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# HARMONIC FORMS ON MANIFOLDS WITH WEIGHTED POINCARÉ INEQUALITY

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**Let $M$ be a complete noncompact manifold. We prove vanishing and finiteness results for harmonic $p$-forms on $M$, assuming both the curvature operator lower bound and the weighted Poincaré inequality on $M$.**

## 1. Introduction

It is interesting to study the structures of noncompact complete manifolds, especially their topological properties at infinity. There are many results [Witten and Yau 1999; Cai and Galloway 1999; Wang 2001; 2001; Wan and Xin 2004] on the topology of conformally compact manifolds. Recently, by assuming that the Ricci curvature is bounded from below in terms of the dimension and of the first eigenvalue, Li and Wang [2001] obtained information on the topology of complete manifolds infinity and in some cases the metric structure of these manifolds, by proving a vanishing-type theorem of $L^2$ harmonic 1-forms. In his thesis, Lam [2007] generalized Li and Wang's result by relaxing the curvature assumptions. He proved that a manifold must have finitely many nonparabolic ends if a similar inequality between the Ricci curvature and the weight function in the weighted Poincaré inequality (see Definition 1.1) is valid outside a compact subset.

In this note, we will consider general harmonic $p$-forms. Working with a complete manifold $M$ satisfying a weighted Poincaré inequality and a curvature operator lower bound expressed in terms of the dimension and the weight function, we prove vanishing and finiteness theorems for the $L^d$ harmonic $p$-forms. Also, on an end of manifold with weighted $p$-Poincaré inequality, we prove that the Green's form satisfies a sharp decay estimate. Let us first recall some definitions.

**Definition 1.1.** Let $M^m$ be an $m$-dimensional complete Riemannian manifold. We say that $M^m$ satisfies a weighted Poincaré inequality [Li and Wang 2006] with a

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nonnegative weight function $\rho(x)$ if the inequality

$$\int_M \rho(x)\phi^2 \le \int_M |\nabla\phi|^2 \quad \text{for } \phi \in C_0^\infty(M)$$

is valid for all compactly supported smooth functions $\phi \in C_0^\infty(M)$.

**Definition 1.2.** Let $M^m$ be an $m$-dimensional complete Riemannian manifold. We say that $M^m$ has property $(\mathcal{P}_\rho)$ if a weighted Poincaré inequality is valid on $M$ with some nonnegative weight function $\rho$ and if the $\rho$-metric, defined by $ds_\rho^2 = \rho ds_M^2$, is complete.

Let $\lambda_1(M)$ denote the greatest lower bound of the spectrum of the Laplacian acting on $L^2$ functions. Then the variation principle for $\lambda_1(M)$ asserts the validity of the Poincaré inequality

$$\lambda_1(M)\int_M \phi^2 \le \int_M |\nabla\phi|^2$$

for all compactly supported functions $\phi \in C_0^\infty(M)$. If $\lambda_1(M)$ is positive, then obviously $M$ has property $(\mathcal{P}_\rho)$ with $\rho(x) = \lambda_1(M)$. Property $(\mathcal{P}_\rho)$ may be seen as a generalization of the assumption $\lambda_1(M) > 0$.

For harmonic $p$-forms, let $C_0^\infty(\bigwedge^p M)$ denote the space of smooth $p$-forms with compact support on $M$. Then we define property $(\mathcal{P}_{p,\rho})$ as follows.

**Definition 1.3.** Let $M^m$ be an $m$-dimensional complete Riemannian manifold. We say that $M^m$ has property $(\mathcal{P}_{p,\rho})$ if a weighted $p$-Poincaré inequality is valid on $M$ with some nonnegative weight function $\rho$, that is,

$$\int_M \rho(x)|\phi|^2 \le \int_M |d\phi|^2 + |\delta\phi|^2 \quad \text{for } \phi \in C_0^\infty(\textstyle\bigwedge^p M),$$

and if the $\rho$-metric, defined by $ds_\rho^2 = \rho ds_M^2$, is complete.

