D \subset N$ with $\partial D \subset F \cup Q$ consisting of $2g$ arcs labeled around ∂D as $\delta_1, \epsilon_1, \dots, \delta_g, \epsilon_g$. The labels are chosen so that $\partial D \cap Q = \bigcup \delta_i$ and $\partial D \cap F = \bigcup \epsilon_i$. We require that each ϵ_i arc is a subarc of some essential simple closed curve in $\eta(a)$ and that the ϵ_i arcs are all mutually parallel as oriented arcs in $F - \partial Q$. Furthermore we require that attaching to Q a rectangle in $F - \partial Q$ containing all the ϵ_i arcs produces an orientable surface.

Example. [Figure 1](#) shows a hypothetical example. The surface outlined with dashed lines is Q . It has boundary components on F . There are two such boundary components pictured. The curve running through Q and F could be the boundary of an *a*-torsion 4-gon. Notice that the arcs ϵ_1 and ϵ_2 are parallel and oriented in the same direction. Attaching the rectangle containing those arcs as two of its edges to Q produces an orientable surface.

Remark. Notice that an *a*-torsion 2-gon is an *a*-boundary compressing disc.

If 2-handles are attached to each curve b_i for $1 \leq i \leq |\mathcal{B}|$, a 3-manifold $N[\mathcal{B}]$ is obtained. Each component of $\partial Q - \partial_0 Q$ bounds a disc in $N[\mathcal{B}]$. Let \overline{Q} be the result of attaching a disc to each component of $\partial Q - \partial_0 Q$. Then $\partial \overline{Q} = \partial_0 Q$. We will usually also attach 3-balls to spherical components of $\partial N[\mathcal{B}]$. Throughout the paper, if a 2-handle $\beta \times I$ is attached to a curve b , the cocore of the 2-handle will be denoted $\overline{\beta}$. Thus, the notations $\overline{\alpha}$, $\overline{\beta}$, and $\overline{\beta}^*$ all refer to arcs in certain 3-manifolds.

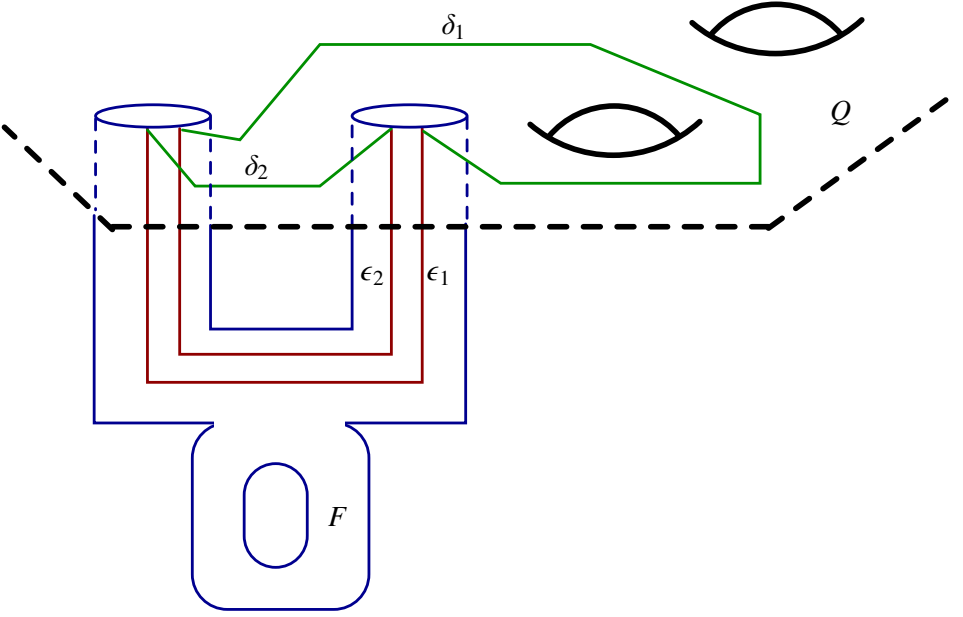


Figure 1. The boundary of an a -torsion 4-gon.

Remark. The term “ a -torsion $2g$ -gon” is chosen because in certain cases (but not all) the presence of an a -torsion $2g$ -gon with $g \geq 2$ guarantees that $N[\mathcal{B}]$ has torsion in its first homology.

Define

$$K(\overline{Q}) = \sum_{i=1}^{|\mathcal{B}|} q_i (\Delta_i - 2) + \Delta_{\partial} - v_{\partial}.$$

We will use the surface Q to study the effects of attaching a 2-handle $\alpha \times I$ to a regular neighborhood of the curve $a \subset F$. Let $N[a]$ denote the resulting 3-manifold. Perform the attachment so that the 2-disc α has boundary a . Let $\bar{\alpha}$ denote the arc that is the cocore of the 2-handle $\alpha \times I$.

We can now state our main sutured manifold theory result. It is an adaptation of [Scharlemann 1989, Theorem 9.1]; see also [Scharlemann 1990, Proposition 4.1].

Main Theorem. *Suppose that $(N[a], \gamma)$ is $\bar{\alpha}$ -taut, that Q is incompressible, and that Q contains no disc or sphere component disjoint from $\gamma \cup a$. Suppose that one of the following holds:*

- $N[a]$ is not \emptyset -taut.
- There is a conditioned $\bar{\alpha}$ -taut surface $S \subset N[a]$ that is not \emptyset -taut.
- $N[a]$ is homeomorphic to a solid torus $S^1 \times D^2$ and $\bar{\alpha}$ cannot be isotoped so that its projection to the S^1 factor is monotonic.

Then at least one of the following holds:

- There is an a -torsion $2g$ -gon for Q for some $g \in \mathbb{N}$.
- $H_1(N[a])$ contains nontrivial torsion.
- $-2\chi(\overline{Q}) \geq K(\overline{Q})$.

Remark. If \overline{a} can be isotoped to be monotonic in the solid torus $N[a]$ then it is, informally, a “braided arc”. The contrapositive of this aspect of the theorem is similar to the conclusion in [Gabai 1989] and [Scharlemann 1990] that if a nontrivial surgery on a knot with nonzero wrapping number in a solid torus produces a solid torus, then the knot is a 0 or 1-bridge braid.

The rest of this section proves the theorem. Following [Scharlemann 1990], define a *Gabai disc* for Q to be an embedded disc $D \subset N[a]$ such that

- $|\overline{a} \cap \mathring{D}| > 0$ and all points of intersection have the same sign of intersection,
- $|Q \cap \partial D| < |\partial Q \cap \eta(a)|$.

The next proposition points out that the existence of a Gabai disc guarantees the existence of an a -boundary compressing disc or an a -torsion $2g$ -gon.

Proposition 3.1. *If there is a Gabai disc for Q , then there is an a -torsion $2g$ -gon.*

Proof. Let D be a Gabai disc for Q . The intersection of Q with D produces a graph Λ on D . The vertices of Λ are ∂D and the points $\overline{a} \cap D$. The latter are called the *interior vertices* of Λ . The edges of Λ are the arcs $Q \cap D$. A *loop* is an edge in Λ with initial and terminal points at the same vertex. A loop is *trivial* if it bounds a disc in D with interior disjoint from Λ .

To show that there is an a -torsion $2g$ -gon for Q , we will show that the graph Λ contains a “Scharlemann cycle” of length g . The interior of the Scharlemann cycle will be the a -torsion $2g$ -gon. In our situation, Scharlemann cycles will arise from a labeling of Λ that is slightly nonstandard. Traditionally, when \overline{a} is a knot instead of an arc, the labels on the endpoints of edges in Λ , which are used to define “Scharlemann cycles”, are exactly the components of ∂Q . In our case, since each component of ∂Q likely intersects a more than once, we need to use a slightly different labeling. After defining the labeling and the revised notion of “Scharlemann cycle”, it will be clear to those familiar with the traditional situation that the new Scharlemann cycles give rise to the same types of topological conclusions as in the traditional setting. The discussion is modeled on [Culler et al. 1987, Section 2.6].

A Scharlemann cycle of length 1 is defined to be a trivial loop at an interior vertex of Λ bounding a disc with interior disjoint from Λ . We now work toward a definition of Scharlemann cycles of length $g > 1$. Without loss of generality, we may assume that $|\overline{a} \cap D| \geq 2$. Recall that the arc \overline{a} always intersects the disc D with the same sign. There is, in F , a regular neighborhood A of a such

that $D \cap F \subset A$. We may choose A so that $\partial A \subset D \cap F$. Let $\partial_{\pm} A$ be the two boundary components of A . The boundary components of Q all have orientations arising from the orientation of \bar{Q} . We may assume by an isotopy that all the arcs $\partial Q \cap A$ are fibers in the product structure on A . Cyclically around A , label the arcs of $\partial Q \cap A$ with labels c_1, \dots, c_{μ} . Let \mathcal{C} be the set of labels. Being a submanifold of ∂Q , each arc is oriented. Say that two arcs are *parallel* if they run through A in the same direction (that is, both from $\partial_- A$ to $\partial_+ A$ or both from $\partial_+ A$ to $\partial_- A$). Call two arcs *antiparallel* if they run through A in opposite directions. Note that since the orientations of $\dot{D} \cap F$ in A are all the same, an arc intersects each component of $\dot{D} \cap F$ with the same algebraic sign.

Call an edge of Λ with at least one endpoint on ∂D a *boundary edge*, and call all other edges *interior edges*. As each edge of Λ is an arc and as all vertices of Λ are parallel oriented curves on ∂W , an edge of Λ must have endpoints on arcs of $\mathcal{C} = \{c_1, \dots, c_{\mu}\}$ that are antiparallel. As in [Culler et al. 1987], we call this the *parity principle*. Label each endpoint of an edge in Λ with the arc in \mathcal{C} on which the endpoint lies.

We will occasionally orient an edge e of Λ ; in which case, let $\partial_- e$ be the tail and $\partial_+ e$ the head. A *cycle* in Λ is a subgraph homeomorphic to a circle. An x -cycle is a cycle which, when each edge e in the cycle is given a consistent orientation, has $\partial_- e$ labeled with $x \in \mathcal{C}$. Let Λ' be a subgraph of Λ , and let x be a label in \mathcal{C} . We say that Λ' satisfies condition $P(x)$ if, for each vertex v of Λ' , there exists an edge of Λ' incident to v with label x connecting v to an interior vertex.

Lemma 3.2 [Culler et al. 1987, Lemma 2.6.1]. *Suppose that Λ' satisfies $P(x)$. Then each component of Λ' contains an x -cycle.*

Proof. The proof is the same as in [Culler et al. 1987]. □

A *Scharlemann cycle* is an x -cycle σ where the interior of the disc in D bounded by σ is disjoint from Λ . See Figure 2. Since each intersection point of $D \cap \bar{\alpha}$ has the same sign, the set of labels on a Scharlemann cycle contains x and precisely one other label y , a component of \mathcal{C} adjacent to x in A . The arc y and the arc x are antiparallel by the parity principle. The *length* of the Scharlemann cycle is the number of edges in the x -cycle.

Lemma 3.3 [Culler et al. 1987, Lemma 2.6.2]. *If Λ contains an x -cycle, then (possibly after a trivial 2-surgery on D), Λ contains a Scharlemann cycle.*

Proof. The proof is again the same as in [Culler et al. 1987]. It uses the assumption that Q is incompressible to eliminate circles of intersection on the interior of an innermost x -cycle. □

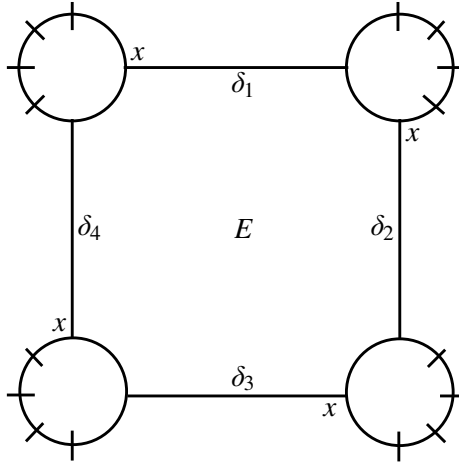


Figure 2. A Scharlemann cycle of length 4 bounding an a -torsion 8-gon.

Remark. The presence of any such disc D with Λ containing a Scharlemann cycle is good enough for our purposes. So, henceforth, we assume that all circles in Λ have been eliminated using the incompressibility of Q .

Remark. In [Culler et al. 1987], there is a distinction between x -cycles and so-called great x -cycles. We do not need this here because all components of $D \cap F$ are parallel in $\eta(\partial\alpha)$ as oriented curves.

The next corollary explains the necessity of considering Scharlemann cycles.

Corollary 3.4 [Culler et al. 1987]. *If ∂D intersects fewer than $|\partial Q \cap A|$ edges of Λ , then Λ contains a Scharlemann cycle.*

Proof. As ∂D contains fewer than $|\partial Q \cap A|$ endpoints of boundary edges in Λ , there is some $x \in \mathcal{C}$ that does not appear as a label on a boundary edge. As every interior vertex of Λ contains an edge with label x at that vertex, none of those edges can be a boundary edge. Consequently, Λ satisfies $P(x)$. Hence, by Lemmas 3.2 and 3.3, Λ contains a Scharlemann cycle of length g (for some g). \square

In A there is a rectangle R with boundary consisting of the arcs x and y and subarcs of ∂A . See Figure 3. Because $\bar{\alpha}$ always intersects D with the same sign, ∂D always crosses R in the same direction. This shows that the arcs ϵ_i are all mutually parallel in F . The arcs x and y are antiparallel, so attaching R to Q produces an orientable surface. Hence, the interior of the Scharlemann cycle is an a -torsion $2g$ -gon. \square

We now proceed with proving the contrapositive of the theorem. Suppose that none of the three possible conclusions of the theorem hold.

