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We prove that the ∂ and $\bar{\partial}$ operators introduced by Gualtieri for a generalized complex structure coincide with the $\breve{d}_*$ and $\breve{\partial}$ operators introduced by Alekseev and Xu for Evens–Lu–Weinstein modules of a Lie bialgebroid.

Introduction

Generalized complex structures [Hitchin 2003; Gualtieri 2003; Cavalcanti 2007] have been extensively studied recently due to their close connection with mirror symmetry. They include both symplectic and complex structures as extreme cases. Gualtieri [2003; 2007] defined the ∂ and $\bar{\partial}$ operators for any twisted generalized complex structure in the same way that these operators are defined in complex geometry. In fact, he proved that an H-twisted generalized complex structure $\mathbb{J}$ determines an alternative grading of differential forms and a splitting $d^H = \partial + \bar{\partial}$, where $d^H = d - H\wedge$ is the de Rham differential twisted by a closed three-form H.

A Lie bialgebroid, as introduced by Mackenzie and Xu [1994], is a pair of Lie algebroids (A, A^*) satisfying some compatibility condition; see also [Kosmann-Schwarzbach 1995]. They appear naturally in many places in Poisson geometry. In [Alekseev and Xu 2001], two differential operators $\breve{d}_*$ and $\breve{\partial}$ were introduced for Evens–Lu–Weinstein modules of a Lie bialgebroid, as follows.

We consider a pair of (real or complex) Lie algebroid structures on a vector bundle A and its dual A^*, and we assume that the (real or complex) line bundle $\mathscr{L} = (\bigwedge^{\mathrm{top}} A^* \otimes \bigwedge^{\mathrm{top}} T^*M)^{1/2}$ exists. Then $\mathscr{L}$ is a module over A^*, as discovered by Evens, Lu and Weinstein [Evens et al. 1999]. The Lie algebroid structures of A^* and A induce two natural differential operators $\breve{d}_* : \Gamma(\bigwedge^k A \otimes \mathscr{L}) \to \Gamma(\bigwedge^{k+1} A \otimes \mathscr{L})$ and $\breve{\partial} : \Gamma(\bigwedge^k A \otimes \mathscr{L}) \to \Gamma(\bigwedge^{k-1} A \otimes \mathscr{L})$; see Equations (2-1) through (2-7).

Since a generalized complex structure $\mathbb{J}$ induces a (complex) Lie bialgebroid $(L, \bar{L})$, where L and $\bar{L}$ are respectively the $+i$ and $-i$ eigenspaces of $\mathbb{J}$, it is tempting to investigate the relations between the operators ∂, $\bar{\partial}$, $\breve{d}_*$ and $\breve{\partial}$. In this

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note, we show that ∂ and $\bar{\partial}$ essentially coincide with $\breve{d}_*$ and $\breve{\partial}$ respectively, under some natural isomorphisms.

1. Courant algebroids and Lie bialgebroids

In this article, all vector bundles are complex vector bundles. Likewise, Lie algebroids are always complex Lie algebroids.

A (complex) Courant algebroid consists of a vector bundle $\pi : E \to M$, a nondegenerate pseudometric $\langle \cdot\,, \cdot \rangle$ on the fibers of π, a bundle map $\rho : E \to T_{\mathbb{C}}M$, called the anchor, and a $\mathbb{C}$-bilinear operation $\circ$ on $\Gamma(E)$ called the Dorfman bracket, which for all $f \in C^\infty(M, \mathbb{C})$ and $z_1, z_2, z_3 \in \Gamma(E)$ satisfy the relations

$$z_1 \circ (z_2 \circ z_3) = (z_1 \circ z_2) \circ z_3 + z_2 \circ (z_1 \circ z_3), \tag{1-1}$$

$$\rho(z_1 \circ z_2) = [\rho(z_1), \rho(z_2)], \tag{1-2}$$

$$z_1 \circ f z_2 = (\rho(x) f) z_2 + f(z_1 \circ z_2), \tag{1-3}$$

$$z_1 \circ z_2 + z_2 \circ z_1 = 2\mathfrak{D}\langle z_1, z_2 \rangle, \tag{1-4}$$

$$\mathfrak{D} f \circ z_1 = 0, \tag{1-5}$$

$$\rho(z_1)\langle z_2, z_3 \rangle = \langle z_1 \circ z_2, z \rangle + \langle z_2, z_1 \circ z_3 \rangle, \tag{1-6}$$

where $\mathfrak{D} : C^\infty(M, \mathbb{C}) \to \Gamma(E)$ is the $\mathbb{C}$-linear map defined by $\langle \mathfrak{D} f, z_1 \rangle = \frac{1}{2}\rho(z_1) f$.

The symmetric part of the Dorfman bracket is given by (1-4). The Courant bracket is defined as the skew-symmetric part $[\![z_1, z_2]\!] = \frac{1}{2}(z_1 \circ z_2 - z_2 \circ z_1)$ of the Dorfman bracket. Thus we have the relation $z_1 \circ z_2 = [\![z_1, z_2]\!] + \mathfrak{D}\langle z_1, z_2 \rangle$.

The definition of a Courant algebroid can be rephrased using the Courant bracket instead of the Dorfman bracket [Roytenberg 1999].

A Dirac structure is a smooth subbundle $A \to M$ of the Courant algebroid E; it is maximally isotropic with respect to the pseudometric and its space of sections is closed under (necessarily both) brackets. Thus a Dirac structure inherits a canonical Lie algebroid structure [Liu et al. 1997].

Let $A \to M$ be a vector bundle. Assume that A and its dual A^* both carry a Lie algebroid structure with anchor maps $a : A \to T_{\mathbb{C}}M$ and $a_* : A^* \to T_{\mathbb{C}}M$, brackets on sections

$$\Gamma(A) \otimes_{\mathbb{C}} \Gamma(A) \to \Gamma(A) : X \otimes Y \mapsto [X, Y],$$

$$\Gamma(A^*) \otimes_{\mathbb{C}} \Gamma(A^*) \to \Gamma(A^*) : \theta \otimes \xi \mapsto [\theta, \xi]_*,$$

and differentials $d : \Gamma(\bigwedge^\bullet A^*) \to \Gamma(\bigwedge^{\bullet+1} A^*)$ and $d_* : \Gamma(\bigwedge^\bullet A) \to \Gamma(\bigwedge^{\bullet+1} A)$.

By [Kosmann-Schwarzbach 1995; Mackenzie and Xu 2000; 1994], this pair (A, A^*) of Lie algebroids is a Lie bialgebroid (or Manin triple) if d_* is a derivation of the Gerstenhaber algebra $(\Gamma(\bigwedge^\bullet A), \wedge, [\cdot\,, \cdot])$ or, equivalently, if d is a derivation of the Gerstenhaber algebra $(\Gamma(\bigwedge^\bullet A^*), \wedge, [\cdot\,, \cdot]_*)$. The link between Courant and Lie bialgebroids is as follows.

