**GRADIENT SHRINKING SOLITONS WITH
VANISHING WEYL TENSOR**

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We will give a local version of the Hamilton–Ivey-type pinching estimate of the gradient shrinking soliton with vanishing Weyl tensor, and then give a complete classification of gradient shrinking solitons with vanishing Weyl tensor.

1. Introduction

Let (M, g, f) be a gradient shrinking soliton, that is, (M, g) is a smooth Riemannian manifold with a smooth function f that satisfies $R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}$, where λ is a positive constant.

Recent work has been directed at classifying the complete gradient shrinking soliton, as it is an important problem in the theory of the Ricci flow. Note that people often do not distinguish between the gradient shrinking soliton and the self-similar solution, which is defined in [Chow and Knopf 2004, Chapter 2]. In fact, the author [Zhang 2008b, Theorem 1] has shown that the complete gradient shrinking soliton solution is in fact the self-similar solution.

In dimension 2, Hamilton [1995] proved that any 2-dimensional complete nonflat ancient solution of bounded curvature must be S^2, RP^2, or the cigar soliton. In dimension 3, Ivey [1993] first showed that the compact 3-dimensional gradient shrinking soliton has constant positive sectional curvature. For the noncompact case, Perelman [2003] showed that a 3-dimensional complete nonflat gradient shrinking soliton that is bounded and has nonnegative sectional curvature, and that is also κ-noncollapsed on all scales, must be the finite quotient of $S^2 \times R$ or S^3. This result of Perelman was improved by Ni and Wallach [2008] and Naber [2008], who the assumption on κ-noncollapsing condition and replaced nonnegative sectional curvature by nonnegative Ricci curvature. In addition, Ni and Wallach allowed the curvature to grow as fast as $e^{a(r(x)+1)}$, where $r(x)$ is the distance function and a is a positive constant. In particular, Ni and Wallach's result implies that

any 3-dimensional complete noncompact nonflat gradient shrinking soliton with nonnegative Ricci curvature and with curvature not growing faster than $e^{a(r(x)+1)}$ must be a quotient of the round infinite cylinder $S^2 \times R$. Recently, by using a local version of the Hamilton–Ivey pinching estimate due to Chen [2007, Corollary 2.4], Cao, Chen, and Zhu [Cao et al. ≥ 2009] obtained a complete classification (without any curvature bound assumption) of 3-dimensional complete gradient shrinking solitons, as follows.

Theorem 1.1 [Cao et al. ≥ 2009, Proposition 4.7]. *The only 3-dimensional complete gradient shrinking solitons are the finite quotients of R^3, $S^2 \times R$, and S^3.*

Classifying complete gradient shrinking solitons in higher dimension is more difficult. Note that a 3-dimensional manifold automatically has vanishing Weyl tensor, so some recent work has focused on complete gradient shrinking solitons with vanishing Weyl tensor in higher dimension. The first classification theorem in dimension $n \geq 4$, given by Gu and Zhu [2008, Proposition 4.1], says that any nonflat κ-noncollapsing rotationally symmetric gradient shrinking soliton with bounded and nonnegative sectional curvature must be the finite quotient of $S^n \times R$ or S^{n+1}. Later, Kotschwar [2007] improved this result by showing that any complete rotationally symmetric gradient shrinking soliton (without any bounds on the curvature) is the finite quotient of R^{n+1}, $S^n \times R$, or S^{n+1}. Note that any rotationally symmetric metric has vanishing Weyl tensor. In the more general case of vanishing Weyl tensor, Ni and Wallach [2008] considered a complete n-dimensional gradient shrinking soliton with vanishing Weyl tensor and nonnegative Ricci curvature that grows no faster than $e^{a(r(x)+1)}$, where $r(x)$ is the distance function and a is a positive constant; they showed its universal cover is either R^n, S^n, or $S^{n-1} \times R$. This result has been improved by Peterson and Wylie [2008, Theorem 1.2 and Remark 1.3], in which they only needed to assume the Ricci curvature is bounded from below and grows no faster than $\exp(\frac{2}{5}cr(x)^2)$ outside of a compact set, where $c < \lambda/2$. We also note that Cao and Wang [2008] had an alternative proof of Ni and Wallach's result [2008].