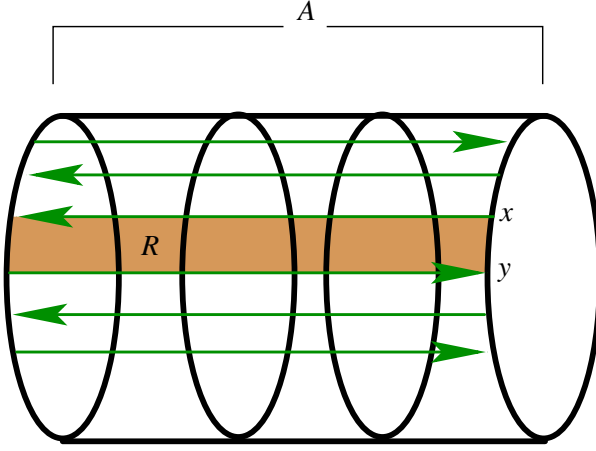


Figure 3. The rectangle R .

First, Q is a parameterizing surface for the $\bar{\alpha}$ -taut sutured manifold $(N[a], \gamma)$. Let

$$(N[a], \gamma) = (N_0, \gamma_0) \xrightarrow{S_1} (N_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (N_n, \gamma_n)$$

be an $\bar{\alpha}$ -taut sutured manifold hierarchy for $(N[a], \gamma)$ that is adapted to Q . The surface S_1 may be obtained from the surface S by performing the double-curve sum of S with k copies of R_+ and l copies of R_- [Scharlemann 1989, Theorem 2.6].

The index $I(Q_i)$ is defined to be

$$I(Q_i) = |\partial Q_i \cap \partial \eta(\bar{\alpha}_i)| + |\partial Q_i \cap \gamma_i| - 2\chi(Q_i),$$

where Q_i is the parameterizing surface in N_i and $\bar{\alpha}_i$ is the remnant of $\bar{\alpha}$ in N_i . Since $-2\chi(\bar{Q}) < K(\bar{Q})$, simple arithmetic shows that $I(Q) < 2|\partial Q \cap \eta(a)|$. Since there is no a -torsion $2g$ -gon for Q , by the previous proposition, there is no Gabai disc for Q . The proof of [Scharlemann 1989, Theorem 9.1] shows that (N_n, γ_n) is also \emptyset -taut, after substituting the assumption that there are no Gabai discs for Q in N wherever [Scharlemann 1989, Lemma 9.3] was used, as in [Scharlemann 1990, Proposition 4.1]. To prove 3, 4, and 11 of [Scharlemann 1989, Theorem 9.1], use the inequality $I(Q) < 2|\partial Q \cap A|$ to derive a contradiction rather than the inequalities stated in the proofs of those claims.

Hence, the hierarchy is \emptyset -taut, $(N[a], \gamma)$ is a \emptyset -taut sutured manifold, and S_1 is a \emptyset -taut surface. Suppose that S is not \emptyset -taut. Then there is a surface S' with the same boundary as S but with smaller Thurston norm. Then the double-curve sum of S' with k copies of R_+ and l copies of R_- has smaller Thurston norm than S_1 , showing that S_1 is not \emptyset -taut. Hence, S is \emptyset -taut.

The proof of [Scharlemann 1989, Theorem 9.1] concludes by noting that at the final stage of the hierarchy, there is a canceling or (nonself) amalgamating disc for each remnant of $\bar{\alpha}$. When $N[a]$ is a solid torus the only \emptyset -taut conditioned surfaces are unions of discs. If S is chosen to be a single disc, then S_1 is isotopic to S . To see this, notice that R_{\pm} is an annulus and so the double-curve sum of S with R_{\pm} is isotopic to S . Hence, the hierarchy has length one and the cancelling and (nonself) amalgamating discs show that $\bar{\alpha}$ is braided in $N[a]$. \square

Remark. The proof proves more than the theorem states. It is actually shown that at the end of the hierarchy, $\bar{\alpha} \cap N_n$ consists of unknotted arcs in 3-balls. This may be useful in future work.

For this theorem to be useful, we need to discuss the placement of sutures γ on ∂N and the construction of a surface Q without a -torsion $2g$ -gons. The next two sections address these issues. In each of them, we restrict F to being a genus 2 surface.

4. Placing sutures

Let N be a compact, orientable, irreducible 3-manifold with $F \subset \partial N$ a component containing an essential simple closed curve a . Suppose that $\partial N - F$ is incompressible in N . For effective application of the [main theorem](#), we need to choose curves γ on $\partial N[a]$ so that $(N[a], \gamma)$ is $\bar{\alpha}$ -taut. With our applications in mind, we restrict our attention to the situation when the boundary component F containing a has genus 2. Define $\partial_1 N[a] = \partial N - F$ and $\partial_0 N[a] = \partial N[a] - \partial_1 N[a]$.

For the moment, we consider only the choice of sutures $\hat{\gamma}$ on $\partial_0 N[a]$. If a is separating, so that $\partial_0 N[a]$ consists of two tori joined by the arc $\bar{\alpha}$, we do not place any sutures on $\partial_0 N[a]$, that is, $\hat{\gamma} = \emptyset$. See [Figure 4A](#). If a is nonseparating, choose $\hat{\gamma}$ to be a pair of disjoint parallel loops on $F - \eta(a)$ that separate the endpoints of $\bar{\alpha}$. See [Figure 4B](#).

If we are in the special situation of “refilling meridians”, we will want to choose the curves $\hat{\gamma}$ more carefully. Recall that in this case $N \subset M$ and F bounds a genus 2 handlebody $W \subset (M - \mathring{N})$. The curves a and b bound in W discs α and β respectively.

Assuming that the discs β and α have been isotoped to intersect minimally and nontrivially, the intersection $\alpha \cap \beta$ is a collection of arcs. An arc of $\alpha \cap \beta$ that is outermost on β cobounds with a subarc ψ of b a disc with interior disjoint from α . This disc is a meridional disc of a (solid torus) component of $W - \mathring{\eta}(a)$. The arc ψ has both endpoints on the same component of $\partial \eta(a) \subset F$. We therefore define a *meridional arc* of $b - a$ to be any arc of $b - \mathring{\eta}(a)$ that together with an arc in $\partial \eta(a) \cap \mathring{W}$ bounds a meridional disc of $W - \mathring{\eta}(a)$. If a is nonseparating, then the existence of meridional arcs shows that every arc of $b - \mathring{\eta}(a)$ with endpoints

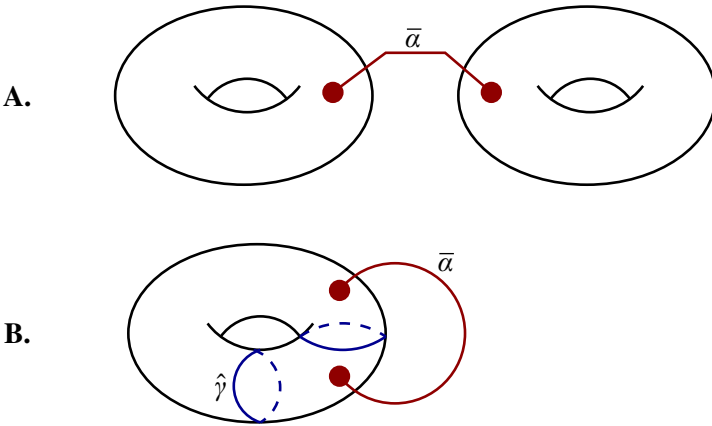


Figure 4. Choosing $\hat{\gamma}$.

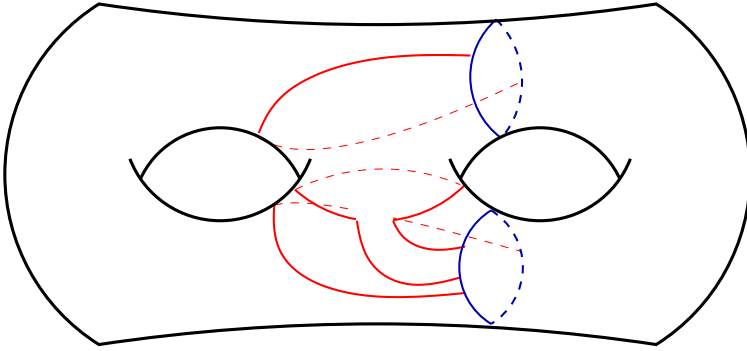


Figure 5. Some meridional arcs on ∂W .

on the same component of $\partial\eta(a) \subset F$ is a meridional arc of $b - a$. An easy counting argument shows that if a is nonseparating, then there are equal numbers of meridional arcs of $b - a$ based at each component of $\partial\eta(a) \subset F$. Hence, when a is nonseparating, the number of meridional arcs of $b - a$, denoted $\mathcal{M}_a(b)$, is even. Some meridional arcs are depicted in Figure 5.

Returning to the definition of the sutures $\hat{\gamma}$, we insist that when “refilling meridians” and when α is nonseparating, the curves $\hat{\gamma}$ be meridional curves of the solid torus $W - \hat{\eta}(\alpha)$ that separate the endpoints of $\bar{\alpha}$ and that are disjoint from the meridional arcs of $b - a$ for a specified b .

We now show how to define sutures $\tilde{\gamma}$ on nontorus components of $\partial_1 N[a]$. Let $T(\gamma)$ be all the torus components of $\partial_1 N[a]$. If $\partial_1 N = T(\gamma)$, then $\tilde{\gamma} = \emptyset$. Otherwise, the next lemma demonstrates how to choose $\tilde{\gamma}$ so that, under certain hypotheses, $(N, \gamma \cup a)$ is taut, where $\gamma = \hat{\gamma} \cup \tilde{\gamma}$.

Lemma 4.1. *Suppose that $F - (\gamma \cup a)$ is incompressible in N . Suppose also that if $\partial_1 N[a] \neq T(\gamma)$, then there is no essential annulus in N with boundary on $\hat{\gamma} \cup a$. Then $\tilde{\gamma}$ can be chosen so that $(N, \gamma \cup a)$ is \emptyset -taut and so that $(N[a], \gamma)$ is \bar{a} -taut. Furthermore, if $c \subset \partial_1 N[a]$ is a collection of disjoint, nonparallel curves such that*

- $|c| \leq 2$;
- all components of c are on the same component of $\partial_1 N[a]$;
- no curve of c cobounds an essential annulus in N with a curve of $\hat{\gamma} \cup a$;
- if $|c| = 2$, then there is no essential annulus in N with boundary c ; and
- if $|c| = 2$ and a is separating, there is no essential thrice punctured sphere in N with boundary $c \cup a$,

then $\tilde{\gamma}$ can be chosen to be disjoint from c .

The main ideas of the proof are contained in [Scharlemann 1990, Section 5] and [Lackenby 1997, Theorem 2.1]. Scharlemann considers “special” collections of curves on a nontorus component of ∂N . These curves cut the component into thrice punctured spheres. Exactly two of the curves in the collection bound once punctured tori. In those tori are two curves of the collection that are called “redundant”. The redundant curves are removed, and the remaining curves form the desired sutures. Scharlemann shows how to construct such a special collection that is disjoint from a set of given curves and that gives rise to a taut-sutured manifold structure on the manifold under consideration. Lackenby uses essentially the same construction (but with fewer initial hypotheses) to construct a collection of curves cutting the nontorus components of ∂N into thrice punctured spheres, but where all the curves are nonseparating. We need to allow the sutures to contain separating curves as c may contain separating curves. By slightly adapting Scharlemann’s work, in the spirit of Lackenby, we can make do with the hypotheses of the lemma, which are slightly weaker than what a direct application of Scharlemann’s work would allow.

Proof. Let τ be the number of once punctured tori in ∂N with boundary some component of $c \cup a$. Since all components of c are on the same component of ∂N , $\tau \leq 4$ with $\tau \geq 3$ only if a is separating.

Say that a collection of curves on ∂N is *pantsless* if, whenever a thrice punctured sphere has its boundary a subset of the collection, all components of the boundary are on the same component of ∂N . If a is nonseparating, then $\tau \leq 2$. Hence, either $\tau \leq 2$ or $c \cup a \cup \hat{\gamma}$ is pantsless.

Scharlemann shows how to extend the set c to a collection Γ such that there is no essential annulus in N with boundary on $\Gamma \cup a \cup \hat{\gamma}$ and the curves Γ cut ∂N into tori, once punctured tori, and thrice punctured spheres. Furthermore, if $c \cup a \cup \hat{\gamma}$ is pantsless, then so is $\Gamma \cup a \cup \hat{\gamma}$. An examination of Scharlemann’s construction

shows that all curves of $\Gamma - c$ may be taken to be nonseparating. Thus, the number of once punctured tori in ∂N with boundary on some component of $\Gamma \cup a$ is still τ . If Γ cannot be taken to be a collection of sutures on ∂N , then, by construction, $|c| = 2$, and one curve of c bounds a once punctured torus in ∂N containing the other curve of c . The component of c in the once punctured torus is “redundant” in Scharlemann’s terminology. If no curve of c is redundant, let $\tilde{\gamma} = \Gamma$; otherwise, form $\tilde{\gamma}$ by removing the redundant curve from Γ . Let $\gamma' = \tilde{\gamma} \cup a \cup \hat{\gamma}$. We now have a sutured manifold (N, γ') . Notice that the number of once punctured torus components of $\partial N - \gamma'$ is equal to τ .