If the greatest lower bound $\lambda_{1,p}$ of the $p$ spectrum satisfies $\lambda_{1,p}(M) > 0$, then $M$ has property $(\mathcal{P}_{p,\rho})$ with the weight function $\rho(x) = \lambda_{1,p}(M)$. Hence property $(\mathcal{P}_{p,\rho})$ can also be viewed as a generalization of the assumption that $\lambda_{1,p}(M) > 0$.

Throughout, we use $H_d^p(M^m)$ to denote the space of $L^d$ harmonic $p$-forms, and $r_\rho(x)$ to denote the geodesic distance from some fixed point to $x$ with respect to the metric $ds_\rho^2$. Our main result is the following.

**Theorem 1.4.** *For $m \ge 3$, let $M^m$ be a complete noncompact Riemannian manifold with properties $(\mathcal{P}_{\rho_0})$ and $(\mathcal{P}_{p,\rho})$. Suppose the volume growth of $M$ satisfies*

$$\int_{B_\rho(2R)\setminus B_\rho(R)} \rho e^{-2(m-p-1)r_\rho} \le cR$$

*and the curvature operator $K_p$ of $M^m$ has the lower bound*

$$K_p > -\frac{m-p}{m-p-1}\rho_0 \quad \text{on } M.$$

*Then the space $H_d^p(M)$ is trivial for $2 \le d \le 2(m-p-1)/(m-p-2)$.*

We will employ some of the arguments from [Li and Wang 2001] to prove the sharp decay estimate for the Green's form in the next section, and prove the main theorem in Section 3.

## 2. Decay estimate

Let $M^m$ be a complete manifold, and let $\Delta$ be the Hodge Laplace–Beltrami operator of $M^m$ acting on differential $p$-forms. The Weitzenböck formula gives

$$\Delta = \nabla^2 - K_p,$$

where $\nabla^2$ is the Bochner Laplacian and $K_p$ is an endomorphism depending upon the curvature tensor of $M^m$. Using an orthonormal basis $\{\theta^1, \dots, \theta^m\}$ dual to $\{e_1, \dots, e_m\}$, one may express the curvature term $K_p$ as

$$\langle K_p(\omega), \omega\rangle = \left\langle \sum_{j,k=1}^{m} \theta^k \wedge i_{e_j} R(e_k, e_j)\omega, \omega \right\rangle.$$

In particular, $\langle K_1(\omega), \omega\rangle = \mathrm{Ric}(\omega^\sharp, \omega^\sharp)$, where $\omega^\sharp$ is the vector dual to the form $\omega$. We say $M^m$ has curvature lower bound $k_p$ if for all $p$-forms $\omega$ on $M^m$,

$$\langle K_p(\omega), \omega\rangle \ge k_p|\omega|^2$$

We recall [Li 2000] that an end is simply an unbounded component of $M\backslash D$, where $D$ is a compact smooth domain of $M$. Write $E(R) = E \cap B_q(R)$, and define $\partial E(R) = \partial E \cup (\partial B_q(R) \cap E)$. Let $\lambda_{1,p}(E(R))$ be the first eigenvalue of $\Delta$ for $p$-forms satisfying Dirichlet boundary conditions on $\partial E(R)$, that is, $\lambda_{1,p}(E) = \inf_{R>0, E(R)\neq\varnothing} E(R)$; see [Donnelly 1984; Donnelly and Xavier 1984]. Therefore we have for all $\omega \in C_0^\infty(\bigwedge^p E)$

$$\lambda_{1,p}(E)\int |\omega|^2 \le \int (|\nabla\omega|^2 + \langle K_p(\omega), \omega\rangle),$$

where $C_0^\infty(\bigwedge^p E)$ is the space of smooth $p$-forms with compact supported on the end $E$. If the $p$-spectrum $\lambda_{1,p}(E)$ is positive, then $E$ has property $(\mathscr{P}_{p,\rho})$ with the weight function $\rho = \lambda_{1,p}(E)$.