Theorem 1.1 [Liu et al. 1997]. *There is a one-to-one correspondence between Lie bialgebroids and pairs of transversal Dirac structures in a Courant algebroid.*

More precisely, if the pair (A, A^*) is a Lie bialgebroid, then the vector bundle $A \oplus A^* \to M$, together with the pseudometric

$$\langle X_1+\xi_1, X_2+\xi_2\rangle = \tfrac{1}{2}(\xi_1(X_2)+\xi_2(X_1)), \tag{1-7}$$

the anchor map $\rho = a + a_*$ (whose dual is given through $\mathcal{D}f = df + d_*f$ for $f \in C^\infty(M,\mathbb{C})$), and the Dorfman bracket

$$\begin{aligned}(X_1+\xi_1)\circ(X_2+\xi_2) = \\ ([X_1, X_2]+\mathcal{L}_{\xi_1}X_2 - \iota_{\xi_2}(d_*X_1)) + ([\xi_1,\xi_2]_* + \mathcal{L}_{X_1}\xi_2 - \iota_{X_2}(d\xi_1)),\end{aligned} \tag{1-8}$$

is a Courant algebroid of which A and A^* are transverse Dirac structures. It is called the double of the Lie bialgebroid (A, A^*). Here X_1 and X_2 denote arbitrary sections of A, and ξ_1 and ξ_2 arbitrary sections of A^*.

An important example is when $A = T_{\mathbb{C}}M$ is the tangent bundle of a manifold M and $A^* = T^*_{\mathbb{C}}M$ takes the trivial Lie algebroid structure. Then $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ has the standard Courant algebroid structure. Severa and Weinstein [2001] observed that the Dorfman bracket on $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ can be twisted by a closed three-form $H \in Z^3(M)$:

$$\begin{aligned}(x_1+\eta_1)\circ_H(x_2+\eta_2) &= (x_1+\eta_1)\circ(x_2+\eta_2) + \iota_{x_2}\iota_{x_1}H \\ &= [x_1,x_2] + \mathcal{L}_{x_1}\eta_2 - \mathcal{L}_{x_2}\eta_1 + \tfrac{1}{2}d\langle\eta_1|x_2\rangle + \iota_{x_2}\iota_{x_1}H.\end{aligned}$$

And $\circ_H$ defines a Courant algebroid structure on $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$, using the same inner product and anchor. The corresponding Courant bracket is also twisted:

$$\begin{aligned}&[\![x_1+\eta_1, x_2+\eta_2]\!]_H \\ &= [\![x_1+\eta_1, x_2+\eta_2]\!] + \iota_{x_2}\iota_{x_1}H \\ &= [x_1,x_2] + \mathcal{L}_{x_1}\eta_2 - \mathcal{L}_{x_2}\eta_1 + \tfrac{1}{2}d(\langle\eta_1|x_2\rangle - \langle\eta_2|x_1\rangle) + \iota_{x_2}\iota_{x_1}H.\end{aligned} \tag{1-9}$$

2. Clifford modules and Dirac generating operators

Let V be a vector space of dimension r endowed with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. Its Clifford algebra $\mathcal{C}(V)$ is defined as the quotient of the tensor algebra $\bigoplus_{k=0}^r V^{\otimes r}$ by the relations $x\otimes y + y\otimes x = 2\langle x, y\rangle$, with $x, y \in V$. It is naturally an associative $\mathbb{Z}_2$-graded algebra. Up to isomorphisms, there exists a unique irreducible module S of $\mathcal{C}(V)$, called spin representation [Chevalley 1997]. The vectors of S are called spinors.

Example 2.1. Let W be a vector space of dimension r. We can endow $V = W \oplus W^*$ with the nondegenerate pairing defined in the same fashion as in Equation (1-7).

The representation of $\mathscr{C}(V)$ on $S = \bigoplus_{k=0}^{r} \bigwedge^k W$ defined by $u \cdot w = u \wedge w$ and $\xi \cdot w = \iota_\xi w$, where $u \in W$, $\xi \in W^*$ and $w \in S$, is the spin representation. Note that S is $\mathbb{Z}$-graded and thus also $\mathbb{Z}_2$-graded.

Now let $\pi : E \to M$ be a vector bundle endowed with a nondegenerate pseudometric $\langle \cdot , \cdot \rangle$ on its fibers, and let $\mathscr{C}(E) \to M$ be the associated bundle of Clifford algebras. Assume there exists a smooth vector bundle $S \to M$ whose fiber S_m over a point $m \in M$ is the spin module of the Clifford algebra $\mathscr{C}(E)_m$. Assume furthermore that S is $\mathbb{Z}_2$-graded, that is, $S = S^0 \oplus S^1$.

An operator O on $\Gamma(S)$ is called even (or of degree 0) if $O(S^i) \subset S^i$ and odd (or of degree 1) if $O(S^i) \subset S^{i+1}$. Here $i \in \mathbb{Z}_2$.

Example 2.2. If the vector bundle E decomposes as the direct sum $A \oplus A^*$ of two transverse Lagrangian subbundles as in Example 2.1, then $S = \bigwedge A$. The multiplication by a function $f \in C^\infty(M, \mathbb{C})$ is an even operator on $\Gamma(S)$, while the Clifford action of a section $e \in \Gamma(E)$ is an odd operator on $\Gamma(S)$.

If O_1 and O_2 are operators of degree d_1 and d_2 respectively, then their commutator is the operator $[O_1, O_2] = O_1 \circ O_2 - (-1)^{d_1 d_2} O_2 \circ O_1$.

Definition 2.3 [Alekseev and Xu 2001]. A Dirac generating operator for $(E, \langle\ ,\ \rangle)$ is an odd operator D on $\Gamma(S)$ satisfying the following properties:

(1) $[D, f] \in \Gamma(E)$ for all $f \in C^\infty(M, \mathbb{C})$. This means that the operator $[D, f]$ is the Clifford action of some section of E.

(2) $[[D, z_1], z_2] \in \Gamma(E)$ for all $z_1, z_2 \in \Gamma(E)$.

(3) The square of D is multiplication by some function on M, that is, D^2 is in $C^\infty(M, \mathbb{C})$.

Note that "deriving operators" of [Kosmann-Schwarzbach 2005] do not require assumption (3).

Theorem 2.4 [Alekseev and Xu 2001]. *Let D be a Dirac generating operator for a vector bundle $\pi : E \to M$. Then there is a canonical Courant algebroid structure on E. The anchor $\rho : E \to T_{\mathbb{C}} M$ is defined by $\rho(z) f = 2\langle [D, f], z\rangle = [[D, f], z]$, while the Dorfman bracket reads $z_1 \circ z_2 = [[D, z_1], z_2]$.*

We follow the same setup as in [Alekseev and Xu 2001; Chen and Stiénon 2009].

