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We prove several sharp Hardy inequalities for domains with a spherical symmetry. In particular, we prove an inequality for domains of the unit n -dimensional sphere with a point singularity, and an inequality for functions defined on the half-space \mathbb{R}_+^{n+1} vanishing on the hyperplane $\{x_{n+1} = 0\}$, with singularity along the x_{n+1} -axis. The proofs rely on a one-dimensional Hardy inequality involving a weight function related to the volume element on the sphere, as well as on symmetrization arguments. The one-dimensional inequality is derived in a general form.

1. Introduction and main results

Sharp Hardy inequalities have attracted a considerable attention in recent years, particularly in view of their applications to differential equations motivated by physics and geometry. Let $1 < p < n$. The classical Hardy inequality states that

$$(1) \quad \int_{\mathbb{R}^n} |Du|_n^p \geq \left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|u|^p}{|x|_n^p}$$

for all smooth functions u compactly supported on \mathbb{R}^n , where $|x|_n^2 = x_1^2 + \cdots + x_n^2$ for all $x \in \mathbb{R}^n$. A considerable effort has been devoted to extending this inequality to manifolds, to special weight functions, and to domains exhibiting particular symmetries. See [Kufner and Persson 2003] for an extensive review. Our aim in this note is to derive some sharp Hardy type inequalities specifically tailored for manifolds with a spherical symmetry. Several recently proved inequalities concern the special case of the sphere; see [Bandle et al. 2005; Brezis and Peletier 2006]. Indeed, certain specific phenomena that do not occur on Euclidean space actually

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do occur on spheres. For example, in [Bandle et al. 2005] it was shown that the Sobolev inequality admits minimizers on sufficiently large spherical caps.

To state our main results, we introduce some notation. Denote by

$$\mathbb{S}^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : |x| = 1\}$$

the unit n -sphere, where we set $|x|^2 = |x|_{n+1}^2 = x_1^2 + \dots + x_{n+1}^2$ for all $x \in \mathbb{R}^{n+1}$. For $1 < p < n$ and for $a > 0$, we define a weight function, which is related to the volume element on \mathbb{S}^n , by

$$(2) \quad \tilde{\eta}_a : (0, a) \rightarrow \mathbb{R}, \quad t \mapsto \frac{(\sin t)^{-(n-1)/(p-1)}}{\int_t^a (\sin s)^{-(n-1)/(p-1)} ds}.$$

Note that $\lim_{t \rightarrow 0^+} \tilde{\eta}_a(t) = \lim_{t \rightarrow a^-} \tilde{\eta}_a(t) = +\infty$; see (14) below. In Lemma 2.4 we will show that there exists a $\tilde{T} \in (0, a)$ such that $\tilde{\eta}_a$ decreases in $(0, \tilde{T})$ and increases in (\tilde{T}, a) . Therefore, the truncated function

$$(3) \quad \tilde{\eta}_{a, \tilde{T}}(t) = \begin{cases} \tilde{\eta}_a(t) & \text{if } t \in (0, \tilde{T}), \\ \tilde{\eta}_a(\tilde{T}) & \text{if } t \in [\tilde{T}, a) \end{cases}$$

is decreasing in $(0, a)$. We denote by $\Theta = (\theta_1, \dots, \theta_{n-1}, \theta_n)$ the angular variables on \mathbb{S}^n and to simplify notation we set $\theta = \theta_n$. The angle $\theta \in [0, \pi]$, satisfying $x_{n+1} = |x| \cos \theta$, will be the only relevant angular variable to our purposes. We denote by g the standard metric on \mathbb{S}^n and by dV the volume element on \mathbb{S}^n . For $\alpha \in (0, \pi]$, we denote by $\mathcal{B}(\alpha)$ the geodesic ball (spherical cap) on \mathbb{S}^n with radius α centered at the north pole $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Namely, we define

$$\mathcal{B}(\alpha) = \{x \in \mathbb{S}^n : 0 \leq \theta < \alpha\}.$$

Let $N \in \Omega \subset \mathbb{S}^n$ be an open set such that $|\Omega| < |\mathbb{S}^n|$, and let $a^* \in (0, \pi)$ be such that $|\mathcal{B}(a^*)| = |\Omega|$. Here $|E|$ denotes the volume of a measurable set $E \subset \mathbb{S}^n$ with respect to the standard Lebesgue measure induced by g on \mathbb{S}^n . In turn, we define the weight function

$$(4) \quad \rho_{a^*} : \mathbb{S}^n \setminus \{N\} \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} \frac{p-1}{n-p} \tilde{\eta}_{a^*, \tilde{T}}(\theta) & \text{if } x \in \mathcal{B}(a^*) \setminus \{N\}, \\ \frac{p-1}{n-p} \tilde{\eta}_{a^*, \tilde{T}}(\tilde{T}) & \text{if } x \in \mathbb{S}^n \setminus \mathcal{B}(a^*), \end{cases}$$

where $\tilde{\eta}_{a^*, \tilde{T}}$ is the weight function defined in (3) with $a = a^*$.

Theorem 1.1. *Let $n \geq 2$ and $1 < p < n$. Let $\Omega \subset \mathbb{S}^n$ be an open set such that $N \in \Omega$ and $|\Omega| < |\mathbb{S}^n|$. Let a^* be such that $|\Omega| = |\mathcal{B}(a^*)|$. Then, for every $u \in W_0^{1,p}(\Omega)$,*

$$(5) \quad \int_{\Omega} |\nabla u|^p dV \geq \left(\frac{n-p}{p}\right)^p \int_{\Omega} |u|^p \rho_{a^*}^p dV,$$

and the constant $((n-p)/p)^p$ is sharp.