In this section, we study the harmonic $p$-forms on the end $E$ of a manifold with weighted $p$-Poincaré inequality and prove the following decay estimate. See [Donnelly 1984; Li and Wang 2001].

**Lemma 2.1.** *Let $M$ be a complete noncompact manifold. If $E$ is an end of $M$ with the property $(\mathcal{P}_{p,\rho})$ for some nonnegative weight function $\rho$, then for any smooth harmonic $p$-form $\omega$ satisfying*

$$\int_{E_\rho(2R)\setminus E_\rho(R)} \rho(x)\exp(-2r_\rho)|\omega|^2 dv = o(R), \tag{1}$$

*we have*

$$\int_{E_\rho(R)} \rho(x)\exp(2r_\rho)|\omega|^2 dv \le CR,$$

$$\int_{E_\rho(R+1)\setminus E_\rho(R)} \rho(x)\exp(2r_\rho)|\omega|^2 dv \le C$$

*for all $R$ sufficiently large and for some constant $C$ depending on $\omega$ and $\rho$.*

*Proof.* Let $\psi$ be a nonnegative cutoff function. Integration by parts gives

$$\begin{aligned}\int_E |\nabla(\psi\omega)|^2 + \langle K_p(\psi\omega), \psi\omega\rangle &= \int_E |\nabla\psi|^2|\omega|^2 + 2\psi\langle\omega\nabla\psi, \nabla\omega\rangle + |\psi|^2|\nabla\omega|^2 + \psi^2\langle K_p(\omega), \omega\rangle \\ &= \int_E |\nabla\psi|^2|\omega|^2 - \int_E \psi^2\langle\omega, \Delta\omega\rangle_M \\ &= \int_E |\nabla\psi|^2|\omega|^2.\end{aligned}$$

By property $(\mathcal{P}_{p,\rho})$,

$$\int_E \rho(x)\psi^2|\omega|^2 dv \le \int_E (|\nabla(\psi\omega)|^2 + \langle K_p(\psi\omega), \psi\omega\rangle),$$

so we have

$$\int_E \rho(x)\psi^2|\omega|^2 dv \le \int_E |\nabla\psi|^2|\omega|^2 dv \tag{2}$$

for any cutoff function $\psi$ on $E$. Let $\psi = \phi(r_\rho(x))\exp(a(r_\rho(x)))$. Then

$$\int_E \phi^2\exp(2a)|\omega|^2 dv \le \int_E \left(|\nabla\phi|^2 + 2\langle\nabla\phi, \nabla a\rangle\phi + |\nabla a|^2\phi^2\right)\exp(2a)|\omega|^2 dv. \tag{3}$$

Choose $\phi$ as the nonnegative cutoff function defined by

$$\phi(r_\rho(x)) = \begin{cases} r_\rho(x) - R_0 & \text{on } E_\rho(R_0+1)\setminus E_\rho(R_0), \\ 1 & \text{on } E_\rho(R)\setminus E_\rho(R_0+1), \\ (2R - r_\rho(x))/R & \text{on } E_\rho(2R)\setminus E_\rho(R), \\ 0 & \text{on } E\setminus E_\rho(2R), \end{cases}$$

and also choose $a(r_\rho(x))$ as

$$a(r_\rho(x)) = \begin{cases} \delta r_\rho(x) & \text{for } r_\rho \le K/(1+\delta), \\ K - r_\rho(x) & \text{for } r_\rho > K/(1+\delta), \end{cases}$$

for some fixed $K > (R_0+1)(1+\delta)$ and $0 < \delta < 1$. If $R \ge K/(1+\delta)$, it is easy to check that