We now desire to show that (N, γ') is \emptyset -taut. If it is not taut, then $R_{\pm}(\gamma)$ is not norm-minimizing in $H_2(N, \eta(\partial R_{\pm}))$. Let J be an essential surface in N with $\partial J = \partial R_{\pm} = \gamma'$. Notice that $\chi_{\emptyset}(R_{\pm}) = -\chi(\partial N)/2$ and that $|\gamma'| = -3\chi(\partial N)/2 - \tau$.

Recall that either $\tau \leq 2$ or γ' is pantsless. Suppose first that $\tau \leq 2$. Since no component of J can be an essential annulus, by the arguments of Scharlemann and Lackenby, $\chi_{\emptyset}(J) \geq |\partial J|/3 = |\gamma'|/3$. Hence, $\chi_{\emptyset}(J) \geq -\chi(\partial N)/2 - \tau/3$. Since $\tau \leq 2$ and since $\chi_{\emptyset}(J)$ and $-\chi(\partial N)/2$ are integers, $\chi_{\emptyset}(J) \geq |\partial N|/2 = \chi_{\emptyset}(R_{\pm})$. Thus, (N, γ') is a \emptyset -taut sutured manifold when $\tau \leq 2$.

Suppose therefore that γ' is pantsless. Recall that $\tau \leq 4$. We first examine the case when each component of J has its boundary contained on a single component of ∂M . Let J_0 be all the components of J with boundary on a single component T of ∂N . Let τ_0 be the number of once punctured torus components of $T - \gamma'$. Notice that $\tau_0 \leq 2$. The proof for the case when $\tau \leq 2$ shows that $\chi_{\emptyset}(J_0) \geq \chi_{\emptyset}(R_{\pm} \cap T)$. Summing over all component of ∂N shows that $\chi_{\emptyset}(J) \geq \chi_{\emptyset}(R_{\pm})$, as desired.

We may therefore assume that some component J_0 of J has boundary on at least two components of ∂N . Since γ' is pantsless, $\chi_{\emptyset}(J_0) \geq (|\partial J_0| + 2)/3$. For the other components of J we have $\chi_{\emptyset}(J - J_0) \geq |\partial(J - J_0)|/3$. Thus

$$\chi_{\emptyset}(J) \geq \frac{|\gamma'| + 2}{3} \geq -\frac{\chi(\partial N)}{2} + \frac{2 - \tau}{3}.$$

Since $\tau \leq 4$ and since $\chi_{\emptyset}(J)$ and $-\chi(\partial N)/2$ are integers, we have as desired that $\chi_{\emptyset}(J) \geq -\chi(\partial N)/2 = \chi_{\emptyset}(R_{\pm})$. Hence $(N, \gamma') = (N, \gamma \cup a)$ is \emptyset -taut. \square

Remark. The assumption that all components of c are contained on the same component of $\partial_1 N[a]$ can be weakened to a hypothesis on the number τ . For what follows, however, our assumption suffices.

We will be interested in when a component of $\partial N - F$ becomes compressible upon attaching a 2-handle to $a \subset F$ and also becomes compressible upon attaching a 2-handle to $b \subset F$. If such occurs, the curves c of the previous lemma will be the boundaries of the compressing discs for that component of ∂N . Obviously, in order to apply the lemma we will need to make assumptions on how that component compresses.

5. Constructing Q

The typical way in which we will apply the [main theorem](#) is as follows. Suppose that a and b are simple closed curves on a genus 2 component $F \subset \partial N$ and that there is an “interesting” surface $\bar{R} \subset N[b]$. We will want to use this surface to show that either $-2\chi(\bar{R}) \geq K(\bar{R})$ or $N[a]$ is taut. *A priori*, though, the surface $R = \bar{R} \cap N$ may have a -boundary compressing discs or a -torsion $2g$ -gons. The purpose of this section is to show how, given the surface \bar{R} , we can construct another surface \bar{Q} which will hopefully have similar properties to \bar{R} but be such that $Q = \bar{Q} \cap N$ does not have a -boundary compressing discs or a -torsion $2g$ -gons. This goal will not be entirely achievable, but [Theorem 5.1](#) shows how close we can come. Throughout we assume that N is a compact, orientable, irreducible 3-manifold with $F \subset \partial N$ a component having genus equal to 2. Let a and b be two essential simple closed curves on F so that a and b intersect minimally and nontrivially. As before, let $\partial_1 N = \partial_1 N[b] = \partial N - F$ and let $\partial_0 N[b] = \partial N[b] - \partial_1 N[b]$. Let T_0 and T_1 be the components of $\partial_0 N[b]$. If b is nonseparating, then $T_0 = T_1$.

Before stating the theorem, we make some important observations about $N[b]$ and surfaces in $N[b]$. If b is nonseparating, there are multiple ways to obtain a manifold homeomorphic to $N[b]$. Certainly attaching a 2-handle to b is one such way. If b^* is any curve in F that cobounds in F with $\partial\eta(b)$ a thrice punctured sphere, then attaching 2-handles to both b^* and b creates a manifold with a spherical boundary component. Filling in that sphere with a 3-ball creates a manifold homeomorphic to $N[b]$. We will often think of $N[b]$ as obtained in this fashion. Say that a surface $\bar{Q} \subset N[b]$ is *suitably embedded* if each component of $\partial Q - \partial \bar{Q}$ is a curve parallel to b or to some b^* . We denote the number of components of $\partial Q - \partial \bar{Q}$ parallel to b by $q = q(\bar{Q})$ and the number parallel to b^* by $q^* = q^*(\bar{Q})$. Let $\tilde{q} = q + q^*$. If b is separating, define $b^* = \emptyset$. Define $\Delta = |b \cap a|$, $\Delta^* = |b^* \cap a|$, $\nu = |b \cap \gamma|$, and $\nu^* = |b^* \cap \gamma|$. We then have

$$K(\bar{Q}) = (\Delta - \nu - 2)q + (\Delta^* - \nu^* - 2)q^* + \Delta_\partial - \nu_\partial.$$

Define a *slope* on a component of $\partial N[b]$ to be an isotopy class of pairwise disjoint, pairwise nonparallel curves on that component. The set of curves is allowed to be the empty set. Place a partial order on the set of slopes on a component of $\partial N[b]$ by declaring $r \leq s$ if there is some set of curves representing r that is contained in a set of curves representing s . Notice that $\emptyset \leq r$ for every slope r . Say that a surface $\bar{R} \subset N[b]$ has boundary slope \emptyset on a component of ∂N if $\partial \bar{Q}$ is disjoint from that component. Say that a surface $\bar{R} \subset N[b]$ has boundary slope $r \neq \emptyset$ on a component of ∂N if each curve of $\partial \bar{R}$ on that component is contained in some representative of r and every curve of a representative of r is isotopic to some component of $\partial \bar{R}$.

Define a surface to be *essential* if it is incompressible, boundary incompressible and has no component that is boundary-parallel or that is a 2-sphere bounding a 3-ball. The next theorem takes as input an essential surface $\bar{R} \subset N[b]$ and gives as output a surface \bar{Q} such that $Q = \bar{Q} \cap N$ can (in many circumstances) be effectively used as a parameterizing surface. The remainder of the section will be spent proving it.

Theorem 5.1. *Suppose that $\bar{R} \subset N[b]$ is a suitably embedded essential surface and suppose either*

- (I) \bar{R} is a collection of essential spheres and discs, or
- (II) $N[b]$ contains no essential sphere or disc.

Then there is a suitably embedded incompressible and boundary-incompressible surface $\bar{Q} \subset N[b]$ with the following properties. (The properties have been organized for convenience. The properties marked with a “” are optional and need not be invoked.)*

- \bar{Q} is no more complicated than \bar{R} :

(C1) $(-\chi(\bar{Q}), \tilde{q}(\bar{Q})) \leq (-\chi(\bar{R}), \tilde{q}(\bar{R}))$ in lexicographic order.

(C2) The sum of the genera components of \bar{Q} is no bigger than the sum of the genera of components of \bar{R} .

(C3) \bar{Q} and \bar{R} represent the same class in $H_2(N[b], \partial N[b])$.

- The options for compressions, a -boundary compressions, and a -torsion $2g$ -gons are limited:

(D0) Q is incompressible.

(D1) Either there is no a -boundary compressing disc for Q or $\tilde{q} = 0$.

(*D2) If no component of \bar{R} is separating and if $\tilde{q} \neq 0$, then there is no a -torsion $2g$ -gon for Q .

(D3) If \bar{Q} is a disc or 2-sphere, then either $N[b]$ has a lens space connected summand or there is no a -torsion $2g$ -gon for Q with $g \geq 2$.

(D4) If \bar{Q} is a planar surface, then either there is no a -torsion $2g$ -gon for Q with $g \geq 2$ or attaching 2 -handles to $\partial N[b]$ along $\partial \bar{Q}$ produces a 3 -manifold with a lens space connected summand.

- The boundaries are not unrelated:

(*B1) Suppose that (II) holds, that we are refilling meridians, that no component of \bar{R} separates, and that $\partial \bar{R}$ has exactly one nonmeridional component on each component of $\partial_0 N[b]$. Then \bar{Q} has exactly one boundary component on each component of $\partial_0 N[b]$ and the slopes are the same as those of $\partial \bar{R} \cap \partial_0 N[b]$.

- (B2) *If $\partial\bar{R} \cap \partial_1 N$ is contained on torus components of $\partial_1 N$ or if neither (*D2) or (*B1) are invoked, then the boundary slope of \bar{Q} on a component of $\partial_1 N[b]$ is less than or equal to the boundary slope of \bar{R} on that component.*
- (B3) *If (*D2) is not invoked and if the boundary slope of \bar{R} on a component of $\partial_0 N[b]$ is nonempty, then the boundary slope of \bar{Q} on that component is less than or equal to the boundary slope of \bar{R} .*

Property (*B1), which is the most unpleasant to achieve, is present to guarantee that if \bar{R} is a Seifert surface for L_β , then \bar{Q} (possibly after discarding components) is a Seifert surface for L_β . This is not used subsequently in this paper, but future work is planned which will make use of it. However, achieving property (*D2), which is used here, requires similar considerations. Here, we will often want to achieve (*D2), though it is incompatible with (B3). However, [Taylor 2008] does not need (*D2), and so we state the theorem in a fairly general form.

The only difficulty in proving the theorem is keeping track of the listed properties of \bar{Q} and \bar{R} . Eliminating a -boundary compressions is psychologically easier than eliminating a -torsion $2g$ -gons, so we first go through the argument that a surface \bar{Q} exists that has all but properties (*D2)–(D4). The argument may be easier to follow if, on a first reading, \bar{R} is considered to be a sphere or essential disc. The proof is based on similar work in [Scharlemann 2008], which restricts \bar{R} to being a sphere or disc.

The main purpose of assumptions (I) and (II) is to easily guarantee that the process for creating \bar{Q} described below terminates. We will show that if $\tilde{q}(\bar{R}) \neq 0$ and there is an a -boundary compressing disc or a -torsion $2g$ -gon for $R = \bar{R} \cap N$, then there is a sequence of operations on \bar{R} each of which reduces a certain complexity but preserves the properties listed above (including essentiality of \bar{R}). If (I) holds, the complexity is $(\tilde{q}(\bar{R}), -\chi(R))$, with lexicographic ordering. If (II) holds, the complexity is $(-\chi(\bar{R}), \tilde{q}(\bar{R}))$, also with lexicographic ordering. If (II) holds, it is clear that $-\chi(\bar{R})$ is always nonnegative. It will be evident that each measure of complexity has a minimum. The process stops either when $\tilde{q} = 0$ or when the minimum complexity is reached.

5.1. Eliminating compressions. Suppose that R is compressible, and let D be a compressing disc. Since \bar{R} is incompressible, ∂D is inessential on \bar{R} . Compress \bar{R} using D . Let \bar{Q} be the new surface. \bar{Q} consists of a surface of the same topological type as \bar{R} and an additional sphere. We have $\tilde{q}(\bar{Q}) = \tilde{q}(\bar{R})$. If we are assuming (II), the sphere component must be inessential in $N[b]$ and so may be discarded. Notice that in both cases (I) and (II) the complexity has decreased. Since R can be compressed only finitely many times, the complexity cannot be decreased arbitrarily far by compressions.