The key to obtaining the above complete classification theorem of 3-dimensional complete gradient shrinking solitons without making a curvature bound assumption is the local version of the Hamilton–Ivey pinching estimate, which in 3-dimension plays a crucial role in the analysis of the Ricci flow. An open question is how to generalize Hamilton and Ivey's work to higher dimension. In [Zhang 2008a], the author obtained the following (global) Hamilton–Ivey-type pinching estimate in higher dimension:

Theorem. *Suppose we have a solution to the Ricci flow on an n-dimension manifold that is complete with bounded curvature and vanishing Weyl tensor for each $t \geq 0$. Assume at $t = 0$ that the smallest eigenvalue of the curvature operator at*

each point is bounded from below by $v \geq -1$. Then at all points and all times $t \geq 0$, we have the pinching estimate

$$R \geq (-v)[\log(-v) + \log(1 + t) - n(n + 1)/2]$$

whenever $v < 0$.

In this paper, we will get a local version of this Hamilton–Ivey-type pinching estimate for the gradient shrinking solitons with vanishing Weyl tensor (without curvature bound). The idea is to use the methods of [Chen 2007], but since the curvature operator is more complicated in higher dimension, we first need to prove an algebraic lemma. Based on this pinching estimate, we will obtain the following complete classification theorem (without any curvature bound assumption).

Theorem 1.2. *Any complete gradient shrinking soliton with vanishing Weyl tensor must be the finite quotient of R^n, $S^{n-1} \times R$, or S^n.*

This rest of this paper is organized as follows. In Section 2, we will prove an algebraic lemma, which will be used to prove the local version of the Hamilton–Ivey-type pinching estimate. In Section 3, we will give some propositions and finish the proof of Theorem 1.2.

2. An algebraic lemma

In this section, we will give an algebraic lemma. Assume $x_1, \ldots, x_n$ for $n \geq 4$ are real numbers, and m is any positive integer. Define $M_{ij} := x_i + x_j$ and a function

$$
\begin{aligned}
f(x_1, \ldots, x_n) := & \\
& -\frac{\sum_{i<j, \{i,j\} \neq \{1,2\}} M_{ij}}{m+1} \left(\sum_{\substack{i<j \\ \{i,j\} \neq \{1,2\}}} M_{ij} + (m+1)M_{12} \right) \\
& - M_{12} \sum_{\substack{i<j \\ \{i,j\} \neq \{1,2\}}} M_{ij} + \left(\sum_{\substack{i<j \\ \{i,j\} \neq \{1,2\}}} (M_{ij}^2 + \sum_{k \neq i,j} M_{ik}M_{jk}) + (m+1) \sum_{k \neq 1,2} M_{1k}M_{2k} \right).
\end{aligned}
$$

Lemma 2.1. *Suppose ρ is a nonnegative constant. Then there exists a nonnegative constant $C(m, n)$ such that $f \geq -C(m, n)\rho^2$ if the following hold:*

(i) $\quad x_1 \leq x_2 \leq \min_{3 \leq i \leq n} x_i.$

(ii) $\quad \displaystyle\sum_{\substack{i<j \\ \{i,j\} \neq \{1,2\}}} M_{ij} + mM_{12} \geq -\rho.$

(iii) $\quad \displaystyle\sum_{\substack{i<j \\ \{i,j\} \neq \{1,2\}}} M_{ij} + (m+1)M_{12} < -(m+1)(m+n-1)\rho.$

Proof. We first claim that $f(\ldots, x_3, x_4, \ldots) \geq f(\ldots, x, x, \ldots)$, where $2x = x_3 + x_4$.

