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The notion of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces is a generalization of classical symmetric spaces, where the group $\mathbb{Z}_2$ is replaced by $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this article, a classification is given of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces G/K where G is an exceptional compact Lie group or Spin(8), complementing recent results of Bahturin and Goze. Our results are equivalent to a classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings on the exceptional simple Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, $\mathfrak{g}_2$ and $\mathfrak{so}(8)$.

1. Introduction and results

The notion of Γ-symmetric spaces introduced by Lutz [1981] is a generalization of the classical notion of a symmetric space.

Definition 1.1. Let Γ be a finite abelian group and let G be a connected Lie group. A homogeneous space G/K is called Γ-*symmetric* if G acts almost effectively on G/K and there is an injective homomorphism $\rho \colon \Gamma \to \operatorname{Aut} G$, such that $G_0^\Gamma \subseteq K \subseteq G^\Gamma$, where G^Γ is the subgroup of elements fixed by $\rho(\Gamma)$ and G_0^Γ its connected component.

In the case $\Gamma = \mathbb{Z}_2$ this is just the classical definition of symmetric spaces, and in the case $\Gamma = \mathbb{Z}_k$ one obtains k-symmetric spaces, as studied in [Wolf and Gray 1968]. In the case $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ we can rephrase the definition as follows. A homogeneous space G/K is $\mathbb{Z}_2 \times \mathbb{Z}_2$-*symmetric* if and only if there are two different commuting involutions on G, that is, there are $\sigma, \tau \in \operatorname{Aut} G \setminus \{\operatorname{id}_G\}$ such that $\sigma^2 = \tau^2 = \operatorname{id}_G$, $\sigma \neq \tau$ and $\sigma\tau = \tau\sigma$ such that $(G^\sigma \cap G^\tau)_0 \subseteq K \subseteq G^\sigma \cap G^\tau$.

It is the purpose of this paper to give a classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces in the case G is a simply connected compact Lie group of isomorphism type Spin(8), E_6, E_7, E_8, F_4 or G_2. This amounts to a classification of pairs of commuting involutions on the Lie algebras of these groups. Recently, such a classification has been obtained by Bahturin and Goze [2008] for the case of simple classical Lie algebras except $\mathfrak{so}(8)$. The results of the classification in the exceptional and Spin(8) case are given in Theorems 1.2 and 1.3, respectively.

Let G/K be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space. Then there is a triple of involutions σ, τ, $\sigma\tau$ of G and we say that the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space is of *type* (X, Y, Z) where X, Y, Z denote the local isomorphism classes of the symmetric spaces G/G^σ, G/G^τ and $G/G^{\sigma\tau}$, respectively. For example, we will show that there are two commuting involutions σ, τ on $G = \mathrm{E}_7$ such that G/G^σ, G/G^τ and $G/G^{\sigma\tau}$ are isomorphic to symmetric spaces of type E V, E VI and E VII, respectively and such that the Lie algebra of $\mathrm{E}_6^\sigma \cap \mathrm{E}_6^\tau$ is isomorphic to $\mathfrak{su}(6) + \mathfrak{sp}(1) + \mathbb{R}$; see Table 2. Thus we say that the corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space is of type (E V, E VI, E VII), abbreviated as E V-VI-VII.

Type	$\mathfrak{g}$	$\mathfrak{k}$
E I-I-II	$\mathfrak{e}_6$	$\mathfrak{so}(6) + \mathbb{R}$
E I-I-III	$\mathfrak{e}_6$	$\mathfrak{sp}(2) + \mathfrak{sp}(2)$
E I-II-IV	$\mathfrak{e}_6$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$
E II-II-II	$\mathfrak{e}_6$	$\mathfrak{su}(3) + \mathfrak{su}(3) + \mathbb{R} + \mathbb{R}$
E II-II-III	$\mathfrak{e}_6$	$\mathfrak{su}(4) + \mathfrak{sp}(1) + \mathfrak{sp}(1) + \mathbb{R}$
E II-III-III	$\mathfrak{e}_6$	$\mathfrak{su}(5) + \mathbb{R} + \mathbb{R}$
E III-III-III	$\mathfrak{e}_6$	$\mathfrak{so}(8) + \mathbb{R} + \mathbb{R}$
E III-IV-IV	$\mathfrak{e}_6$	$\mathfrak{so}(9)$
E V-V-V	$\mathfrak{e}_7$	$\mathfrak{so}(8)$
E V-V-VI	$\mathfrak{e}_7$	$\mathfrak{su}(4) + \mathfrak{su}(4) + \mathbb{R}$
E V-V-VII	$\mathfrak{e}_7$	$\mathfrak{sp}(4)$
E V-VI-VII	$\mathfrak{e}_7$	$\mathfrak{su}(6) + \mathfrak{sp}(1) + \mathbb{R}$
E VI-VI-VI	$\mathfrak{e}_7$	$\mathfrak{so}(8) + \mathfrak{so}(4) + \mathfrak{sp}(1)$
	$\mathfrak{e}_7$	$\mathfrak{u}(6) + \mathbb{R}$
E VI-VII-VII	$\mathfrak{e}_7$	$\mathfrak{so}(10) + \mathbb{R} + \mathbb{R}$
E VII-VII-VII	$\mathfrak{e}_7$	$\mathfrak{f}_4$
E VIII-VIII-VIII	$\mathfrak{e}_8$	$\mathfrak{so}(8) + \mathfrak{so}(8)$
E VIII-VIII-IX	$\mathfrak{e}_8$	$\mathfrak{su}(8) + \mathbb{R}$
E VIII-IX-IX	$\mathfrak{e}_8$	$\mathfrak{so}(12) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$
E IX-IX-IX	$\mathfrak{e}_8$	$\mathfrak{e}_6 + \mathbb{R} + \mathbb{R}$
F I-I-I	$\mathfrak{f}_4$	$\mathfrak{u}(3) + \mathbb{R}$
F I-I-II	$\mathfrak{f}_4$	$\mathfrak{sp}(2) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$
F II-II-II	$\mathfrak{f}_4$	$\mathfrak{so}(8)$
G	$\mathfrak{g}_2$	$\mathbb{R} + \mathbb{R}$

Table 1. Exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces.

The problem of commuting involutions has a geometric interpretation in terms of classical $\mathbb{Z}_2$-symmetric spaces. Let G be a simple compact connected Lie group and let H, L be symmetric subgroups of G (see Definition 2.1 below) such that $(G^\sigma)_0 \subseteq H \subseteq G^\sigma$ and $(G^\tau)_0 \subseteq L \subseteq G^\tau$, where $\sigma, \tau \in \mathrm{Aut}(G)$ are involutions. Then $M = G/L$ endowed with a left G-invariant metric is a symmetric space in the classical sense and there is a natural isometric action of H on M which is hyperpolar, that is, there exists a flat submanifold which meets all the orbits and such that its intersections with the orbits are everywhere orthogonal; see [Heintze et al. 1995] and [Kollross 2002]. These actions have first been considered by Hermann [1960]; they are called *Hermann actions*.

Hermann [1962] proved that the H–orbit through $g\,L \in M$, $g \in G$ is totally geodesic if and only if the involutions $i_g\,\sigma\,i_g^{-1}$ and τ commute where we use the notation i_g to denote the automorphism of a Lie group G given by $i_g(x) = g\,x\,g^{-1}$; this means that the classification of commuting involutions on G is equivalent to a classification of totally geodesic orbits of Hermann actions.

Conlon [1969] determined all pairs of conjugacy classes of involutions σ, τ for which there is a $g \in G$ such that $i_g\sigma i_g^{-1}$ and τ commute. The proof relies on the unpublished notes [Conlon 1968]. (For inner involutions $\sigma = i_a$, $\tau = i_b$ there is obviously always such a g since there is a $g \in G$ such that $g\,a\,g^{-1}$ and b are contained in the same maximal torus of G.) Our classification however uses a direct approach which does not rely on any of the previously mentioned results.