$$|\nabla\phi|^2(x) = \begin{cases} \rho(x) & \text{on } E_\rho(R_0+1)\setminus E_\rho(R_0), \\ 0 & \text{on } E_\rho(R)\setminus E_\rho(R_0+1), \\ \rho(x)/R^2 & \text{on } E_\rho(2R)\setminus E_\rho(R), \\ 0 & \text{on } E\setminus E_\rho(2R), \end{cases}$$

$$|\nabla a|^2(x) = \begin{cases} \delta^2\rho(x) & \text{for } r \le K/(1+\delta), \\ \rho(x) & \text{for } r > K/(1+\delta), \end{cases}$$

and then

$$\langle\nabla\phi, \nabla a\rangle\,(x) = \begin{cases} \delta\rho(x) & \text{on } E_\rho(R_0+1)\setminus E_\rho(R_0), \\ \rho(x)/R & \text{on } E_\rho(2R)\setminus E_\rho(R), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore (3) becomes

$$\begin{aligned} &\int_{E_\rho(2R)} \rho\phi^2 \exp(2a)|\omega|^2 dv \\ &\quad\le \int_{E_\rho(2R)} \left(|\nabla\phi|^2 + 2\langle\nabla\phi, \nabla a\rangle\phi + |\nabla a|^2\phi^2\right)\exp(2a)|\omega|^2 dv \\ &\quad\le \int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} \rho\exp(2a)|\omega|^2 dv + \frac{1}{R^2}\int_{E_\rho(2R)\setminus E_\rho(R)} \rho\exp(2a)|\omega|^2 dv \\ &\qquad + 2\delta\int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} \rho\phi\exp(2a)|\omega|^2 dv + \frac{2}{R}\int_{E_\rho(2R)\setminus E_\rho(R)} \rho\phi\exp(2a)|\omega|^2 dv \\ &\qquad + \delta^2\int_{E_\rho(\frac{K}{1+\delta})\setminus E_\rho(R_0)} \rho\phi^2\exp(2a)|\omega|^2 dv + \int_{E_\rho(2R)\setminus E_\rho(\frac{K}{1+\delta})} \rho\phi^2\exp(2a)|\omega|^2 dv. \end{aligned}$$

Hence

$$\begin{aligned} (1-\delta^2)&\int_{E_\rho(\frac{K}{1+\delta})\setminus E_\rho(R_0+1)} \rho\exp(2a)|\omega|^2 dv \\ &\le (\delta^2+2\delta+1)\int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} \rho\exp(2a)|\omega|^2 dv \\ &\qquad + \left(\frac{1}{R^2}+\frac{2}{R}\right)\int_{E_\rho(2R)\setminus E_\rho(R)} \rho\exp(2a)|\omega|^2 dv. \end{aligned}$$

By the definition of $a(x)$ and the growth estimate of $w$ (see (1)), the last term on the right side tends to 0 as $R \to \infty$. Thus we obtain the estimate

$$\frac{1-\delta^2}{(\delta+1)^2} \int_{E_\rho(\frac{K}{1+\delta})\setminus E_\rho(R_0+1)} \rho \exp(2\delta r_\rho)|w|^2 dv \leq \int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} \rho \exp(2\delta r_\rho)|w|^2 dv.$$

Since the right side of this inequality is independent of $K$, by letting $K \to \infty$ we conclude that

$$\int_{E\setminus E_\rho(R_0+1)} \rho \exp(2\delta r_\rho)|w|^2 dv \leq C \quad \text{for some constant } 0 < C < \infty. \tag{4}$$

Next we improve this estimate by setting $a(r_\rho(x)) = r_\rho(x)$ in the preceding argument. For $R_0 < R_1 < R$, let us choose $\phi$ to be

$$\phi(x) = \begin{cases} (r_\rho(x) - R_0)/(R_1 - R_0) & \text{on } E_\rho(R_1)\setminus E_\rho(R_0), \\ (R - r_\rho(x))/(R - R_1) & \text{on } E_\rho(R)\setminus E_\rho(R_1). \end{cases}$$