Let $(A, [\cdot , \cdot], a)$ and $(A^*, [\cdot , \cdot]_*, a_*)$ be a pair of Lie algebroids, where A is of rank r and the base manifold M is of dimension m. Then the line bundle $\bigwedge^r A^* \otimes \bigwedge^m T^*_{\mathbb{C}} M$ is a module over the Lie algebroid A^* [Evens et al. 1999]: A section $\alpha \in \Gamma(A^*)$ acts on $\Gamma(\bigwedge^r A^* \otimes \bigwedge^m T^*_{\mathbb{C}} M)$ by

$$\nabla_\alpha(\alpha_1 \wedge \cdots \wedge \alpha_r \otimes \mu) = \sum_{i=1}^{r} \big(\alpha_1 \wedge \cdots \wedge [\alpha, \alpha_i]_* \wedge \cdots \wedge \alpha_r \otimes \mu\big) + \alpha_1 \wedge \cdots \wedge \alpha_r \otimes \mathscr{L}_{a_*(\alpha)} \mu. \tag{2-1}$$

If it exists, the square root $\mathscr{L} = (\bigwedge^r A^* \otimes \bigwedge^m T^*_{\mathbb{C}} M)^{1/2}$ of this line bundle is also a module over A^*. Here $\mathscr{L}$ is a (complex) vector bundle whose square, $\mathscr{L}^2 = \mathscr{L} \otimes \mathscr{L}$, is isomorphic to $\bigwedge^r A^* \otimes \bigwedge^m T^*_{\mathbb{C}} M$. The A^*-module structure of $\mathscr{L}$ is illustrated in [Evens et al. 1999, Proposition 4.3].

One can thus define a differential operator

$$\breve{d}_* : \Gamma(\textstyle\bigwedge^k A \otimes \mathscr{L}) \to \Gamma(\bigwedge^{k+1} A \otimes \mathscr{L}). \tag{2-2}$$

Similarly, $(\bigwedge^r A \otimes \bigwedge^m T^*_{\mathbb{C}} M)^{1/2}$ is — provided it exists — a module over A. Hence we obtain a differential operator

$$\Gamma(\textstyle\bigwedge^k A^* \otimes (\bigwedge^r A \otimes \bigwedge^m T^*_{\mathbb{C}} M)^{1/2}) \to \Gamma(\bigwedge^{k+1} A^* \otimes (\bigwedge^r A \otimes \bigwedge^m T^*_{\mathbb{C}} M)^{1/2}). \tag{2-3}$$

But the isomorphisms of vector bundles

$$\textstyle\bigwedge^k A^* \cong \bigwedge^k A^* \otimes \bigwedge^{r-k} A^* \otimes \bigwedge^{r-k} A \cong \bigwedge^{r-k} A \otimes \bigwedge^r A^* \tag{2-4}$$

and

$$\textstyle\bigwedge^r A^* \otimes (\bigwedge^r A \otimes \bigwedge^m T^*_{\mathbb{C}} M)^{1/2} \cong (\bigwedge^r A^* \otimes \bigwedge^m T^*_{\mathbb{C}} M)^{1/2} \tag{2-5}$$

imply that

$$\begin{aligned}\textstyle\bigwedge^k A^* \otimes (\bigwedge^r A \otimes \bigwedge^m T^*_{\mathbb{C}} M)^{1/2} &\cong \textstyle\bigwedge^{r-k} A \otimes \bigwedge^r A^* \otimes (\bigwedge^r A \otimes \bigwedge^m T^*_{\mathbb{C}} M)^{1/2} \\ &\cong \textstyle\bigwedge^{r-k} A \otimes (\bigwedge^r A^* \otimes \bigwedge^m T^*_{\mathbb{C}} M)^{1/2}.\end{aligned} \tag{2-6}$$

Therefore, one ends up with a differential operator

$$\breve{\partial} : \Gamma(\textstyle\bigwedge^k A \otimes \mathscr{L}) \to \Gamma(\bigwedge^{k-1} A \otimes \mathscr{L}). \tag{2-7}$$

Theorem 2.5 [Chen and Stiénon 2009]. *The pair of Lie algebroids (A, A^*) is a Lie bialgebroid if and only if $\breve{D}^2 \in C^\infty(M, \mathbb{C})$, that is, the square of the operator*

$$\breve{D} = \breve{d}_* + \breve{\partial} : \Gamma(\textstyle\bigwedge A \otimes \mathscr{L}) \to \Gamma(\bigwedge A \otimes \mathscr{L})$$

is multiplication by some function $\breve{f} \in C^\infty(M, \mathbb{C})$. Moreover $\breve{D}_*^2 = \breve{f}$, where $\breve{D}_* = \breve{d} + \breve{\partial}_*$ is defined analogously to $\breve{D}$ by exchanging the roles of A and A^*.

3. Generalized complex geometry

In this section, we fix a real $2n$-dimensional manifold M and denote the tangent and cotangent bundle of M by T and T^*, respectively. Let $T_{\mathbb{C}}$ and $T^*_{\mathbb{C}}$ be respectively the complexification of T and T^*. The first vital ingredient in $T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ is the natural pairing:

$$\langle x_1+\eta_1, x_2+\eta_2\rangle = \tfrac{1}{2}(\langle x_1 \mid \eta_2\rangle + \langle x_2 \mid \eta_1\rangle) \quad \text{for all } x_i \in T_{\mathbb{C}} \text{ and } \eta_i \in T^*_{\mathbb{C}}. \tag{3-1}$$

Here on the right side, $\langle x \mid \eta\rangle$ is the natural pairing between $T_{\mathbb{C}}$ and $T^*_{\mathbb{C}}$.

Thus we have the Clifford algebra $\mathscr{C}(T_{\mathbb{C}} \oplus T^*_{\mathbb{C}})$, which acts on the spinor bundle $\mathcal{M} \triangleq \bigoplus_{i=0}^{2n} \bigwedge^i T^*_{\mathbb{C}}$ via $(x+\eta)\cdot\rho = \iota_x\rho + \eta\wedge\rho$ for all $\rho \in \mathcal{M}$.

Introduce a $\mathbb{C}$-linear map $(\cdot)^T : \mathcal{M} \to \mathcal{M}$ by $(\eta_1 \wedge \cdots \wedge \eta_j)^T = \eta_j \wedge \cdots \wedge \eta_1$. The Mukai pairing $(\ ,\ ) : \mathcal{M} \times \mathcal{M} \to \bigwedge^{2n}(T^*_{\mathbb{C}})$ is defined by $(\chi, \omega) = [\chi^T \wedge \omega]^{2n}$, where $[\ ]^{2n}$ indicates the top degree component of the product. Explicitly, if $\chi = \sum_{i=0}^{2n} \chi_i$ and $\omega = \sum_{i=0}^{2n} \omega_i$, where $\chi_i, \omega_i \in \bigwedge^i T^*_{\mathbb{C}}$, then

$$(\chi, \omega) = \sum_{i=0}^{2n} (-1)^{i(i-1)/2} \chi_i \wedge \omega_{2n-i}.$$

For all $\chi, \omega \in \mathcal{M}$ and $\phi \in \bigwedge^2 T^*_{\mathbb{C}}$, these properties are standard [Gualtieri 2003]:

$$(\chi, \omega) = (-1)^n (\omega, \chi), \tag{3-2}$$

$$(\phi \wedge \chi, \omega) + (\chi, \phi \wedge \omega) = 0. \tag{3-3}$$

Consider a real, closed 3-form $H \in Z^3(M)$ and the twisted differential operator $d^H = d + H \wedge (\cdot)$ it induces.

