THE MÖBIUS CHARACTERIZATIONS OF WILLMORE TORI
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Suppose M is a m-dimensional submanifold without umbilic points in the $(m + p)$-dimensional unit sphere S^{m+p}. Four basic invariants of M^m under the Möbius transformation group of S^{m+p} are a symmetric positive definite 2-form g called the Möbius metric, a section B of the normal bundle called the Möbius second fundamental form, a 1-form Φ called the Möbius form, and a symmetric $(0, 2)$ tensor A called the Blaschke tensor. In the Möbius geometry of submanifolds, the most important examples of Möbius minimal submanifolds (also called Willmore submanifolds) are Willmore tori and Veronese submanifolds. In this paper, several fundamental inequalities of the Möbius geometry of submanifolds are established and the Möbius characterizations of Willmore tori and Veronese submanifolds are presented by using Möbius invariants.

1. Introduction

Let $x : M \to S^{m+p}$ be a m-dimensional submanifold without umbilic point in a $(m + p)$-dimensional unit sphere S^{m+p}. Let $\{e_i\}$ be a local orthonormal tangent frame field of x for the standard metric $I = dx \cdot dx$ with the dual frame field $\{\theta_i\}$, let $\{e_\alpha\}$ be the local orthonormal normal frame field of x, let $II = \sum_{i,j,\alpha} h_{ij}^\alpha \theta_i \theta_j e_\alpha$ be the second fundamental form, and let $\boldsymbol{H} = \sum_\alpha H^\alpha e_\alpha$ be the mean curvature vector of x. Let $\rho^2 = (m/(m-1))|II - (1/m) \operatorname{tr}(II)I|^2$. Then the positive definite 2-form $\boldsymbol{g} = \rho^2 I$ is invariant under the Möbius transformations of S^{m+p} and is called the Möbius metric of x. In [Wang 1998], the author gave the structure equations for Möbius geometry of submanifolds in the unit sphere. Three fundamental forms $\boldsymbol{\Phi}$, $\boldsymbol{A}$ and $\boldsymbol{B}$ appear naturally in the structure equations; together with $\boldsymbol{g}$, these determine the submanifold up to Möbius transformations of S^{m+p}. $\boldsymbol{\Phi} = \sum_{\alpha,i} C_i^\alpha \theta_i e_\alpha$ is called the Möbius form, $\boldsymbol{A} = \rho^2 \sum_{i,j} A_{ij} \theta_i \theta_j$ the Blaschke

tensor, and $\boldsymbol{B} = \rho B_{ij}^\alpha \theta_i \theta_j e_\alpha$ the Möbius second fundamental form. The relations among $\boldsymbol{\Phi}$, $\boldsymbol{A}$, and $\boldsymbol{B}$ and Euclidean invariants of x are given by [Wang 1998]

$$(1\text{-}1) \qquad C_i^\alpha = -\rho^{-2}\big(H_{,i}^\alpha + \textstyle\sum_j (h_{ij}^\alpha - H^\alpha \delta_{ij}) e_j (\log \rho)\big),$$

$$(1\text{-}2) \qquad A_{ij} = -\rho^{-2}\big(\text{Hess}_{ij}(\log \rho) - e_i(\log \rho) e_j(\log \rho) - \textstyle\sum_\alpha H^\alpha h_{ij}^\alpha\big)$$
$$\qquad\qquad\qquad\qquad - \tfrac{1}{2}\rho^{-2}\big(\textstyle\sum_i ((\log \rho)_i)^2 - 1 + |\boldsymbol{H}|^2\big)\delta_{ij},$$

$$(1\text{-}3) \qquad B_{ij}^\alpha = \rho^{-1}(h_{ij}^\alpha - H^\alpha \delta_{ij}),$$

where Hess_{ij} and $H_{,i}^\alpha$ are the Hessian matrix of $dx \cdot dx$ and the covariant derivative of the mean curvature vector field of x in the normal bundle $N(M)$. Let

$$\|A\|^2 = \textstyle\sum_{i,j}(A_{ij})^2 \quad \text{and} \quad \|\boldsymbol{\Phi}\|^2 = \textstyle\sum_{\alpha,i}(C_i^\alpha)^2.$$

Then both $\|A\|^2$ and $\|\boldsymbol{\Phi}\|^2$ are Möbius invariants.

In the Euclidean geometry of submanifolds in the unit sphere, a famous result is this rigidity theorem of minimal submanifolds: If x is a minimal immersion and the second fundamental form satisfies $|II|^2 \le m/(2 - 1/p)$, then either $|II|^2 = 0$ or $|II|^2 = m/(2 - 1/p)$ and x is Euclidean equivalent to the geodesic sphere, a Clifford torus or a Veronese surface; see [Chern et al. 1970; Lawson 1969; Simons 1968]. Recently, Li [2001; 2002] modified this result for the case of a Willmore submanifold x (satisfying the Euler–Lagrange equation of the Willmore functional). He proved that if x satisfies $\rho^2 \le m/(2 - 1/p)$, then $\rho^2 = m/(2 - 1/p)$ and x is Euclidean equivalent to a Willmore torus or a Veronese surface. However, ρ^2 is not conformally invariant under the Möbius group of S^{m+p}. A natural question is, Do conformal invariants characterize Willmore tori and Veronese submanifolds in Möbius geometry? In this paper, we give a positive answer to this question. It should be noted that we don't need the assumption of Willmore submanifold for our theorems here. To state our results, we first introduce traceless Blaschke tensor $\tilde{A} = \rho^2 \sum_{i,j} \tilde{A}_{ij}\theta_i\theta_j$ and traceless tensor $\check{B} = \rho^2 \sum_{i,j} \check{B}_{ij}\theta_i\theta_j$, where

$$\tilde{A}_{ij} = A_{ij} - \tfrac{1}{m}\big(\textstyle\sum_k A_{kk}\big)\delta_{ij} \quad \text{and} \quad \check{B}_{ij} = \frac{m-1}{m^2}\delta_{ij} - \textstyle\sum_{k,\alpha} B_{ik}^\alpha B_{kj}^\alpha.$$

We let $\|\tilde{A}\|^2 = \sum_{i,j}(\tilde{A}_{ij})^2$. We denote by κ the normalized Möbius scalar curvature of $x : M \to S^{m+p}$ (see (2-8)).