Note that, since $\theta = d_g(x, N)$, we have

$$(6) \quad \lim_{x \rightarrow N} d_g(x, N) \rho_{a^*}(x) = 1;$$

see (27) below. Thus ρ_{a^*} is a natural extension of the classical singularity $|x|_n^{-p}$ appearing in (1).

Theorem 1.1, together with a Steiner symmetrization with respect to the angular variables, yields an inequality for functions defined on the half-space

$$\mathbb{R}_+^{n+1} = \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

with singularity along the x_{n+1} -axis. More precisely, let $x' = (x_1, \dots, x_n, 0)$ for every $x \in \mathbb{R}^{n+1}$. We take $a^* = \pi/2$ in (2)–(4) and we define the singularity ζ as

$$\zeta : \mathbb{R}_+^{n+1} \setminus \{x' = 0\} \rightarrow \mathbb{R}, \quad x \mapsto \rho_{\pi/2}(x/|x|).$$

Note that ζ is singular on the x_{n+1} -axis.

Theorem 1.2. *For every $u \in W_0^{1,p}(\mathbb{R}_+^{n+1})$, we have*

$$(7) \quad \int_{\mathbb{R}_+^{n+1}} |D_{\Theta}u|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}_+^{n+1}} |u|^p \frac{\zeta^p(x)}{|x|^p} dx.$$

Here $D_{\Theta}u(x)$ is the projection of the gradient $Du(x)$ on the sphere $\partial B(0, |x|)$. The constant $((n-p)/p)^p$ is sharp.

For special values of p and n , the singularities appearing in (5) and (7) take particularly simple and explicit forms: Let $p = (n+1)/2$ and suppose that $\Omega \subset \mathbb{S}^n$ is such that $|\Omega| = |\mathbb{S}^n|/2$. Then, $a^* = \pi/2$, $(p-1)/(n-p) = 1$, $(n-1)/(p-1) = 2$ and therefore $\tilde{\eta}_{\pi/2}(t) = \sin^{-2} t (\int_t^{\pi/2} \sin^{-2} \sigma d\sigma)^{-1} = (\sin t \cos t)^{-1}$. Consequently, $\tilde{T} = \pi/4$ and

$$\rho_{\pi/2}(x) = \begin{cases} (\sin \theta \cos \theta)^{-1} & \text{if } x \in \mathcal{B}(\pi/4) \setminus \{N\}, \\ 2 & \text{if } x \in \mathbb{S}^n \setminus \mathcal{B}(\pi/4). \end{cases}$$

Note also that $(\sin \theta \cos \theta)^{-1} > \theta^{-1}$ for any $\theta \in (0, \pi/4)$ and $2 > \theta^{-1}$ for any $\theta \in (\pi/4, \pi)$. Therefore, inequality (5) implies that

$$\int_{\Omega} |\nabla u|^p dV \geq \left(\frac{n-p}{p}\right)^p \int_{\Omega} |u|^p \left(\frac{1}{d_g(x, N)^p} + h\right) dV,$$

where h is a positive quantity. This shows that (1) is improved on the sphere in this case. When $p = (n+1)/2$ the same arguments also yield a simple form for (7).

Indeed, in this case (7) may be written in the form

$$\int_{\mathbb{R}_+^{n+1}} |D_{\Theta} u|^p dx \geq \left(\frac{p-1}{p}\right)^p \left(\int_{\mathbb{R}_+^{n+1} \cap \{0 \leq \theta < \pi/4\}} \frac{|u|^p}{|x'|^p (\cos \theta)^p} dx + 2^p \int_{\mathbb{R}_+^{n+1} \cap \{\pi/4 \leq \theta < \pi/2\}} \frac{|u|^p}{|x|^p} dx \right).$$

This special case was also shown to be of interest in [Bandle et al. 2005].

An outline of our proofs is as follows. Our starting point is the one-dimensional Hardy inequality

$$(8) \quad \int_0^a |u'|^p \sin^{n-1}(t) dt \geq \left(\frac{p-1}{p}\right)^p \int_0^a |u|^p \tilde{\eta}_{a, \tilde{T}}(t)^p \sin^{n-1}(t) dt,$$

for all u such that $u(a) = 0$, where $\tilde{\eta}_{a, \tilde{T}}$ is the function defined in (3). In fact, in Section 2 we shall prove some sharp weighted one-dimensional Hardy inequalities involving a *general* weight ϕ ; these reduce to (8) when $\phi(t) = \tilde{\phi}(t) = \sin^{n-1}(t)$; see Propositions 2.3 and 2.5 below. To this end, we extend a method described in [Hardy et al. 1952] — but see also [Tartar 1998] — for the special case $\phi(t) = 1$. In fact, one of our efforts is to determine the general conditions on ϕ so that this method applies. This technique was also employed in [Brandolini et al. 2007] in the special case $\phi(t) = (2\pi)^{-1/2} \exp\{-t^2/2\}$ in the context of symmetrization with respect to Gaussian measure. On the other hand, our sharpness considerations as in Proposition 2.5 are new even in these special cases. In Section 3, we use spherical symmetrization to reduce Theorem 1.1 to (8). In turn, Theorem 1.1 together with a Steiner symmetrization with respect to the angular variables concludes the proof of Theorem 1.2.