Theorem 1.2. *Let $\mathfrak{g}$ be a compact exceptional Lie algebra of type $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, or $\mathfrak{g}_2$. If $\sigma, \tau \in \mathrm{Aut}(\mathfrak{g})$ are two commuting involutions such that $\sigma \neq \tau$ then the pair $(\mathfrak{g}, \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau)$ is one of the pairs $(\mathfrak{g}, \mathfrak{k})$ given by Table 1; the conjugacy classes of σ, τ and $\sigma\tau$ are given by the first row of Table 1. Conversely, for any pair of Lie algebras $(\mathfrak{g}, \mathfrak{k})$ in Table 1 there exists a pair (σ, τ) of commuting involutions of $\mathfrak{g}$ such that $\mathfrak{k} \cong \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$.*

The data in Table 1 in all cases but one determines the conjugacy class of the subalgebra $\mathfrak{k} \subset \mathfrak{g}$; see Remark 4.2. To prove Theorem 1.2, we first exhibit various standard examples for pairs of involutions on exceptional Lie algebras in Section 3. Then it is shown in Section 4 that these constructions already exhaust all possibilities for $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces.

Theorem 1.3. *Let $\mathfrak{k} \subset \mathfrak{g} = \mathfrak{so}(8)$ be a subalgebra such that there are two commuting involutions σ and τ, $\sigma \neq \tau$, of $\mathfrak{g}$ with $\mathfrak{k} = \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$. Then there is an automorphism φ of $\mathfrak{so}(8)$ such that $\varphi(\mathfrak{k})$ is one of the following subalgebras of $\mathfrak{so}(8)$.*

(i) $\mathfrak{so}(n_1) + \mathfrak{so}(n_2) + \mathfrak{so}(n_3)$, $n_1 + n_2 + n_3 = 8$, $n_i \geq 1$;

(ii) $\mathfrak{so}(n_1) + \mathfrak{so}(n_2) + \mathfrak{so}(n_3) + \mathfrak{so}(n_4)$, $n_1 + n_2 + n_3 + n_4 = 8$, $n_i \geq 1$;

(iii) $\mathfrak{u}(3) + \mathfrak{u}(1)$.

Note that for the case of $\mathfrak{so}(8)$ there are less cases than there are in the classification of [Bahturin and Goze 2008] for the Lie algebras $\mathfrak{so}(n)$, $n \neq 8$, since some of the $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetric subalgebras for general $\mathfrak{so}(n)$ which appear as distinct cases in [Bahturin and Goze 2008] are conjugate by some automorphism of $\mathfrak{so}(8)$. Theorem 1.3 is proved in Section 5.

It should be mentioned that recently several authors announced results on the classification of gradings on exceptional algebras [Draper and Martín 2007; Draper et al. 2008; Draper and Viruel 2007]. As was remarked in [Bahturin and Goze 2008], the result on $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings on Spin(8) can also be obtained from [Draper and Martín 2006]. $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces are of some interest in Riemannian geometry since they are examples of spaces possessing families of nonnegatively curved homogeneous metrics; see [Schwachhöfer and Tapp 2008].

2. Preliminaries

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra. We say that $\mathfrak{k} \subset \mathfrak{g}$ is a *symmetric subalgebra* of $\mathfrak{g}$ if there is a nontrivial automorphism σ of $\mathfrak{g}$ with $\sigma^2 = \mathrm{id}_{\mathfrak{g}}$ such that $\mathfrak{k} = \mathfrak{g}^{\sigma} := \{X \in \mathfrak{g} \mid \sigma(X) = X\}$. If G is a Lie group, we say that a closed subgroup $K \subset G$ is a *symmetric subgroup* if the Lie algebra of K is a symmetric subalgebra in the Lie algebra of G.

Proposition 2.2. *Let $\mathfrak{g}$ be a real Lie algebra. Let σ, τ be automorphisms of $\mathfrak{g}$ such that $\sigma^2 = \tau^2 = \mathrm{id}_{\mathfrak{g}}$, $\sigma\tau = \tau\sigma$, $\sigma \neq \tau$. Then the Lie algebra $\mathfrak{g}$ splits as a direct sum of vector spaces*

$$(2\text{-}1) \qquad \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{\sigma} \oplus \mathfrak{g}_{\tau} \oplus \mathfrak{g}_{\sigma\tau},$$

where $\mathfrak{g}_1 = \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\tau}$, $\mathfrak{g}_1 + \mathfrak{g}_{\sigma} = \mathfrak{g}^{\sigma}$, $\mathfrak{g}_1 + \mathfrak{g}_{\tau} = \mathfrak{g}^{\tau}$, $\mathfrak{g}_1 + \mathfrak{g}_{\sigma\tau} = \mathfrak{g}^{\sigma\tau}$ such that the following hold:

  (i) *We have $[\mathfrak{g}_{\varphi}, \mathfrak{g}_{\psi}] \subseteq \mathfrak{g}_{\varphi\psi}$ for all φ, $\psi \in \Gamma$, that is, the Lie algebra $\mathfrak{g}$ is Γ-graded, where $\Gamma = \{1, \sigma, \tau, \sigma\tau\}$ is the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by σ and τ.*

 (ii) *For all $\varphi \in \Gamma \setminus \{1\}$ we have that $\mathfrak{g}_1$ is a symmetric subalgebra of $\mathfrak{g}_1 + \mathfrak{g}_{\varphi}$.*

(iii) *For all $\varphi \in \Gamma \setminus \{1\}$ we have that $\mathfrak{g}_1 + \mathfrak{g}_{\varphi}$ is a symmetric subalgebra of $\mathfrak{g}$.*

Proof. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{g}^{\sigma} + \mathfrak{p}^{\sigma}$. Since τ commutes with σ, it leaves this decomposition invariant and hence we have further splittings

$$\mathfrak{g}^{\sigma} = (\mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\tau}) + (\mathfrak{g}^{\sigma} \cap \mathfrak{p}^{\tau}) \quad \text{and} \quad \mathfrak{p}^{\sigma} = (\mathfrak{p}^{\sigma} \cap \mathfrak{g}^{\tau}) + (\mathfrak{p}^{\sigma} \cap \mathfrak{p}^{\tau}),$$

where $\mathfrak{g} = \mathfrak{g}^{\tau} + \mathfrak{p}^{\tau}$ is the Cartan decomposition with respect to τ. Define $\mathfrak{g}_{\sigma} = \mathfrak{g}^{\sigma} \cap \mathfrak{p}^{\tau}$, $\mathfrak{g}_{\tau} = \mathfrak{g}^{\tau} \cap \mathfrak{p}^{\sigma}$ and $\mathfrak{g}_{\sigma\tau} = \mathfrak{p}^{\sigma} \cap \mathfrak{p}^{\tau}$. Now the assertions of the proposition are easily checked.   □

Conversely, given a $\mathbb{Z}_2 \times \mathbb{Z}_2$–grading on a Lie algebra $\mathfrak{g}$, one can define a pair of commuting involutions σ, τ on $\mathfrak{g}$ such that (2-1) agrees with the grading. Thus our results are equivalent to a classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings on $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$ and $\mathfrak{so}(8)$.