In the Möbius geometry of submanifolds, Willmore tori

$$\big\{W_k^n = S^k(\sqrt{(m-k)/m}) \times S^{m-k}(\sqrt{k/m}), \ 1 \le k \le m-1\big\},$$

which can also be obtained by exchanging the radii in the Clifford tori

$$M_k^n = S^k(\sqrt{k/m}) \times S^{m-k}(\sqrt{(m-k)/m}),$$

are the chief examples of stable Willmore submanifolds; see [Guo et al. 2001; Li 2001; 2002]. The main purpose of this paper is to obtain some geometric inequalities expressed in Möbius invariants, which then characterize Willmore tori and Veronese submanifolds when equalities hold in the inequalities. The main results are as follows.

Theorem 1.1. *Let $x : M \to S^{m+p}$ be an m-dimensional compact submanifold in the unit sphere S^{m+p}. Then*

(1-4)
$$\int_M \left(\kappa - m\|\boldsymbol{\Phi}\|^2 - \frac{m}{4(m-1)}\|\tilde{\boldsymbol{A}} + \check{\boldsymbol{B}}\|^2 - \frac{1}{m^2}\left(m - 2 + \tfrac{1}{2}(m-1)\operatorname{sgn}(p-1)\right)\right) dM \leq 0.$$

In particular, if

(1-5) $$\kappa - m\|\boldsymbol{\Phi}\|^2 - \frac{m}{4(m-1)}\|\tilde{\boldsymbol{A}} + \check{\boldsymbol{B}}\|^2 \geq \frac{1}{m^2}\left(m - 2 + \frac{m-1}{2}\operatorname{sgn}(p-1)\right),$$

then M is Möbius equivalent to either

  (i) *a Willmore torus $W_k^m = S^k(\sqrt{(m-k/m)}) \times S^{m-k}(\sqrt{k/m})$ in S^{m+1}, or*

 (ii) *a Veronese surface in S^4.*

Theorem 1.2. *Let $x : M \to S^{m+p}$ be an m-dimensional compact submanifold in the unit sphere S^{m+p}. Let $K(x)$ be the function that assigns to each point of M the infimum of the Möbius sectional curvatures of M at that point. Then*

$$\int_M \left(2(m+1)K - m\kappa - 1/m - m(m+2)\|\boldsymbol{\Phi}\|^2 - m\|\tilde{\boldsymbol{A}}\|\right) dM \leq 0.$$

In particular, if

(1-6) $$2(m+1)K - m\kappa - m(m+2)\|\boldsymbol{\Phi}\|^2 - m\|\tilde{\boldsymbol{A}}\| \geq 1/m,$$

then $1/K = m(m+2)$ and M is Möbius equivalent to the Veronese submanifold $S^m(\sqrt{2(m+1)/m})$.

Corollary 1.3. *Let $x : M^m \to S^{m+p}$ be an m-dimensional compact submanifold with constant Möbius sectional curvature c and vanishing Möbius form. Then*

$$c \leq \frac{1}{m(m+2)},$$

with equality holding if and only if M is Möbius equivalent to the Veronese submanifold $S^m(\sqrt{2(m+1)/m})$.

Remark 1.4. The functions on the right side of inequalities (1-5) and (1-6) are Möbius invariants.

Remark 1.5. For related results about Möbius submanifolds in an unit sphere, see [Akivis and Goldberg 1996; 1997; Guo and Guo 2006; Hu and Li 2003; 2004; Li and Wang 2003a; 2003b; Li et al. 2001].

In addition, in Section 4 we prove two Möbius sectional curvature pinching results, Theorem 4.1 and Theorem 4.2.

2. Möbius invariants for submanifolds in S^{m+p}

In this section we define the Möbius invariants for submanifolds in S^{m+p}. For more detail we refer to [Wang 1998] or [Guo et al. 2001].

Let R_1^{m+p+2} be the Lorentzian space with inner product

$$\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_{m+p+1} y_{m+p+1},$$

where $x = (x_0, x_1, \ldots, x_{m+p+1})$ and $y = (y_0, y_1, \ldots, y_{m+p+1})$. Let $x : M \to S^{m+p}$ be a submanifold of S^{m+p} without umbilic points. The second fundamental form and the mean curvature vector are denoted by II and $\boldsymbol{H}$ respectively. Define the positive function $\rho = \sqrt{(m-1)/m} |II - \boldsymbol{H} I|$, and define the Möbius position vector $Y : M \to R_1^{m+p+2}$ of x by $Y = \rho(1, x)$.

Theorem 2.1 [Wang 1998]. *Two submanifolds $x, \tilde{x} : M \to S^{m+p}$ are Möbius equivalent if and only if there exists a T in the Lorentz group $O(m+p+2, 1)$ on R_1^{m+p+2} such that $Y = \tilde{Y} T$.*

Since the Möbius group in S^{m+p} is isomorphic to the subgroup $O^+(m+p+2, 1)$ of $O(m+p+2, 1)$, which preserves the positive part of the light cone in R_1^{m+p+2}, we know from Theorem 2.1 that the 2-form

$$\boldsymbol{g} = \langle dY, dY \rangle = \rho^2 dx \cdot dx$$

is a Möbius invariant; see [Blaschke 1929; Chen 1984; Pedit and Willmore 1988; Wang 1998]. We call $\boldsymbol{g}$ the Möbius metric or the Möbius first fundamental form induced by x. Let Δ denote the Laplacian of $\boldsymbol{g}$. Then $\langle \Delta Y, \Delta Y \rangle = 1 + m^2 \kappa$, where κ is the normalized scalar curvature of $\boldsymbol{g}$. By defining

$$N = -\frac{1}{m} \Delta Y - \frac{1}{2m^2}(1 + m^2 \kappa) Y,$$

we have

$$\langle Y, Y \rangle = \langle N, N \rangle = 0 \quad \text{and} \quad \langle Y, N \rangle = 1.$$

Moreover, if we take a local orthonormal basis $\{E_i\}$ with respect to $\boldsymbol{g}$ with dual basis $\{\omega_i\}$ and let $Y_i := E_i(Y)$, then

$$\langle Y_i, Y_j \rangle = \delta_{ij} \quad \text{and} \quad \langle Y_i, Y \rangle = \langle Y_i, N \rangle = 0 \quad \text{for } 1 \le i, j \le m.$$

Let V be the orthogonal complement of span$\{Y, N, Y_i\}$ in R_1^{m+p+2}. Then V is a spacelike subspace of R_1^{m+p+2}, and we have the orthogonal decomposition

$$R_1^{m+p+2} = \text{span}\{Y, N\} \oplus \text{span}\{Y_1, \dots, Y_m\} \oplus V.$$