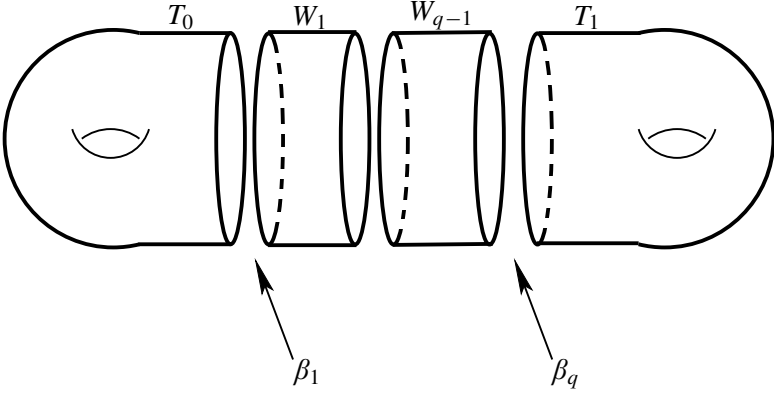


Figure 6. The tori and 1-handles W_j .

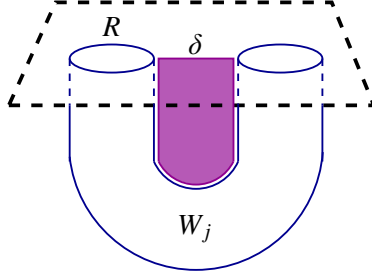


Figure 7. The disc D describes an isotopy of \bar{R} .

5.2. Eliminating a -boundary compressions. Assume that $\tilde{q} \neq 0$ and that there is an a -boundary compressing disc D for R with $\partial D = \delta \cup \epsilon$, where ϵ is a subarc of some essential circle in $\eta(a)$. There is no harm in considering $\epsilon \subset a - \partial R$.

Case 1: b separates W . In this case, $\eta(\bar{\beta}) - \text{int } \bar{R}$ consists of $q-1$ copies of $D^2 \times I$, labeled W_1, \dots, W_{q-1} , and $\partial_0 N[b] = \partial N[b] - \partial N$ has two components T_0 and T_1 , both tori. The frontiers of the W_j in $\eta(\bar{\beta})$ are discs β_1, \dots, β_q , each parallel to β , the core of the 2-handle attached to b . Each 1-handle W_j lies between β_j and β_{j+1} . The torus T_0 is incident to β_1 and the torus T_1 is incident to β_q . See Figure 6.

The interior of the arc $\epsilon \subset F$ is disjoint from ∂R . Consider the options for how ϵ could be positioned on W :

Case 1.1: ϵ lies in $\partial W_j \cap F$ for some $1 \leq j \leq q-1$. In this case, ϵ must span the annulus $\partial W_j \cap F$. The 1-handle W_j can be viewed as a regular neighborhood of the arc ϵ . The disc D can then be used to isotope W_j through $\partial D \cap R$, reducing $|\bar{R} \cap \bar{\beta}|$ by 2. See Figure 7. This maneuver decreases $\tilde{q}(\bar{R})$. Alternatively, the disc E describes an isotopy of \bar{R} to a surface \bar{Q} in $N[b]$ reducing \tilde{q} . Clearly, \bar{Q} satisfies the (C) and (B) properties.

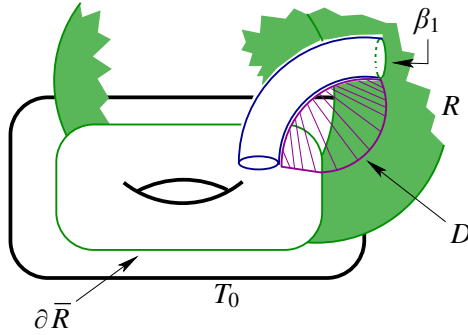


Figure 8. The disc D describes an isotopy of \bar{R} .

Suppose, then, that ϵ is an arc on T_0 or T_1 . Without loss of generality, we may assume it is on T_0 .

Case 1.2: ϵ lies in T_0 and has both endpoints on $\partial\bar{R}$. This is impossible since \bar{R} was assumed to be essential in $N[b]$ and $\tilde{q} > 0$.

Case 1.3: ϵ lies in T_0 and has one endpoint on $\partial\beta_1$ and the other on $\partial\bar{R}$. The disc D guides a proper isotopy of \bar{R} to a surface \bar{Q} in $N[b]$ that reduces \tilde{q} . See Figure 8. Clearly, the (C) and (D) properties are satisfied.

Case 1.4: ϵ lies in T_0 and has endpoints on $\partial\beta_1$. Boundary-compressing $\bar{R} - \beta_1$ produces a surface \bar{J} with two new boundary components on T_0 , both of which are essential curves. They are oppositely oriented and bound an annulus containing β_1 . If $\partial\bar{R} \cap T_0 \neq \emptyset$, then these two new components have the same slope on T_0 as $\partial\bar{R}$, showing that property (B2) is satisfied. It is easy to check that $\chi(\bar{J}) = \chi(\bar{R})$ and that $\tilde{q}(\bar{J}) = \tilde{q}(\bar{R}) - 1$, so that (C1) is satisfied. Clearly, (C2), (C3), and (B3) are also satisfied.

If \bar{J} were compressible, there would be a compressing disc for \bar{R} by an outermost arc/innermost disc argument. Thus, \bar{J} is incompressible. Suppose that E is a boundary-compressing disc for \bar{J} in $N[b]$ with $\partial E = \kappa \cup \lambda$, where κ is an arc in $\partial N[b]$ and λ is an arc in \bar{J} . Since \bar{R} is boundary-incompressible, the arc κ must lie on T_0 (and not on T_1). Since T_0 is a torus, either some component of \bar{J} is a boundary-parallel annulus or \bar{J} (and therefore \bar{R}) is compressible. We may assume the former. If \bar{J} has other components apart from the boundary-parallel annulus, discarding the boundary-parallel annulus leaves a surface \bar{Q} satisfying the (C) and (B) properties. We may therefore assume that \bar{J} in its entirety is a boundary-parallel annulus.

Since $\chi(\bar{R}) = \chi(\bar{J})$, since \bar{J} is a boundary-parallel annulus, and since $\partial\bar{J}$ has two more components than $\partial\bar{R}$, \bar{R} is an essential torus. However, using D to

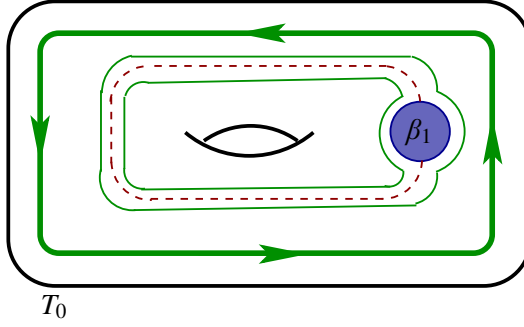


Figure 9. The annulus A lies between $\partial \bar{R}$ and one of the new boundary components of \bar{L} .

isotope $\eta(\delta) \subset \bar{R}$ into T_0 and then isotoping \bar{J} into T_0 gives a homotopy of \bar{R} into T_0 , showing that it is not essential, a contradiction.

Thus, after possibly discarding a boundary-parallel annulus from \bar{J} to obtain \bar{L} , we obtain a nonempty essential surface in $N[b]$ satisfying the first five required properties. If we do not desire property (*B1) to be satisfied, take $\bar{Q} = \bar{L}$. Notice that this step may, for example, convert an essential sphere into two discs or an essential disc with boundary on $\partial_1 N[b]$ into an annulus and a disc with boundary on $\partial_0 N[b]$. This fact accounts for the delicate phrasing of the (B) properties.

Suppose therefore that we wish to satisfy (*B1). Among other properties, we assume that \bar{R} has a single boundary component on T_0 .

There is an annulus $A \subset T_0$ that is disjoint from $\beta_1 \subset T_0$, that has interior disjoint from $\partial \bar{L}$, and that has its boundary two of the two or three components of $\partial \bar{L}$. See Figure 9, in which the dashed line represents the arc ϵ . The two circles formed by joining ϵ to $\partial \beta_1$ are the two new boundary components of \bar{L} . Since they came from a boundary-compression, they are oppositely oriented. If $\partial \bar{R}$ has a single component on T_0 (indicated by the curve with arrows in the figure), it must be oriented in the opposite direction from one of the new boundary components of $\partial \bar{L}$. Attaching A to \bar{L} creates an orientable surface and does not increase negative Euler characteristic or \tilde{q} .

Thus, $\bar{L} \cup A$ is well defined if $|\partial \bar{R} \cap T_0| \leq 1$. It may however be compressible or boundary-compressible. Since it represents the homology class $[\bar{R}]$ in $H_2(N[b], \partial N[b])$, as long as that class is nonzero we may thoroughly compress and boundary-compress it, obtaining a surface \bar{J} . Discard all null-homologous components of \bar{J} to obtain a surface \bar{Q} . By assumption (II), we never discard an essential sphere or disc. Note that since $\partial \bar{R}$ has a single boundary component on T_1 , the surface \bar{Q} will also have a single boundary component on T_1 . That is, discarding separating components of \bar{J} does not discard the component with boundary on T_1 .

Boundary-compressing \bar{J} may change the slope of $\partial\bar{J}$ on nontorus components of $\partial_1 N[b]$. Discarding separating components may convert a slope on a torus component to the empty slope. Nevertheless, properties (B2) and (B3) still hold.

If a component of \bar{J} is an inessential sphere, then either \bar{L}_A contained an inessential sphere or the sphere arose from compressions of \bar{L}_A . Suppose that the latter happened. Then after some compressions \bar{L}_A contains a solid torus and compressing that torus creates a sphere component. Discarding the torus instead of the sphere shows that this process does not increase negative Euler characteristic. If \bar{L}_A contains an inessential sphere, this component is either a component of \bar{L} and therefore of \bar{R} or it arose by attaching A to two disc components, D_1 and D_2 , of \bar{L} . The first is forbidden by the assumption that \bar{R} is essential and the second by (II). Consequently, negative Euler characteristic is not increased.

Notice that, in general, compressing \bar{L}_A may increase \tilde{q} , but because $-\chi(\bar{Q})$ is decreased, property (C1) is still preserved and complexity is decreased. Since we assume (II) for the maneuver, if (I) holds at the end of this case, we can still conclude that \tilde{q} was decreased. (This is an observation needed to show that the construction of \bar{Q} for the conclusion of the theorem terminates.)

Case 2: b is nonseparating and $q^ \neq 0$.* This is very similar to Case 1. In what follows, only the major differences are highlighted.

Since $q^* \neq 0$, the cocore $\bar{\beta}^*$ of the 2-handle attached to b^* and the cocore $\bar{\beta}$ form an arc with a loop at one end. Let $U = \eta(\bar{\beta}^* \cup \bar{\beta})$. Then $U - \bar{R}$ consists of a solid torus, $q^* - 1$ copies of $D^2 \times I$ labeled $W_1^*, \dots, W_{q^*-1}^*$, a 3-ball \mathcal{P} , and $q - 1$ copies of $D^2 \times I$ labeled W_1, \dots, W_{q-1} . The cylinders $W_1^*, \dots, W_{q^*-1}^*$ have frontiers in U consisting of discs $\beta_1^*, \dots, \beta_{q^*}^*$ all parallel to β^* (the core of the 2-handle attached to b^*). The ball \mathcal{P} has frontier in U consisting of 3 discs $\beta_{q^*}^*$, β_1 , and β_q . The cylinders W_1, \dots, W_{q-1} have with frontiers β_1, \dots, β_q consisting of discs $\beta_1, \dots, \beta_{q^*}$ all parallel to β . See Figure 10. $\partial_0 N[b]$ consists of a single torus T_0 .

Case 2.1: ϵ is not located in \mathcal{P} . This is nearly identical to Case 1. To achieve (*B1), an annulus attachment trick like that in Case 1.4 is necessary.

Case 2.2: ϵ is located in \mathcal{P} . Since $\partial\bar{R}$ is essential in $N[b]$ and since \bar{R} is embedded, $\partial\bar{R}$ is disjoint from \mathcal{P} . The arc ϵ has its endpoints on exactly two of $\{\partial\beta_{q^*}^*, \partial\beta_1, \partial\beta_q\}$. Denote by x and y the two discs containing $\partial\epsilon$, and denote the third by z . That is, $\{\partial x, \partial y, \partial z\} = \{\partial\beta_{q^*}^*, \partial\beta_1, \partial\beta_q\}$. Boundary-compressing $\text{cl}(\bar{Q} - (x \cup y))$ along D removes ∂x and ∂y as boundary-components of R and adds another boundary-component parallel to ∂z . Attach a disc in F parallel to z to this new component, forming \bar{J} . Such \bar{J} is isotopic in $N[b]$ to \bar{R} (see Figure 11) and is therefore essential and satisfies the (C) and (B) properties.

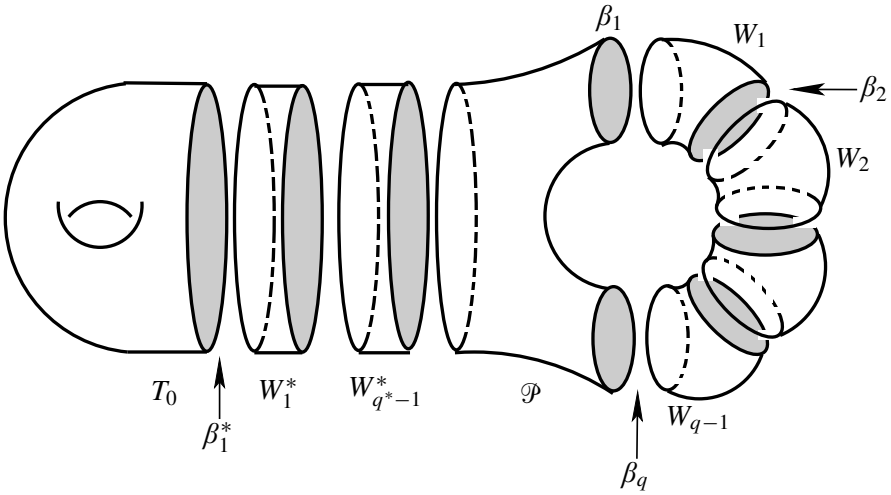


Figure 10. The torus, pair of pants, and 1-handles.