In order to classify $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces, we want to find all possibilities for a pair of involutions (σ, τ) such that σ and τ commute. It follows from Proposition 2.2 that in this case we have

$$(2\text{-}2) \qquad \dim \mathfrak{g} - \dim \mathfrak{g}^\sigma - \dim \mathfrak{g}^\tau = \dim \mathfrak{g}^{\sigma\tau} - 2 \dim \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau.$$

This gives a necessary condition for the existence of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces of certain types. The left-hand side in (2-2) depends only on the conjugacy classes of σ and τ; we define

$$d := d(\sigma, \tau) := \dim \mathfrak{g} - \dim \mathfrak{g}^\sigma - \dim \mathfrak{g}^\tau.$$

To find candidates for the pair $(\mathfrak{g}^{\sigma\tau}, \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau)$, we list all symmetric subalgebras of symmetric subalgebras in exceptional compact Lie algebras in Table 3 on the next two pages; the table gives for each pair $(\mathfrak{h}\mathfrak{k})$, the number $c(\mathfrak{h}, \mathfrak{k}) := \dim \mathfrak{h} - 2 \dim \mathfrak{k}$. By (2-2), the only candidates for a pair $(\mathfrak{g}^{\sigma\tau}, \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau)$ are those pairs $(\mathfrak{h}, \mathfrak{k})$, where the number $c(\mathfrak{h}, \mathfrak{k})$ equals d. Furthermore, we may eliminate immediately all pairs $(\mathfrak{h}, \mathfrak{k})$ from the list of candidates which do not fulfil the necessary condition that $\mathfrak{g}^\sigma$ and $\mathfrak{g}^\tau$ both contain a symmetric subalgebra isomorphic to $\mathfrak{k}$. However, these are only necessary conditions and we have to check in each case if a decomposition (2-1) with $\mathfrak{g}^{\sigma\tau} \cong \mathfrak{h}$, $\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau \cong \mathfrak{k}$ actually exists.

For the convenience of the reader we list the symmetric subalgebras $\mathfrak{k}$ of simple compact exceptional Lie algebras $\mathfrak{g}$ in Table 2; in Tables 4 (page 121) and 5 (page 127) we list respectively the pairs of (conjugacy classes of) involutions on the exceptional Lie algebras and $\mathfrak{so}(8)$, together with the numbers $d = d(\sigma, \tau)$.

	$\mathfrak{g}$	$\mathfrak{k}$		$\mathfrak{g}$	$\mathfrak{k}$
E I	$\mathfrak{e}_6$	$\mathfrak{sp}(4)$	E VII	$\mathfrak{e}_7$	$\mathfrak{e}_6 + \mathbb{R}$
E II	$\mathfrak{e}_6$	$\mathfrak{su}(6) + \mathfrak{sp}(1)$	E VIII	$\mathfrak{e}_8$	$\mathfrak{so}(16)$
E III	$\mathfrak{e}_6$	$\mathfrak{so}(10) + \mathbb{R}$	E IX	$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$
E IV	$\mathfrak{e}_6$	$\mathfrak{f}_4$	F I	$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$
E V	$\mathfrak{e}_7$	$\mathfrak{su}(8)$	F II	$\mathfrak{f}_4$	$\mathfrak{so}(9)$
E VI	$\mathfrak{e}_7$	$\mathfrak{so}(12) + \mathfrak{sp}(1)$	G	$\mathfrak{g}_2$	$\mathfrak{sp}(1) + \mathfrak{sp}(1)$

Table 2. Exceptional symmetric spaces of type I.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{k}$	$c(\mathfrak{h}, \mathfrak{k})$
$\mathfrak{e}_6$	$\mathfrak{sp}(4)$	$\mathfrak{sp}(4-n)+\mathfrak{sp}(n),\ n=1,2$	$-12, -4$
$\mathfrak{e}_6$	$\mathfrak{sp}(4)$	$\mathfrak{su}(4)+\mathbb{R}$	4
$\mathfrak{e}_6$	$\mathfrak{su}(6)+\mathfrak{sp}(1)$	$\mathfrak{s}(\mathfrak{u}(6-n)+\mathfrak{u}(n))+\mathbb{R}$ $n=0,\ldots,3$	$-34, -14,$ $-2, 2$
$\mathfrak{e}_6$	$\mathfrak{su}(6)+\mathfrak{sp}(1)$	$\mathfrak{s}(\mathfrak{u}(6-n)+\mathfrak{u}(n))+\mathfrak{sp}(1)$ $n=1,\ldots,3$	$-18, -6, -2$
$\mathfrak{e}_6$	$\mathfrak{su}(6)+\mathfrak{sp}(1)$	$\mathfrak{sp}(3)+\mathfrak{sp}(1)$	-10
$\mathfrak{e}_6$	$\mathfrak{su}(6)+\mathfrak{sp}(1)$	$\mathfrak{sp}(3)+\mathbb{R}$	-6
$\mathfrak{e}_6$	$\mathfrak{su}(6)+\mathfrak{sp}(1)$	$\mathfrak{so}(6)+\mathfrak{sp}(1)$	2
$\mathfrak{e}_6$	$\mathfrak{su}(6)+\mathfrak{sp}(1)$	$\mathfrak{so}(6)+\mathbb{R}$	6
$\mathfrak{e}_6$	$\mathfrak{so}(10)+\mathbb{R}$	$\mathfrak{so}(10-n)+\mathfrak{so}(n)$ $n=0,\ldots,5$	$-44, -26, -12,$ $-2, 4, 6$
$\mathfrak{e}_6$	$\mathfrak{so}(10)+\mathbb{R}$	$\mathfrak{so}(10-n)+\mathfrak{so}(n)+\mathbb{R}$ $n=1,\ldots,5$	$-28, -14,$ $-4, 2, 4$
$\mathfrak{e}_6$	$\mathfrak{so}(10)+\mathbb{R}$	$\mathfrak{u}(5)+\mathbb{R}$	-6
$\mathfrak{e}_6$	$\mathfrak{so}(10)+\mathbb{R}$	$\mathfrak{u}(5)$	-4
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathfrak{sp}(3)+\mathfrak{sp}(1)$	4
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathfrak{so}(9)$	-20
$\mathfrak{e}_7$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(8-n)+\mathfrak{u}(n))$ $n=1,\ldots,4$	$-35, -15,$ $-3, 1$
$\mathfrak{e}_7$	$\mathfrak{su}(8)$	$\mathfrak{sp}(4)$	-9
$\mathfrak{e}_7$	$\mathfrak{su}(8)$	$\mathfrak{so}(8)$	7
$\mathfrak{e}_7$	$\mathfrak{so}(12)+\mathfrak{sp}(1)$	$\mathfrak{so}(12-n)+\mathfrak{so}(n)+\mathbb{R}$ $n=0,\ldots,6$	$-65, -43, -25,$ $-11, -1, 5, 7$
$\mathfrak{e}_7$	$\mathfrak{so}(12)+\mathfrak{sp}(1)$	$\mathfrak{so}(12-n)+\mathfrak{so}(n)+\mathfrak{sp}(1),$ $n=1,\ldots,6$	$-47, -29,$ $-15, -5, 1, 3$
$\mathfrak{e}_7$	$\mathfrak{so}(12)+\mathfrak{sp}(1)$	$\mathfrak{u}(6)+\mathfrak{sp}(1)$	-9
$\mathfrak{e}_7$	$\mathfrak{so}(12)+\mathfrak{sp}(1)$	$\mathfrak{u}(6)+\mathbb{R}$	-5
$\mathfrak{e}_7$	$\mathfrak{e}_6+\mathbb{R}$	$\mathfrak{sp}(4)+\mathbb{R}$	5
$\mathfrak{e}_7$	$\mathfrak{e}_6+\mathbb{R}$	$\mathfrak{sp}(4)$	7
$\mathfrak{e}_7$	$\mathfrak{e}_6+\mathbb{R}$	$\mathfrak{so}(10)+\mathbb{R}+\mathbb{R}$	-15
$\mathfrak{e}_7$	$\mathfrak{e}_6+\mathbb{R}$	$\mathfrak{so}(10)+\mathbb{R}$	-13
$\mathfrak{e}_7$	$\mathfrak{e}_6+\mathbb{R}$	$\mathfrak{su}(6)+\mathfrak{sp}(1)+\mathbb{R}$	1
$\mathfrak{e}_7$	$\mathfrak{e}_6+\mathbb{R}$	$\mathfrak{su}(6)+\mathfrak{sp}(1)$	3
$\mathfrak{e}_7$	$\mathfrak{e}_6+\mathbb{R}$	$\mathfrak{f}_4+\mathbb{R}$	-27
$\mathfrak{e}_7$	$\mathfrak{e}_6+\mathbb{R}$	$\mathfrak{f}_4$	-25