Definition 3.1 [Gualtieri 2003; Cavalcanti 2006]. A twisted generalized complex structure with respect to H is determined by any of the following three equivalent objects:

(i) A real automorphism $\mathbb{J}$ of $T \oplus T^*$ that squares to -1, is orthogonal with respect to the natural pairing (3-1), and has vanishing Nijenhuis tensor, meaning that for all $z_1, z_2 \in \Gamma(T \oplus T^*)$,

$$N(z_1, z_2) \triangleq -[\![\mathbb{J}z_1, \mathbb{J}z_2]\!]_H + \mathbb{J}[\![\mathbb{J}z_1, z_2]\!]_H + \mathbb{J}[\![z_1, \mathbb{J}z_2]\!]_H + [\![z_1, z_2]\!]_H = 0.$$

Here $[\![\ ,\ ]\!]_H$ is the twisted Courant bracket defined in (1-9).

(ii) A Dirac structure $L \subset T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ that is twisted with respect to H and satisfies $L \cap \bar{L} = \{0\}$.

(iii) A line subbundle N of $\mathcal{M} = \bigwedge^\bullet(T^*_{\mathbb{C}})$ generated at each point by a form u, such that $L = \{X \in T_{\mathbb{C}} \oplus T^*_{\mathbb{C}} \mid X \cdot u = 0\}$ is maximally isotropic, $(u, \bar{u}) \neq 0$, and $d^H u = e \cdot u$ for some $e \in \Gamma(T \oplus T^*)$.

The line bundle in (iii) is called the pure spinor line bundle corresponding to L.

Remark 3.2. See also [Alekseev and Xu 2001] for the relation between Dirac structures and Dirac generating operators.

To generalize the usual ∂ and $\bar\partial$ operators in complex geometry, Gualtieri [2003] introduced ∂ and $\bar\partial$ operators for any twisted generalized complex structure. We recall its construction briefly below.

Let $\mathbb{J}$ be a twisted generalized complex structure, and let $L \subset T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ be its $+i$ eigenspace. L is a twisted Dirac structure and satisfies $L \cap \bar{L} = \{0\}$. We will regard $\bar{L} = L^*$ by defining the canonical pairing between L and $\bar{L}$ by

$$\langle X \,|\, \theta\rangle = 2\langle X, \theta\rangle \quad \text{for all } X \in L \text{ and } \theta \in \bar{L}. \tag{3-4}$$

The coefficient 2 is due to the natural pairing; see Equation (3-1).

Set $N_0 = N$ and $N_k = \bigwedge^k \bar{L} \cdot N$ for $k = 1, \dots, 2n$. Then $\bar{N}_k = N_{2n-k}$ and specifically $N_{2n} = \bar{N}$ is the pure spinor of $\bar{L}$. We have a decomposition

$$\mathcal{M} = N_0 \oplus N_1 \oplus \cdots \oplus N_{2n}.$$

Gualtieri [2003] proves that one can decompose $d^H = \partial + \bar\partial$. Here

$$\partial : \Gamma(N_\bullet) \to \Gamma(N_{\bullet-1}), \qquad \bar\partial : \Gamma(N_\bullet) \to \Gamma(N_{\bullet+1})$$

(or $\Gamma(\bar{N}_\bullet) \to \Gamma(\bar{N}_{\bullet-1})$) are defined by

$$\partial(n_k) \triangleq pr_{N_{k-1}}(d^H n_k), \qquad \bar\partial(n_k) \triangleq pr_{N_{k+1}}(d^H n_k) \quad \text{for all } n_k \in \Gamma(N_k).$$

Using the identification $N_k = (\bigwedge^k \bar{L}) \otimes N$ (see (4-5)), Gualtieri observed that $\bar\partial$ is determined by the rule

$$\bar\partial(W \otimes s) = (d_L W) \otimes s + (-1)^k W \otimes d^H s,$$

where $W \in \Gamma(\bigwedge^k \bar{L})$ and s is a local nonvanishing section of N. It is clear from this description that N must be a module of the Lie algebroid L. Moreover, under the additional assumption that N admits a global nowhere vanishing section s, Gualtieri showed that $d^H s = \bar\partial s = e \cdot s$, where $e \in C^\infty(\bar{L})$, and e actually defines a cohomology class in the Lie algebroid cohomology $H^1(L)$.

4. The main theorem

Given a generalized complex structure $\mathbb{J}$ as above, it is clear that $(L, \bar{L})$ is a Lie bialgebroid (regarding $L^* = \bar{L}$ by (3-4)). We prove in Theorem 4.3 that the operators $\breve{d}_*$ and $\breve{\partial}$ for this particular situation are essentially ∂ and $\bar\partial$.

We continue the notations in Section 3. The following lemma is easy.

Lemma 4.1. *For all $W \in \bigwedge^j \bar{L}$ and $X \in \bigwedge^i L$, and for $i \leq j \leq 2n$, one has*

$$X \cdot W \cdot u = (-1)^{i(i-1)/2} (\imath_X W) \cdot u. \tag{4-1}$$

Here $\imath$ denotes the generalized interior product defined by

$$\langle \imath_X W \,|\, Y \rangle = \langle W \,|\, X \wedge Y \rangle \quad \textit{for all } Y \in \textstyle\bigwedge^{j-i} L. \tag{4-2}$$

Let u be a nowhere vanishing local section of N. Assume that $V \in \Gamma(\bigwedge^{2n} L)$ satisfies $V \cdot \bar{u} = u$. Hence $\overline{V} \cdot u = \bar{u}$ and by (4-1),

$$\begin{aligned} \langle V \,|\, \overline{V} \rangle u &= (\imath_V \overline{V}) u = (-1)^n V \cdot \overline{V} \cdot u = (-1)^n u, \\ \langle V \,|\, \overline{V} \rangle &= (-1)^n. \end{aligned} \tag{4-3}$$

Here $\langle V \,|\, \overline{V} \rangle$ is the natural pairing. Therefore the dual section of V is given by $\Omega = (-1)^n \overline{V} \in \Gamma(\bigwedge^{2n} \bar{L})$.

Proposition 4.2 [Gualtieri 2003, Proposition 2.22]; also [Chevalley 1997, III.3.2]. *The line bundle $\mathscr{L} = (\bigwedge^{2n} \bar{L} \otimes \bigwedge^{2n} T^*_{\mathbb{C}})^{1/2}$ and $N_{2n} = \overline{N}$ are canonically isomorphic. The isomorphism can be explicitly described by*

$$\begin{aligned} \overline{N} \otimes \overline{N} &\to \mathscr{L}^2 = \textstyle\bigwedge^{2n} \bar{L} \otimes \bigwedge^{2n} T^*_{\mathbb{C}}, \\ \omega_1 \otimes \omega_2 &\mapsto \Omega \otimes (V \cdot \omega_1, \omega_2). \end{aligned} \tag{4-4}$$

This isomorphism does not depend on the choice of u and V.