Let $\{E_\alpha : m+1 \leq \alpha \leq m+p\}$ be an orthogonal basis of V. Then

$$\{Y, N, Y_1, \dots, Y_m, E_{m+1}, \dots, E_{m+p}\}$$

forms a moving frame in R_1^{m+p+2} along M. For ranges of indices, we use the conventions $1 \leq i, j, k, \dots \leq m$ and $m+1 \leq \alpha, \beta, \dots \leq m+p$, and we sum each repeated index over its respective range unless it also appears unrepeated in the same context. Then the structure equations are given by

$$dY = \omega_i Y_i,$$
$$dN = \psi_i Y_i + \phi_\alpha E_\alpha,$$
$$dY_i = -\psi_i Y - \omega_i N + \omega_{ij} Y_j + \omega_{i\alpha} E_\alpha,$$
$$dE_\alpha = -\phi_\alpha Y - \omega_{i\alpha} Y_i + \omega_{\alpha\beta} E_\beta,$$

where $\{\psi_i, \omega_{ij}, \omega_{i\alpha}, \phi_\alpha, \omega_{\alpha\beta}\}$ are 1-forms on M satisfying $\omega_{ij} = -\omega_{ji}$, $\omega_{i\alpha} = -\omega_{\alpha i}$ and $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. Let

$$\psi_i = A_{ij}\omega_j, \quad \omega_{i\alpha} = B_{ij}^\alpha \omega_j, \quad \phi_\alpha = C_i^\alpha \omega_i.$$

It is clear that $A = A_{ij}\omega_i \otimes \omega_j$, $B = B_{ij}^\alpha \omega_i \otimes \omega_j E_\alpha$, and $\Phi = C_i^\alpha \omega_i E_\alpha$ are Möbius invariants; these are called the Blaschke tensor, the Möbius second fundamental form, and the Möbius form, respectively.

Remark 2.2. The relations among A, B, Φ and the Euclidean invariants of x are given by (1-1), (1-2) and (1-3).

Taking the exterior derivations of structure equations, we obtain the integrability conditions for the structure equations as follows:

$$(2\text{-}1) \qquad A_{ij} = A_{ji}, \quad B_{ij}^\alpha = B_{ji}^\alpha,$$

$$(2\text{-}2) \quad A_{ij,k} - A_{ik,j} = (B_{ik}^\alpha C_j^\alpha - B_{ij}^\alpha C_k^\alpha),$$

$$(2\text{-}3) \quad C_{i,j}^\alpha - C_{j,i}^\alpha = (B_{ik}^\alpha A_{kj} - B_{jk}^\alpha A_{ki}),$$

$$(2\text{-}4) \quad B_{ij,k}^\alpha - B_{ik,j}^\alpha = \delta_{ij}C_k^\alpha - \delta_{ik}C_j^\alpha,$$

$$(2\text{-}5) \qquad R_{ijkl} = (B_{ik}^\alpha B_{jl}^\alpha - B_{il}^\alpha B_{jk}^\alpha) + (\delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}),$$

$$(2\text{-}6) \qquad R_{\alpha\beta ij} = (B_{ik}^\alpha B_{kj}^\beta - B_{jk}^\alpha B_{ki}^\beta),$$

$$(2\text{-}7) \qquad R_{ij} = -B_{ik}^\alpha B_{kj}^\alpha + \text{tr}(A)\delta_{ij} + (m-2)A_{ij},$$

$$(2\text{-}8) \qquad \text{tr}(A) = \frac{1}{2m}(1 + m^2\kappa), \quad \sum_i B_{ii}^\alpha = 0, \quad B_{ij}^\alpha B_{ij}^\alpha = (m-1)/m,$$

where R_{ijkl} and $R_{\alpha\beta ij}$ denote the curvature tensor of $\boldsymbol{g}$ and the normal curvature tensor of the normal connection, respectively. We let $\kappa = 1/(m(m-1)) \sum_{i,j} R_{ijij}$ be the normalized scalar curvature of $x : M \to S^{m+p}$. The first covariant derivatives of A_{ij} and B_{ij}^α are defined through

$$(2\text{-}9) \qquad A_{ij,k}\omega_k = dA_{ij} + A_{kj}\omega_{ki} + A_{ik}\omega_{kj},$$

$$(2\text{-}10) \qquad B_{ij,k}^\alpha \omega_k = dB_{ij}^\alpha + B_{kj}^\alpha \omega_{ki} + B_{ik}^\alpha \omega_{kj} + B_{ij}^\beta \omega_{\beta\alpha},$$

From (2-4) and (2-8) we have

$$(2\text{-}11) \qquad B_{ij,i}^\alpha = -(m-1)C_j^\alpha.$$

The second covariant derivative of A_{ij} and B_{ij}^α are defined through

$$A_{ij,kl}\omega_l = dA_{ij,k} + A_{lj,k}\omega_{li} + A_{il,k}\omega_{lj} + A_{ij,l}\omega_{lk},$$

$$B_{ij,kl}^\alpha \omega_l = dB_{ij,k}^\alpha + B_{lj,k}^\alpha \omega_{li} + B_{il,k}^\alpha \omega_{lj} + B_{ij,l}^\alpha \omega_{lk} + B_{ij,k}^\beta \omega_{\beta\alpha}.$$

We have the Ricci identities

$$(2\text{-}12) \qquad A_{ij,kl} - A_{ij,lk} = A_{tj}R_{tikl} + A_{it}R_{tjkl},$$

$$(2\text{-}13) \qquad B_{ij,kl}^\alpha - B_{ij,lk}^\alpha = B_{tj}^\alpha R_{tikl} + B_{it}^\alpha R_{tjkl} + B_{ij}^\beta R_{\beta\alpha kl}.$$

3. The proofs of Theorems 1.1 and 1.2

From the definition of the Laplacian, by using (2-8), (2-4), (2-13) and (2-11) we have

$$(3\text{-}1) \qquad 0 = \tfrac{1}{2}\Delta(B_{ij}^\alpha B_{ij}^\alpha) = B_{ij,k}^\alpha B_{ij,k}^\alpha + B_{ij}^\alpha B_{ij,kk}^\alpha$$
$$= \|\nabla\boldsymbol{B}\|^2 + B_{ij}^\alpha(B_{lk}^\alpha R_{lijk} + B_{il}^\alpha R_{lkjk} + B_{ik}^\beta R_{\beta\alpha jk})$$
$$- m(B_{ij}^\alpha C_i^\alpha)_j - m(m-1)\|\boldsymbol{\Phi}\|^2.$$