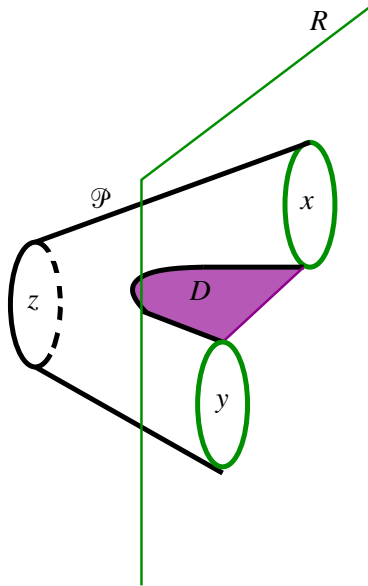


Figure 11. The disc D in [Case 2.2](#).

Case 3: b is nonseparating and $q^ = 0$.* Since b is nonseparating, $\eta(\bar{\beta}) - \bar{Q}$ consists of copies of $D^2 \times I$ labeled W_1, \dots, W_{q-1} that are separated by discs β_1, \dots, β_q each parallel to β so that each W_i is adjacent to β_i and β_{i+1} , where the indices run mod q . $\partial_0 N[b]$ is a single torus T_0 . See [Figure 12](#).

We need only consider two more cases, as the others are similar to prior cases.

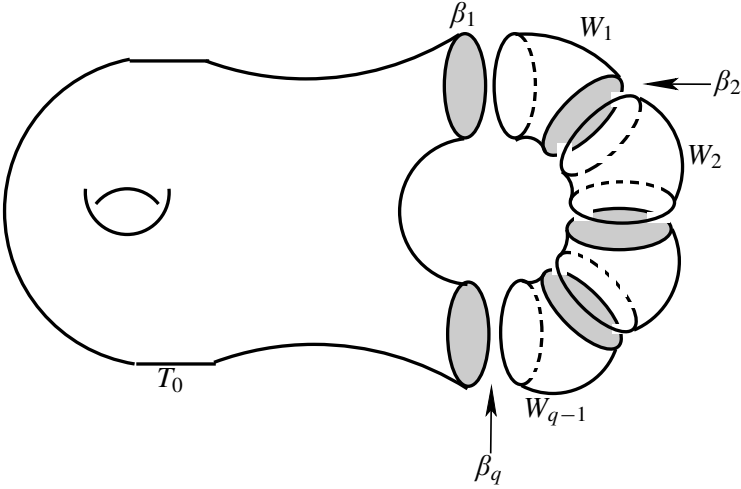


Figure 12. The solid torus and 1-handles W_j .

Case 3.4: ϵ is located on T_0 and either both endpoints are on $\partial\beta_1$ or both are on $\partial\beta_q$. The arc ϵ is a meridional arc. Suppose without loss of generality that $\partial\epsilon \subset \partial\beta_1$. Boundary-compress $\bar{R} - \beta_1$ along D . This creates a surface \bar{J} with boundary on T_0 . After possibly discarding a boundary-parallel annulus, \bar{J} is essential and the (C) properties hold as well as (B2) and (B3). We need to show that (*B1) can be achieved, if desired.

Suppose that we are in the situation of refilling meridians, so that $N \subset M$ and F bounds a genus 2 handlebody W in $M - N$ with a and b bounding discs in W . Then since the endpoints of ϵ are on the same component of $\partial\eta(a) \subset F$, ϵ is a meridional arc of $b - a$. If $\partial\bar{R}$ is not meridional on T_0 , this case therefore cannot occur. Thus, the (C) and (B) properties hold.

Case 3.5: ϵ is located on T_0 and has one endpoint on β_1 and the other on β_q . The disc D guides an isotopy of \bar{R} to a surface \bar{Q} that is suitably embedded in $N[b]$ and has $q^*(\bar{Q}) = 1$. We have $\tilde{q}(\bar{Q}) = \tilde{q}(\bar{R}) - 1$. The surface \bar{Q} can also be created by boundary-compressing $\bar{R} - (\beta_1 \cup \beta_q)$ with D and then adding a disc β^* to the new boundary component. See Figure 11. Clearly, the (C) and (B) properties hold.

The previous cases have each described an operation on \bar{R} that produces an essential surface \bar{Q} having the (C) and (B) properties. Furthermore, the maneuver described in each case strictly decreases complexity. Thus, after repeating the operation enough times, either the surface \bar{Q} will have $\tilde{q}(\bar{Q}) = 0$ or there will be no a -boundary compressions for Q . That is, the (C) and (B) properties hold, and in addition, (D0) and (D1) hold.

5.3. Eliminating a -torsion $2g$ -gons. We may now assume there is an a -torsion $2g$ -gon D for Q with $g \geq 2$ (since an a -torsion 2 -gon is an a -boundary compressing disc). For ease of notation, relabel and let $\bar{R} = \bar{Q}$ and $R = Q$. By the definition of a -torsion $2g$ -gon, there is a rectangle E containing the parallel arcs $\partial D \cap F$ which, when attached to R , creates an orientable surface. Two opposite edges of ∂E lie on ∂R and the other two are parallel (as unoriented arcs) to the arcs of $\partial D \cap F$. Denote the components of ∂R containing the two edges of ∂E by ∂_x and ∂_y . It is entirely possible that $\partial_x = \partial_y$. If ∂_x is a component of $\partial R - \partial \bar{R}$, let β_x denote the disc in $\bar{R} - R$ that it bounds. Similarly define β_y .

Suppose that \bar{R} is a planar surface or 2 -sphere. Let \hat{N} be the 3 -manifold obtained from $N[b]$ by attaching 2 -handles to $\partial N[b]$ so that each component of $\partial \bar{J}$ but one bounds a disc in \hat{N} . Attach these discs to \bar{R} , forming a surface \hat{R} . Since \bar{R} was a planar surface or 2 -sphere, \hat{R} is a disc or 2 -sphere. A regular neighborhood of $\hat{R} \cup E$ is a solid torus, and the disc D is in the exterior of that solid torus and winds longitudinally around it $n \geq 2$ times. Thus $\eta(\hat{R} \cup E \cup D)$ is a lens space connected summand of \hat{N} . Hence, redefining $\bar{Q} = \bar{J}$ we satisfy the (C), (B), and (D) properties.

We may therefore assume that \bar{R} is not a planar surface or 2 -sphere. We need to show that we can achieve (*D2) in addition to the (C), (B), (D0), and (D1) properties. The surface $\bar{R}' = (\bar{R} - (\beta_x \cup \beta_y)) \cup E$ is compressible by the disc D . Compress it to obtain an orientable surface \bar{J} . Notice that

$$(-\chi(\bar{J}), \tilde{q}(\bar{J})) < (-\chi(\bar{R}), \tilde{q}(\bar{R})).$$

Analyzing the position of E as we did the position of ϵ in the previous section and possibly performing the annulus attachment trick, we can guarantee that the (C) and (B) properties are satisfied. If the ends of E are both on $\partial \bar{R}$, the boundary of \bar{J} may have different slope from the boundary of \bar{R} . Whether or not we perform the annulus attachment trick, the surface \bar{J} may be inessential. Compressing, boundary compressing, and discarding null-homologous components produces a nonempty essential surface \bar{Q} satisfying properties (B) and (C). Considerations similar to those necessary for achieving (*B1) in Case 1.4 explain why (B2) is phrased as it is. (B3) is incompatible with (*D2) since discarding components may discard $\partial \bar{R} \cap \partial_0 N[b]$, converting a nonempty slope to an empty slope. A future attempt to eliminate an a -boundary compressing disc or a -torsion $2g$ -gon may then introduce new boundary components on $\partial_0 N[b]$ of different slope.

As before, complexity has been strictly decreased under both assumptions (I) and (II). Of course, we may now have additional compressing discs, a -boundary compressing discs, or a -torsion $2g$ -gons to eliminate as in the previous sections. Since all these operations lower complexity, the process terminates with the required surface \bar{Q} . \square

The surface \bar{Q} produced by the previous theorem may be disconnected. (For example, if b is separating it is possible we could start with \bar{R} being a disc with boundary on T_0 and end up with \bar{Q} being the union of an annulus with boundary on $T_0 \cup T_1$ and a disc with boundary on T_1 .) The next corollary puts our minds at rest by elucidating when we can discard components to arrive at a connected surface \bar{Q} .

Corollary 5.2. • *If \bar{R} is a collection of spheres or discs then after discarding components of the surface \bar{Q} created by [Theorem 5.1](#), we may assume that \bar{Q} is an essential sphere or disc such that $\tilde{q}(\bar{Q}) \leq \tilde{q}(\bar{R})$ and conclusions (B2), (B3), (D0), (D1), (D3), and (D4) hold.*

- *If $N[b]$ does not contain an essential disc or sphere, then we may assume the \bar{Q} produced by [Theorem 5.1](#) to be connected and conclusions (C1), (C2), (B2), and (D0)–(D4) hold. Furthermore, if \bar{R} is nonseparating, so is \bar{Q} .*

Proof. Suppose that \bar{R} is a collection of spheres or a discs, and let \bar{Q} be the surface produced by [Theorem 5.1](#). Notice that each component of $\bar{Q} \cap N$ is incompressible. Since $-\chi(\bar{R}) < 0$, by conclusions (C1) and (C2) of that theorem, $-\chi(\bar{Q}) < 0$ and each component of \bar{Q} is a planar surface or \bar{Q} is a sphere. Indeed, at least one component \bar{Q} of \bar{Q} is a sphere or disc. By conclusion (D1), either \bar{Q} is disjoint from $\bar{\beta}$ or there is no a -boundary compressing disc for $\bar{Q} \cap N$. If there is an a -boundary compressing disc for $\bar{Q} \cap N$, then an outermost arc argument shows that there would be one for $\bar{Q} \cap N$. Thus, either \bar{Q} is disjoint from $\bar{\beta}$ or there is no a -boundary compressing disc for \bar{Q} . As argued in the proof of [Theorem 5.1](#), if there is an a -torsion $2g$ -gon for \bar{Q} , then $N[b]$ contains a lens space connected summand. It is clear, therefore, that the required conclusions hold.

Suppose that $N[b]$ contains no essential disc or sphere. Let \bar{Q} be the surface produced by [Theorem 5.1](#). Notice that \bar{Q} contains no disc or sphere components, and also that each component of $\bar{Q} \cap N$ is incompressible. Choose a component \bar{Q}_0 of \bar{Q} and discard the other components. Neither negative Euler characteristic nor \tilde{q} are raised. If \bar{R} was nonseparating, choose \bar{Q}_0 to be nonseparating. Either \bar{Q}_0 satisfies the conclusion of the corollary or $\tilde{q}(\bar{Q}_0) > 0$, and there is an a -boundary compressing disc or a -torsion $2g$ -gon for $\bar{Q}_0 \cap N$. Apply the theorem with $\bar{R} = \bar{Q}_0$, and notice that the surface \bar{Q}_1 produced has strictly smaller complexity. Thus, repeating this process, each time discarding all but one component, we eventually obtain the connected surface \bar{Q} promised by corollary. \square

6. Refilling meridians

We now turn to applying the [main theorem](#) to “refilling meridians”. For the remainder, suppose that M is a 3-manifold containing an embedded genus 2 handlebody W . Let $N = M - \mathring{W}$. Let α and β be two essential discs in W isotoped to intersect

minimally and nontrivially. Let $a = \partial\alpha$, $b = \partial\beta$, $b^* = \partial\beta^*$, $M[\alpha] = N[a]$, and $M[\beta] = N[b]$. Recall that L_α and L_β are the cores of the solid tori produced by cutting W along α and β respectively. If we need to place sutures $\hat{\gamma}$ on $F = \partial W$ we will do so as described in Section 4. We begin by briefly observing that for any suitably embedded surface $\bar{Q} \subset M[\beta]$, with boundary disjoint from $\gamma \cap \partial M$, $K(\bar{Q}) \geq 0$.

If α is separating,

$$K(\bar{Q}) = q(\Delta - 2) + q^*(\Delta^* - 2) + \Delta_\partial.$$

Since b , b^* , and a all bound discs in W , Δ is at least two. If $q^* \neq 0$, then Δ^* is also at least two. Thus, $K(\bar{Q}) \geq 0$.