Table 3. Symmetric subalgebras of symmetric subalgebras of simple exceptional compact Lie algebras.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{k}$	$c(\mathfrak{h}, \mathfrak{k})$
$\mathfrak{e}_8$	$\mathfrak{so}(16)$	$\mathfrak{so}(16-n) + \mathfrak{so}(n)$ $n = 1, \ldots, 8$	$-90, -64, -42,$ $-24, -10, 0, 6, 8$
$\mathfrak{e}_8$	$\mathfrak{so}(16)$	$\mathfrak{u}(8)$	-8
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{e}_7 + \mathbb{R}$	-132
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{su}(8) + \mathfrak{sp}(1)$	4
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{su}(8) + \mathbb{R}$	8
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{so}(12) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$	-8
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{so}(12) + \mathfrak{sp}(1) + \mathbb{R}$	-4
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{e}_6 + \mathbb{R} + \mathfrak{sp}(1)$	-28
$\mathfrak{e}_8$	$\mathfrak{e}_7 + \mathfrak{sp}(1)$	$\mathfrak{e}_6 + \mathbb{R} + \mathbb{R}$	-24
$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$\mathfrak{sp}(3) + \mathbb{R}$	-20
$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$\mathfrak{sp}(2) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$	-8
$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$\mathfrak{sp}(2) + \mathfrak{sp}(1) + \mathbb{R}$	-4
$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$\mathfrak{u}(3) + \mathfrak{sp}(1)$	0
$\mathfrak{f}_4$	$\mathfrak{sp}(3) + \mathfrak{sp}(1)$	$\mathfrak{u}(3) + \mathbb{R}$	4
$\mathfrak{f}_4$	$\mathfrak{so}(9)$	$\mathfrak{so}(9-n) + \mathfrak{so}(n)$ $n = 1, \ldots, 4$	$-20, -8, 0, 4$
$\mathfrak{g}_2$	$\mathfrak{so}(4)$	$\mathfrak{u}(2)$	-2
$\mathfrak{g}_2$	$\mathfrak{so}(4)$	$\mathfrak{so}(3)$	0
$\mathfrak{g}_2$	$\mathfrak{so}(4)$	$\mathbb{R} + \mathbb{R}$	2

Table 3 (continued).

3. Involutions of reductive Lie algebras and construction of examples

We will now discuss some constructions of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces. We start with a list of several possibilities how one can define involutive automorphisms of a reductive complex Lie algebra from a given root space decomposition. Let $\mathfrak{g}$ be a reductive complex Lie algebra with Cartan subalgebra $\mathfrak{g}_0$ and let

$$(3\text{-}1) \qquad \mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be the root space decomposition with respect to $\mathfrak{g}_0$. Also assume that we have chosen once and for all a set $\{\alpha_1, \ldots, \alpha_n\}$ of simple roots.

Type 1. Let $\mathfrak{h} \subset \mathfrak{g}$ be a symmetric subalgebra of maximal rank. Then we may assume that $\mathfrak{h}$ contains $\mathfrak{g}_0$. Hence there is a subset $S \subset \Phi$ such that $\mathfrak{h} = \mathfrak{g}_0 + \sum_{\alpha \in S} \mathfrak{g}_\alpha$ and we may define an involutive automorphism σ of $\mathfrak{g}$ by requiring that $\sigma(X) = X$

if $X \in \mathfrak{h}$ and $\sigma(X) = -X$ if $X \in \sum_{\alpha \in \Phi \setminus S} \mathfrak{g}_\alpha$. The corresponding symmetric spaces are exactly those where the involution is an inner automorphism.

Type 2. Outer involutions induced from automorphisms of the Dynkin diagram (see [Helgason 1978, Chapter X, Section 5]): The corresponding simply connected irreducible symmetric spaces with simple compact isometry group are the spaces $SU(2n+1)/SO(2n+1)$, $SU(2n)/Sp(n)$, $SO(2n+2)/SO(2n+1)$ and E_6/F_4.

Type 3. Let $\mathfrak{g}$ be a reductive complex Lie algebra and let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra. Then there is an automorphism σ of $\mathfrak{g}$ which acts as minus identity on $\mathfrak{t}$ and sends each root to its negative. The symmetric spaces G/H given in this way are exactly those where $\operatorname{rk} G/H = \operatorname{rk} G$; the Satake diagram of G/H is given by the Dynkin diagram of G with uniform multiplicity one. The automorphisms obtained in this way may be inner or outer. The corresponding irreducible compact symmetric space are: A I, $SO(2n+1)/SO(n+1) \times SO(n)$, $Sp(n)/U(n)$, $SO(2n)/SO(n) \times SO(n)$, E I, E V, E VIII, F I, G.

The distinction of the three types of involutions above pertains to the action of the involution with respect to one fixed Cartan subalgebra. While Type 1 involutions obviously are never conjugate to involutions of type Type 2, a Type 3 involution may be conjugate to a Type 1 or Type 2 involution.

In the examples below, we show how to construct various $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces mainly by combining two commuting involutions as defined above with respect to one fixed root space decomposition (3-1). We start with the examples where both involutions are of Type 1.

Example 3.1. Let G be a connected compact Lie group and let σ_1 and σ_2 be two involutions of G defined by conjugation with the elements g_1 and g_2, respectively, of G, such that the fixed point sets G^{σ_1} and G^{σ_2} are nonisomorphic. After conjugation, we may assume that g_1 and g_2 are both contained in one and the same maximal torus T of G; in particular, σ_1 and σ_2 commute. Consider the root space decomposition (3-1) where $\mathfrak{g}_0$ is the complexification of the Lie algebra of T. Then the automorphisms of $\mathfrak{g}_\mathbb{C}$ induced by σ and τ are both of Type 1 and the complexification of the Lie algebra of $G^\sigma \cap G^\tau$ is given by $\mathfrak{g}_0 + \sum_{\alpha \in S_1 \cap S_2} \mathfrak{g}_\alpha$, where S_i is the root system of G^{σ_i}, $i = 1, 2$, with respect to $\mathfrak{g}_0$.

The maximal subgroups of maximal rank in a simple compact Lie group G can be conveniently described by using extended Dynkin diagrams; see [Onishchik 1994, 1.3.11]. The nodes of the extended Dynkin diagram correspond to roots $\alpha_0, \alpha_1, \ldots, \alpha_n$, where $\alpha_1, \ldots, \alpha_n$ are the simple roots and $\alpha_0 = -\delta$ where δ is highest root. The simple roots of a maximal subgroup of maximal rank are then given by certain subsets of the set of nodes of the extended Dynkin diagram. For the groups E_6, E_7 and E_8 these subsets are given below by the black nodes • for the symmetric ones among the subgroups of maximal rank. The root systems S_i of

these subgroups are given as the union of all roots which are integral linear combinations of the simple roots corresponding to black nodes. Now we may consider two diagrams depicting two symmetric subgroups G^{σ_1} and G^{σ_2} of maximal rank. Then the root system of $G^{\sigma_1} \cap G^{\sigma_2}$ with respect to $\mathfrak{g}_0$ is given by $S_1 \cap S_2$ and it is a straightforward task to explicitly determine this set. However, to completely avoid these computations, we will use a simplified approach. We know from [Onishchik 1994, 1.3, Proposition 15] that both G^{σ_1} and G^{σ_2} contain a subgroup whose simple roots with respect to $\mathfrak{g}_0$ are those whose corresponding nodes are marked black in both diagrams. Thus we obtain the set of simple roots of a regular subgroup of G contained in the intersection $G^{\sigma} \cap G^{\tau}$. This information, together with a dimension count using Table 3 (pages 118–119) and Table 4, turns out to be sufficient in all cases given below to identify the connected component of the intersection $G^{\sigma} \cap G^{\tau}$, which is a symmetric subgroup of both G^{σ} and G^{τ}.

| Type	d	Type	d	Type	d
E I-I	6	E III-IV	-20	E VII-VII	-25
E I-II	4	E IV-IV	-26	E VIII-VIII	8
E I-III	-4	E V-V	7	E VIII-IX	-8
E I-IV	-10	E V-VI	1	E IX-IX	-24
E II-II	2	E V-VII	-9	F I-I	4
E II-III	-6	E VI-VI	-5	F I-II	-8
E II-IV	-12	E VI-VII	-15	F II-II	-20
E III-III	-14			G	2

Table 4. Pairs of involutions on exceptional groups.