2. Some Hardy inequalities on intervals

In this section we prove some weighted one-dimensional Hardy inequalities, as stated in Propositions 2.3 and 2.5. As already mentioned in Section 1, we will exploit a technique from [Hardy et al. 1952, Theorem 253, page 175]; see also [Brandolini et al. 2007; Tartar 1998]. Assume $a > 0$ and $p > 1$, and choose $\phi \in C^1((0, a)) \cap C^0([0, a])$ so that

$$(9) \quad \phi(0) = 0, \quad \phi(t) > 0 \text{ in } (0, a], \quad c_1 t^{p-1+\delta} \leq \phi(t) \leq c_2 t^{p-1+\delta},$$

for some $c_1, c_2, \delta > 0$. We let $W^{1,p}(0, a; \phi)$ be the set of maps $u : [0, a] \rightarrow \mathbb{R}$ such that $u \in L^1_{loc}[0, a]$ and $\int_0^a |u'|^p \phi dt < +\infty$, where u' denotes the distributional derivative of u . In it, we consider the subspace $\mathcal{E} = \{u \in W^{1,p}(0, a; \phi) : u(a) = 0\}$, endowed with the norm $\|u\| = (\int_0^a |u'|^p \phi)^{1/p}$. We note that if $u \in \mathcal{E}$, then u is absolutely continuous in $[\epsilon, a]$ for all $\epsilon \in (0, a)$. On the other hand, u is in general

unbounded near the origin. Nevertheless, u may be approximated in \mathcal{E} by functions that vanish at 0. More precisely we have the following.

Lemma 2.1. $C_0^1([0, a])$ is dense in \mathcal{E} .

Proof. Let $u \in \mathcal{E}$. By standard properties of Sobolev spaces, we may assume that $u \in C^1([\epsilon, a])$ for all $\epsilon \in (0, a)$. We first show that $C_0^1([0, a])$ is dense in $\mathcal{E} \cap L^\infty(0, a)$. Let $u \in \mathcal{E} \cap L^\infty(0, a)$. For $k \in \mathbb{N}$, we consider the sequence

$$u_k(t) = \begin{cases} u(k^{-1})kt & \text{if } t \in [0, k^{-1}], \\ u(t) & \text{if } t \in (k^{-1}, a]. \end{cases}$$

By the elementary inequality $|\alpha + \beta|^p \leq 2^{p-1} (|\alpha|^p + |\beta|^p)$ for all $\alpha, \beta \in \mathbb{R}$,

$$(10) \quad \int_0^a |u' - u'_k|^p \phi \, dt = \int_0^{k^{-1}} |u' - u(k^{-1})k|^p \phi \, dt \leq 2^{p-1} \left(\int_0^{k^{-1}} |u'|^p \phi \, dt + \|u\|_\infty^p k^p \int_0^{k^{-1}} \phi \, dt \right).$$

Since $u' \in L^p(0, a; \phi)$, the absolute continuity of the Lebesgue integral implies that

$$(11) \quad \int_0^{k^{-1}} |u'|^p \phi \, dt = o(1) \quad \text{as } k \rightarrow +\infty.$$

Then by (9), we have $\max_{t \in [0, k^{-1}]} \phi \leq c_2 \max_{t \in [0, k^{-1}]} t^{p-1+\delta} = c_2 k^{-p+1-\delta}$. Thus

$$(12) \quad k^p \int_0^{k^{-1}} \phi \, dt \leq k^{p-1} \max_{t \in [0, k^{-1}]} \phi \leq c_2 k^{-\delta} = o(1) \quad \text{as } k \rightarrow +\infty.$$

From (10), (11) and (12), it follows that $\int_0^a |u' - u'_k|^p \phi \, dt = o(1)$ as $k \rightarrow +\infty$. We conclude by a standard regularization argument. Suppose that $u \in \mathcal{E}$. The sequence of bounded functions

$$\hat{u}_k(t) = \begin{cases} u(k^{-1}) & \text{if } t \in [0, k^{-1}], \\ u(t) & \text{if } t \in (k^{-1}, a], \end{cases}$$

satisfies

$$\int_0^a |u' - \hat{u}'_k|^p \phi \, dt = \int_0^{k^{-1}} |u'|^p \phi \, dt = o(1) \quad \text{as } k \rightarrow +\infty.$$

We have reduced to the case where u is bounded, and the claim is established. \square

Fix $a > 0$. Let

$$(13) \quad \eta_a(t) = \frac{\phi(t)^{-1/(p-1)}}{\int_t^a \phi(\sigma)^{-1/(p-1)} \, d\sigma} \quad \text{for } t \in (0, a).$$

We note that $\eta_a > 0$ and furthermore the following holds.

Lemma 2.2. *The function η_a defined by (13) satisfies*

$$(14) \quad \left(\frac{c_1}{c_2}\right)^{1/(p-1)} \frac{\delta}{p-1} \frac{a^{\delta/(p-1)}}{t(a^{\delta/(p-1)} - t^{\delta/(p-1)})} \\ \leq \eta_a(t) \leq \left(\frac{c_2}{c_1}\right)^{1/(p-1)} \frac{\delta}{p-1} \frac{a^{\delta/(p-1)}}{t(a^{\delta/(p-1)} - t^{\delta/(p-1)})}$$

for all $t \in (0, a)$, where $c_1, c_2, \delta > 0$ are the constants defined in (9).

Proof. We have

$$\int_t^a (\sigma^{p-1+\delta})^{-1/(p-1)} d\sigma = \int_t^a \sigma^{-1-\delta/(p-1)} d\sigma = \frac{p-1}{\delta} (t^{-\delta/(p-1)} - a^{-\delta/(p-1)}).$$

Consequently,

$$\frac{(t^{p-1+\delta})^{-1/(p-1)}}{\int_t^a (\sigma^{p-1+\delta})^{-1/(p-1)} d\sigma} = \frac{\delta}{p-1} \frac{a^{\delta/(p-1)}}{t(a^{\delta/(p-1)} - t^{\delta/(p-1)})}.$$

On the other hand, by the assumption (9) on ϕ , we have

$$\left(\frac{c_1}{c_2}\right)^{1/(p-1)} \frac{(t^{p-1+\delta})^{-1/(p-1)}}{\int_t^a (\sigma^{p-1+\delta})^{-1/(p-1)} d\sigma} \leq \eta_a(t) \leq \left(\frac{c_2}{c_1}\right)^{1/(p-1)} \frac{(t^{p-1+\delta})^{-1/(p-1)}}{\int_t^a (\sigma^{p-1+\delta})^{-1/(p-1)} d\sigma}$$

and the asserted estimate follows. \square

For later use, we also note that η_a satisfies a Riccati equation

$$(15) \quad \eta_a \phi' / \phi + (p-1) \eta_a' = (p-1) \eta_a^2 \quad \text{in } (0, a).$$

The following Hardy inequality holds.