The inequality (3) asserts that

$$\begin{aligned}\int_{E_\rho(R)} \rho\phi^2 \exp(2r_\rho)|w|^2 dv &\leq \int_{E_\rho(R)} |\nabla(\phi \exp(r_\rho))|^2 |w|^2 dv \\ &= \frac{1}{(R-R_1)^2} \int_{E_\rho(R)\setminus E_\rho(R_1)} \rho \exp(2r_\rho)|w|^2 dv \\ &\quad + \frac{1}{(R_1-R_0)^2} \int_{E_\rho(R_1)\setminus E_\rho(R_0)} \rho \exp(2r_\rho)|w|^2 dv \\ &\quad - \frac{2}{(R-R_1)^2} \int_{E_\rho(R)\setminus E_\rho(R_1)} (R - r_\rho(x))\rho \exp(2r_\rho)|w|k2 dv \\ &\quad + \frac{2}{(R_1-R_0)^2} \int_{E_\rho(R_1)\setminus E_\rho(R_0)} (r_\rho(x) - R_0)\rho \exp(2r_\rho)|w|^2 dv \\ &\qquad + \int_{E_\rho(R)} \rho\phi^2 \exp(2r_\rho)|w|^2 dv.\end{aligned}$$

Then

$$\begin{aligned}\frac{2}{(R-R_1)^2} &\int_{E_\rho(R)\setminus E_\rho(R_1)} (R - r_\rho(x))\rho \exp(2r_\rho)|w|^2 dv \\ &\leq \frac{1}{(R-R_1)^2} \int_{E_\rho(R)\setminus E_\rho(R_1)} \rho \exp(2r_\rho)|w|^2 dv \\ &\quad + \frac{1}{(R_1-R_0)^2} \int_{E_\rho(R_1)\setminus E_\rho(R_0)} \rho \exp(2r_\rho)|w|^2 dv \\ &\quad + \frac{2}{(R_1-R_0)} \int_{E_\rho(R_1)\setminus E_\rho(R_0)} \frac{r_\rho(x) - R_0}{R_1 - R_0} \rho \exp(2r_\rho)|w|^2 dv.\end{aligned}$$

On the other hand, for any fixed $0 < t < R - R_1$,

$$\frac{t}{(R-R_1)^2}\int_{E_\rho(R-t)\setminus E_\rho(R_1)} \rho\exp(2r_\rho)|w|^2dv \le \frac{1}{(R-R_1)^2}\int_{E_\rho(R)\setminus E_\rho(R_1)} (R-r_\rho(x))\rho\exp(2r_\rho)|w|^2dv,$$

we deduce that

$$\frac{2t}{(R-R_1)^2}\int_{E_\rho(R-t)\setminus E_\rho(R_1)} \rho\exp(2r_\rho)|w|^2dv \le \Big(\frac{1}{(R_1-R_0)^2}+\frac{2}{R_1-R_0}\Big)\int_{E_\rho(R_1)\setminus E_\rho(R_0)} \rho\exp(\sqrt{2}r_\rho)|w|^2dv + \frac{1}{(R-R_1)^2}\int_{E_\rho(R)\setminus E_\rho(R_1)} \rho\exp(\sqrt{2}r_\rho)|w|^2dv. \tag{5}$$

Observe that if $R_1 = R_0 + 1$, if $t = 1$, and if

$$g(R) = \int_{E_\rho(R)\setminus E_\rho(R_0+1)} \rho\exp(2r_\rho)|w|^2dv,$$

then the inequality (5) can be written as

$$g(R-1) \le C_1R^2 + \tfrac{1}{2}g(R),$$

where $C_1 = 2\int_{E_\rho(R_0+1)\setminus E_\rho(R_0)} \rho\exp(2r_\rho)|w|^2dv$ is independent of $R$. Iterating this inequality, we show that for any positive integer $k$, $R \ge 1$, and constant $C_2$,

$$\begin{aligned} g(R) &\le C_1\sum_{i=1}^{k}\frac{(R+i)^2}{2^{i-1}} + 2^{-k}g(R+k) \\ &\le C_1R^2\sum_{i=1}^{\infty}\frac{(1+i)^2}{2^{i-1}} + 2^{-k}g(R+k) \le C_2R^2 + 2^{-k}g(R+k). \end{aligned}$$