Without loss of generality, we may assume $x_3 < x_4$. Let $2\delta = x_4 - x_3$. Then by direct computation we get $(M_0 = x)$

$$
f(\ldots, x_3, x_4, \ldots) - f(\ldots, x, x, \ldots) =
$$

$$
\sum_{k \neq 3,4} (M_{3k} M_{4k} - M_{0k}^2) + \sum_{\substack{i < j, \ \{i,j\} \neq \{1,2\} \\ \{i,j\} \neq \{3,4\}}} (M_{3i} M_{3j} + M_{4i} M_{4j} - 2 M_{0i} M_{0j})
$$

$$
+ \sum_{i \neq 3,4} \left(M_{3i}^2 + \sum_{k \neq 3,i} M_{3k} M_{ik} - M_{0i}^2 - \sum_{k \neq 3,i} M_{0k} M_{ik} \right)
$$

$$
+ \sum_{i \neq 3,4} \left(M_{4i}^2 + \sum_{k \neq 4,i} M_{4k} M_{ik} - M_{0i}^2 - \sum_{k \neq 4,i} M_{0k} M_{ik} \right)
$$

$$
+ (m+1)(M_{13} M_{23} + M_{14} M_{24} - 2 M_{10} M_{20})
$$

$$
= \sum_{k \neq 3,4} (-\delta^2) + \sum_{\substack{i < j, \{i,j\} \neq \{1,2\} \\ \{i,j\} \neq \{3,4\}}} 2\delta^2 + \sum_{i \neq 3,4} 2\delta^2 + (m+1) 2\delta^2 \geq 0.
$$

Note that f and the assumptions are symmetric with respect to $x_3, \ldots, x_n$. By the claim above, we only need to prove the special case that $x_3 = \cdots = x_n$. In this case,

$$
\sum_{\substack{i < j \\ \{i,j\} \neq \{1,2\}}} M_{ij} = (n-2)\left(M_{12} + \tfrac{1}{2}(n-1) M_{33}\right),
$$

and

$$
f(x_1, x_2, x_3, \ldots, x_3) = -\frac{(n-2)\left(M_{12} + \frac{n-1}{2} M_{33}\right)}{m+1} \left(\sum_{\substack{i < j \\ \{i,j\} \neq \{1,2\}}} M_{ij} + (m+1) M_{12} \right)
$$

$$
+ \frac{n-2}{2} \Big((n-1) M_{13}^2 + (n-1) M_{23}^2 + (n-1)(n-3) M_{33}^2
$$

$$
+ (n-3) M_{12} M_{33} + 2(m+1) M_{13} M_{23} \Big)
$$

$$
= I + II.
$$

Clearly, $-(m+1)(m+n-1)\rho \leq -2\rho \leq 0$, and we have some estimates in the following assertion.

Claim. *The following inequalities hold.*

(1) $M_{12} < -\rho \leq 0$.

(2) $M_{33} > 0$.

(3) $M_{12} + \tfrac{1}{2}(n-1) M_{33} > 0$.

(4) $(m + n - 1)(-M_{12}) \geq \frac{1}{2}(n - 1)(n - 2)M_{33}$.

(5) $\frac{1}{2}(n - 1)(n - 2)M_{33} \geq -\rho - (m + n - 2)M_{12}$.

(6) $(n - 2)(M_{12} + \frac{1}{2}(n - 1)M_{33}) \geq (m - 1)(-M_{12})$.

Proof of claims. Obviously, by combining the assumptions (ii) and (iii), we get (1)–(3). Then by (iii), we have $(n - 2)(M_{12} + \frac{1}{2}(n - 1)M_{33}) + (m + 1)M_{12} \leq 0$, so we get (4). Now by (ii), $(n - 2)(M_{12} + \frac{1}{2}(n - 1)M_{33}) \geq -\rho - mM_{12}(\geq 0)$, which gives (5). Then (6) follows from (5) immediately. □

By the claim we know that I is always nonnegative. In the following, we will divide the argument into two cases.

Case 1: $m = 1, 2$. In this case, we have

$$II \geq \frac{1}{2}(n - 2)\big(3M_{13}^2 + 3M_{23}^2 - 6|M_{13}M_{23}|$$
$$+ (n - 3)\big(M_{12} + \frac{1}{2}(n - 1)M_{33}\big)M_{33} + \frac{1}{2}(n - 1)(n - 3)M_{33}^2\big) \geq 0.$$

In this case, Lemma 2.1 is proved.