The symmetric subgroups of maximal rank in E_6, namely $SU(6) \cdot Sp(1)$ and $Spin(10) \cdot U(1)$ are given by the following diagrams:
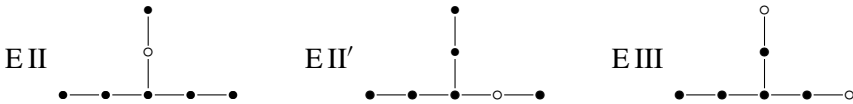
Comparing E-II and E-III, we see that there is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space G/K such that $\mathfrak{k}$ contains a subalgebra isomorphic to $\mathfrak{su}(5)+\mathbb{R}+\mathbb{R}$. A simple dimension count, facilitated by Tables 3 and 4, shows that the corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space is of type E II-III-III with $\mathfrak{k} \cong \mathfrak{su}(5) + \mathbb{R} + \mathbb{R}$. Using an alternative embedding of $SU(6) \cdot Sp(1)$ into E_6, shown as E II$'$, and combining it with E II, we obtain a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space with $\mathfrak{k} \cong \mathfrak{su}(4) + \mathfrak{sp}(1) + \mathfrak{sp}(1) + \mathbb{R} \cong \mathfrak{so}(6) + \mathfrak{so}(4) + \mathbb{R}$ of type E II-II-III. Combining E III and E III$'$ (as given in Example 3.6 below) we construct a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E III-III-III with $\mathfrak{k} = \mathfrak{so}(8) + \mathbb{R} + \mathbb{R}$.

The connected symmetric subgroups of E_7, namely $SU(8)$, $Spin(12) \cdot Sp(1)$ and $E_6 \cdot U(1)$, are given by the following diagrams.

E V   E VI

E VI'   E VII
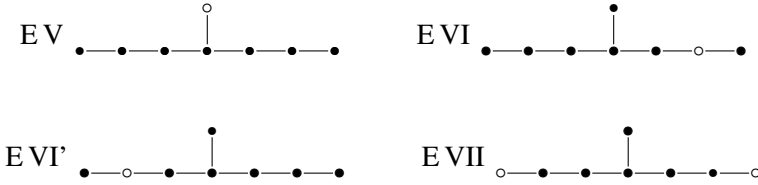
Combining E V and E VI we obtain a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space with $\mathfrak{k} = \mathfrak{su}(6) + \mathfrak{sp}(1) + \mathbb{R}$ of type E V-VI-VII. Combining E VI and E VII we obtain a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space with $\mathfrak{k}$ containing a subalgebra isomorphic to $\mathfrak{so}(10) + \mathbb{R} + \mathbb{R}$; using Tables 3 and 4 it follows that the corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space is of type E VI-VII-VII with $\mathfrak{k} \cong \mathfrak{so}(10) + \mathbb{R} + \mathbb{R}$. Combining E VI with E VI', we obtain a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space such that $\mathfrak{k}$ contains a subalgebra isomorphic to $\mathfrak{so}(8) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$. We see from the same tables that the corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space is of type E VI-VI-VI with $\mathfrak{k} \cong \mathfrak{so}(8) + \mathfrak{so}(4) + \mathfrak{sp}(1)$.

The connected symmetric subgroups of E_8, namely $SO'(16)$ and $E_7 \cdot Sp(1)$ are given by the following diagrams.

E VIII   E IX

Combining the two diagrams and using Tables 4 and 3 proves the existence of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E VIII-IX-IX with $\mathfrak{k} \cong \mathfrak{so}(12) + \mathfrak{sp}(1) + \mathfrak{sp}(1)$.

Example 3.2. Let now G be simple and let $g \in G$ be an element with $i_g^2 = \mathrm{id}_G$. Let T be a maximal torus of G containing g. Let $w = k\, Z_G(T)$, $k \in N_G(T)$, be an element of the Weyl group $W_G = N_G(T)/Z_G(T)$ of G such that the action of w on T does not leave the root system of $Z_G(\{g\})$ invariant. Define $h = w \cdot g = k\, g\, k^{-1}$. Then i_g and i_h are two conjugate commuting involutions of G whose fixed point sets do not agree. Hence they define a $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetric space.

We now use this construction to prove the existence of various $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces. First, let $G = F_4$ and let i_g be an inner involution of type F II, that is, the connected component of the fixed point set of i_g is isomorphic to $Spin(9)$. Define another involution i_h as above; it is also of type F II, but the fixed point sets of i_g and i_h do not agree. From Table 4 we see that $d = -20$ for a pair of involutions of Type F II-II. In Table 3 there are two entries with $\mathfrak{g} = \mathfrak{f}_4$ and $c(\mathfrak{h}, \mathfrak{k}) = -20$, namely $\mathfrak{k} = \mathfrak{sp}(3) + \mathbb{R}$ and $\mathfrak{k} = \mathfrak{so}(8)$. Since $\mathfrak{sp}(3)$ is not subalgebra of $\mathfrak{so}(9)$, it follows that the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space constructed in this way is of type F II-II-II. This argument also shows that there is no $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type F I-II-II.

Now let $G = E_8$ and let σ be an involution of type E IX as given in Example 3.1 and let $\mathfrak{g}^\sigma$ be the Lie algebra of the fixed point set of i_g. Consider the subalgebra $\mathfrak{h} \subset \mathfrak{g} = \mathfrak{e}_8$ isomorphic to $\mathfrak{su}(3) + \mathfrak{e}_6$ given by the diagram
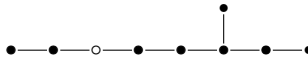
Let W be the Weyl group of $\mathfrak{e}_8$ with respect to $\mathfrak{g}_0 = \mathfrak{t}$. The Weyl groups of $\mathfrak{g}^\sigma$ and of $\mathfrak{h}$ are subgroups of W in a natural way. Comparing the two diagrams corresponding to $\mathfrak{g}^\sigma$ and $\mathfrak{h}$, respectively, we see that there is at least one root space of $\mathfrak{g}$ which is contained in the $\mathfrak{su}(3)$-summand of $\mathfrak{h}$, but not in $\mathfrak{g}^\sigma$. Hence there is a Weyl group element $w \in W \subset \mathrm{Aut}(\mathfrak{g})$ whose action on $\mathfrak{g}$ leaves the $\mathfrak{e}_6$-summand of $\mathfrak{h}$ fixed, but which does not leave $\mathfrak{g}^\sigma$ invariant. This implies that there is an involution $\tau \neq \sigma$ such that $\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$ contains a subalgebra isomorphic to $\mathfrak{e}_6$. By a dimension count, this shows the existence of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E IX-IX-IX with $\mathfrak{k} \cong \mathfrak{e}_6 + \mathbb{R} + \mathbb{R}$.