Integrating this over M gives a lemma:

Lemma 3.1. *Suppose M is an m-dimensional compact submanifold in the unit sphere S^{m+p}. Then we have*

$$(3\text{-}2) \quad 0 = \int_M \big(\|\nabla\boldsymbol{B}\|^2 - m(m-1)\|\boldsymbol{\Phi}\|^2$$
$$+ B_{ij}^\alpha(B_{lk}^\alpha R_{lijk} + B_{il}^\alpha R_{lkjk} + B_{ik}^\beta R_{\beta\alpha jk})\big)dM.$$

By using (2-5), (2-6) and (2-7), we have the calculation

$$(3\text{-}3) \quad B_{ij}^\alpha(B_{lk}^\alpha R_{lijk} + B_{il}^\alpha R_{lkjk} + B_{ik}^\beta R_{\beta\alpha jk})$$
$$= -\operatorname{tr}\big((B^\alpha B^\beta - B^\beta B^\alpha)(B^\beta B^\alpha - B^\alpha B^\beta)\big)$$
$$- (\operatorname{tr} B^\alpha B^\beta)(\operatorname{tr} B^\alpha B^\beta) + ((m-1)/m)\operatorname{tr}(A) + m\operatorname{tr}(B^\alpha B^\alpha A),$$

where B^α denotes the matrix (B_{ij}^α). Since at least one element of B^α is nonzero, we have the following lemma, which is due to Li A.-M. and Li J.-M. [1992] for $p \geq 2$, and which is obvious for $p = 1$.

Lemma 3.2. *We have*

$$(3\text{-}4) \quad \operatorname{tr}\left((B^\alpha B^\beta - B^\beta B^\alpha)(B^\beta B^\alpha - B^\alpha B^\beta)\right) + (\operatorname{tr} B^\alpha B^\beta)(\operatorname{tr} B^\alpha B^\beta)$$
$$\leq (1 + \tfrac{1}{2}\operatorname{sgn}(p-1))\|\boldsymbol{B}\|^4,$$

with equality holding if and only if either

(i) $p = 1$, *or*

(ii) $p = 2$ *and B^{m+1} and B^{m+2} can be transformed simultaneously by an orthogonal matrix into $\lambda \tilde{B}^{n+1}$ and $\mu \tilde{B}^{n+2}$, respectively, where*

$$(3\text{-}5) \qquad \begin{aligned} \tilde{B}_{12}^{m+1} &= \tilde{B}_{21}^{m+1} = 1 \quad \text{and } \tilde{B}_{ij}^{m+1} = 0 \text{ otherwise,} \\ \tilde{B}_{11}^{m+2} &= -\tilde{B}_{22}^{m+2} = 1 \quad \text{and } \tilde{B}_{ij}^{m+2} = 0 \text{ otherwise,} \end{aligned}$$

and $\lambda^2 = \mu^2$.

Proposition 3.3. *Let M be an m-dimensional compact submanifold in the unit sphere S^{m+p}. If*

$$(3\text{-}6) \quad \kappa - m\|\boldsymbol{\Phi}\|^2 - \frac{m}{4(m-1)}\|\tilde{A} + \check{B}\|^2 \geq \frac{1}{m^2}\left(m - 2 + \tfrac{1}{2}(m-1)\operatorname{sgn}(p-1)\right),$$

then

$$(3\text{-}7) \quad \kappa - m\|\boldsymbol{\Phi}\|^2 - \frac{m}{4(m-1)}\|\tilde{A} + \check{B}\|^2 = \frac{1}{m^2}\left(m - 2 + \tfrac{1}{2}(m-1)\operatorname{sgn}(p-1)\right),$$

and only the following two cases happen:

(i) $p = 1$,

$$(3\text{-}8) \quad \boldsymbol{\Phi} = 0, \quad B_{ij}^{m+1} = \begin{cases} (\sqrt{m-1}/m)\sqrt{(m-k)/k}\,\delta_{ij} & \text{if} \quad 1 \leq i \leq k, \\ -(\sqrt{m-1}/m)\sqrt{k/(m-k)}\,\delta_{ij} & \text{if } k+1 \leq i \leq m, \end{cases}$$

and

$$(3\text{-}9) \qquad A_{ij} = \begin{cases} \dfrac{m-1}{m^2} \cdot \dfrac{3km - k^2 - m^2}{2k(m-k)}\delta_{ij} & \text{if} \quad 1 \leq i \leq k, \\ \dfrac{m-1}{m^2} \cdot \dfrac{m^2 - k^2 - km}{2k(m-k)}\delta_{ij} & \text{if } k+1 \leq i \leq m. \end{cases}$$

(ii) $p = 2$, $m = 2$, *and*

$$(3\text{-}10) \qquad \boldsymbol{\Phi} = 0, \quad A = \frac{3}{16}\boldsymbol{g}, \quad B^3 = \frac{\sqrt{2}}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B^4 = \frac{\sqrt{2}}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof. We consider two traceless tensors

$$\check{\boldsymbol{B}} = \frac{m-1}{m^2}\boldsymbol{g} - B^\alpha B^\alpha \quad \text{and} \quad \tilde{\boldsymbol{A}} = \boldsymbol{A} - \frac{1}{m}\operatorname{tr}(\boldsymbol{A})\boldsymbol{g}.$$