Case 2: $m \geq 3$. It suffices to prove that $M_{13}M_{23} < 0$, that is, $M_{13} < 0$ and $M_{23} > 0$, which implies $-M_{12}M_{33} \geq -M_{13}M_{23} > 0$. (Indeed, if $M_{13}M_{23} \geq 0$, it is easy to see that II is positive; therefore we have proved Lemma 2.1.)

Then we have

$$I \geq \frac{1}{m+1}(m - 1)(-M_{12})(m + 1)(m + n - 1)\rho$$
$$\geq (m - 1)\frac{1}{2}(n - 1)(n - 2)\rho M_{33}$$
$$\geq (n - 3)\rho M_{33},$$

$$II \geq \frac{1}{2}(n - 2)\Big((n - 1)M_{13}^2 + (n - 1)M_{23}^2 + 2(m + 1)M_{13}M_{23}$$
$$+ \frac{n-3}{n-2}M_{33}[-\rho - mM_{12}] + \frac{n-3}{n-2}M_{33}[-\rho - (m + n - 2)M_{12}]\Big)$$
$$\geq \frac{1}{2}(n - 2)\Big((n - 1)M_{13}^2 + (n - 1)M_{23}^2 + 2(m + 1)M_{13}M_{23}$$
$$- 2(m + 1)\frac{n-3}{n-2}M_{12}M_{33} - 2\frac{n-3}{n-2}\rho M_{33}\Big)$$
$$\geq \frac{1}{2}(n - 2)\Big((n - 1)M_{13}^2 + (n - 1)M_{23}^2 + \frac{2(m+1)}{n-2}M_{13}M_{23}\Big) - (n - 3)\rho M_{33}.$$

Therefore

$$f \geq \frac{1}{2}(n - 1)(n - 2)(M_{13}^2 + M_{23}^2 + 2\frac{m+1}{(n-1)(n-2)}M_{13}M_{23}).$$

If

$$\frac{(m+1)}{(n-1)(n-2)} \leq 1,$$

then $f \geq 0$; if the same quantity is greater than 1 and

$$M_{23} + 2 \frac{m+1}{(n-1)(n-2)} M_{13} \geq 0,$$

then we have $f \geq 0$ again.

Otherwise, $m + 1 > (n-1)(n-2)$, and

$$M_{13} < - \frac{(n-1)(n-2)}{2(m+1)} M_{23}.$$

Since $M_{12} + \frac{1}{2}(n-1)M_{33} = M_{13} + M_{23} + \frac{1}{2}(n-3)M_{33}$, by (ii), we get

$$\frac{(n-1)(n-2)}{2} \frac{2(m+1)-(n-1)(n-2)}{2(m+1)} M_{23} + \left(m - \frac{1}{2}(n-2)(n-3)\right) M_{12} \geq -\rho,$$

So

$$-M_{12} \leq \frac{(n-1)(n-2)}{2(m+1)} \frac{2(m+1)-(n-1)(n-2)}{2m-(n-2)(n-3)} M_{23} + \frac{2}{2m-(n-2)(n-3)} \rho$$

$$\leq \frac{(n-1)(n-2)}{2(m+1)} M_{23} + \frac{\rho}{n},$$

and then

$$f \geq \frac{1}{2}(n-1)(n-2) \left(\left(\frac{(n-1)(n-2)}{2(m+1)} M_{23} \right)^2 - \frac{2(m+1)}{n(n-1)(n-2)} \rho M_{23} \right)$$

$$\geq -C(m,n)\rho^2,$$

where $C(m,n)$ is a constant depending only on m and n. $\square$

3. The proof of Theorem 1.2

Let (M, g, f) be a smooth gradient shrinking soliton. Then by using the contracted second Bianchi identity, we get the equation $R + |\nabla f|^2 - 2\lambda f = \text{const}$. Obviously, by rescaling g and changing f by a constant, we can assume that $\lambda = 1/2$ and that $R + |\nabla f|^2 - f = 0$. We call such a soliton normalized, and f a normalized soliton function.