Proposition 2.3. *Let $a > 0$ and $p > 1$, and suppose that ϕ satisfies (9). Let η_a be correspondingly defined by (13). Then, for every $u \in \mathcal{E}$,*

$$(16) \quad \int_0^a |u'|^p \phi dt \geq \left(\frac{p-1}{p}\right)^p \int_0^a |u|^p \eta_a^p \phi dt.$$

The constant $((p-1)/p)^p$ is sharp.

Proof. In view of Lemma 2.1 we may assume that $u \in C_0^1((0, a))$. We recall the elementary convexity inequality $|\alpha|^p \geq |\beta|^p + p|\beta|^{p-2}\beta(\alpha - \beta)$ for all $\alpha, \beta \in \mathbb{R}$. Taking $\alpha = u'$ and $\beta = -((p-1)/p)u\eta_a$, we derive

$$(17) \quad |u'|^p \geq \left| \frac{p-1}{p} u \eta_a \right|^p - p \left| \frac{p-1}{p} u \eta_a \right|^{p-2} \frac{p-1}{p} u \eta_a \left(u' + \frac{p-1}{p} u \eta_a \right).$$

Multiplying by ϕ and integrating over $[0, a]$, we obtain

$$\int_0^a |u'|^p \phi \, dt \geq \left(\frac{p-1}{p}\right)^p (1-p) \int_0^a |u|^p \eta_a^p \phi \, dt - p \left(\frac{p-1}{p}\right)^{p-1} \int_0^a |u|^{p-2} u u' \eta_a^{p-1} \phi \, dt.$$

Integration by parts yields

$$\int_0^a |u'|^p \phi \, dt \geq \left(\frac{p-1}{p}\right)^p (1-p) \int_0^a |u|^p \eta_a^p \phi \, dt + \left(\frac{p-1}{p}\right)^{p-1} \int_0^a |u|^p \left(\eta_a^{p-1} \phi\right)' \, dt - \left(\frac{p-1}{p}\right)^{p-1} |u|^p \eta_a^{p-1} \phi \Big|_0^a.$$

Now by (14) and the fact that $u \in C_0^1([0, a])$, we have $u \eta_a \in L^\infty(0, a)$. Therefore the boundary terms vanish and we obtain

$$\begin{aligned} &\int_0^a |u'|^p \phi \, dt \\ &\geq (1-p) \left(\frac{p-1}{p}\right)^p \int_0^a |u|^p \eta_a^p \phi \, dt + \left(\frac{p-1}{p}\right)^{p-1} \int_0^a |u|^p (\eta_a^{p-1} \phi' + (p-1) \eta_a^{p-2} \eta_a' \phi) \, dt \\ &= (1-p) \left(\frac{p-1}{p}\right)^p \int_0^a |u|^p \eta_a^p \phi \, dt + \left(\frac{p-1}{p}\right)^{p-1} \int_0^a |u|^p \eta_a^{p-2} (\eta_a \phi' + (p-1) \eta_a') \phi \, dt. \end{aligned}$$

In view of (15), we have

$$\int_0^a |u|^p \eta_a^{p-2} (\eta_a \phi' + (p-1) \eta_a') \phi \, dt = (p-1) \int_0^a |u|^p \eta_a^p \phi \, dt.$$

It follows that

$$\begin{aligned} \int_0^a |u'|^p \phi \, dt &\geq \left(\left(\frac{p-1}{p}\right)^p - \frac{(p-1)^p}{p^{p-1}}\right) \int_0^a |u|^p \eta_a^p \phi \, dt + \frac{(p-1)^p}{p^{p-1}} \int_0^a |u|^p \eta_a^p \phi \, dt \\ &= \left(\frac{p-1}{p}\right)^p \int_0^a |u|^p \eta_a^p \phi \, dt. \end{aligned}$$

Hence, (16) is satisfied.

Now we verify sharpness. Consider the sequence of functions $\{U_k\}_{k \in \mathbb{N}} \subset \mathcal{E}$ defined by

$$(18) \quad U_k(t) = \begin{cases} \left(\int_0^a \phi^{-1/(p-1)} \, d\sigma\right)^{(p-1)/p} & \text{if } t \in [0, 1/k], \\ \left(\int_t^{1/k} \phi^{-1/(p-1)} \, d\sigma\right)^{(p-1)/p} & \text{if } t \in [1/k, a]. \end{cases}$$

Then,

$$U'_k(t) = \begin{cases} 0 & \text{if } t \in [0, 1/k), \\ -\frac{p-1}{p} \left(\int_t^a \phi^{-1/(p-1)} d\sigma \right)^{-1/p} \phi^{-1/(p-1)}(t) & \text{if } t \in [1/k, a]. \end{cases}$$

We claim that

$$\lim_{k \rightarrow +\infty} \frac{\int_0^a |U'_k|^p \phi dt}{\int_0^a U_k^p \eta_a^p \phi dt} = \left(\frac{p-1}{p} \right)^p.$$

Indeed, note that

$$(\eta_a^p \phi)(t) = \frac{\phi^{-1/(p-1)}(t)}{\left(\int_t^a \phi^{-1/(p-1)} d\sigma \right)^p} \quad \text{for all } t \in (0, a).$$