From now on we will identify $\overline{N}$ with $(\bigwedge^{2n} \bar{L} \otimes \bigwedge^{2n} T^*_{\mathbb{C}})^{1/2}$. As a consequence of the $\bar{L}$-module structure on the latter, we have two differential operators (see Equations (2-2) and (2-7))

$$\check{d}_* : (\textstyle\bigwedge^{\bullet} L) \otimes \overline{N} \to (\bigwedge^{\bullet+1} L) \otimes \overline{N} \quad \text{and} \quad \check{\partial} : (\bigwedge^{\bullet} L) \otimes \overline{N} \to (\bigwedge^{\bullet-1} L) \otimes \overline{N}.$$

It is also shown in [Gualtieri 2003] that $(\bigwedge^k \bar{L}) \otimes N \cong N_k$ and $(\bigwedge^k L) \otimes \overline{N} \cong \overline{N}_k$ respectively by the two isomorphisms

$$I : (\textstyle\bigwedge^k \bar{L}) \otimes N \to N_k, \quad W \otimes p \mapsto W \cdot p \quad \text{for all } W \in \bigwedge^k \bar{L}, \ p \in N, \tag{4-5}$$

$$\bar{I} : (\textstyle\bigwedge^k L) \otimes \overline{N} \to \overline{N}_k, \quad X \otimes \bar{p} \mapsto X \cdot \bar{p} \quad \text{for all } X \in \bigwedge^k L, \ \bar{p} \in \overline{N}. \tag{4-6}$$

We now give our main theorem, whose proof is in Section 6.

Theorem 4.3. *The following two diagrams are commutative.*

$$\begin{CD} (\textstyle\bigwedge^k L) \otimes \overline{N} @>{\check{d}_*}>> (\bigwedge^{k+1} L) \otimes \overline{N} \\ @V{I}VV @VV{I}V \\ \overline{N}_k @>>{\partial}> \overline{N}_{k+1} \end{CD} \tag{4-7}$$

$$
\begin{array}{ccc}
(\bigwedge^k L)\otimes \overline{N} & \xrightarrow{\breve{\partial}} & (\bigwedge^{k-1} L)\otimes \overline{N} \\
\bar{I}\big\downarrow & & \big\downarrow \bar{I} \\
\overline{N}_k & \xrightarrow[\bar{\partial}]{} & \overline{N}_{k-1}
\end{array}
\tag{4-8}
$$

In retrospect, the existence of such a result has likely been suggested by earlier works of Gualtieri. In [2007], he constructed an L-module structure on N and an $\overline{L}$-module structure on $\overline{N}$, respectively by

$$\nabla_X p \triangleq X\cdot d^H p = X\cdot \bar{\partial} p, \tag{4-9}$$

$$\nabla_W \bar{p} \triangleq W\cdot d^H \bar{p} = W\cdot \partial \bar{p}, \tag{4-10}$$

for all $p\in\Gamma(N)$, $X\in\Gamma(L)$ and $W\in\Gamma(\bar{L})$.

Here is a special situation of $k=0$ in diagram (4-7):

Proposition 4.4. *The above $\overline{L}$-module structure defined by (4-10) coincides with the $\overline{L}$-module structure defined by (2-1), under the isomorphism (4-4).*

5. Modular cocycles of Lie algebroids

In this section we establish a list of important identities valid in any Lie bialgebroid (A, A^*) and generalized complex structure; we use these in Section 6 to prove the statements of Section 4.

We continue the setup of Section 2: Let $(A, [\cdot,\cdot], a)$ and $(A^*, [\cdot,\cdot]_*, a_*)$ be a pair of rank-r Lie algebroids over dimension-m base manifold M.

Assume there exists a volume form $s\in\Gamma(\bigwedge^m T^*_{\mathbb{C}}M)$ and a nowhere vanishing section $\Omega\in\Gamma(\bigwedge^r A^*)$, so that $\mathscr{L}$ is the trivial line bundle over M. Let $V\in\Gamma(\bigwedge^r A)$ be the section dual to Ω, that is, $\langle\Omega\,|\,V\rangle=1$. These induce bundle isomorphisms

$$\Omega^\sharp : \textstyle\bigwedge^k A \;\to \bigwedge^{r-k} A^* : X\mapsto \iota_X\Omega, \tag{5-1}$$

$$V^\sharp : \textstyle\bigwedge^k A^* \to \bigwedge^{r-k} A \;\; : \xi\mapsto \iota_\xi V. \tag{5-2}$$

Here we adopt similar conventions as that of Equation (4-2). These two operations are essentially inverse to each other:

$$(V^\sharp\circ\Omega^\sharp)(X) = (-1)^{k(r-1)}X \quad \text{for all } X\in\textstyle\bigwedge^k A; \tag{5-3}$$

$$(\Omega^\sharp\circ V^\sharp)(\varphi) = (-1)^{k(r-1)}\varphi \quad \text{for all } \varphi\in\textstyle\bigwedge^k A^*. \tag{5-4}$$

Consider the operator $\mathfrak{d}$ dual to d with respect to $V^\sharp$, as defined by the diagram

$$\begin{array}{ccc} \Gamma(\bigwedge^k A^*) & \xrightarrow{V^\sharp} & \Gamma(\bigwedge^{r-k} A) \\ {\scriptstyle -(-1)^k d}\downarrow & & \downarrow{\scriptstyle \mathfrak{d}} \\ \Gamma(\bigwedge^{k+1} A^*) & \xrightarrow[V^\sharp]{} & \Gamma(\bigwedge^{r-k-1} A), \end{array} \tag{5-5}$$

or by the relation

$$-V^\sharp d\alpha = (-1)^k \mathfrak{d} V^\sharp \alpha \quad \text{for all } \alpha \in \Gamma(\textstyle\bigwedge^k A^*), \tag{5-6}$$

which, due to (5-3) and (5-4), can be rewritten as

$$\Omega^\sharp \mathfrak{d} \beta = (-1)^l d \Omega^\sharp \beta \quad \text{for all } \beta \in \Gamma(\textstyle\bigwedge^l A). \tag{5-7}$$

The operator $\mathfrak{d}$ is a Batalin–Vilkovisky operator for the Lie algebroid A; see [Kosmann-Schwarzbach 2000; Koszul 1985; Xu 1999; Michéa and Novitchkov 2005]. Similarly, we have the operator $\mathfrak{d}_*$ dual to d_*, as defined by

$$\begin{array}{ccc} \Gamma(\bigwedge^{r-k} A) & \xleftarrow{V^\sharp} & \Gamma(\bigwedge^k A^*) \\ {\scriptstyle (-1)^k d_*}\downarrow & & \downarrow{\scriptstyle \mathfrak{d}_*} \\ \Gamma(\bigwedge^{r-k+1} A) & \xleftarrow[V^\sharp]{} & \Gamma(\bigwedge^{k-1} A^*), \end{array}$$

or by the relation

$$d_* V^\sharp \alpha = (-1)^k V^\sharp \mathfrak{d}_* \alpha \quad \text{for all } \alpha \in \Gamma(\textstyle\bigwedge^k A^*).$$

According to [Evens et al. 1999], there exists a unique $X_0 \in \Gamma(A)$ such that

$$\mathscr{L}_\theta(\Omega \otimes s) = (\mathscr{L}_\theta \Omega) \otimes s + \Omega \otimes (\mathscr{L}_{a_*(\theta)} s) = \langle X_0 | \theta \rangle \Omega \otimes s \quad \text{for all } \theta \in \Gamma(A^*). \tag{5-8}$$

Similarly, there exists a unique $\xi_0 \in \Gamma(A^*)$ such that

$$\mathscr{L}_X(s \otimes V) = (\mathscr{L}_{a(X)} s) \otimes V + s \otimes (\mathscr{L}_X V) = \langle \xi_0 | X \rangle s \otimes V \quad \text{for all } X \in \Gamma(A). \tag{5-9}$$

These sections X_0 and ξ_0 are called *modular cocycles*, and their cohomology classes are called modular classes [Evens et al. 1999].