Now consider $\mathfrak{g} = \mathfrak{e}_7$ and let σ be an involution as given by the diagram E VI in Example 3.1. Consider the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ isomorphic to $\mathfrak{su}(6) + \mathfrak{su}(3)$ given by
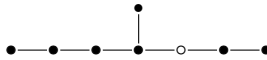
With an analogous argument as above we may construct a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space such that $\mathfrak{k}$ contains a subalgebra isomorphic to $\mathfrak{su}(6)$. Using Table 3 we see that the corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space is of type E VI-VI-VI with $\mathfrak{k} \cong \mathfrak{u}(6) + \mathbb{R}$.

For $G = E_6$, consider the subgroup locally isomorphic to $SU(3) \cdot SU(3) \cdot SU(3)$ given by the diagram
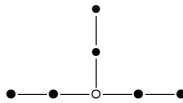
Let i_g be an inner involution corresponding to the diagram E II from Example 3.1. Then there is a Weyl group element $w = k\, Z_G(T)$ such that the involution i_h with $h = k\, g\, k^{-1}$ is such that the Lie algebra of the common fixed point set of i_g and i_h contains a subalgebra isomorphic to $\mathfrak{su}(3) + \mathfrak{su}(3) + \mathbb{R} + \mathbb{R}$. It follows from Table 3 that we have constructed a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E II-II-II with $\mathfrak{k} = \mathfrak{su}(3) + \mathfrak{su}(3) + \mathbb{R} + \mathbb{R}$.

Example 3.3. Here we combine two involutions of Type 3 and Type 1, respectively. Thus let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a Cartan subalgebra. Let σ be an involution of Type 3 on $\mathfrak{g}$. Let τ be an inner involution

given as conjugation by an element $t = \exp X$, where $X \in \mathfrak{g}_0$. Then the connected component of $Z_G(\{t\})$ is the connected Lie subgroup of G corresponding to $\mathfrak{g}_0 + \sum_{\alpha \in S} \mathfrak{g}_\alpha$, where S is a certain subset of the root system Φ of $\mathfrak{g}$ with the property that for each $\alpha \in S$ we have also $-\alpha \in S$. Obviously, the two involutions commute and the involution induced by σ on the fixed point set $\mathfrak{g}^\tau$ of τ is the automorphism which acts as minus identity on $\mathfrak{g}_0$ and sends each root $\alpha \in S$ of $\mathfrak{g}^\tau$ to its negative. Moreover, the fixed point set of $\sigma\tau$ will be isomorphic to the fixed point set of σ, since the corresponding symmetric spaces will have isomorphic Satake diagrams. Hence the data of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space obtained in this way can be immediately deduced from the classification of ordinary symmetric spaces. This shows the existence of the exceptional $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of types E I-I-II, E I-I-III, E V-V-V, E V-V-VI, E V-V-VII, E VIII-VIII-VIII, E VIII-VIII-IX, F I-I-I, F I-I-II, G as given in Table 1.

Example 3.4. Let now $G = E_6$ and choose a Cartan subalgebra $\mathfrak{g}_0 \subset \mathfrak{e}_6$. Let σ be an involution of Type 2, that is, E IV and let τ be an involution of Type 3, that is, E I. Then the two involutions obviously commute. A dimension count using Tables 4 and 3 shows that $\sigma\tau$ is of type E II and hence the corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space is of type E I-II-IV with $\mathfrak{k} \cong \mathfrak{sp}(3) + \mathfrak{sp}(1)$.

Example 3.5. Let $G = E_6$. Let σ be the involution of Type 2 given by the permutation of the simple roots $\alpha_1 \mapsto \alpha_5$, $\alpha_2 \mapsto \alpha_4$, $\alpha_3 \mapsto \alpha_3$, $\alpha_4 \mapsto \alpha_2$, $\alpha_5 \mapsto \alpha_1$, $\alpha_6 \mapsto \alpha_6$ and let τ be the involution of Type 1 given by the following diagram, where the numbering of the roots $\alpha_0, \ldots, \alpha_6$ is given.
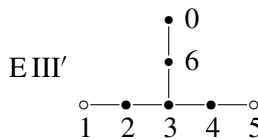
Then obviously σ and τ commute and since $\sigma\tau$ is an outer automorphism of E_6, it follows from Lemma 4.1 that we have constructed a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E III-IV-IV with $\mathfrak{k} \cong \mathfrak{so}(9)$.

Example 3.6. Let $G = E_7$. By [Wolf 1968, Theorem 3.1] there is an isotropy irreducible homogeneous space E_7/L, where L is locally isomorphic to $\mathrm{Sp}(1) \cdot F_4$ such that the 78-dimensional isotropy representation ρ is equivalent to the tensor product of the adjoint representation of $\mathrm{Sp}(1)$ and the 26-dimensional irreducible representation of F_4. In particular, we may view the isotropy representation as a representation of $\mathrm{SO}(3) \times F_4$ and choose two elements $g, h \in L$ such that the action of $\rho(g)$ and $\rho(h)$ is given by $\mathrm{diag}(-1, -1, +1) \otimes e$ and $\mathrm{diag}(1, -1, -1) \otimes e$, respectively, where e stands for the identity element of F_4. Then the involutions i_g and i_h generate a subgroup $\Gamma \subset \mathrm{Aut}(E_7)$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ whose fixed point

set has Lie algebra $\mathfrak{k} \cong \mathfrak{f}_4$. Since the fixed point sets of i_g, i_h and i_{gh} are all 79-dimensional, it follows that we have constructed a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E VII-VII-VII.

4. The classification in the exceptional case

Lemma 4.1. *There are no $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces of type F I-II-II, E I-III-IV, E II-IV-IV, E V-VI-VI, E VI-VI-VII or E V-VII-VII.*

Proof. The nonexistence of type F I-II-II was already shown in Example 3.2. The nonexistence of type E I-III-IV or E II-IV-IV spaces follows immediately from Tables 4 and 3. For a pair of involutions of type E VI-VI on E_7 we have $d = -5$, see Table 4. It follows from Table 3 that there is no symmetric subalgebra $\mathfrak{k}$ of $\mathfrak{h} = \mathfrak{su}(8)$ or $\mathfrak{h} = \mathfrak{e}_6 + \mathbb{R}$ such that $c(\mathfrak{su}(8), \mathfrak{k}) = -5$; this shows there are no $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces of type E V-VI-VI or E VI-VI-VII. Now consider a pair of involutions of type E VII-VII on E_7; by Table 4 we have $d = -25$ and it follows from Table 3 that $\mathfrak{k} \cong \mathfrak{so}(10) + \mathbb{R} + \mathbb{R}$ or $\mathfrak{k} \cong \mathfrak{f}_4$; since neither occurs as a subalgebra of $\mathfrak{su}(8)$, we have proved the nonexistence of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E V-VII-VII. □

Remark 4.2. In each row of Table 1, only the isomorphism type of the subalgebra $\mathfrak{k} \subset \mathfrak{g}$ and the type of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space (that is, the entry in the first column defines the isomorphism type of the three symmetric subalgebras $\mathfrak{g}^\sigma$, $\mathfrak{g}^\tau$, $\mathfrak{g}^{\sigma\tau}$ of $\mathfrak{g}$) is given. However, this information is sufficient to determine the conjugacy class of the subalgebra $\mathfrak{k} \subset \mathfrak{g}$ in most cases, since $\mathfrak{k} \subset \mathfrak{h}$ is also symmetric subalgebra for $\mathfrak{h} = \mathfrak{g}^\sigma$, $\mathfrak{g}^\tau$, $\mathfrak{g}^{\sigma\tau}$: Assume that $\mathfrak{g}$ is a compact Lie algebra and let $\mathfrak{k} \subset \mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{k}' \subset \mathfrak{h}' \subset \mathfrak{g}$ be such that all inclusions are of symmetric subalgebras and such that $\mathfrak{k} \cong \mathfrak{k}'$, $\mathfrak{h} \cong \mathfrak{h}'$. Then there is an automorphism $\eta \in \mathrm{Aut}(\mathfrak{g})$ such that $\eta(\mathfrak{h}') = \mathfrak{h}$. In most cases (see [Kollross 2002, 3.1]) there is an inner automorphism θ_0 of $\mathfrak{h}$ such that $\theta_0(\eta(\mathfrak{k}')) = \mathfrak{k}$. Then obviously θ_0 can be extended to an inner automorphism θ of $\mathfrak{g}$ such that $\theta \circ \eta(\mathfrak{k}') = \mathfrak{k}$.