In terms of moving frames [Hamilton 1986] of the Ricci flow, the curvature operator $M_{\alpha\beta}$ has the evolution equation

$$\frac{\partial}{\partial t} M_{\alpha\beta} = \triangle M_{\alpha\beta} + M_{\alpha\beta}^2 + M_{\alpha\beta}^{\#},$$

where $M_{\alpha\beta}^{\#}$ is the Lie algebra adjoint of $M_{\alpha\beta}$. In general, we know little about $M_{\alpha\beta}^{\#}$. However, when the metric is conformally flat, we know this:

Proposition 3.1. *Suppose we have a smooth solution $g_{ij}(x,t)$ of the Ricci flow on an n-dimensional manifold M, and suppose at $t = t_0$ that the metric $g_{ij}(x, t_0)$ has vanishing Weyl tensor. Then at $t = t_0$, for any point p, there exist an orthonormal*

basis $\{e_i\}$ and n real numbers M_i such that $\{\phi^\alpha = \sqrt{2}e_i \wedge e_j\}$ for $i < j$ is an orthonormal basis of $\bigwedge^2 T_pM$, and we have

(i) $M_{\alpha\beta} = \begin{cases} M_{ij} := M_i + M_j & \text{if } \phi^\alpha = \phi^\beta = \sqrt{2}e_i \wedge e_j, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$

(ii) $M_{\alpha\beta}^\# = \begin{cases} \sum_{k \neq i,j} M_{ik}M_{jk} & \text{if } \phi^\alpha = \phi^\beta = \sqrt{2}e_i \wedge e_j, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$

Proof. Suppose $\{e_i\}$ is an orthonormal basis that diagonalizes the Ricci tensor, that is, $\text{Ric}(e_i) = \lambda_i e_i$.

Because the Weyl tensor vanishes, we have

$$R_{ijkl} = \frac{1}{n-2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}) - \frac{1}{(n-1)(n-2)}R(g_{ik}g_{jl} - g_{il}g_{jk}).$$

Thus

$$R_{ijij} = \frac{\lambda_i + \lambda_j}{n-2} - \frac{1}{(n-1)(n-2)}R,$$

and $R_{ijkl} = 0$ if three of the indices are mutually distinct. We then prove (i) by letting $M_i = 2\lambda_i/(n-2) - R/((n-1)(n-2))$.

See that $M_{\alpha\beta}^\# = C_\alpha^{\gamma\eta}C_\beta^{\delta\theta}M_{\gamma\delta}M_{\eta\theta} = C_\alpha^{\gamma\eta}C_\beta^{\gamma\eta}M_{\gamma\gamma}M_{\eta\eta}$, where $[\phi^\alpha, \phi^\beta] = C_\gamma^{\alpha\beta}\phi^\gamma$.

Let A_{ij} for $i \neq j$ denote by the matrix with $(A_{ij})_{ij} = 1$, $(A_{ij})_{ji} = -1$ and all other elements zero. Then $[A_{ij}, A_{jk}] = A_{ik}$ if $i < j < k$.

By direct computation, we have $M_{\alpha\beta}^\# = 0$ if $\alpha \neq \beta$. If $\alpha = \beta = \sqrt{2}e_i \wedge e_j (i < j)$, we have

$$\begin{aligned} M_{\alpha\alpha}^\# &= (C_\alpha^{\gamma\delta})^2 M_{\gamma\gamma}M_{\delta\delta} = \langle[\tfrac{1}{\sqrt{2}}A_{ij}, \phi^\gamma], \phi^\delta\rangle^2 M_{\gamma\gamma}M_{\delta\delta} \\ &= \sum_{k \neq i,j} \langle[\tfrac{1}{\sqrt{2}}A_{ij}, \tfrac{1}{\sqrt{2}}A_{ki}], \phi^\delta\rangle^2 M_{ki}M_{\delta\delta} + \sum_{k \neq i,j} \langle[\tfrac{1}{\sqrt{2}}A_{ij}, \tfrac{1}{\sqrt{2}}A_{kj}], \phi^\delta\rangle^2 M_{kj}M_{\delta\delta} \\ &= \tfrac{1}{2}\sum_{k \neq i,j} \langle\tfrac{1}{\sqrt{2}}A_{jk}, \phi^\delta\rangle^2 M_{ki}M_{\delta\delta} + \tfrac{1}{2}\sum_{k \neq i,j} \langle\tfrac{1}{\sqrt{2}}A_{ik}, \phi^\delta\rangle^2 M_{kj}M_{\delta\delta} \\ &= \sum_{k \neq i,j} M_{ik}M_{jk}. \qquad \square \end{aligned}$$