Therefore, we may write

$$\int_0^a U_k^p \eta_a^p \phi dt = A_k + B_k,$$

where

$$(19) \quad A_k \equiv \int_0^{k^{-1}} U_k^p \eta_a^p \phi dt = \left(\int_{k^{-1}}^a \phi^{-1/(p-1)} d\sigma \right)^{p-1} \int_0^{k^{-1}} \frac{\phi^{-1/(p-1)}(t)}{\left(\int_t^a \phi^{-1/(p-1)} d\sigma \right)^p} dt$$

and

$$(20) \quad B_k \equiv \int_{k^{-1}}^a U_k^p \eta_a^p \phi dt = \int_{k^{-1}}^a \left(\int_t^a \phi^{-1/(p-1)} d\sigma \right)^{p-1} \frac{\phi^{-1/(p-1)}(t)}{\left(\int_t^a \phi^{-1/(p-1)} d\sigma \right)^p} dt \\ = \int_{k^{-1}}^a \frac{\phi^{-1/(p-1)}(t)}{\int_t^a \phi^{-1/(p-1)} d\sigma} dt.$$

We claim that

$$(21) \quad \lim_{k \rightarrow +\infty} A_k = 1/(p-1).$$

Indeed, we first observe that in view of (9), we have

$$(22) \quad \int_{1/k}^a \frac{\phi^{-1/(p-1)}(t)}{\int_t^a \phi^{-1/(p-1)} d\sigma} dt \geq \left(\frac{c_2}{c_1} \right)^{-1/(p-1)} \int_{1/k}^a \frac{t^{-1-\delta/(p-1)}}{\int_t^a \sigma^{-1-\delta/(p-1)} d\sigma} dt \rightarrow +\infty$$

as $k \rightarrow \infty$. Hence, by l'Hôpital's rule, $\lim_{k \rightarrow +\infty} A_k$ is equal to the $k \rightarrow +\infty$ limit of either

$$\frac{\int_0^{k^{-1}} \frac{\phi^{-1/(p-1)}}{\left(\int_t^a \phi^{-1/(p-1)} d\sigma \right)^p} dt}{\left(\int_{k^{-1}}^a \phi^{-1/(p-1)} dt \right)^{-(p-1)}} \quad \text{or} \quad \frac{\frac{\phi(k^{-1})^{-1/(p-1)}}{\left(\int_{k^{-1}}^a \phi^{-1/(p-1)} d\sigma \right)^p}}{(p-1) \left(\int_{k^{-1}}^a \phi^{-1/(p-1)} dt \right)^{-p} \phi(k^{-1})^{-1/(p-1)}},$$

which then gives (21). We conclude that

$$\int_0^a U_k^p \eta_a^p \phi dt = \int_{k^{-1}}^a \frac{\phi^{-1/(p-1)}(t)}{\int_t^a \phi^{-1/(p-1)} d\sigma} dt + \frac{1}{p-1} + o(1).$$

On the other hand, we have

$$\int_0^a |U'_k|^p \phi dt = \left(\frac{p-1}{p}\right)^p \int_{k^{-1}}^a \frac{\phi^{-1/(p-1)}(t)}{\int_t^a \phi^{-1/(p-1)} d\sigma} dt.$$

Hence, recalling (22), we obtain

$$\lim_{k \rightarrow +\infty} \frac{\int_0^a |U'_k|^p \phi dt}{\int_0^a U_k^p \eta_a^p \phi dt} = \left(\frac{p-1}{p}\right)^p \lim_{k \rightarrow +\infty} \frac{\int_{k^{-1}}^a \frac{\phi^{-1/(p-1)} dt}{\int_t^a \phi^{-1/(p-1)} d\sigma}}{\int_{k^{-1}}^a \frac{\phi^{-1/(p-1)} dt}{\int_t^a \phi^{-1/(p-1)} d\sigma} + \frac{1}{p-1} + o(1)},$$

which is equal to $((p-1)/p)^p$. Hence, the sharpness is also established. □

Now we show that under a simple extra assumption for ϕ , the corresponding function η_a defined by (13) has exactly one critical point, corresponding to the absolute minimum of η_a in $(0, a)$.

Lemma 2.4. *Suppose that $\phi : [0, a] \rightarrow \mathbb{R}$ satisfies (9). Also suppose that ϕ is twice differentiable in $(0, a)$ and that*

$$(23) \quad (\log \phi)''(t) = (\phi'/\phi)'(t) < 0 \quad \text{for all } t \in (0, a).$$

Then there exists a unique $T \in (0, a)$ such that $\eta'_a(t) < 0$ in $(0, T)$ and $\eta'_a(t) > 0$ in (T, a) .

Proof. Differentiating the Riccati equation (15) we obtain

$$(24) \quad \eta'_a \phi' / \phi + \eta_a (\phi' / \phi)' + (p-1)\eta''_a = 2(p-1)\eta_a \eta''_a.$$

Suppose that $\eta'_a(\hat{t}) = 0$. Then (24) implies that

$$\eta''_a(\hat{t}) = -\frac{1}{p-1} \eta_a(\hat{t}) (\phi' / \phi)'(\hat{t}) > 0.$$

It follows that any critical point for η_a is necessarily a strict minimum point. It then follows from Lemma 2.2 that η_a admits a unique minimum point, and the existence of T is established. □

Let ϕ be twice differentiable and suppose that ϕ satisfies (9) and (23). Then, by truncating η_a at the point T , we obtain the nonincreasing function

$$(25) \quad \eta_{a,T}(t) = \begin{cases} \eta_a(t) & \text{for } t \in (0, T], \\ \eta_a(T) & \text{for } t \in (T, a). \end{cases}$$

Since $\eta_{a,T} \leq \eta_a$ pointwise, it is clear that [Proposition 2.3](#) still holds with η_a replaced by $\eta_{a,T}$. On the other hand, it is not clear a priori whether, with such a replacement, the constant $((p-1)/p)^p$ is still sharp. This is indeed the case:

Proposition 2.5. *Suppose that ϕ is twice differentiable and satisfies [\(9\)](#) and [\(23\)](#). Let $\eta_{a,T}$ be defined by [\(25\)](#). Then*

$$\int_0^a |u'|^p \phi \, dt \geq \left(\frac{p-1}{p} \right)^p \int_0^a |u|^p \eta_{a,T}^p \phi \, dt \quad \text{for all } u \in \mathcal{E}.$$

The constant $((p-1)/p)^p$ is sharp.