Recall from Section 4 that if α is nonseparating, any arc of $b - \hat{\eta}(a)$ with endpoints on the same component of $\partial\eta(a)$ is a meridional arc of $b - a$. The number of these meridional arcs is denoted $\mathcal{M}_a(b)$, and it is always even and always at least two since there are the same number of meridional arcs based at each component of $\partial\eta(a) \subset F$. The sutures $\hat{\gamma}$ are disjoint from these meridional arcs. Since any arc of $b - a$ that is not a meridional arc intersects exactly one suture exactly once, we have

$$\Delta - \nu = \mathcal{M}_a(b) \geq 2 \quad \text{and} \quad \Delta^* - \nu^* \geq \mathcal{M}_a(b^*) \geq 2.$$

Since $\partial\bar{Q}$ is disjoint from $b \cup b^*$, it is also disjoint from the meridional arcs of $b - a$. Consequently, each arc of $\partial\bar{Q} - a$ intersects $\hat{\gamma}$ at most once. Hence, $\Delta_\partial - \nu_\partial \geq 0$. When α is nonseparating, we therefore have

$$K(\bar{Q}) \geq q(\mathcal{M}_a(b) - 2) + q^*(\mathcal{M}_a(b^*) - 2) + \Delta_\partial - \nu_\partial \geq 0.$$

6.1. Scharlemann's conjecture. Studying the operation of refilling meridians in [2008], Scharlemann was led to the following definitions and conjecture.

Define (M, W) to be *admissible* if

- (A0) every sphere in M separates,
- (A1) M contains no lens space connected summands,
- (A2) any two curves in ∂M which compress in M are isotopic in ∂M ,
- (A3) $M - W$ is irreducible, and
- (A4) ∂M is incompressible in N .

Conjecture (Scharlemann). *If (M, W) is admissible, then one of the following occurs:*

- $M = S^3$ and W is unknotted (that is, N is a handlebody).
- At least one of $M[\alpha]$ and $M[\beta]$ is irreducible and boundary-irreducible.
- α and β are “aligned” in W .

The definition of “aligned” is rather complicated and is not needed for what follows, so I will not define it here. I will only remark that it is a notion that is independent of the embedding of W in M .

Scharlemann proved the following for admissible pairs (M, W) :

Theorem.

- *If ∂W compresses in N , then the conjecture is true.*
- *If $\Delta \leq 4$, then the conjecture is true.*
- *If α is separating and M contains no summand that is a nontrivial rational homology sphere, then one of $M[\alpha]$ and $M[\beta]$ is irreducible and boundary-irreducible.*
- *If both α and β are separating, then the conjecture is true. If in addition $\Delta \geq 6$, then one of $M[\alpha]$ and $M[\beta]$ is irreducible and boundary-irreducible.*

With a slight variation on the notion of admissible, Scharlemann’s conjecture can now be completed for a large class of manifolds.

Define the pair (M, W) to be *licit* if the following hold:

- (L0) $H_2(M) = 0$.
- (L1) $H_1(M)$ is torsion-free.
- (L2) No curve on a nontorus component of ∂M that compresses in M bounds an essential annulus in N with a meridional curve of ∂W (that is, a curve on ∂W that bounds a disc in W).
- (L3) N is irreducible.
- (L4) ∂M is incompressible in N .

The major improvement provided by the next theorem is that the case of non-separating meridians can be effectively dealt with. The theorem nearly completes Scharlemann’s conjecture for pairs (M, W) that are both licit and admissible. The one major aspect of Scharlemann’s conjecture that is not covered by this theorem is the question of whether or not both of $M[\alpha]$ and $M[\beta]$ can be solid tori. In [Taylor 2008], this case is resolved.

Theorem 6.1 (Modified Scharlemann conjecture). *Suppose that (M, W) is licit and that α and β are two essential discs in W . Suppose ∂W is incompressible in N . Then either α and β can be isotoped to be disjoint or all of the following hold:*

- *One of $M[\alpha]$ or $M[\beta]$ is irreducible.*
- *If one of $M[\alpha]$ or $M[\beta]$ is reducible, then no curve on ∂M compresses in the other.*
- *No curve on ∂M compresses in both $M[\alpha]$ and $M[\beta]$.*

- If one of $M[\alpha]$ or $M[\beta]$ is a solid torus, then the other is not reducible.

Conditions (L0) and (L1) are stronger than conditions (A0) and (A1) but are used to guarantee that $H_1(M[\alpha])$ and $H_1(M[\beta])$ are torsion-free; this is required for the application of the [main theorem](#). Condition (L2) is neither stronger nor weaker than condition (A2) since we allow multiple curves on ∂M to compress in M but forbid the existence of certain annuli. To show that some condition like (A2) was required, Scharlemann points out an example:

Example. Let M be a genus 2-handlebody, and let $W \subset M$ so that $M - \mathring{W}$ is a collar on ∂W . (That is, M is a regular neighborhood of W .) Then conditions (A0), (A1), (A3), (A4), (L0), (L1), (L3), and (L4) are all satisfied. But given any essential disc $\alpha \subset W$, $M[\alpha]$ is obviously boundary-reducible. Both (A2) and (L2) rule out this example.

The modified Scharlemann conjecture is simply a “symmetrized” version of the following theorem, in which the incompressibility assumption has been weakened for later applications.

Theorem 6.2. *Suppose that (M, W) is licit and that α and β are two essential discs, isotoped to intersect minimally, with $\Delta > 0$. Suppose that $M[\beta]$ is reducible or boundary-reducible. If α is separating, assume that $\partial W - a$ is incompressible in N . If β is nonseparating, assume that there is no essential disc in $M[\beta]$ that is disjoint from both $\bar{\beta}$ and a . Then*

- $M[\alpha]$ is irreducible;
- if $M[\beta]$ is reducible, no essential curve in ∂M compresses in $M[\alpha]$; and
- if $M[\beta]$ is boundary-reducible, no essential curve of ∂M compresses in both $M[\beta]$ and $M[\alpha]$.

Proof. We begin by showing that $H_1(M[\alpha])$ is torsion-free. Consider M as the union of $V = W - \mathring{\eta}(\alpha)$ and $M[\alpha]$. Using assumption (L0) that $H_2(M) = 0$, we see that the Mayer–Vietoris sequence gives the exact sequence

$$0 \rightarrow H_1(\partial V) \xrightarrow{\phi} H_1(M[\alpha]) \oplus H_1(V) \xrightarrow{\psi} H_1(M) \rightarrow 0.$$

Suppose that x is an element of $H_1(M[\alpha])$ and that $n \in \mathbb{N}$ is such that $nx = 0$. Then $n\psi(x, 0) = \psi(nx, 0) = 0$. Since $H_1(M)$ is torsion-free, $\psi(x, 0) = 0$. Thus, by exactness, $(x, 0)$ is in the image of ϕ . Let $y \in H_1(\partial V)$ be in the preimage of $(x, 0)$. Also, $\phi(ny) = n\phi(y) = (nx, 0) = (0, 0)$. From exactness, we know that ϕ is injective. Hence, $ny = 0 \in H_1(\partial V)$. The boundary of V is a collection of tori, and therefore $H_1(\partial V)$ is torsion-free. Consequently, $y = 0$. Therefore, $x = 0$ and $H_1(M[\alpha])$ is torsion-free.

We now proceed with the theorem by choosing appropriate sutures on ∂M . If ∂M is compressible in $M[\beta]$, let c_β be a curve on ∂M that compresses in $M[\beta]$. If $c_\beta = \emptyset$, let c be any curve on ∂M that compresses in M ; otherwise let $c = c_\beta$.

By [Lemma 4.1](#), we may choose sutures γ on $\partial M[\alpha]$ so that $\hat{\gamma} = \gamma \cap \partial_0 M[\alpha]$ is chosen as usual and so that $\gamma \cap c = \emptyset$ and $(M[\alpha], \gamma)$ is an $\bar{\alpha}$ -taut sutured manifold. Let \bar{R} be either an essential sphere, an essential disc with boundary $c_\beta = c$, or an essential disc with boundary on $\partial_0 M[\beta]$. Let \bar{Q} be the result of applying [Corollary 5.2](#) to \bar{R} . \bar{Q} is an essential sphere, an essential disc with boundary c_β , or an essential disc with boundary on $\partial_0 M[\beta]$.

If \bar{Q} is a sphere or disc with boundary c_β , then, since N is irreducible and ∂M is incompressible in N , $\tilde{q} > 0$. By [Corollary 5.2](#), there is no compressing disc, a -boundary compressing disc, or a -torsion $2g$ -gon for $Q = \bar{Q} \cap N$. Suppose, for the moment, that \bar{Q} is a disc with boundary on ∂W . If $\tilde{q} > 0$, then Q is not disjoint from a . By [Corollary 5.2](#) there is no compressing disc, a -boundary compressing disc or a -torsion $2g$ -gon for Q . If $\tilde{q} = 0$, then by hypothesis $Q = \bar{Q}$ is not disjoint from a . Since $Q = \bar{Q}$ is a disc, there are no essential arcs in Q and so there is no compressing disc, a -boundary compressing disc, or a -torsion $2g$ -gon in this case either.

Since in all cases $\partial \bar{Q}$ is disjoint from the sutures on ∂M , $K(\bar{Q}) \geq 0$ as noted in the introduction to this section. Since \bar{Q} is a sphere or disc, we also have $-2\chi(\bar{Q}) < 0$. Hence, by the [main theorem](#), $(M[\alpha], \gamma)$ is \emptyset -taut. This implies that $M[\alpha]$ is irreducible and that $R_\pm(\gamma)$ does not compress in $M[\alpha]$. Consequently, c does not compress in $M[\alpha]$. \square

Remark. At the cost of adding hypotheses on the embedding of W in M , the conditions for being licit can be significantly weakened. For example, the hypotheses on the curves c , a , and b of [Lemma 4.1](#) can be substituted for [\(L2\)](#). An examination of the homology argument at the beginning of the proof shows that [\(L0\)](#) can be replaced with the assumption that L_α and L_β are null-homologous in M .

7. Rational tangle replacement

In this section, we show how the [main theorem](#) combined with [Theorem 5.1](#) can be used to give new proofs of several theorems concerning rational tangle replacement. Following [\[Eudave-Muñoz 1988\]](#), we define a few relevant terms.

A *tangle* (B, τ) is a properly embedded pair of arcs τ in a 3-ball B . Two tangles (B, τ) and (B, τ') are *equivalent* if they are homeomorphic as pairs. They are *equal* if there is a homeomorphism of pairs that is the identity on ∂B . The *trivial tangle* is the pair $(D^2 \times I, \{.25, .75\} \times I)$. A *rational tangle* is a tangle equivalent to the trivial tangle. Each rational tangle (B, r) has a disc $D_r \subset B$ separating the strands of r (each of which is isotopic into ∂B). The disc D_r is called a *trivializing disc*.

for (B, r) . The *distance* $d(r, s)$ between two rational tangles (B, r) and (B, s) is simply the minimal intersection number $|D_r \cap D_s|$. We will often write $d(D_r, D_s)$ instead of $d(r, s)$. A *prime* tangle (B, τ) is one without local knots (that is, every meridional annulus is boundary-parallel) and where no disc in B separates the strands of τ .

Given a knot $L_\beta \subset M$ and a 3-ball B' intersecting L_β in two arcs such that $(B', B' \cap L_\beta) = (B', r_\beta)$ is a rational tangle, to replace (B', r_β) with a rational tangle (B', r_α) is to do a *rational tangle replacement* on L_β . Note that $\eta(L_\beta) \cup B'$ is a genus 2 handlebody W . The knots or links L_β and L_α can be obtained by refilling the meridians β and α respectively. If $M = S^3$, then $(B, \tau) = (S^3 - \mathring{B}', L_\beta - \mathring{B}')$ is a tangle. We assume that no component of L_β is disjoint from B .

Before stating the applications, we state and prove some lemmas that allow the terminology of tangle sums and rational tangle replacement to be converted into the terminology of boring.

7.1. Boring and rational tangle replacement.

Lemma 7.1. *Let (B, τ) be a tangle and $N = B - \mathring{\eta}(\tau)$. Suppose that c is an essential curve on $\partial B - \tau$ that separates ∂N . If $\partial N - c$ is compressible in N then c compresses in N .*

Proof. Let d be an essential curve in $\partial N - c$ that bounds a disc $D \subset N$. Since c is separating and ∂N has genus 2, d is a curve in a once punctured torus. Thus, it is either nonseparating or parallel to c . In the latter case, we are done, so suppose that d is nonseparating. Let D_+ and D_- be parallel copies of D so that d is contained in an annulus between ∂D_+ and ∂D_- . Use a loop that intersects d exactly once to band together D_+ and D_- , forming a disc D' . The boundary of D' is an essential separating curve in the once punctured torus. $\partial D'$ is therefore parallel to c . Hence, c compresses in N . \square

Lemma 7.2. *Suppose that (B, τ) and (B', r_α) are tangles embedded in S^3 with (B', r_α) a rational tangle such that $\partial B = \partial B'$ and $\partial \tau = \partial r_\alpha$. Suppose that (B', r_β) is rational tangle of distance at least one from (B', r_α) . Define the sutures $\gamma \cup a$ on ∂N as before. If*

- α is nonseparating in the handlebody $W = B' \cup \eta(\tau)$, or
- if (B, τ) is a prime tangle, or
- if (B, τ) is a rational tangle and $\partial \alpha$ does not bound a trivializing disc for (B, τ) , or
- if $\partial \alpha$ does not compress in (B, τ) ,

then $\partial W - (\gamma \cup a)$ is incompressible in N . Consequently, $(N, \gamma \cup a)$ is \emptyset -taut and $(N[a], \gamma)$ is $\bar{\alpha}$ -taut.