Noting that

$$
\begin{aligned}
(3\text{-}11) \qquad m \cdot \operatorname{tr}(B^\alpha B^\alpha A) &= \frac{m-1}{m}\operatorname{tr}(A) - m\operatorname{tr}(\tilde{\boldsymbol{A}}\check{\boldsymbol{B}}) \\
&= \frac{m-1}{m}\operatorname{tr}(A) - \tfrac{1}{4}m(\|\tilde{\boldsymbol{A}} + \check{\boldsymbol{B}}\|^2 - \|\tilde{\boldsymbol{A}} - \check{\boldsymbol{B}}\|^2) \\
&\geq \frac{m-1}{m}\operatorname{tr}(A) - \tfrac{1}{4}m\|\tilde{\boldsymbol{A}} + \check{\boldsymbol{B}}\|^2,
\end{aligned}
$$

we see that the equality holds if and only if $\check{\boldsymbol{B}} = \tilde{\boldsymbol{A}}$. From (3-1), (3-3) and (3-4), we have

$$
\begin{aligned}
0 \geq \int_M \Big(&\|\nabla \boldsymbol{B}\|^2 - m(m-1)\|\boldsymbol{\Phi}\|^2 - (1 + \tfrac{1}{2}\operatorname{sgn}(p-1))\|\boldsymbol{B}\|^4 \\
&\qquad\qquad + \frac{m-1}{m}\operatorname{tr}(A) + m\operatorname{tr}(B^\alpha B^\alpha A)\Big)dM \\
\geq \int_M \Big(&\|\nabla \boldsymbol{B}\|^2 - m(m-1)\|\boldsymbol{\Phi}\|^2 \\
&\qquad - \tfrac{1}{4}m\|\tilde{\boldsymbol{A}} + \check{\boldsymbol{B}}\|^2 - (1 + \tfrac{1}{2}\operatorname{sgn}(p-1))\|\boldsymbol{B}\|^4 + 2\frac{m-1}{m}\operatorname{tr}(A)\Big)dM \\
\geq \frac{m-1}{m}\int_M \Big(&2\operatorname{tr}(A) - m^2\|\boldsymbol{\Phi}\|^2 - \frac{m^2}{4(m-1)}\|\tilde{\boldsymbol{A}} + \check{\boldsymbol{B}}\|^2 \\
&\qquad\qquad\qquad - (1 + \tfrac{1}{2}\operatorname{sgn}(p-1))\frac{m-1}{m}\Big)dM.
\end{aligned}
$$

Thus, if (3-6) holds all over M, then from (2-8) we know that (3-7) holds all over M and $\nabla \boldsymbol{B} = 0$. The latter implies $\boldsymbol{\Phi} = 0$. At the same time, equality holds in (3-11), and so we have

$$(3\text{-}12) \qquad\qquad \boldsymbol{A} = -B^\alpha B^\alpha + \frac{1}{m}\Big(\operatorname{tr}(A) + \frac{m-1}{m}\Big)\boldsymbol{g}.$$

Noting that the equality holds in (3-5), from Lemma 3.2 we know that only the case $p = 1$ or $p = 2$ is possible. So we consider the cases $p = 1$ and $p = 2$, separately.

Case: $p = 1$. Let $B = B^{m+1}$. We choose our local frame field so that $B_{ij} = B_{ii}\delta_{ij}$. (Here and elsewhere in this case, repeated indexes such as those in B_{ii} are not summed over.) Then from (3-12) we have $A_{ij} = A_{ii}\delta_{ij}$. Since $\nabla \boldsymbol{B} = 0$ and $\boldsymbol{\Phi} = 0$, from (2-4) we know that B_{ii} is constant, $1 \leq i \leq m$, and

$$0 = B_{ij,k}\omega_k = dB_{ij} + B_{ik}\omega_{kj} + B_{kj}\omega_{ki} = (B_{ii} - B_{jj})\omega_{ij},$$

This shows that $\omega_{ij} = 0$ if $B_{ii} \neq B_{jj}$. Hence

$$(3\text{-}13) \qquad -\tfrac{1}{2} R_{ijkl} \omega_k \wedge \omega_l = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = 0.$$

Then (3-13) and the Gauss Equation (2-5) implies that if $B_{ii} \neq B_{jj}$, then

$$(3\text{-}14) \qquad B_{ii} B_{jj} + A_{ii} + A_{jj} = 0.$$

From (3-12), we have

$$(3\text{-}15) \qquad A_{ii} = -(B_{ii})^2 + \tfrac{1}{m}(\operatorname{tr}(A) + (m-1)/m).$$

Then (3-14) and (3-15) yield

$$B_{ii} B_{jj} - (B_{ii})^2 - (B_{jj})^2 + 2t = 0,$$

where $mt = \operatorname{tr}(A) + (m-1)/m$. We will show that there are only two distinct B_{ii}. In fact, if three are distinct — say B_{ii}, B_{jj} and B_{kk} — then we have

$$B_{ii} B_{jj} - (B_{ii})^2 - (B_{jj})^2 + 2t = 0,$$
$$B_{ii} B_{kk} - (B_{ii})^2 - (B_{kk})^2 + 2t = 0,$$
$$B_{jj} B_{kk} - (B_{jj})^2 - (B_{kk})^2 + 2t = 0.$$

This implies

$$(B_{jj} - B_{kk})(B_{ii} - (B_{jj} + B_{kk})) = 0,$$
$$(B_{ii} - B_{kk})(B_{jj} - (B_{ii} + B_{kk})) = 0,$$
$$(B_{ii} - B_{jj})(B_{kk} - (B_{ii} + B_{jj})) = 0,$$

which yields

$$B_{ii} - (B_{jj} + B_{kk}) = 0,$$
$$B_{jj} - (B_{ii} + B_{kk}) = 0,$$
$$B_{kk} - (B_{ii} + B_{jj}) = 0.$$

These equations lead to $B_{ii} = 0$, $B_{jj} = 0$, and $B_{kk} = 0$, a contradiction.

Thus we may assume $B_{11} = \cdots = B_{kk} = \lambda$ and $B_{k+1 k+1} = \cdots = B_{mm} = \mu$. We see that λ, μ, A_{ii} and A_{jj} satisfy the following relations for $1 \leq i \leq k$ and $k+1 \leq j \leq m$:

$$\begin{cases} \lambda\mu - \lambda^2 - \mu^2 + \tfrac{2}{m}(\operatorname{tr}(A) + (m-1)/m) = 0, \\ k\lambda + (m-k)\mu = 0, \\ k\lambda^2 + (m-k)\mu^2 = (m-1)/m, \\ A_{ii} = -\lambda^2 + \tfrac{1}{m}(\operatorname{tr}(A) + (m-1)/m) \quad \text{for } 1 \leq i \leq k, \\ A_{jj} = -\mu^2 + \tfrac{1}{m}(\operatorname{tr}(A) + (m-1)/m) \quad \text{for } k+1 \leq j \leq m, \\ \operatorname{tr}(A) = \sum_l A_{ll} = k A_{ii} + (m-k) A_{jj}. \end{cases}$$

From this system of equations, we get

$$\lambda = \frac{\sqrt{m-1}}{m}\sqrt{\frac{m-k}{k}}, \qquad \mu = -\frac{\sqrt{m-1}}{m}\sqrt{\frac{k}{m-k}},$$

$$A_{ii} = \frac{m-1}{m^2} \cdot \frac{3mk-k^2-m^2}{2k(m-k)}, \qquad A_{jj} = \frac{m-1}{m^2} \cdot \frac{m^2-k^2-mk}{2k(m-k)},$$

where $1 \le i \le k$ and $k+1 \le j \le m$.