Proof. We need only check sharpness. For $k \in \mathbb{N}$ with $1/k < T$, we consider the sequence $\{V_k\}_{k \in \mathbb{N}} \subset \mathcal{E}$ defined by

$$(26) \quad V_k(t) = \begin{cases} U_k(t) & \text{if } t \in [0, T), \\ \frac{2t-a-T}{T-a} \left(\int_T^a \phi^{-1/(p-1)} \, d\sigma \right)^{(p-1)/p} & \text{if } t \in [T, (a+T)/2), \\ 0 & \text{if } t \in [(a+T)/2, a], \end{cases}$$

where $\{U_k\}_{k \in \mathbb{N}}$ is the sequence defined in [\(18\)](#). Then

$$V_k'(t) = \begin{cases} U_k'(t) & \text{if } t \in [0, T), \\ \frac{2}{T-a} \left(\int_T^a \phi^{-1/(p-1)} \, d\sigma \right)^{(p-1)/p} & \text{if } t \in [T, (a+T)/2), \\ 0 & \text{if } t \in [(a+T)/2, a]. \end{cases}$$

We claim that

$$\lim_{k \rightarrow +\infty} \frac{\int_0^a |V_k'|^p \phi \, dt}{\int_0^a |V_k|^p \eta_{a,T}^p \phi \, dt} = \left(\frac{p-1}{p} \right)^p.$$

Let $C_1, C_2 > 0$ be defined by

$$C_1 = \left(\frac{p}{p-1} \right)^p \int_T^{(T+a)/2} |V_k'|^p \phi \, dt \quad \text{and} \quad C_2 = \int_T^{(T+a)/2} |V_k|^p \eta_{a,T}^p \phi \, dt.$$

Note that C_1 and C_2 are independent of k . Then, we have

$$\begin{aligned} \frac{\int_0^a |V_k'|^p \phi \, dt}{\int_0^a |V_k|^p \eta_{a,T}^p \phi \, dt} &= \left(\frac{p-1}{p} \right)^p \frac{\int_{k-1}^T \frac{\phi^{-1/(p-1)}}{\int_t^a \phi^{-1/(p-1)} \, d\sigma} \, dt + C_1}{A_k + \int_{k-1}^T \frac{\phi^{-1/(p-1)}}{\int_t^a \phi^{-1/(p-1)} \, d\sigma} \, dt + C_2} \\ &= \left(\frac{p-1}{p} \right)^p \frac{\int_{k-1}^T \frac{\phi^{-1/(p-1)}}{\int_t^a \phi^{-1/(p-1)} \, d\sigma} \, dt + C_1}{\frac{1}{p-1} + o(1) + \int_{k-1}^T \frac{\phi^{-1/(p-1)}}{\int_t^a \phi^{-1/(p-1)} \, d\sigma} \, dt + C_2}, \end{aligned}$$

where A_k is defined in (19). This establishes the claim. □

3. Proofs of Theorems 1.1 and 1.2

In this section we apply Proposition 2.5 to prove Theorems 1.1 and 1.2. In what follows, we assume that $1 < p < n$. We let $a \in (0, \pi)$ and we take $\phi = \tilde{\phi}$, where $\tilde{\phi}(t) = \sin^{n-1}(t)$. We note that $\tilde{\phi}$ satisfies assumptions (9) with $\delta = n - p$. The weight function corresponding to $\tilde{\phi}$ defined by (13) is given by (2), namely

$$\tilde{\eta}_a(t) = \frac{(\sin t)^{-(n-1)/(p-1)}}{\int_t^a (\sin \sigma)^{-(n-1)/(p-1)} d\sigma}.$$

Furthermore, $\tilde{\phi}$ is twice differentiable and we have

$$(\log \tilde{\phi})''(t) = -\frac{n-1}{\sin^2 t} \quad \text{for all } t \in (0, \pi).$$

In particular, $\tilde{\phi}$ satisfies assumption (23). Using l'Hôpital's rule,

$$\begin{aligned} (27) \quad \frac{p-1}{n-p} \lim_{t \rightarrow 0^+} t \tilde{\eta}_{a, \tilde{T}}(t) &= \frac{p-1}{n-p} \lim_{t \rightarrow 0^+} \frac{t (\sin t)^{-(n-1)/(p-1)}}{\int_t^a (\sin \sigma)^{-(n-1)/(p-1)} d\sigma} \\ &= \frac{p-1}{n-p} \lim_{t \rightarrow 0^+} \frac{(1 + o(1))t^{-(n-p)/(p-1)}}{\int_t^a \sin \sigma^{-(n-1)/(p-1)} d\sigma} = 1, \end{aligned}$$

and therefore (6) follows. The following elementary facts will be used in the sequel. Recall that for $x = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$, we set $x_{n+1} = |x| \cos \theta$.

Lemma 3.1. *Let $\Omega \subset \mathbb{S}^n$ and suppose that $u : \Omega \rightarrow \mathbb{R}$ depends on θ only. Then*

$$(28) \quad |\nabla u|^2 = \left(\frac{\partial u}{\partial \theta}\right)^2 \quad \text{and} \quad \int_{\mathcal{B}(a)} u(\theta) dV = \omega_{n-1} \int_0^a u(\theta) \tilde{\phi}(\theta) d\theta,$$

where $\omega_{n-1} = (2\pi)^{n/2} / \Gamma(n/2)$ denotes the volume of \mathbb{S}^{n-1} .