Proof. If α is nonseparating, then any compressing disc for $\partial W - (\gamma \cup \partial\alpha)$ would have meridional boundary, implying that S^3 had a nonseparating 2-sphere. Thus, we may suppose that α is separating. If (B, τ) is prime, there is no disc separating the strands of τ . Similarly, if (B, τ) is a rational tangle but a does not bound a trivializing disc, then a does not compress in (B, τ) . Thus, for the remaining three hypotheses, we may assume that a does not compress in (B, τ) . By Lemma 7.1, $\partial N - a$ is incompressible in N , as desired. By Lemma 4.1, $(N, \gamma \cup a)$ is taut and $(N[a], \gamma)$ is \bar{a} -taut. \square

One pleasant aspect of working with rational tangle replacements is that we can make explicit calculations of $K(\bar{Q})$. Here are two lemmas which we jointly call the *Tangle Calculations*.

Tangle Calculations I (β separating). *Suppose that L_β is a link obtained from L_α by a rational tangle replacement of distance d using W . Let \bar{Q} be a suitably embedded surface in the exterior $S^3[\beta]$ of L_β . Let $\partial_1 \bar{Q}$ be the components of $\partial \bar{Q}$ on one component of $\partial S^3[\beta]$, and let $\partial_2 \bar{Q}$ be the components on the other. Let n_i be the minimum number of times a component of $\partial_i \bar{Q}$ intersects a meridian of $\partial S^3[\beta]$.*

- *If L_α is a link, then*

$$K(\bar{Q}) \geq 2q(d-1) + d(|\partial_1 \bar{Q}|n_1 + |\partial_2 \bar{Q}|n_2).$$

- *If L_α is a knot, then*

$$K(\bar{Q}) \geq 2q(d-1) + (d-1)(|\partial_1 \bar{Q}|n_1 + |\partial_2 \bar{Q}|n_2).$$

Proof. Since L_β is a link, β is separating. Thus, $q^* = 0$. Since a and b are contained in $\partial B' = \partial B$, every arc of $b - a$ is an meridional arc. Hence, $v = 0$. By definition $2d = \Delta$.

Let T be a component of $\partial S^3[\beta]$. Without loss of generality, suppose that the components of $\partial \bar{Q}$ on T are $\partial_1 \bar{Q}$. Since every arc of $a - b$ is meridional, there exist d meridional arcs on each component of $\partial S^3[\beta]$. Thus, each component of $\partial_1 \bar{Q}$ intersects a at least dn_1 times. Each component of $\partial_2 \bar{Q}$ intersects a at least dn_2 times. Consequently, $|\partial_1 \bar{Q} \cap a| \geq |\partial_1 \bar{Q}|n_1d$. Similarly, $|\partial_2 \bar{Q} \cap a| \geq |\partial_2 \bar{Q}|n_2d$. Hence

$$\Delta_\partial \geq d(|\partial_1 \bar{Q}|n_1 + |\partial_2 \bar{Q}|n_2).$$

If α is nonseparating, the curves γ are also meridian curves of L_β . Thus, γ is intersected n_i times by each component of $\partial_i \bar{Q}$. Hence, if L_α is a knot,

$$v_\partial = |\partial_1 \bar{Q}|n_1 + |\partial_2 \bar{Q}|n_2.$$

The result follows. \square

Tangle Calculations II (β nonseparating). Suppose that L_β is a knot obtained from L_α by a rational tangle replacement of distance d using W . Let \bar{Q} be a suitably embedded surface in the exterior $S^3[\beta]$ of L_β . Suppose that each component of $\partial\bar{Q}$ intersects n times a meridian of $\partial S^3[\beta]$.

- If L_α is a link, then

$$K(\bar{Q}) \geq 2q(d-1) + 2q^*(2d-1) + 2d|\partial\bar{Q}|n.$$

- If L_β is a knot, then

$$K(\bar{Q}) \geq 2(d-1)(q+2q^*) + 2(d-1)|\partial\bar{Q}|n.$$

Proof. These calculations are similar to the calculations of the previous lemma, so we make only a few remarks. First, since b^* and $\partial\eta(b)$ cobound a thrice punctured sphere, every meridional arc of $a-b$ intersects b^* at least twice. Since every arc of $a-b$ is meridional, there are Δ such arcs. Hence $\Delta^* \geq 4d$. Second, if L_α is a knot, then b^* intersects γ twice and b intersects γ not at all. Thus

$$q(\Delta - v - 2) + q^*(\Delta^* - v^* - 2) \geq q(2d - 2) + q^*(4d - 4).$$

The given inequality follows. □

7.2. Discs, spheres, and meridional planar surfaces. In [1988], Eudave-Muñoz states six related theorems. In this section, we give new proofs for three of them. Gordon and Luecke [1994] have also given different proofs for some of them. The new proofs will follow from the following generalization. Using completely different sutured manifold theory techniques, [Taylor 2008] further extends this theorem.

Theorem 7.3. *Suppose that L_β is a knot or link obtained by a rational tangle replacement of distance $d \geq 1$ on the split link L_α . Suppose that $\partial W - \partial\alpha$ does not compress in N . Then L_β is not a split link or unknot. Furthermore, if L_β has an essential properly embedded meridional planar surface with m boundary components, it contains such a surface \bar{Q} with $|\partial\bar{Q}| \leq m$ such that either \bar{Q} is disjoint from $\bar{\beta}$ or*

$$|\bar{Q} \cap \bar{\beta}|(d-1) \leq |\partial\bar{Q}| - 2.$$

Proof. By Lemma 7.2, $(N, \gamma \cup a)$ is a taut sutured manifold. Notice that the pair (S^3, W) is licit and that since L_α and L_β are related by rational tangle replacement, no essential disc in $S^3[\beta]$ is disjoint from a . Thus by Theorem 6.2, L_β is neither a split link nor an unknot.

Suppose therefore that $S^3[\beta]$ contains an essential meridional surface \bar{R} with m boundary components. Use Corollary 5.2 to obtain the connected planar surface $\bar{Q} \subset S^3[\beta]$, and assume that \bar{Q} is not disjoint from $\bar{\beta}$. That is, assume that $\tilde{q} > 0$.

Since \bar{Q} is connected and has Euler characteristic not lower than our original planar surface, $|\partial\bar{Q}| \leq m$. The boundary of \bar{Q} is meridional, by construction, since each arc of $a - b$ is meridional. [Corollary 5.2](#) allows us to conclude that there is no compressing disc, a -boundary compressing disc, or a -torsion $2g$ -gon for Q . Also, $S^3[\alpha]$ is reducible and $H_1(S^3[\alpha])$ is torsion-free.

The [main theorem](#) implies, therefore, that $K(\bar{Q}) \leq -2\chi(\bar{Q})$. Since $\partial\bar{Q}$ is disjoint from $a \cup \gamma$ and since L_α is a link, the tangle calculations tell us that

$$2q(d-1) + 2q^*(2d-1) \leq -2\chi(\bar{Q}).$$

Since $4q^*(d-1) \leq 2q^*(2d-1)$, we conclude that $2(q+2q^*)(d-1) \leq -2\chi(\bar{Q})$. Because \bar{Q} is a planar surface with $|\partial\bar{Q}|$ boundary components, we conclude that $-2\chi(\bar{Q}) = 2|\partial\bar{Q}| - 4$. Plugging into our inequality and dividing by two, we obtain

$$(q+2q^*)(d-1) \leq |\partial\bar{Q}| - 2.$$

A slight isotopy pushing the discs in \bar{Q} with boundary parallel to b^* converts each such disc into two discs each with boundary parallel to b . Hence, after the isotopy $|\bar{Q} \cap \bar{\beta}| = q + 2q^*$. Consequently,

$$|\bar{Q} \cap \bar{\beta}|(d-1) \leq |\partial\bar{Q}| - 2. \quad \square$$

As corollaries, we have two classical results.

Theorem [Eudave-Muñoz 1988]. *If (B, τ) is prime, if L_α is a split link, and if L_β is composite, then $d(\alpha, \beta) \leq 1$.*

Proof. Suppose that $d \geq 1$. Since (B, τ) is prime and α is separating, [Lemma 7.2](#) shows that $\partial W - a$ is incompressible in N . Since L_β contains an essential meridional annulus, we may apply [Theorem 7.3](#) with $m = 2$. Since there are no meridional discs, \bar{Q} is also a meridional annulus. Since (B, τ) is prime, \bar{Q} is not disjoint from $\bar{\beta}$. The inequality from the theorem shows that $d = 1$. \square

Theorem [Eudave-Muñoz 1988]. *If (B, τ) is any tangle and if L_α and L_β are split links, then $r_\alpha = r_\beta$.*

Proof. It suffices to show that α and β are disjoint. Suppose not, so that $d \geq 1$. If $\partial W - a$ is incompressible in N , then by [Theorem 7.3](#) L_β is not a split link. Thus $\partial W - a$ compresses in N . By reversing the roles of α and β , we can also conclude that $\partial W - b$ compresses in N . Since both α and β are separating, [Lemma 7.1](#) shows that both a and b compress in N .

There is therefore a disc D_a in B with boundary a separating the strings of τ . Similarly, there is a disc D_b in B with boundary $b = \partial\beta$ separating the strings of τ . An easy innermost disc/outmost arc argument shows that D_a and D_b are isotopic. In particular, a and b are isotopic in $\partial B - \tau$, which implies that $r_\alpha = r_\beta$.

Thus we may assume without loss of generality that $\partial W - \partial\alpha$ is not compressible in N . Let \bar{R} be an essential sphere in $S^3[\beta]$, and apply [Corollary 5.2](#) to obtain an essential sphere or disc \bar{Q} . Since $a - b$ consists of meridional arcs, \bar{Q} is not disjoint from $\eta(a)$. If \bar{Q} were a disc disjoint from $\bar{\beta}$, there would be no a -boundary compressing disc for \bar{Q} . If \bar{Q} is a sphere, $\tilde{q} > 0$. Thus, we may apply the [main theorem](#) to conclude that $S^3[\alpha]$ is irreducible or that α and β are disjoint. If the latter is true, $r_\alpha = r_\beta$. \square

Theorem [[Scharlemann 1985](#)]. *If (B, τ) is any tangle and L_β is a trivial knot and L_α a split link, then (B, τ) is a rational tangle and $d \leq 1$.*

Proof. Suppose $d \geq 1$. If $\partial W - a$ were incompressible in N , then L_β would not be the unknot by [Theorem 7.3](#). Hence $\partial W - a$ is compressible in N . Since α is separating, [Lemma 7.1](#) shows that a compresses in N . Since L_β is the unknot, τ has no local knots. Thus, (B, τ) is a rational tangle with trivializing disc having boundary a .

It remains to prove that $d = 1$. Since L_β is the unknot, a double-branched cover of S^3 with branch set L_β is S^3 . The preimage \tilde{B} of B is an unknotted solid torus. There is a correspondence between rational tangle replacement and Dehn-surgery in the double-branched cover. Replacing (B', r_β) with (B', r_α) converts the double-branched cover to a lens space, S^3 or $S^1 \times S^2$. In the double branched cover, the Dehn surgery is achieved by making a curve in $\partial\tilde{B}$ that intersects a meridian of \tilde{B} d times bound a disc in the complementary solid torus. Since L_α is a split link, the double branched cover of S^3 over L_α is reducible. Thus, it must be $S^1 \times S^2$ and d must be one, as desired. \square

Remark. In the proof of the previous theorem, note that even without proving $d \leq 1$, we have provided a new proof of Scharlemann's band sum theorem [[1985](#)]: If $K = K_1 \#_b K_2$ is the unknot, then the band sum is the connected sum of unknots. To see this note that W is $\eta(K_1 \cup K_2 \cup b)$, where b is the band. The tangle (B, τ) is $(S^3 - \dot{\eta}(b), (K_1 \cup K_2) - \dot{\eta}(b))$. Since $\partial\beta$ is a loop that encircles the band, $\partial\beta$ only bounds a disc in (B, τ) when the band sum is a connected sum and K_1 and K_2 are unknots.

[[Taylor 2008](#)] gives other significant applications of sutured manifold theory to problems involving rational tangle replacement.

8. Intersections of \emptyset -taut surfaces

The [main theorem](#) is useful for studying a homology class in $H_2(N[a], \partial N[a])$ that is not represented by a surface disjoint from $\bar{\beta}$. The propositions of this section consist of observations that can dramatically simplify the combinatorics of such a situation. Let N be a compact, orientable 3-manifold with $F \subset \partial M$ a genus 2

boundary component. Let $a, b \subset F$ be essential curves that cannot be isotoped to be disjoint, and suppose that $(N[a], \gamma)$ is \bar{a} -taut, as in [Section 4](#).