A simple computation yields the following formulas, which are also given in [Alekseev and Xu 2001].

Proposition 5.1. *With the above notations, the differential operators defined by Equations (2-2) and (2-7) are given respectively by*

$$\breve{d}_*(X \otimes l) = (d_* X + \tfrac{1}{2} X_0 \wedge X) \otimes l \tag{5-10}$$

and

$$\breve{\partial}(X \otimes l) = (-\mathfrak{d}X + \tfrac{1}{2}\iota_{\xi_0}X) \otimes l \tag{5-11}$$

for all $X \in \Gamma(\wedge A)$ *and* $l \in \Gamma(\mathscr{L})$.

Hence the operator $\breve{D}$ in Theorem 2.5 reads

$$\breve{D} = \breve{d}_* + \breve{\partial} = d_* - \mathfrak{d} + \tfrac{1}{2}(X_0 \wedge \cdot + \iota_{\xi_0}).$$

This construction of Dirac generating operators using modular cocycles appeared in [Alekseev and Xu 2001] and [Chen and Stiénon 2009].

Now we consider a twisted generalized complex structure $\mathbb{J}$ on a $2n$-dimensional manifold M and let L and $\bar{L}$ be respectively the $+i$ and $-i$ eigenspace of $\mathbb{J}$. Again we assume that u is a nowhere vanishing local section of N, the pure spinor bundle of L.

Lemma 5.2 [Gualtieri 2007]. *There exists some* $e = x + \eta \in \Gamma(\bar{L})$ *such that*

$$d^H u = \bar{\partial} u = du + H \wedge u = e \cdot u = \iota_x u + \eta \wedge u, \tag{5-12}$$

$$d^H \bar{u} = \partial \bar{u} = d\bar{u} + H \wedge \bar{u} = \bar{e} \cdot \bar{u} = \iota_{\bar{x}} \bar{u} + \bar{\eta} \wedge \bar{u}. \tag{5-13}$$

The main result in this section is the following.

Proposition 5.3. *Let* $V \in \Gamma(\bigwedge^{2n} L)$ *such that* $V \cdot \bar{u} = u$. *Then the modular cocycle of* L *with respect to the top form* V *and the volume form* $s = (u, \bar{u})$ *is* $2e$, *where* $e \in \Gamma(\bar{L})$ *is given by Lemma 5.2.*

Similarly, the modular cocycle of $\bar{L}$ *with respect to* $\Omega = (-1)^n \bar{V} \in \Gamma(\bigwedge^{2n} \bar{L})$ *and* s *is* $2\bar{e}$.

Before the proof, we need a couple of identities and lemmas. Since L is a Lie algebroid and $L^* = \bar{L}$, we have the differential

$$d_L : \Gamma(\textstyle\bigwedge^{\bullet} \bar{L}) \to \Gamma(\textstyle\bigwedge^{\bullet+1} \bar{L}).$$

We also have the equality

$$\begin{aligned} \bar{\partial}(W \cdot u) &= (d_L W) \cdot u + (-1)^k W \cdot \bar{\partial} u \\ &= (d_L W) \cdot u + (-1)^k (W \wedge e) \cdot u \\ &= (d_L W + e \wedge W) \cdot u \quad \text{for all } W \in \Gamma(\textstyle\bigwedge^k \bar{L}), \end{aligned} \tag{5-14}$$

which encodes the L-module structure on N defined by Equation (4-9).

Similarly, one has

$$\begin{aligned} \partial(X \cdot \bar{u}) &= (d_{\bar{L}} X) \cdot \bar{u} + (-1)^i X \cdot \partial \bar{u} \\ &= (d_{\bar{L}} X + \bar{e} \wedge X) \cdot \bar{u} \quad \text{for all } X \in \Gamma(\textstyle\bigwedge^i L). \end{aligned} \tag{5-15}$$

Lemma 5.4. *For any* $X = a + \zeta \in \Gamma(L)$, *we have*

$$(\mathscr{L}_X \overline{V}) \cdot u = \bar{\partial}(X \cdot \bar{u}) - \langle e \,|\, X \rangle \bar{u}, \tag{5-16}$$

$$\mathscr{L}_a u = \langle e \,|\, X \rangle u - (d\zeta + \iota_a H) \wedge u, \tag{5-17}$$

$$\mathscr{L}_a \bar{u} = \bar{\partial}(X \cdot \bar{u}) + (d_{\bar{L}} X) \cdot \bar{u} - (d\zeta + \iota_a H) \wedge \bar{u}. \tag{5-18}$$

Proof. A basic fact is that

$$0 = X \cdot u = \iota_a u + \zeta \wedge u \tag{5-19}$$

for any $X = a + \zeta \in \Gamma(L)$. Hence

$$\begin{aligned} \bar{\partial}(X \cdot \bar{u}) &= \bar{\partial}(X \cdot \overline{V} \cdot u) = \bar{\partial}((\iota_X \overline{V}) \cdot u) \\ &= (d_L \iota_X \overline{V}) \cdot u - (\iota_X \overline{V} \wedge e) \cdot u \qquad \text{(by (5-14))} \\ &= (d_L \iota_X \overline{V} + \iota_X d_L \overline{V}) \cdot u + (\langle e \,|\, X \rangle \overline{V}) \cdot u \\ &= (\mathscr{L}_X \overline{V}) \cdot u + \langle e \,|\, X \rangle \bar{u}. \end{aligned}$$

This proves (5-16). For (5-17), we have

$$\begin{aligned} \mathscr{L}_a u &= \iota_a du + d\iota_a u \\ &= \iota_a(\iota_x u + \eta \wedge u - H \wedge u) - d(\zeta \wedge u) \qquad \text{(by (5-12) and (5-19))} \\ &= -\iota_x \iota_a u + \langle a \,|\, \eta \rangle u - \eta \wedge \iota_a u - \iota_a H \wedge u + H \wedge \iota_a u - d\zeta \wedge u + \zeta \wedge du \\ &= \iota_x(\zeta \wedge u) + \langle a \,|\, \eta \rangle u + (\eta - H) \wedge (\zeta \wedge u) \\ &\quad - \iota_a H \wedge u - d\zeta \wedge u + \zeta \wedge (\iota_x u + \eta \wedge u - H \wedge u) \quad \text{(by (5-12) and (5-14))} \\ &= (\langle x \,|\, \zeta \rangle + \langle a \,|\, \eta \rangle) u - \iota_a H \wedge u - d\zeta \wedge u. \end{aligned}$$