We shall also need some basic facts concerning spherical rearrangements; see for example [Baernstein and Taylor 1976; Sperner 1973]. For every $a \in [0, \pi]$, let

$$(29) \quad A(a) = |\mathcal{B}(a)| = \omega_{n-1} \int_0^a \tilde{\phi}(\theta) d\theta.$$

Let $\Omega \subset \mathbb{S}^n$ be an open set, and let $u : \Omega \rightarrow \mathbb{R}$ be a measurable function. For $t > 0$, let

$$\mu(t) = |\{x \in \Omega : |u(x)| > t\}|$$

denote the distribution function of u . Then the decreasing rearrangement u^* of u is defined in [Bennett and Sharpley 1988, page 39] by

$$(30) \quad u^*(s) = \inf \{t \geq 0 : \mu(t) \leq s\} \quad \text{for every } s \in [0, |\Omega|].$$

Let $\Omega^* = \mathcal{B}(a^*)$, where $a^* = A^{-1}(|\Omega|)$. Then the spherical rearrangement u^* of u is defined by $u^*(x) = u^*(A(\theta))$ for $x \in \Omega^*$. It follows that u^* is a decreasing function of θ , and that its level sets are geodesic balls (spherical caps) centered at $N = (0, 0, \dots, 1) \in \mathbb{S}^n$. Since $|u|$ and u^* have the same distribution function, we have

$$\int_{\Omega} |u|^q dV = \int_{\Omega^*} (u^*)^q dV \quad \text{for all } q \geq 1.$$

We shall use two standard inequalities involving rearrangements. The following lemma is a special case of the well-known Hardy–Littlewood inequality and may be found, for example, in [Bennett and Sharpley 1988, Theorem 2.2, page 44].

Lemma 3.2. *Let $\Omega \subset \mathbb{S}^n$ be an open set and suppose that $u, v : \Omega \rightarrow \mathbb{R}$ are measurable and finite almost everywhere. Then*

$$(31) \quad \int_{\Omega} uv dV \leq \int_{\Omega^*} u^*v^* dV.$$

The next inequality is a special case of the Pólya–Szegő principle, which may be found in [Aubin 1998, Proposition 2.17, page 41]; see also [Sperner 1973, theorem on page 325].

Lemma 3.3. *Let $q \geq 1$ and let $u \in W^{1,q}(\mathbb{S}^n)$. Then*

$$(32) \quad \int_{\mathbb{S}^n} |\nabla u|^q dV \geq \int_{\mathbb{S}^n} |\nabla u^*|^q dV.$$

Proof of Theorem 1.1. For every $u \in W_0^{1,p}(\Omega)$, by Lemmas 3.1 and 3.3, we have

$$\int_{\Omega} |\nabla u|^p dV \geq \int_{\Omega^*} |\nabla u^*|^p dV = \omega_{n-1} \int_0^{a^*} \left| \frac{\partial u^*}{\partial \theta} \right|^p \tilde{\phi}(\theta) d\theta.$$

On the other hand, in view of Lemma 3.2, we have

$$\int_{\Omega} |u|^p \rho_{a^*}^p dV \leq \int_{\Omega^*} |u^*|^p \rho_{a^*}^p dV = \omega_{n-1} \int_0^{a^*} |u^*|^p \rho_{a^*}^p \tilde{\phi}(\theta) d\theta.$$

Therefore, it suffices to show that

$$\int_0^{a^*} \left| \frac{\partial u^*}{\partial \theta} \right|^p \tilde{\phi}(\theta) d\theta - \left(\frac{n-p}{p} \right)^p \int_0^{a^*} |u^*|^p \rho_{a^*}^p \tilde{\phi}(\theta) d\theta \geq 0.$$

This inequality holds by that definition of ρ_{a^*} , as in (4), and by Proposition 2.5.

To show that the constant $((n-p)/p)^p$ is sharp, it suffices to use the sequence $\{\tilde{V}_k\}_{k \in \mathbb{N}}$ of test functions obtained by setting $\phi = \tilde{\phi}$, $a = a^*$ and $T = \tilde{T}$ in (26).

Namely,

$$(33) \quad \tilde{V}_k(\theta) = \begin{cases} \left(\int_{k^{-1}}^{a^*} \tilde{\phi}^{-1/(p-1)} d\sigma \right)^{(p-1)/p} & \text{if } \theta \in [0, k^{-1}), \\ \left(\int_t^{a^*} \tilde{\phi}^{-1/(p-1)} d\sigma \right)^{(p-1)/p} & \text{if } \theta \in [k^{-1}, \tilde{T}), \\ \frac{2t - a^* - \tilde{T}}{\tilde{T} - a^*} \left(\int_{\tilde{T}}^{a^*} \tilde{\phi}^{-1/(p-1)} d\sigma \right)^{(p-1)/p} & \text{if } \theta \in [\tilde{T}, (a^* + \tilde{T})/2), \\ 0 & \text{if } \theta \in [(a^* + \tilde{T})/2, a^*]. \end{cases}$$