Case: $p = 2$. In this case, $B^{m+1} = \lambda \tilde{B}^{m+1}$ and $B^{m+2} = \mu \tilde{B}^{m+2}$, where $\tilde{B}^{m+1}$ and $\tilde{B}^{m+2}$ are defined in [Lemma 3.2](#) and $\lambda^2 = \mu^2$. Noting that $2(\lambda^2 + \mu^2) = \|B\|^2 = (m-1)/m$, we have $2\lambda = 2\mu = \sqrt{(m-1)/m}$. We get

$$(3\text{-}16) \qquad B^{m+1} = \frac{1}{2}\sqrt{\frac{m-1}{m}}\tilde{B}^{m+1} \quad \text{and} \quad B^{m+2} = \frac{1}{2}\sqrt{\frac{m-1}{m}}\tilde{B}^{m+2}.$$

Since $B^\alpha_{ij,k} = 0$, we have from [(2-10)](#)

$$(3\text{-}17) \qquad dB^\alpha_{ij} = -B^\alpha_{kj}\omega_{ki} - B^\alpha_{ik}\omega_{kj} - B^\beta_{ij}\omega_{\beta\alpha}.$$

Setting $\alpha = m+1$, $i = 1$ and $j \ge 3$, we see that by using [(3-16)](#) and [(3-17)](#) we get

$$\omega_{2j} = 0 \quad \text{and} \quad j \ge 3.$$

Setting $\alpha = m+1$, $i = 2$ and $j \ge 3$, we see from [(3-16)](#) and [(3-17)](#) that

$$\omega_{1j} = 0 \quad \text{and} \quad j \ge 3.$$

Thus if $j \ge 3$, then

$$0 = d\omega_{1j} = \omega_{1k} \wedge \omega_{kj} - (A_{11} + A_{jj})\omega^1 \wedge \omega^j = -(A_{11} + A_{jj})\omega^1 \wedge \omega^j.$$

Since $\omega^1, \ldots, \omega^m$ are orthonormal, $\omega^1 \wedge \omega^j = 0$ implies $\omega^j = 0$ for $j \ge 3$. Thus this shows that $m = 2$. Thus we have

$$A = \tfrac{3}{16}g, \quad B^3 = \frac{\sqrt{2}}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B^4 = \frac{\sqrt{2}}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This completes the proof of [Proposition 3.3](#). □

Proof of [Theorem 1.1](#). We first consider some examples.

Example 3.4. First, the Willmore torus $W^m_k = S^k(\sqrt{(m-k)/m}) \times S^{m-k}(\sqrt{k/m})$ in S^{m+1}. In [Guo et al. 2001], we proved that the Möbius invariants Φ, A and B of W^m_k locally coincide with [(3-8)](#) and [(3-9)](#).

Example 3.5. Second, the Veronese surface in S^4. Let (x, y, z) be the natural coordinate system in R^5. The mapping defined by

$$u^1 = \tfrac{1}{\sqrt{3}}yz, \ u^2 = \tfrac{1}{\sqrt{3}}zx, \ u^3 = \tfrac{1}{\sqrt{3}}xy, \ u^4 = \tfrac{1}{2\sqrt{3}}(x^2 - y^2), \ u^5 = \tfrac{1}{6}(x^2 + y^2 - 2z^2)$$

gives an immersion of $S^2(\sqrt{3})$ into S^4. We can choose the local frame field $\{e_1, e_2, e_3, e_4\}$ so that

$$(h_{ij}^3) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad (h_{ij}^4) = -\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus $H = 0$ and $\rho^2 = 8/3$. Putting these into (1-1), (1-2) and (1-3), we have

$$\boldsymbol{\Phi} = 0, \quad A = \tfrac{3}{16} \boldsymbol{g}, \quad B^3 = \frac{\sqrt{2}}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B^4 = \frac{\sqrt{2}}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is known that if two submanifolds in S^{m+p} have the same Möbius metric, Möbius second fundamental form, and Blaschke tensor, then they are locally Möbius equivalent. Since we assume M is compact in Theorem 1.1, we complete the proof by comparing (3-8), (3-9) and (3-10) with Examples 3.4 and 3.5. $\quad\square$

Proposition 3.6. *Let $x : M \to S^{m+p}$ be an m-dimensional compact submanifold in the unit sphere S^{m+p}. Let $K(x)$ be the function that assigns to each point of M the infimum of the Möbius sectional curvatures of M at that point. Then*

$$(3\text{-}18) \qquad \int_M \left(2(m+1)K - m\kappa - 1/m - m(m+2)\|\boldsymbol{\Phi}\|^2 - m\|\tilde{A}\| \right) dM \le 0.$$

In particular, if

$$(3\text{-}19) \qquad 2(m+1)K - m\kappa - m(m+2)\|\boldsymbol{\Phi}\|^2 - m\|\tilde{A}\| \ge 1/m,$$

then

$$(3\text{-}20) \qquad 2(m+1)K - m\kappa - m(m+2)\|\boldsymbol{\Phi}\|^2 - m\|\tilde{A}\| = 1/m,$$

and

$$K = \kappa = \frac{1}{m(m+2)}, \quad \boldsymbol{\Phi} = 0, \quad A = \tfrac{1}{m} \operatorname{tr}(A) \boldsymbol{g}.$$

Proof. In the proof, we will use Yau's technique; see [Yau 1975; Itoh 1975; 1978]. By defining

$$(3\text{-}21) \qquad \|R^{\perp}\|^2 = \operatorname{tr}((B^{\alpha} B^{\beta} - B^{\beta} B^{\alpha})(B^{\beta} B^{\alpha} - B^{\alpha} B^{\beta}))$$

and noting that $B_{jl}^{\alpha} B_{lk}^{\beta} R_{\beta\alpha jk} = -\frac{1}{2}\|R^{\perp}\|^2$, from (3-2) and (3-3) we have

$$(3\text{-}22) \quad 0 = \int_M \big(\|\nabla \boldsymbol{B}\|^2 - m(m-1)\|\boldsymbol{\Phi}\|^2 + (1+a)(B_{kj}^{\alpha} B_{ml}^{\alpha} R_{mkjl} + B_{jk}^{\alpha} B_{km}^{\alpha} R_{ljlm})$$
$$+ a \operatorname{tr}(B^{\alpha} B^{\beta}) \operatorname{tr}(B^{\alpha} B^{\beta}) - a(2(m-1)/m) \operatorname{tr}(A)$$
$$- am \operatorname{tr}(B^{\alpha} B^{\alpha} \tilde{A}) - \tfrac{1}{2}(1-a)\|R^{\perp}\|^2 \big) dM,$$

where a is a real number.