However, the previous estimate in (4) asserts that

$$\int_E \rho\exp(2\delta r_\rho)|w|^2dv \le C \quad \text{for any } \delta < 1.$$

This implies that

$$\begin{aligned} g(R+k) &= \int_{E_\rho(R+2k)\setminus E_\rho(R_0+1)} \rho\exp(2r_\rho)|w|^2 \\ &\le \exp(2(1-\delta)(R+k))\int_{E_\rho(R+k)\setminus E_\rho(R_0+1)} \rho\exp(2\delta r_\rho)|w|^2 \\ &\le C\exp(2(1-\delta)(R+k)). \end{aligned}$$

Hence $2^{-k}g(R+k) \to 0$ as $k \to \infty$ by choosing $2(1-\delta) < \ln 2$. This proves the estimate that $g(R) \le C_2 R^2$. Adjusting the constant, we have

$$\int_{E_\rho(R)} \rho \exp(2r_\rho)|w|^2 dv \le C_3 R^2 \quad \text{for all } R \ge R_0. \tag{6}$$

Using inequality (5) again and choosing $R_1 = R_0 + 1$, $R > 2R_1$ and $t = R/2$, we conclude that

$$R \int_{E_\rho(R/2)\setminus E_\rho(R_0+1)} \rho \exp(2r_\rho)|w|^2 dv \le C_4 R^2 + 2\int_{E_\rho(R)\setminus E_\rho(R_0+1)} \rho \exp(2r_\rho)|w|^2 dv.$$

However, applying the estimate (6) to the second term on the right side, we have

$$\int_{E_\rho(R/2)\setminus E_\rho(R_0+1)} \rho \exp(2r_\rho)|w|^2 dv \le C_5 R.$$

Therefore

$$\int_{E_\rho(R)} \rho \exp(2r_\rho)|w|^2 dv \le C_6 R \quad \text{for } R \ge R_0. \tag{7}$$

We are ready to prove the theorem by using (7). Setting $t = 2$ and $R_1 = R - 4$ in (5), we have

$$\begin{aligned}\int_{E_\rho(R-2)\setminus E_\rho(R-4)} &\rho \exp(2r_\rho)|w|^2 dv \\ &\le \Big(\frac{4}{(R-R_0-4)^2} + \frac{8}{R-R_0-4}\Big) \int_{E_\rho(R-4)\setminus E_\rho(R_0)} \rho \exp(2r_\rho)|w|^2 dv \\ &\qquad + \frac{1}{4}\int_{E_\rho(R)\setminus E_\rho(R-4)} \rho \exp(2r_\rho)|w|^2 dv.\end{aligned}$$

According to (7), the first term of the right side is bounded by a constant. Hence this inequality can be rewritten as

$$\int_{E_\rho(R-2)\setminus E_\rho(R-4)} \rho \exp(2r_\rho)|w|^2 dv \le C_7 + \frac{1}{2}\int_{E_\rho(R)\setminus E_\rho(R-4)} \rho \exp(2r_\rho)|w|^2 dv.$$

Iterating this inequality $k$ times, we have

$$\begin{aligned}\int_{E_\rho(R+2)\setminus E_\rho(R)} &\rho \exp(2r_\rho)|w|^2 dv \\ &\le C_7 \sum_{i=0}^{k-1} 2^{-i} + \frac{1}{2^k}\int_{E_\rho(R+2(k+1))\setminus E_\rho(R)} \rho \exp(2r_\rho)|w|^2 dv.\end{aligned}$$