Now, combing Lemma 2.1 and Proposition 3.1, we are ready to prove the local version of the Hamilton–Ivey-type pinching estimate. The basic idea is to use the methods of [Chen 2007].

Proposition 3.2. *For any nonnegative integer m, there is a constant C_m depending only on m and n with the following property. Suppose we have a complete gradient shrinking soliton $(M^n, g_{ij}(x,t))$ for $n \geq 4$ on $[0, T]$ with vanishing Weyl tensor. Also assume that $\text{Ric}(x,t) \leq (n-1)r_0^{-2}$ for $x \in B_t(x_0, Ar_0)$ and $t \in [0, T]$ and*

that $R + m\nu \geq -K_m (K_m \geq 0)$ on $B_0(x_0, Ar_0)$ at $t = 0$, where ν is the smallest eigenvalue of the curvature operator. Then we have

$$R(x, t) + m\nu \geq \min\left\{-\frac{C_m}{t + 1/K_m}, -\frac{C_m}{Ar_0^2}\right\} \quad \text{if } A \geq 2$$

whenever $x \in B_t(x_0, \frac{1}{2}Ar_0)$, with $t \in [0, T]$.

Proof. By [Perelman 2002], we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) d_t(x_0, x) \geq -\frac{5(n-1)}{3} r_0^{-1},$$

whenever $d_t(x_0, x) > r_0$ in the sense of support functions.

We will argue by induction on m to prove the estimate holds on ball of radius $(1/2 + 1/2^{m+2})Ar_0$. The case $m = 0$ follows from [Zhang 2008b, Theorem 1]. Suppose we have proved the result for some nonnegative m_0, that is, there is a constant C_{m_0} such that

$$R(x, t) + m_0\nu \geq \min\left\{-\frac{C_{m_0}}{t + 1/K_{m_0}}, -\frac{C_{m_0}}{Ar_0^2}\right\}$$

whenever $x \in B_t(x_0, (1/2 + 1/2^{m_0+2})Ar_0)$ and $t \in [0, T]$. We are going to prove the result for $m = m_0 + 1$ on ball of radius $(1/2 + 1/2^{m_0+3})Ar_0$. Without loss of generality, we may assume $K_0 \leq K_1 \leq K_2 \leq \cdots$.

Define a function $C_{m_0}(t) := \max\{C_{m_0}/(t + 1/K_{m_0}), C_{m_0}/Ar_0^2\}$.

Under the moving frame, let

$$N_{\alpha\beta} := Rg_{\alpha\beta} + (m_0 + 1)M_{\alpha\beta} \quad \text{and} \quad P_{\alpha\beta} := \varphi\left(\frac{d_t(x, x_0)}{Ar_0}\right)N_{\alpha\beta},$$

where φ is a smooth nonnegative decreasing function that is 1 on the interval $(-\infty, 1/2 + 1/2^{m_0+3}]$ and 0 on $[1/2 + 1/2^{m_0+2}, \infty)$.