This shows that for most cases the subalgebra $\mathfrak{k}$ is uniquely defined by the data in Table 1. The only exception occurs for the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E VI-VI-VI with $\mathfrak{k} \cong \mathfrak{u}(6) + \mathbb{R}$. Indeed, there are two conjugacy classes of subalgebras isomorphic to $\mathfrak{a}_5$ in $\mathfrak{e}_7$. These two classes are denoted by $[A_5]'$ and $[A_5]''$ in [Dynkin 1957]. They are given by the two diagrams below.

$$[A_5]' \quad \bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet\!-\!\!-\!\overset{\circ}{\underset{|}{}}\!-\!\!-\!\bullet\!-\!\!-\!\circ\!-\!\!-\!\circ \qquad\qquad [A_5]'' \quad \bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet\!-\!\!-\!\overset{\bullet}{\underset{|}{}}\!-\!\!-\!\circ\!-\!\!-\!\circ\!-\!\!-\!\circ$$

These two subalgebras of $\mathfrak{e}_7$ are both contained in $\mathfrak{so}(12) \subset \mathfrak{e}_7$ and, as subalgebras of $\mathfrak{so}(12)$, they are conjugate by an outer automorphism of $\mathfrak{so}(12)$; however, this automorphism cannot be extended to an automorphism of $\mathfrak{e}_7$. It follows from

[Dynkin 1957, Tables 25 and 26, pages 204, 207] that only the subalgebra $[A_5]''$ gives rise to a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space.

Proof of Theorem 1.2. For each entry in Table 1 we have given a construction in Section 3 thus showing the existence of a corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space. Lemma 4.1 shows that the types of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space not listed in Table 1 indeed do not exist. (Note that on E_6, the involutions of type E I and E IV are outer, while the involutions of type E II and E III are inner; in particular, there cannot be $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces of type E I-I-I or of type E I-II-II etc.)

However, it remains to be shown that the type of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space as given in Table 1 indeed determines the conjugacy class of the fixed point set $\mathfrak{k} = \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau$ except for spaces of type E VI-VI-VI. In most cases, it can be seen from Tables 4 and 3 that the conjugacy class of the subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is uniquely determined by the type of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space, that is, in most cases, the conjugacy class of the subalgebra $\mathfrak{k}$ is uniquely determined by the triple $(\mathfrak{g}, \mathfrak{h}, c(\mathfrak{h}, \mathfrak{k}))$ where $c(\mathfrak{h}, \mathfrak{k}) = d$. There are three exceptions to this general rule:

For $\mathfrak{g} = \mathfrak{e}_6$ we have two entries in Table 3 with $\mathfrak{h} = \mathfrak{su}(6) + \mathfrak{sp}(1)$ and $c(\mathfrak{h}, \mathfrak{k}) = 2$, namely $\mathfrak{k} = \mathfrak{su}(3) + \mathfrak{su}(3) + \mathbb{R} + \mathbb{R}$ and $\mathfrak{k} = \mathfrak{so}(6) + \mathfrak{sp}(1)$. For the first possibility we have already shown the existence of a corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E II-II-II. The second possibility would imply the existence of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E II-II-II with $\mathfrak{k} = \mathfrak{so}(6) + \mathfrak{sp}(1) \subset \mathfrak{su}(6) + \mathfrak{sp}(1)$ such that the isotropy representation of $\mathfrak{k}$ on $\mathfrak{g}/\mathfrak{k}$ restricted to $\mathfrak{so}(6)$ consists of a sum of three isotypical summands; however this is not the case according to [Dynkin 1957, Tables 25 and 26, pages 200, 206].

There are also two entries in Table 3 with $\mathfrak{h} = \mathfrak{su}(6) + \mathfrak{sp}(1)$ and $c(\mathfrak{h}, \mathfrak{k}) = -6$, namely $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) + \mathfrak{sp}(1)$ and $\mathfrak{k} = \mathfrak{sp}(3) + \mathbb{R}$. For the first we have already shown the existence of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E II-II-III in Example 3.1; the second one can be ruled out since $\mathfrak{sp}(3) + \mathbb{R}$ is not a symmetric subalgebra of $\mathfrak{so}(10) + \mathbb{R}$.

Finally, for $\mathfrak{g} = \mathfrak{e}_7$ there are two items in Table 3 with $\mathfrak{h} = \mathfrak{so}(12) + \mathfrak{sp}(1)$ and $c(\mathfrak{h}, \mathfrak{k}) = -5$, namely $\mathfrak{k} = \mathfrak{so}(8) + \mathfrak{so}(4) + \mathfrak{sp}(1)$ and $\mathfrak{k} = \mathfrak{u}(6) + \mathbb{R}$. We have shown the existence of a corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of type E VI-VI-VI for both cases in Examples 3.1 and 3.2, respectively. $\square$

5. The classification in the Spin(8) case

In [Bahturin and Goze 2008], the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space of the classical groups have been determined except for $G \cong \mathrm{Spin}(8)$. This case requires special attention as the group of outer automorphisms $\mathrm{Out}(\mathrm{Spin}(8)) = \mathrm{Aut}(\mathrm{Spin}(8))/\mathrm{Inn}(\mathrm{Spin}(8))$ is isomorphic to the symmetric group on three letters, whereas for all other simple compact Lie groups the group $\mathrm{Out}(G)$ is either trivial or isomorphic to $\mathbb{Z}_2$. In this

section, we will classify the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces with $G = \text{Spin}(8)$. It was remarked in [Bahturin and Goze 2008] that the result can be obtained from [Draper and Martín 2006]. We give an independent proof, using an analogous method as in the exceptional case.

We recall [Kollross 2002] that all connected symmetric subgroups of Spin(8) are conjugate by some automorphism of Spin(8) to one of the groups

$$\text{Spin}(8 - n) \cdot \text{Spin}(n), \qquad n = 1, 2, 3, 4.$$

However, if one considers conjugacy classes with respect to inner automorphisms, there are three conjugacy classes of symmetric subgroups locally isomorphic to one of Spin(7), Spin(6) · Spin(2), Spin(5) · Spin(3), respectively, and only one conjugacy class of subgroups locally isomorphic to Spin(4)·Spin(4); see [Kollross 2002, Proposition 3.3]. Hence there are ten conjugacy classes (with respect to inner automorphisms) of connected symmetric subgroups in Spin(8); under the covering map Spin(8) → SO(8), they correspond to the conjugacy classes (with respect to inner automorphisms) of connected (locally) symmetric subgroups in SO(8) given by SO(7), Spin(7)$^+$, Spin(7)$^-$, SO(6) × SO(2), U(4), α(U(4)), SO(5) × SO(3), Sp(2) · Sp(1), α(Sp(2) · Sp(1)), SO(4) × SO(4), where $\alpha = i_{\text{diag}(-1,1,\dots,1)}$.

G^σ	G^τ	d
Spin(7)	Spin(7)	−14
Spin(7)	Spin(6) · Spin(2)	−9
Spin(7)	Spin(5) · Spin(3)	−6
Spin(7)	Spin(4) · Spin(4)	−5
Spin(6) · Spin(2)	Spin(6) · Spin(2)	−4
Spin(6) · Spin(2)	Spin(5) · Spin(3)	−1
Spin(6) · Spin(2)	Spin(4) · Spin(4)	0
Spin(5) · Spin(3)	Spin(5) · Spin(3)	2
Spin(5) · Spin(3)	Spin(4) · Spin(4)	3
Spin(4) · Spin(4)	Spin(4) · Spin(4)	4

Table 5. Pairs of involutions on Spin(8).