For each fixed $\bar{\alpha}$, let α_i be the eigenvalues of the matrix $(B_{ij}^{\bar{\alpha}})$. By the definition of K, we have

$$
\begin{aligned}
(B_{kj}^{\bar{\alpha}} B_{ml}^{\bar{\alpha}} R_{mkjl} + B_{jk}^{\bar{\alpha}} B_{km}^{\bar{\alpha}} R_{ljlm}) &= \tfrac{1}{2}(\alpha_i - \alpha_j)(\alpha_i - \alpha_j) R_{ijij} \\
&\geq \tfrac{1}{2}(\alpha_i - \alpha_j)(\alpha_i - \alpha_j) K \\
&= m K B_{ij}^{\bar{\alpha}} B_{ij}^{\bar{\alpha}}.
\end{aligned}
$$
(3-23)

The following equality is called the Lincoln equality (also see [Itoh 1978]):

$$
\|R^{\perp}\|^2 \leq m \operatorname{tr}(B^{\alpha} B^{\beta}) \operatorname{tr}(B^{\alpha} B^{\beta}),
$$
(3-24)

where the equality implies

$$
(B_{ik}^{\alpha} B_{jl}^{\alpha} - B_{il}^{\alpha} B_{jk}^{\alpha}) = -\frac{1}{m^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).
$$
(3-25)

Hence, for $a \geq 0$, by putting (3-23) and (3-24) into (3-22), we have

$$
\begin{aligned}
0 \geq \int_M \big(\|\nabla \boldsymbol{B}\|^2 &- m(m-1)\|\boldsymbol{\Phi}\|^2 + (1+a)(m-1)K \\
&- a(2((m-1)/m)\operatorname{tr}(A) + m \operatorname{tr}(B^{\alpha} B^{\alpha} \tilde{A})) \\
&\qquad\qquad + (a/m - \tfrac{1}{2}(1-a))\|R^{\perp}\|^2 \big) dM.
\end{aligned}
$$
(3-26)

By Schwarz's inequality, we have

$$
\operatorname{tr}(\tilde{A} B^{\alpha} B^{\alpha}) \leq \|\tilde{A}\| \|B^{\alpha}\|^2,
$$
(3-27)

where the equality holds when there exist real numbers a^{α} such that

$$
A = a^{\alpha}(B^{\alpha})^2 \quad \text{for } m+1 \leq \alpha \leq m+p.
$$

Now by taking $a = m/(m+2)$ in (3-25), and (2-8) and (3-27), we have

$$
\begin{aligned}
0 \geq \int_M \Big(\|\nabla \boldsymbol{B}\|^2 &- m(m-1)\|\boldsymbol{\Phi}\|^2 \\
&+ \frac{m-1}{m+2}\big(2(m+1)K - m\kappa - \tfrac{1}{m} - m\|\tilde{A}\|\big)\Big) dM
\end{aligned}
$$
(3-28)

Thus we get (3-18). If (3-19) holds, then we have (3-20), $\nabla \boldsymbol{B} = 0$ and the equality holding in (3-6). These implies $\boldsymbol{\Phi} = 0$ and $\tilde{A} = 0$. From (2-2), we get $A_{ij} = \lambda \delta_{ij}$, $\lambda = \operatorname{tr}(A)/m = \text{constant}$. Now (3-25) holds since equality holds in (3-24). Hence, from (2-5) we know that the Möbius sectional curvature is constant. This implies that $K = \kappa$. Substituting these into (3-20), we have $K = \kappa = 1/(m(m+2))$. □

Proof of Theorem 1.2. Under the assumptions of Theorem 1.2, from Proposition 3.6 we have $\boldsymbol{\Phi} = 0$, $\tilde{A} = 0$ and $\operatorname{tr}(A) = (1 + m^2\kappa)/(2m) > 0$. According to the classification theorem on Möbius isotropic submanifolds, proved by Liu, Wang, and Zhao [2001], M is Möbius equivalent to a minimal (Euclidean) submanifold $\overline{M}$ with constant Euclidean scalar curvature in S^{m+p}. Since $\overline{M}$ shares same A, $\boldsymbol{B}$

and K with M, we can use the same symbols to denote the Möbius quantities of $\overline{M}$. Now we can compute the Euclidean sectional curvature K_E of $\overline{M}$. In fact, since $\overline{M}$ is Euclidean minimal and its Euclidean scalar curvature is constant, we have $\rho^2 = $ constant. From (1-2), we have $A_{ij} = \frac{1}{2}\rho^{-2}\delta_{ij}$. From $g = \rho^2 dx \cdot dx$, we have

$$(3\text{-}29) \qquad\qquad K = \rho^{-2}K_E,$$

Noting that $\rho^{-2} = 2/m\,\mathrm{tr}\,(A)$ and $\mathrm{tr}(A) = (1 + m^2\kappa)/(2m)$, we have

$$(3\text{-}30) \qquad\qquad \rho^{-2} = 1/m^2 + \kappa.$$

From Proposition 3.6 we have $\kappa = K = 1/(m(m+2))$. Hence, from (3-29) and (3-30) we get $K_E = m/(2(m+1))$. Since $\overline{M}$ is a Euclidean minimal submanifold with sectional curvature K_E in S^{m+p}, Itoh's theorem [1978] shows that $\overline{M}$ is isometric to the Veronese submanifold $S^m(\sqrt{2(m+1)/m})$. Thus M is Möbius equivalent to Veronese submanifold