To prove (5-18), we observe that, on one hand

$$\begin{aligned} d^H(X \cdot \bar{u}) &= \bar{\partial}(X \cdot \bar{u}) + \partial(X \cdot \bar{u}) \\ &= \bar{\partial}(X \cdot \bar{u}) + (d_{\bar{L}} X) \cdot \bar{u} - (X \wedge \bar{e}) \cdot \bar{u} \quad \text{(by (5-15))}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} d^H(X \cdot \bar{u}) &= d(\iota_a \bar{u} + \zeta \wedge \bar{u}) + H \wedge (X \cdot \bar{u}) \\ &= d\iota_a \bar{u} + d\zeta \wedge \bar{u} - \zeta \wedge d\bar{u} + H \wedge (X \cdot \bar{u}) \\ &= (d\iota_a \bar{u} + \iota_a d\bar{u}) + d\zeta \wedge \bar{u} - (\iota_a + \zeta \wedge) d\bar{u} + H \wedge (X \cdot \bar{u}) \\ &= \mathscr{L}_a \bar{u} + d\zeta \wedge \bar{u} - X \cdot (\bar{e} \cdot \bar{u} - H \wedge \bar{u}) + H \wedge (X \cdot \bar{u}) \qquad \text{(by (5-13))} \\ &= \mathscr{L}_a \bar{u} + d\zeta \wedge \bar{u} - (X \wedge \bar{e}) \cdot \bar{u} + X \cdot (H \wedge \bar{u}) + H \wedge (X \cdot \bar{u}) \\ &= \mathscr{L}_a \bar{u} + d\zeta \wedge \bar{u} - (X \wedge \bar{e}) \cdot \bar{u} + (\iota_a + \zeta \wedge)(H \wedge \bar{u}) + H \wedge (\iota_a \bar{u} + \zeta \wedge \bar{u}) \\ &= \mathscr{L}_a \bar{u} + d\zeta \wedge \bar{u} - (X \wedge \bar{e}) \cdot \bar{u} + (\iota_a H) \wedge \bar{u}. \end{aligned}$$

□

Lemma 5.5 [Cavalcanti 2006]. *The Mukai pairing vanishes in $N_i \times N_k$, unless $i+k=2n$, in which case it is nondegenerate.*

Proof of Proposition 5.3. For an $X = a+\zeta \in \Gamma(L)$, we assume that $\bar\partial(X\cdot\bar u) = f\bar u$ for some function $f \in C^\infty(M,\mathbb{C})$. Then (5-16) implies that $\mathcal{L}_X \overline{V} = (f - \langle e\,|\,X\rangle)\overline{V}$. Since the paring between V and $\overline{V}$ is a constant (see (4-3)),

$$\mathcal{L}_X V = (\langle e\,|\,X\rangle - f)V. \tag{5-20}$$

According to (5-17) and (5-18), we also have

$$\begin{aligned}\mathcal{L}_{\rho_L(X)} s &= \mathcal{L}_a(u,\bar u) = (\mathcal{L}_a u, \bar u) + (u, \mathcal{L}_a \bar u)\\ &= (\langle e\,|\,X\rangle u - (d\zeta + \iota_a H)\wedge u, \bar u) + (u,\, f\bar u + (d_{\bar L}X)\cdot \bar u - (d\zeta+\iota_a H)\wedge \bar u)\\ &= (\langle e\,|\,X\rangle + f)(u,\bar u) = (\langle e\,|\,X\rangle + f)s.\end{aligned}$$

In the last step, we have applied (3-3) and Lemma 5.5. In turn, we get

$$\mathcal{L}_X V \otimes s + V \otimes \mathcal{L}_{\rho_L(X)} s = 2\langle e\,|\,X\rangle V \otimes s.$$

This proves the first claim. By symmetry, we also have

$$\mathcal{L}_W \overline{V} \otimes \bar s + \overline{V} \otimes \mathcal{L}_{\rho_{\bar L}(W)} \bar s = 2\langle \bar e\,|\,W\rangle \overline{V}\otimes \bar s$$

for all $W \in \Gamma(\bar L)$. By (3-2), we know $\bar s = (\bar u, u) = (-1)^n (u,\bar u) = (-1)^n s$. Thus

$$\mathcal{L}_W \Omega \otimes s + \Omega \otimes \mathcal{L}_{\rho_{\bar L}(W)} s = 2\langle \bar e\,|\,W\rangle \Omega \otimes s,$$

which implies that $2\bar e$ is the modular cocycle of $\bar L$ with respect to Ω and s. □

6. Proof of the main theorem

Proof of Proposition 4.4. $\overline{N}$ has an induced $\bar L$-module structure arising from the $\bar L$-module $\mathcal{L}$; see (2-1). According to the second statement of Proposition 5.3, we know that this module structure is determined by the equation

$$\mathcal{L}_W \bar u = \langle \bar e\,|\,W\rangle \bar u \quad \text{for all } W \in \Gamma(\bar L).$$

This coincides with the standard $\bar L$-module structure defined by (4-10) because

$$W \cdot \partial \bar u = W\cdot \bar e \cdot \bar u = \langle \bar e\,|\,W\rangle \bar u,$$

by Lemma 4.1. □

Proof of Theorem 4.3. By Proposition 5.3 and Equations (5-10) and (5-11), we conclude that

$$\begin{aligned}\breve d_*(X\otimes \bar u) &= (d_{\bar L}X + \bar e \wedge X)\otimes \bar u,\\ \breve\partial(X\otimes\bar u) &= (-\mathfrak{d}X + \iota_e X)\otimes \bar u\end{aligned}$$

for all $X \in \Gamma(\bigwedge^i L)$.

Comparing with the expression for ∂ in (5-15), we immediately know that diagram (4-7) is commutative. To prove the commutativity of diagram (4-8), it suffices to prove that

$$\bar{\partial}(X \cdot \bar{u}) = (-\mathfrak{d}X + \iota_e X) \cdot \bar{u} \quad \text{for all } X \in \Gamma(\textstyle\bigwedge^i L). \tag{6-1}$$

In fact, by (5-14),

$$\begin{aligned}\text{left side of (6-1)} &= \bar{\partial}(X \cdot \overline{V} \cdot u) = (-1)^{i(i-1)/2}\bar{\partial}((\iota_X \overline{V}) \cdot u) \\ &= (-1)^{i(i-1)/2}(d_L \iota_X \overline{V} + e \wedge \iota_X \overline{V}) \cdot u.\end{aligned}$$

We also have

$$(\iota_e X) \cdot \bar{u} = e \cdot X \cdot \overline{V} \cdot u = (-1)^{i(i-1)/2} e \cdot (\iota_X \overline{V}) \cdot u = (-1)^{i(i-1)/2}(e \wedge \iota_X \overline{V}) \cdot u.$$