Example 5.1. We may construct standard examples of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces for $\mathfrak{g} = \mathfrak{so}(8)$ using a pair of commuting involutions both given by conjugation with diagonal matrices of the form diag $(\pm 1, \dots, \pm 1)$. In this way obtain we $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces with $\mathfrak{k} = \mathfrak{so}(n_1) + \mathfrak{so}(n_2) + \mathfrak{so}(n_3)$, $n_1 + n_2 + n_3 = 8$, $n_i \geq 1$ or $\mathfrak{k} = \mathfrak{so}(n_1) + \mathfrak{so}(n_2) + \mathfrak{so}(n_3) + \mathfrak{so}(n_4)$, $n_1 + n_2 + n_3 + n_4 = 8$, $n_i \geq 1$.

Example 5.2. Now assume that the subalgebra $\mathfrak{so}(6) + \mathfrak{so}(2) \cong \mathfrak{u}(4) \subset \mathfrak{so}(8)$ is embedded as

$$\left\{ \begin{pmatrix} A & -B^t \\ B & A \end{pmatrix} \,\middle|\, A + iB \in \mathfrak{u}(4) \right\}.$$

We may define a pair of commuting involutions of $\mathfrak{so}(8)$ using conjugation with the matrices

$$J = \begin{pmatrix} & -I \\ I & \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix},$$

respectively, where I is the 4×4-identity matrix and D is either $\mathrm{diag}(-1, 1, 1, 1)$ or $\mathrm{diag}(-1, -1, 1, 1)$. We obtain $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces with $\mathfrak{k} = \mathfrak{u}(1) + \mathfrak{u}(3)$ or $\mathfrak{k} = \mathfrak{u}(2) + \mathfrak{u}(2)$. Note however that the subalgebra $\mathfrak{u}(2) + \mathfrak{u}(2)$ of $\mathfrak{so}(8)$ is conjugate to $\mathfrak{so}(4) + \mathfrak{so}(2) + \mathfrak{so}(2)$ via an outer automorphism.

Proof of Theorem 1.3. Assume σ and τ are two commuting involutions of Spin(8). Then the $\mathfrak{g}^{\sigma\tau}$ is a symmetric subalgebra of $\mathfrak{g} = \mathfrak{so}(8)$ by Proposition 2.2. Hence we may assume (by applying an automorphism of $\mathfrak{so}(8)$) that $\mathfrak{g}^{\sigma\tau}$ is one of the subalgebras $\mathfrak{h}$ of $\mathfrak{so}(8)$ as given in Table 6. Since $\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau \subset \mathfrak{h}$ is a symmetric subalgebra of $\mathfrak{h}$, it is conjugate (by some automorphism θ_0 of $\mathfrak{h}$) to one of the subalgebras $\mathfrak{k}$ as from Table 6 such that $c(\mathfrak{h}, \mathfrak{k}) = d(\sigma, \tau)$.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{k}$	$c(\mathfrak{h}, \mathfrak{k})$
$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{so}(7-n) + \mathfrak{so}(n)$ $n = 1, 2, 3$	$-9, -1, 3$
$\mathfrak{so}(8)$	$\mathfrak{so}(6) + \mathfrak{so}(2)$	$\mathfrak{so}(6-n) + \mathfrak{so}(n) + \mathfrak{so}(2)$ $n = 1, 2, 3$	$-6, 0, 2$
$\mathfrak{so}(8)$	$\mathfrak{so}(6) + \mathfrak{so}(2)$	$\mathfrak{so}(6-n) + \mathfrak{so}(n)$ $n = 0, 1, 2, 3$	$-14, -4, 2, 4$
$\mathfrak{so}(8)$	$\mathfrak{so}(6) + \mathfrak{so}(2)$	$\mathfrak{u}(3) + \mathfrak{u}(1)$	-4
$\mathfrak{so}(8)$	$\mathfrak{so}(6) + \mathfrak{so}(2)$	$\mathfrak{u}(3)$	-2
$\mathfrak{so}(8)$	$\mathfrak{so}(5) + \mathfrak{so}(3)$	$\mathfrak{so}(5-n) + \mathfrak{so}(n) + \mathfrak{so}(3)$ $n = 1, 2$	$-5, -1$
$\mathfrak{so}(8)$	$\mathfrak{so}(5) + \mathfrak{so}(3)$	$\mathfrak{so}(5-n) + \mathfrak{so}(n) + \mathfrak{so}(2)$ $n = 0, 1, 2$	$-9, -1, 3$
$\mathfrak{so}(8)$	$\mathfrak{so}(4) + \mathfrak{so}(4)$	$(4-n) \cdot \mathfrak{sp}(1) + n \cdot \mathfrak{so}(2)$ $n = 1, 2, 3, 4$	$-8, -4, 0, 4$
$\mathfrak{so}(8)$	$\mathfrak{so}(4) + \mathfrak{so}(4)$	$\mathfrak{so}(3) + \mathfrak{so}(4)$	-6
$\mathfrak{so}(8)$	$\mathfrak{so}(4) + \mathfrak{so}(4)$	$\mathfrak{so}(3) + \mathfrak{so}(3)$	0
$\mathfrak{so}(8)$	$\mathfrak{so}(4) + \mathfrak{so}(4)$	$\mathfrak{so}(3) + \mathfrak{u}(2)$	-2
$\mathfrak{so}(8)$	$\mathfrak{so}(4) + \mathfrak{so}(4)$	$\mathfrak{so}(3) + \mathfrak{so}(2) + \mathfrak{so}(2)$	2

Table 6. Symmetric subalgebras of symmetric subalgebras of $\mathfrak{so}(8)$.

Now we show that any automorphism θ_0 of $\mathfrak{h}$ can be extended to an automorphism θ of $\mathfrak{so}(8)$ such that $\theta(\mathfrak{h}) = \mathfrak{h}$ and $\theta|\mathfrak{h} = \theta_0$: This is obvious for inner automorphisms of $\mathfrak{h}$ and there is only one case where $\mathfrak{h}$ has outer automorphisms, namely $\mathfrak{h} = \mathfrak{so}(4) + \mathfrak{so}(4) \cong 4 \cdot \mathfrak{a}_1$. After choosing a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{h}$, and a set of simple roots $\alpha_1, \ldots, \alpha_4$ we can define diagram automorphisms which act by permuting the simple roots $\alpha_1, \alpha_2, \alpha_3$ and hence by permuting the corresponding three of the four $\mathfrak{a}_1$-summands of $\mathfrak{h}$. In addition, consider the inner automorphism i_g with

$$g = \begin{pmatrix} & I \\ I & \end{pmatrix}, \quad I = \mathrm{diag}(1, 1, 1, 1).$$

This acts on $\mathfrak{h} = \mathfrak{so}(4) + \mathfrak{so}(4)$ by interchanging the two $\mathfrak{so}(4)$-summands. These automorphisms together with the inner automorphisms of $\mathfrak{h}$ extended to $\mathfrak{g}$ generate a subgroup of $\mathrm{Aut}(\mathfrak{g})$ isomorphic to $\mathrm{Aut}(\mathfrak{h})$.

Hence there is an automorphism φ of $\mathfrak{so}(8)$ such that $\varphi(\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau)$ is one the subalgebras $\mathfrak{k} \subset \mathfrak{so}(8)$ as given in Table 6. Conversely, it is straightforward to see that for each candidate $(\mathfrak{h}, \mathfrak{k})$ one can construct a corresponding $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric space using Examples 5.1 or 5.2. $\square$

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