$$x_m : S^m(\sqrt{2(m+1)/m}) \to S^{m+p},$$

where $p = \frac{1}{2}(m-1)(m+2)$. This proves Theorem 1.2. $\square$

Proof of Corollary 1.3. We assume $c \geq 1/(m(m+2))$. Under the assumptions of Corollary 1.3, we have $\Phi = 0$, $\kappa = c$ and $K = c \geq 1/(m(m+2))$. Writing $A_{ij} = \lambda_i\delta_{ij}$, from (2-2) and (2-12), we have

$$(3\text{-}31) \qquad \tfrac{1}{2}\Delta\|\tilde{A}\|^2 = A_{ijk}A_{ijk} + \tfrac{1}{2}\sum_{i,j}(\lambda_i - \lambda_j)^2 R_{ijij},$$

where $\tilde{A}_{ij} = A_{ij} - (\mathrm{tr}\,A/m)\delta_{ij}$ and $\|\tilde{A}\|^2 = \tilde{A}_{ij}\tilde{A}_{ij}$. Thus, from (3-31) we have

$$\tilde{A} = 0.$$

In this case, (1-6) is equivalent to $c \geq 1/(m(m+2))$. Thus from Theorem 1.2, M is Möbius equivalent to Veronese submanifold $S^m(\sqrt{2(m+1)/m})$. $\square$

4. Möbius sectional curvature pinching theorems

Theorem 4.1. *Let $x : M \to S^{m+p}$ be an m-dimensional compact submanifold in the unit sphere S^{m+p}. Let $K(x)$ be the function that assigns to each point of M the infimum of the Möbius sectional curvatures of M at that point. Then*

$$(4\text{-}1) \qquad \int_M \left(K - \frac{m-1}{2m^2}\left(1 - \tfrac{1}{p} + \tfrac{1}{2}\,\mathrm{sgn}(p-1)\right) - m\|\boldsymbol{\Phi}\|^2 - 2\|\tilde{A}\|\right)dM \leq 0.$$

In particular, if

$$(4\text{-}2) \qquad K \geq \frac{m-1}{2m^2}\left(1 - \tfrac{1}{p} + \tfrac{1}{2}\,\mathrm{sgn}(p-1)\right) + m\|\boldsymbol{\Phi}\|^2 + 2\|\tilde{A}\|,$$

then either

(i) $p = 1$, $K = 0$ and M is Möbius equivalent to the Clifford minimal torus $M_k^m = S^k(\sqrt{k/(m-k)}) \times S^{m-k}(\sqrt{(m-k)/m})$ in S^{m+1}, or

(ii) $m = 2$, $p = 2$, $K = 1/8$ and M is Möbius equivalent to a Veronese surface in S^4.

Theorem 4.2. *Let* $x : M \to S^{m+p}$ *for* $p \geq 2$ *be an* m*-dimensional compact submanifold in the unit sphere* S^{m+p}. *Let* $K(x)$ *be the function that assigns to each point of* M *the infimum of the Möbius sectional curvatures of* M *at that point. Then*

$$(4\text{-}3) \qquad \int_M \left(K - \frac{(m-1)(3p-2)}{4pm^2} - m\|\Phi\|^2 \right) dM \leq 0.$$

In particular, if

$$(4\text{-}4) \qquad K \geq (m-1)(3p-2)/(4pm^2) + m\|\Phi\|^2,$$

then $m = 2$, $p = 2$, $K = 1/8$ *and* M *is Möbius equivalent to a Veronese surface in* S^4.

Proof of Theorem 4.1. Choosing $a = 1$ in (3-22), using (3-23) and (2-8) we have

$$
\begin{aligned}
0 = \int_M \Big(& \|\nabla \boldsymbol{B}\|^2 - m(m-1)\|\Phi\|^2 \\
& + 2(B_{kj}^\alpha B_{ml}^\alpha R_{mkjl} + B_{jk}^\alpha B_{km}^\alpha R_{ljlm}) \\
& + \operatorname{tr}(B^\alpha B^\beta)\operatorname{tr}(B^\alpha B^\beta) - \tfrac{1}{m}2(m-1)\operatorname{tr}(A) - m\operatorname{tr}(B^\alpha B^\alpha \tilde{A}) \Big) dM \\
\geq \int_M \Big(& \|\nabla \boldsymbol{B}\|^2 - m(m-1)\|\Phi\|^2 + 2(m-1)K \\
& + \frac{1}{p}\left(\frac{m-1}{m}\right)^2 - \frac{2(m-1)}{m}\left(\frac{1}{2m}(1+m^2\kappa)\right) - (m-1)\|\tilde{A}\| \Big) dM \\
= \int_M \Big(& \|\nabla \boldsymbol{B}\|^2 - m(m-1)\|\Phi\|^2 + 2(m-1)K \\
& + \frac{m-1}{m^2}\left(\frac{m-1}{p} - 1\right) - (m-1)\kappa - (m-1)\|\tilde{A}\| \Big) dM.
\end{aligned}
$$

Using (1-4), we get (4-1). If (4-2) holds, we have $\|\nabla \boldsymbol{B}\| = 0$ on M. The conclusion of Theorem 4.1 follows in the same way as in the proof of Theorem 1.1. □

Proof of Theorem 4.2. Choosing $a = 0$ in (3-22) and using (3-23) and (2-8), we have

$$(4\text{-}5) \qquad 0 \geq \int_M \left(\|\nabla \boldsymbol{B}\|^2 - m(m-1)\|\Phi\|^2 + (m-1)K - \tfrac{1}{2}\|R^\perp\|^2 \right) dM.$$

Combining (4-5), (3-21) with Lemma 3.2, we have

$$
\begin{aligned}
0 \geq \int_M \Big(& \|\nabla \boldsymbol{B}\|^2 - m(m-1)\|\boldsymbol{\Phi}\|^2 + (m-1)K \\
& - \tfrac{1}{2}\big(\tfrac{3}{2}\|\boldsymbol{B}\|^4 - \operatorname{tr}(B^\alpha B^\beta B^\alpha B^\beta)\big)\Big)dv \\
\geq \int_M \Big(& \|\nabla \boldsymbol{B}\|^2 - m(m-1)\|\boldsymbol{\Phi}\|^2 + (m-1)K \\
& - \frac{3}{4}\Big(\frac{m-1}{m}\Big)^2 + \frac{1}{2p}\Big(\frac{m-1}{m}\Big)^2\Big)dM.
\end{aligned}
$$

Thus we get (4-3). If (4-4) holds on M, then we have $\|\nabla \mathbf{B}\| = 0$ on M. The conclusion of Theorem 4.2 then follows as for Theorem 1.1.   □

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