8.1. Intersection graphs.

Proposition 8.1. *Let $(N[a], \gamma)$ and b be as above, and let $z \in H_2(N[a], \partial N[a])$ be a nontrivial homology class. Suppose that $N[a]$ does not contain an essential disc disjoint from \bar{a} . Then z is represented by an embedded conditioned \bar{a} -taut surface \bar{P} . Furthermore, for any such \bar{P} , either \bar{P} is disjoint from \bar{a} or $P = \bar{P} \cap N$ has no compressing discs, b -boundary compressing discs or b -torsion $2g$ -gons.*

Proof. Let \bar{P} be a conditioned \bar{a} -taut surface. (Such a surface is guaranteed to exist by [[Scharlemann 1989](#), Theorem 2.6].) Suppose that \bar{P} is not disjoint from \bar{a} . Recall from the definition of “ \bar{a} -taut” that \bar{a} intersects \bar{P} always with the same sign. Because \bar{P} is \bar{a} -taut, P is incompressible. Suppose that D is a b -torsion $2g$ -gon for P . If $g = 1$, D is a b -boundary compressing disc for P . Let ϵ_i be the arcs $\partial D \cap F$. Let R be the rectangle containing the ϵ_i from the definition of b -torsion $2g$ -gon. Suppose that the ends of R are on components of $\partial P - \partial \bar{P}$. The endpoints of the ϵ_i have signs arising from the intersection of ∂D with ∂P . Since \bar{a} always intersects \bar{P} with the same sign, an arc ϵ_i has the same sign of intersection at both its head and tail. Since the arcs are all parallel, all heads and tails of all the ϵ_i have the same sign of intersection. However, an arc of $\partial D \cap P$ must have opposite signs of intersection, arising as it does from the intersection of two surfaces. This implies that the head of some ϵ_i has a sign different from the tail of some ϵ_i , a contradiction. Hence, at least one end of R must lie on a component of $\partial \bar{P}$.

If one end of R is on $\partial P - \partial \bar{P}$, denote that component by α_1 and denote by α_1 the disc that it bounds in \bar{P} . If both ends of R are on $\partial \bar{P}$, let $\alpha_1 = \emptyset$. Attach R to $\bar{P} - \alpha_1$, creating a surface \tilde{P} . The disc D is contained in N and, therefore, had interior disjoint from \bar{a} . Compress \tilde{P} using D and continue to call the result \tilde{P} .

An easy calculation shows that

$$\begin{aligned} \text{if } \alpha_1 \neq \emptyset, \quad & \text{then } \chi(\tilde{P}) = \chi(\bar{P}) \text{ but } |\bar{a} \cap \tilde{P}| = |\bar{a} \cap \bar{P}| - 1; \\ \text{if } \alpha_1 = \emptyset, \quad & \text{then } -\chi(\tilde{P}) = -\chi(\bar{P}) - 1 \text{ and } |\bar{a} \cap \tilde{P}| = |\bar{a} \cap \bar{P}|. \end{aligned}$$

If $\chi_{\bar{a}}(\tilde{P}) \neq |\bar{a} \cap \tilde{P}| - \chi(\tilde{P})$, then a component of \tilde{P} is a disc disjoint from \bar{a} or a sphere intersected by \bar{a} once. Either of these contradict our hypotheses on $N[a]$. Suppose therefore that $\chi_{\bar{a}}(\tilde{P}) = |\bar{a} \cap \tilde{P}| - \chi(\tilde{P})$.

Similarly, $\chi_{\bar{a}}(\tilde{P}) = |\bar{a} \cap \tilde{P}| - \chi(\tilde{P})$. Hence $\chi_{\bar{a}}(\tilde{P}) = \chi_{\bar{a}}(\bar{P}) - 1$. Since \bar{a} always intersects \tilde{P} with the same sign, \tilde{P} is not \bar{a} -taut, a contradiction. Hence, there are no b -torsion $2g$ -gons for P . \square

Remark. As Scharlemann notes [[2008](#)], when a and b are nonseparating it can be difficult to use combinatorial methods to analyze the intersection of surfaces

in $N[a]$ and $N[b]$. The primary reason for this is the need to work with a^* and b^* boundary components on the surfaces. The previous proposition shows that when the surfaces in question are $\bar{\alpha}$ -taut and $\bar{\beta}$ -taut and not disjoint from $\bar{\alpha}$ and $\bar{\beta}$, respectively, there is no need to consider a^* and b^* curves.

The remainder of this section develops notation for studying the intersection graphs of such surfaces. Let $\bar{P} \subset N[a]$ be an $\bar{\alpha}$ -taut surface, and let $\bar{Q} \subset N[b]$ be a $\bar{\beta}$ -taut surface. Suppose that \bar{P} and \bar{Q} are not disjoint from $\bar{\alpha}$ and $\bar{\beta}$, respectively. Suppose also that there is no b -torsion $2g$ -gon for $P = \bar{P} \cap N$ and no a -torsion $2g$ -gon for $Q = \bar{Q} \cap N$. It is clear that P and Q are incompressible.

In [Section 3](#), we defined intersection graphs between \bar{Q} and a disc D . We now define, in a similar fashion, intersection graphs between \bar{P} and \bar{Q} . Orient P (respectively, Q) so that all boundary components of $\partial P - \partial \bar{P}$ (respectively, $\partial Q - \partial \bar{Q}$) are parallel on $\eta(\bar{\alpha})$ (respectively, $\eta(\bar{\beta})$). The intersection of P and Q forms graphs Λ_α and Λ_β on \bar{P} and \bar{Q} . A component of $\partial P - \partial \bar{P}$ or $\partial Q - \partial \bar{Q}$ is called an *interior boundary component*. The vertex of Λ_α or Λ_β to which it corresponds is called an *interior vertex*.

Label the components of $\partial Q \cap \eta(a)$ as $1, \dots, \mu_Q$ and those of $\partial P \cap \eta(b)$ as $1, \dots, \mu_P$. The labels should be in order around $\eta(a)$ and $\eta(b)$. An endpoint of an edge on an interior vertex of Λ_α corresponds to an arc of $\partial Q \cap \partial \eta(a)$. Give the endpoint of the edge the label associated to that arc. Similarly, label all endpoints of edges on interior vertices of Λ_β . A *Scharlemann cycle* is a type of cycle that bounds a disc in \bar{P} (\bar{Q} , respectively). The interior of the disc must be disjoint from Λ_α (Λ_β) and all of the vertices of the cycle must be interior vertices. Furthermore, the cycle can be oriented so that the tail end of each edge has the same label. This is the same notion of Scharlemann cycle as in [Section 3](#), but adapted to the possibly nonplanar surfaces \bar{P} and \bar{Q} .

Lemma 8.2. *There is no Scharlemann cycle in Λ_α or Λ_β .*

Proof. Were there a trivial loop at an interior vertex or a Scharlemann cycle in Λ_α or Λ_β , the interior would be an a or b -torsion $2g$ -gon, which contradicts [Proposition 8.1](#). □

Although we will not use it here, the next lemma may be a useful observation in the future.

Lemma 8.3. *If \bar{P} is a disc, then every loop in Λ_α is based at $\partial \bar{P}$.*

Proof. Suppose that \bar{P} is a disc and that there is a loop based at an interior vertex of Λ_α . A component X of the complement of the loop in \bar{P} does not contain $\partial \bar{P}$. The loop is an x -cycle and [Lemma 3.3](#) then guarantees the existence of a Scharlemann cycle in X , contrary to [Lemma 8.2](#). □

8.2. When the exterior of W is anannular. We conclude this section with an application to refilling meridians of a genus 2 handlebody whose exterior is irreducible, boundary-irreducible, and anannular. It is based on the ideas in [Scharlemann and Wu 1993]. Suppose that M is the exterior of a link in S^3 . Suppose that $W \subset M$ is a genus 2 handlebody embedded in M . Let $N = M - \mathring{W}$.

Theorem 8.4. *Suppose that N is irreducible, boundary-irreducible and anannular. Suppose that α and β are nonseparating meridians of W such that $\Delta > 0$. Suppose that neither $M[\alpha]$ nor $M[\beta]$ contain an essential disc or sphere. Suppose also that in $H_2(M[\alpha], \partial M)$ there is a homology class z_a that cannot be represented by a surface disjoint from $\bar{\alpha}$ and that in $H_2(M[\beta], \partial M)$ there is a homology class z_b that cannot be represented by a surface disjoint from $\bar{\beta}$. Then there is a \emptyset -taut surface $\bar{P} \subset M[\alpha]$ representing z_a intersecting $\bar{\alpha}$ p times and an \emptyset -taut surface $\bar{Q} \subset M[\beta]$ representing z_b intersecting $\bar{\beta}$ q times such that one of the following occurs:*

- (1) $-2\chi(\bar{P}) \geq p(\mathcal{M}_b(a) - 2)$.
- (2) $-2\chi(\bar{Q}) \geq q(\mathcal{M}_a(b) - 2)$.
- (3) All of the following occur:
 - \bar{Q} is $\bar{\beta}$ -taut.
 - \bar{P} is $\bar{\alpha}$ -taut.
 - $pq\Delta \leq 18(p - \chi(\bar{P}))(q - \chi(\bar{Q}))$.
 - $\Delta < \frac{9}{2}\mathcal{M}_a(b)\mathcal{M}_b(a)$.

Proof. Notice that the right hand side of the inequalities in (1) and (2) are $K(\bar{P})$ and $K(\bar{Q})$, respectively. Choose a taut representative in $M[\beta]$ for z_b and apply Theorem 5.1, obtaining \bar{Q} . Since negative Euler characteristic is not increased and $M[\beta]$ does not contain an essential disc or sphere, \bar{Q} is also taut. If (1) holds, we are done, so assume that $-2\chi(\bar{Q}) < K(\bar{Q})$. Recall that \bar{Q} is not disjoint from $\bar{\beta}$. Apply the main theorem to obtain a surface $\bar{P} \subset M[\alpha]$ representing z_a . (The surface \bar{P} is the surface S in the statement of that theorem.) \bar{P} is both $\bar{\alpha}$ -taut and \emptyset -taut. If (2) holds, we are done, so assume $-2\chi(\bar{P}) < K(\bar{P})$. Applying the main theorem again, with α and β reversed, we find a $\bar{\beta}$ -taut and \emptyset -taut surface in $M[\beta]$ representing z_b . We may call this surface \bar{Q} , forgetting the previous one. Consider the graphs formed by the intersection of P and Q ; let Λ_α be the graph on \bar{P} and Λ_β the graph on \bar{Q} . Lemma 8.2 assures us that there is no trivial loop based at an interior vertex of either graph.

Lemma 8.5. $pq\Delta \leq 18(p - \chi(\bar{P}))(q - \chi(\bar{Q}))$.

Proof of Lemma 8.5. By [Scharlemann and Wu 1993, Lemma 2.1], if two edges of $P \cap Q$ are parallel in both Λ_α and Λ_β , there is an essential annulus in N , contrary

to our assumption that N is anannular. The proof proceeds as in [Scharlemann and Wu 1993].

Each interior boundary component of P intersects ∂Q at $q\Delta$ places. Therefore $|\partial Q \cap \partial P| \geq pq\Delta$. Thus, Λ_α and Λ_β each have at least $pq\Delta/2$ edges.

Claim. Λ_α has at least $pq\Delta/(6(p - \chi(\bar{P})))$ mutually parallel edges.

This claim is similar to work in [Gordon and Litherland 1984]. Let Λ' be the graph obtained by combining each set of parallel edges of Λ_α into a single edge. Since Λ' has no loops at interior vertices and no parallel edges, by applying the formula for the Euler characteristic of a closed surface we obtain

$$\begin{aligned}\chi(\bar{P}) + |\partial \bar{P}| &= V - E + F \\ &\leq p + |\partial \bar{P}| - E + (2/3)E \\ &= p + |\partial \bar{P}| - (1/3)E,\end{aligned}$$

where V , E , and F represent the number of vertices, edges, and faces of Λ' . Thus, $E \leq 3(p - \chi(\bar{P}))$. Let n be the largest number of mutually parallel edges in Λ_α . Then, since there are at least $pq\Delta/2$ edges in Λ_α , we have

$$pq\Delta/(2n) \leq E \leq 3(p - \chi(\bar{P})).$$

The claim follows.

A similar argument shows that if a graph in \bar{Q} has more than $3(q - \chi(\bar{Q}))$ edges, then two of them are parallel. Hence, since there are no mutually parallel edges in Λ_α and Λ_β we must have

$$\frac{pq\Delta}{6(p - \chi(\bar{P}))} \leq 3(q - \chi(\bar{Q})),$$

whence the lemma and the first inequality of conclusion (3) follow. \square

We now proceed with the proof of the theorem. Since we are assuming that neither (1) nor (2) hold, we have

$$\begin{aligned}-\chi(\bar{P}) &< K(\bar{P})/2 = p(\mathcal{M}_b(a) - 2)/2, \\ -\chi(\bar{Q}) &< K(\bar{Q})/2 = q(\mathcal{M}_a(b) - 2)/2\end{aligned}$$

Plugging into the inequality from the lemma, we obtain

$$pq\Delta < 18pq \left(1 + \frac{\mathcal{M}_b(a) - 2}{2}\right) \left(1 + \frac{\mathcal{M}_a(b) - 2}{2}\right).$$

Since neither p nor q is zero, we divide and simplify to obtain

$$\Delta < 9\mathcal{M}_b(a)\mathcal{M}_a(b)/2. \quad \square$$

Remark. The point of the previous theorem is that, under the specified conditions, either we obtain a bound on the Euler characteristic of surfaces representing the homology classes z_a or z_b or we obtain a restriction on the number of nonmeridional arcs of $a - b$ and $b - a$. For example, suppose that discs α and β are chosen so that z_a is represented by a once punctured torus, and so that $\mathcal{M}_b(a) = \mathcal{M}_a(b) = 6$. Then $-2\chi(\bar{P}) = 2 < 4p = K(\bar{P})$. Then if z_b is also represented by a once punctured torus, we have $\Delta < 162$. Since Δ is even, this implies $\Delta \leq 160$.

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