By direct computation, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) P_{\alpha\beta} = -2\nabla_l\varphi\nabla_l N_{\alpha\beta} + Q_{\alpha\beta},$$

where

$$Q_{\alpha\beta} = \left(\varphi'\frac{1}{Ar_0}\left(\frac{\partial}{\partial t} - \Delta\right)d_t - \varphi''\frac{1}{(Ar_0)^2}\right)N_{\alpha\beta}$$

$$+ \varphi\left(g_{\alpha\beta}\left(\frac{\partial}{\partial t} - \Delta\right)R + (m_0 + 1)\left(\frac{\partial}{\partial t} - \Delta\right)M_{\alpha\beta}\right)$$

$$= \left(\varphi'\frac{1}{Ar_0}\left(\frac{\partial}{\partial t} - \Delta\right)d_t - \varphi''\frac{1}{(Ar_0)^2}\right)N_{\alpha\beta}$$

$$+ \varphi g_{\alpha\beta}\left(\sum_{i<j}\left(M_{ij}^2 + \sum_{k\neq i,j} M_{ik}M_{jk}\right) + (m_0 + 1)\left(M_{i_0 j_0}^2 + \sum_{k\neq i_0, j_0} M_{i_0 k}M_{j_0 k}\right)\right),$$

where $\phi^\alpha = \sqrt{2}e_{i_0} \wedge e_{j_0}$ and the second equality follows from Proposition 3.1.

Note that the smallest eigenvalue of $N_{\alpha\beta}$ is $R + (m_0 + 1)\nu$.

Let
$$u(t) := \min_{x \in M}\{(R + (m_0 + 1)\nu)\varphi(x, t)\}.$$

For fixed $t_0 \in [0, T]$, assume
$$(R + (m_0 + 1)\nu)\varphi(x_0, t_0) = u(t_0) < -(m_0 + 2 + 1)(m_0 + 2 + n - 1)C_{m_0}(t_0).$$

Otherwise, we have the estimate at time t_0.

Let V be the corresponding unit eigenvector of the smallest eigenvalue of $N_{\alpha\beta}$ at (x_0, t_0). Let $\{\lambda_i\}$ be the eigenvalues of the Ricci tensor, where $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Then by Lemma 2.1 and Proposition 3.1, we have

$$
\begin{aligned}
&Q(V, V)(x_0, t_0) \\
&= \left(\varphi'\frac{1}{Ar_0}\left(\frac{\partial}{\partial t} - \Delta\right)d_t - \varphi''\frac{1}{(Ar_0)^2}\right)\frac{u(t_0)}{\varphi} \\
&\quad + \varphi\left(\sum_{i<j}(M_{ij}^2 + \sum_{k\neq i,j}M_{ik}M_{jk}) + (m_0 + 1)(M_{12}^2 + \sum_{k\neq 1,2}M_{1k}M_{2k})\right) \\
&= \left(\varphi'\frac{1}{Ar_0}\left(\frac{\partial}{\partial t} - \Delta\right)d_t - \varphi''\frac{1}{(Ar_0)^2}\right)\frac{u(t_0)}{\varphi} \\
&\quad + \varphi\left(\frac{\left(\sum_{i<j,\ \{i,j\}\neq\{1,2\}}M_{ij} + (m_0 + 2)M_{12}\right)^2}{m_0 + 2} + f(M_1, \ldots, M_n)\right) \\
&\ge \left(\varphi'\frac{1}{Ar_0}\left(-\frac{5(n-1)}{3}r_0^{-1}\right) - \varphi''\frac{1}{(Ar_0)^2}\right)\frac{u(t_0)}{\varphi} \\
&\qquad\qquad\qquad\qquad + \frac{1}{(m_0+2)\varphi}u(t_0)^2 - \varphi C(m_0)C_{m_0}(t_0)^2 \\
&= \frac{1}{(m_0+2)\varphi}\left(u(t_0)^2 - \left(\varphi'\frac{1}{Ar_0^2}\frac{5(n-1)(m_0+2)}{3} + \varphi''\frac{m_0+2}{(Ar_0)^2}\right)u(t_0)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad - C(m_0)C_{m_0}(t_0)^2.
\end{aligned}
$$

where $m = m_0 + 1$ in the second equality.