By (5-3) and (5-7),

$$\begin{aligned}\mathfrak{d}X &= (-1)^{(i-1)(2n-1)} V^\sharp \Omega^\sharp \mathfrak{d} X \\ &= (-1)^{(i-1)(2n-1)+i} V^\sharp d_L \Omega^\sharp X = -V^\sharp d_L \Omega^\sharp X.\end{aligned}$$

Hence

$$\begin{aligned}-(\mathfrak{d}X) \cdot \bar{u} &= (V^\sharp d_L \Omega^\sharp X) \cdot \bar{u} \\ &= (-1)^{(2n-i+1)(2n-i)/2}(d_L \iota_X \Omega) \cdot V \cdot \bar{u} = (-1)^{i(i-1)/2}(d_L \iota_X \overline{V}) \cdot u.\end{aligned}$$

This proves (6-1), and the proof of Theorem 4.3 is thus completed. □

7. Some corollaries

The first obvious result is that, by the isomorphisms

$$\textstyle\bigwedge^k L \otimes \mathscr{L} \cong (\bigwedge^k L) \cdot \overline{N} = \overline{N}_k = N_{2n-k},$$

the Dirac generating operator constructed by Theorem 2.5 for $E = L \oplus \overline{L}$ is exactly

$$\check{D} = \check{d}_* + \check{d} = \partial + \bar{\partial} = d^H,$$

and specifically $\check{f} = \check{D}^2 = 0$.

In Section 5, we defined for any Lie bialgebroid (A, A^*) a pair of operators d_* and $\mathfrak{d}$ on $\Gamma(\bigwedge A)$, and similarly d and $\mathfrak{d}_*$ on $\Gamma(\bigwedge A^*)$. Let $D = d_* + \mathfrak{d}$ and $D_* = d + \mathfrak{d}_*$. Their squares yield the pair of Laplacian operators

$$\Delta = D^2 = d_*\mathfrak{d} + \mathfrak{d}d_* : \Gamma(\textstyle\bigwedge^k A) \to \Gamma(\bigwedge^k A), \tag{7-1}$$

$$\Delta_* = D_*^2 = d\mathfrak{d}_* + \mathfrak{d}_* d : \Gamma(\textstyle\bigwedge^k A^*) \to \Gamma(\bigwedge^k A^*). \tag{7-2}$$

See [Alekseev and Xu 2001].

Theorem 7.1 [Chen and Stiénon 2009, Theorem 3.4]. *If (A, A^*) is a Lie bialgebroid, then*

$$\Delta = \tfrac{1}{2}(\mathcal{L}_{X_0} + \mathcal{L}_{\xi_0}) : \Gamma(\textstyle\bigwedge A) \to \Gamma(\bigwedge A),$$
$$\Delta_* = \tfrac{1}{2}(\mathcal{L}_{X_0} + \mathcal{L}_{\xi_0}) : \Gamma(\textstyle\bigwedge A^*) \to \Gamma(\bigwedge A^*),$$

where X_0 and ξ_0 are modular cocycles defined by (5-8) and (5-9).

As an immediate corollary, we have:

Corollary 7.2. *Let $(L, \bar{L})$ be the Lie bialgebroid coming from a twisted generalized complex structure $\mathbb{J}$. The Laplacian operators Δ and Δ_* defined by (7-1) and (7-2) are given by*

$$\Delta f = \Delta_* f = \tfrac{1}{2} pr_T(e + \bar{e})(f) \qquad \text{for all } f \in C^\infty(M, \mathbb{C}),$$
$$\Delta X = (\bar{e} + \tfrac{1}{2} e) \circ_H X - \tfrac{i}{2}\mathbb{J}(e \circ_H X) \quad \text{for all } X \in \Gamma(L),$$
$$\Delta_* W = (e + \tfrac{1}{2}\bar{e}) \circ_H W + \tfrac{i}{2}\mathbb{J}(\bar{e} \circ_H W) \quad \text{for all } W \in \Gamma(\bar{L}).$$

Proof. For $e \in \Gamma(\bar{L})$ and $\bar{e} \in \Gamma(L)$, the Lie derivations $\mathcal{L}_e$ and $\mathcal{L}_{\bar{e}}$ on $\Gamma(L)$ are given by

$$\mathcal{L}_e X = pr_L(e \circ_H X) \quad \text{and} \quad \mathcal{L}_{\bar{e}} = \bar{e} \circ_H X \quad \text{for all } X \in \Gamma(L).$$

The projections of $T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ to L and $\bar{L}$ are given respectively by

$$pr_L(z) = \tfrac{1}{2}(z - i\mathbb{J}z) \quad \text{and} \quad pr_{\bar{L}}(z) = \tfrac{1}{2}(z + i\mathbb{J}z) \quad \text{for all } z \in T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}.$$

The claim then follows directly from Theorem 7.1. □

It is well known [Mackenzie and Xu 1994] that, if a and a_* denote the anchor maps of a Lie bialgebroid (A, A^*), the bundle map

$$\pi^\sharp = a \circ (a_*)^* : T^*_{\mathbb{C}} M \to T_{\mathbb{C}} M \tag{7-3}$$

defines a (complex) Poisson structure on M.

In particular, for the Lie bialgebroid $(L, \bar{L})$ coming from the twisted generalized complex structure $\mathbb{J}$, the map $-i\pi$ is a real Poisson structure. In fact, up to a factor of 2, it is given by [Barton and Stiénon 2008; Gualtieri 2007]

$$P(\xi, \eta) = \langle \mathbb{J}\xi, \eta \rangle. \tag{7-4}$$

Let us briefly recall the definition of the modular vector field of a Poisson manifold (M, π) from [Weinstein 1997]. Let $\omega \in \Omega^{\text{top}}(M)$ be a volume form. The modular vector field with respect to ω is the derivation X_ω of the algebra of functions $C^\infty(M)$ characterized by

$$\mathcal{L}_{\pi^\sharp(df)}\omega = X_\omega(f)\omega. \tag{7-5}$$

For the Poisson structure induced from a Lie bialgebroid, the relation between modular cocycles and modular vector fields is as follows.

Lemma 7.3 [Chen and Stiénon 2009, Corollary 3.8]. *Suppose M is an orientable manifold with volume form $s \in \Omega^{\mathrm{top}}(M)$, and let (A, A^*) be a real Lie bialgebroid over M with associated Poisson bivector π defined by (7-3). Then the modular vector field of the Poisson manifold (M, π) with respect to s is*

$$X_s = \tfrac{1}{2}(a_*(\xi_0) - a(X_0)), \tag{7-6}$$

where ξ_0 and X_0 are modular cocycles defined by (5-8) and (5-9) (choosing arbitrary V and Ω).

As an immediate consequence, we obtain [Gualtieri 2007, Proposition 3.27].

Corollary 7.4. *The modular vector field of the Poisson structure P defined in (7-4) is given by $\frac{i}{2}\, pr_{T_{\mathbb{C}}}(e - \bar{e})$ with respect to the volume form $s = (u, \bar{u})$.*

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