This completes the proof of [Theorem 1.1](#). □

To prove [Theorem 1.2](#), we use a Steiner-type symmetrization (see for example [[Alvino et al. 1996](#); [Kawohl 1985](#)]) on \mathbb{R}_+^{n+1} with respect to the angular variables. Let $u \in C_0^1(\mathbb{R}_+^{n+1})$. For every fixed $r > 0$, we consider the function obtained by restricting u to $\mathbb{S}^n \cap \mathbb{R}_+^{n+1}$. Namely, we consider the function

$$(34) \quad \Theta \in \mathbb{S}^n \cap \mathbb{R}_+^{n+1} \rightarrow u(r, \Theta),$$

where $\Theta = (\theta_1, \dots, \theta_{n-1}, \theta)$ is the set of all angular variables. We denote by $u^*(r, \cdot)$ the decreasing rearrangement of the function in (34), according to the definition given in (30). Finally we introduce the Steiner rearrangement u^\sharp of u by

$$(35) \quad u^\sharp(r, \theta) = u^*(r, A(\theta)) \quad \text{for } \theta \in [0, \pi/2],$$

where $A(\theta)$ is defined in (29). We denote by g_r the standard metric on \mathbb{S}_r^n and by dV_r the volume element on \mathbb{S}_r^n . Then, we have $D_{\Theta}u = \nabla_{g_r}u$ and therefore, in view of [Lemma 3.1](#) and a rescaling argument,

$$(36) \quad |D_{\Theta}u^\sharp(r, \theta)|^p = \frac{1}{r^p} \left| \frac{\partial u^\sharp}{\partial \theta} \right|^p.$$

We claim that

$$(37) \quad \frac{\zeta^p(x)}{|x|^p} \in L^1(\mathbb{R}_+^{n+1} \cap B_R) \quad \text{for every } R > 0,$$

where ζ is the weight function appearing in the statement of [Theorem 1.2](#). Indeed, writing $x = (x', x_{n+1})$, in view of (27), we have for some $C > 0$

$$\frac{\zeta(x)}{|x|} \leq \frac{C}{\theta r} \leq \frac{C}{r \sin \theta} = \frac{C}{|x'|}.$$

Consequently, for any $R > 0$ we have

$$\int_{|x'|, |x_{n+1}| < R} \frac{\zeta^p(x)}{|x|^p} dx = \int_0^R dx_{n+1} \int_{|x'| < R} \frac{\zeta^p(x)}{|x|^p} dx'.$$

Now (37) follows from the assumption $p < n$.

Proof of Theorem 1.2. By density, it suffices to consider $u \in C_0^1(\mathbb{R}_+^{n+1})$. If $\Omega \subset \mathbb{S}_r^n$ and $u : \Omega \rightarrow \mathbb{R}$ depends on θ only, then rescaling (28) gives

$$(38) \quad |\nabla_{g_r} u|^2 = \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} \right)^2 \quad \text{and} \quad \int_{\mathbb{B}_r(\alpha)} u(\theta) dV_r = \omega_{n-1} r^n \int_0^\alpha u(\theta) \tilde{\phi}(\theta) d\theta,$$

By Fubini’s theorem and Lemma 3.3, we have

$$(39) \quad \int_{\mathbb{R}_+^{n+1}} |D_{\Theta} u|^p dx = \int_0^{+\infty} \int_{\mathbb{S}_r^n} |\nabla_{g_r} u|^p d\sigma_r \geq \int_0^{+\infty} \int_{\mathbb{S}_r^n} |\nabla_{g_r} u^\sharp|^p d\sigma_r,$$

where $u^\sharp = u^\sharp(r, \theta)$ is defined in (35). Consequently, from (36), (39) and by Theorem 1.1 with $a^* = \pi/2$, we derive

$$\begin{aligned} \int_{\mathbb{R}_+^{n+1}} |D_{\Theta} u|^p dx &\geq \omega_{n-1} \int_0^{+\infty} \left(\int_0^{\pi/2} \frac{1}{r^p} \left(\frac{\partial u^*}{\partial \theta} \right)^p \sin^{n-1} \theta d\theta \right) r^n dr \\ &\geq \omega_{n-1} \left(\frac{n-p}{p} \right)^p \int_0^{+\infty} \left(\frac{1}{r^p} \int_0^{\pi/2} |u^*|^p \zeta^p \sin^{n-1} \theta d\theta \right) r^n dr \\ &\geq \left(\frac{n-p}{p} \right)^p \int_{\mathbb{R}_+^{n+1}} \frac{|u|^p}{r^p} \zeta^p dx. \end{aligned}$$

We are left to prove sharpness. We consider the sequence $u_k(\theta, r) = \Theta_k(\theta) R_k(r)$ for $k \in \mathbb{N}$, where $R_k \in C_0((0, +\infty))$ satisfies $R_k > 0$ and $R_k^p(r) \xrightarrow{*} \delta_1(r)$, weakly in the sense of measures. Here δ_1 denotes the Dirac mass on $(0, +\infty)$ centered at $r = 1$, and $\Theta_k(\theta) = \tilde{V}_k(\theta)$, where \tilde{V}_k is the sequence defined in (33), with $a^* = \pi/2$. We have

$$\lim_{k \rightarrow +\infty} \int_0^{+\infty} R_k^p(r) r^n dr = \lim_{k \rightarrow +\infty} \int_0^{+\infty} R_k^p(r) r^{n-p} dr = 1.$$

Now, the claim follows since

$$\begin{aligned} \frac{\int_{\mathbb{R}_+^{n+1}} |D_{\Theta} u_k|^p dx}{\int_{\mathbb{R}_+^{n+1}} |u_k|^p \frac{\zeta^p}{r^p} dx} &= \frac{\int_0^{+\infty} R_k^p(r) r^n dr}{\int_0^{+\infty} R_k^p(r) r^{n-p} dr} \frac{\int_{\mathbb{S}_r^n \cap \mathbb{R}_+^{n+1}} |\Theta'_k(\theta)|^p dv_{g_r}}{\int_{\mathbb{S}_r^n \cap \mathbb{R}_+^{n+1}} |\Theta_k(\theta)|^p \zeta^p dv_{g_r}} \\ &= \frac{\int_0^{\pi/2} |\Theta'_k(\theta)|^p \tilde{\phi} d\theta}{\int_0^{\pi/2} |\Theta_k(\theta)|^p \zeta^p \tilde{\phi} d\theta} + o(1) = \left(\frac{n-p}{p} \right)^p + o(1), \end{aligned}$$

where $o(1)$ vanishes as $k \rightarrow \infty$. □

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