Since $|\varphi'| \le C(m_0)$, $|\varphi''| \le C(m_0)$ and $(\varphi')^2/\varphi \le C(m_0)$, by applying the maximum principle, we have

$$\frac{d^-}{dt}u\bigg|_{t=t_0} \ge Q(V, V)(x_0, t_0) + \frac{2}{(Ar_0)^2}\frac{(\varphi')^2}{\varphi^2}u(t_0) \ge \frac{1}{2(m_0+2)}u(t_0)^2,$$

provided $|u|(t_0) \ge \max\{C(m_0)C_{m_0}(t_0), C(m_0)/(Ar_0^2)\}$. By integrating the differential inequality above, we get the estimate

$$u(t) \ge \min\left\{\frac{1}{1/u(0) - t/(2(m_0+2))}, -C(m_0)C_{m_0}(t), -\frac{C(m_0)}{Ar_0^2}\right\}.$$

Clearly, there is a constant C_{m_0+1} such that

$$u(t) \geq \left\{ -\frac{C_{m_0+1}}{t+1/K_{m_0+1}}, -\frac{C_{m_0+1}}{Ar_0^2} \right\}. \qquad \square$$

Remark. In fact, the case $m = 0$, we do not need to suppose that the soliton has vanishing Weyl tensor, since Chen [2007] has already proved this result.

Corollary 3.3. *Any gradient shrinking soliton* (*not necessarily having bounded curvature*) *with vanishing Weyl tensor must have nonnegative curvature operator.*

Proof. Let (M, g, f) be a gradient shrinking soliton. Then we have a solution $g(t)$ for $t \in (-\infty, 0]$ of the Ricci flow with $g(0) = g$.

The case $n = 3$ has done by Chen [2007]. Therefore we only need to prove the case $n \geq 4$.

Fix a point x_0 on M. For any $T > 0$, there is a small r_0 such that whenever $t \in [-T, 0]$ and $x \in B_t(x_0, r_0)$, we have $|R_m|(x, t) \leq r_0^2$. Let $A \to \infty$ in Proposition 3.2. Then we get

$$(R+mv)(x, 0) \geq -\frac{C_m}{T-0+1/K_m} \geq -\frac{C_m}{T}.$$

Since C_m does not depend on T, we get that $(R+mv)(x, 0) \geq 0$ for any m. This implies $v \geq 0$, that is, the curvature operator is nonnegative. $\qquad \square$

Proposition 3.4. *Any gradient shrinking soliton*(*not necessarily having bounded curvature*) *with vanishing Weyl tensor must have the properties that*

(i) Ric ≥ 0 *and*

(ii) $|R_{ijkl}| \leq \exp(a(d(p, x) + 1))$ *for some $a > 0$ and fixed point p.*

Proof. Property (i) follows from Corollary 3.3 immediately.

For property (ii), it clearly suffices to prove the result under the condition that the soliton is normalized. So $R + |\nabla f|^2 - f = 0$. Combining the soliton equation $R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}$ and (i), we get that $\nabla_i \nabla_j f \leq \frac{1}{2} g_{ij}$.

For any point $x \in M$, let $\gamma(t) : [0, d(p, x)] \to M$ be the shortest normal geodesic connecting p and x, and denote by $h(t) = f(\gamma(t))$. Then

$$h''(t) = \langle \nabla f, \dot{\gamma} \rangle' = \dot{\gamma} \langle \nabla f, \dot{\gamma} \rangle = \nabla^2 f(\dot{\gamma}, \dot{\gamma}) \leq \frac{1}{2},$$

By integrating inequality above, we have

$$\begin{aligned}
f(x) &= h(d(p, x)) \\
&\leq \tfrac{1}{4} d(p, x)^2 - \langle \nabla f, \dot{\gamma} \rangle(0) d(p, x) - h(0) \\
&\leq \tfrac{1}{4} d(p, x)^2 + |\nabla f|(p) d(p, x) + |f|(p).
\end{aligned}$$

Since the right side of this inequality just depends on local properties of f at p, that $R + |\nabla f|^2 - f = 0$ implies that $R \leq f \leq \exp(a(d(p, x) + 1))$ for some $a > 0$; hence $|R_{ijkl}| \leq \exp(a(d(p, x) + 1))$, because of the nonnegativity of the curvature operator. $\square$

Finally, by [Ni and Wallach 2008] or [Petersen and Wylie 2008], we get the classification theorem, Theorem 1.2.

4. Acknowledgments

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