Using (7) again, we conclude the second term is bounded by

$$\frac{1}{2^k}\int_{E_\rho(R+2(k+1))\setminus E_\rho(R)} \rho\exp(2r_\rho)|w|^2 dv \le \frac{C(R+2(k+1))}{2^k},$$

and the upper bound tends to zero as $k\to\infty$. Hence

$$\int_{E_\rho(R+2)\setminus E_\rho(R)} \rho\exp(2r_\rho)|w|^2 dv \le C_8$$

for some constant $C_8>0$ independent of $R$. □

**Corollary 2.2.** *Let $M$ be a complete manifold. If $E$ is an end of $M$ with positive $\lambda(E)$, where $\lambda(E)$ is equal to either $\lambda_{1,p}(E)$ or $\lambda_1(E)+K_p$, then for any smooth harmonic $p$-form $w$ satisfying*

$$\int_{E(2R)\setminus E(R)} \exp(-2\sqrt{\lambda(E)}r)|w|^2 dv = o(R),$$

*we have*

$$\int_{E(R)} \exp(2\sqrt{\lambda(E)}r)|w|^2 dv \le CR,$$

$$\int_{E(R+1)\setminus E(R)} \exp(2\sqrt{\lambda(E)}r)|w|^2 dv \le C,$$

*for all $R$ sufficiently large.*

## 3. Vanishing and finiteness theorems of harmonic $p$-forms

Let $w$ be a harmonic $p$-form on an $m$-dimensional manifold $M$. Then $w$ satisfies the Kato inequality [Wan and Xin 2004; Calderbank et al. 2000; Herzlich 2000]

$$|\nabla w|^2 \ge \frac{m-p+1}{m-p}\big|\nabla|w|\big|^2,$$

and equality holds if and only if there exists a 1-form $\alpha$ with $\alpha\wedge w=0$ such that

$$\nabla w = \alpha\otimes w - \frac{1}{m+1-p}\sum_{j=1}^{m}\theta^j\otimes(\theta^j\wedge i_{\alpha^\sharp}w), \tag{8}$$

where $\{\theta^1,\dots,\theta^m\}$ is an orthonormal basis for the cotangent bundle and $\alpha^\sharp$ is the vector dual to $\alpha$.

Now we are ready to prove vanishing and finiteness theorems for harmonic $p$-forms using the decay estimate Lemma 2.1 and the Kato inequality. To simplify our statement, we will assume the function $\rho$ is bounded in the rest of the section.

**Theorem 3.1.** *For $m \geq 3$, let $M^m$ be a complete noncompact Riemannian manifold with properties $(\mathcal{P}_{\rho_0})$ and $(\mathcal{P}_{p,\rho})$. Suppose the volume growth of $M$ satisfies*

$$\int_{B_\rho(2R)\setminus B_\rho(R)} \rho e^{-2(m-p-1)r_\rho} \leq c\,R$$

*and the curvature operator $K_p$ of $M^m$ has the lower bound*

$$K_p \geq -\frac{m-p}{m-p-1}\rho_0 \quad \text{on } M.$$

*Then any harmonic $p$-form $\omega$ in $L^d$ with $2 \leq d \leq 2(m-p-1)/(m-p-2)$ must either vanish or satisfy Equation (8).*

*Proof.* The theorem is obviously true if $d = 2$ as $(\mathcal{P}_{p,\rho})$ holds on $M$. So we assume $d > 2$. Let $\omega$ be a smooth harmonic $p$-form. By the Kato inequality, the Bochner formula becomes $|\omega|\Delta|\omega| \geq \frac{1}{m-p}|\nabla|\omega||^2 + K_p|\omega|^2$.

Let $g = |\omega|^{(m-p-1)/(m-p)}$. Then this inequality can be rewritten as

$$\Delta g \geq \frac{m-p-1}{m-p} K_p g. \tag{9}$$

We first show that $g$ satisfies the integral estimate $\int_{B_\rho(2R)\setminus B_\rho(R)} \rho g^2 \leq CR$. To see this, using the Schwarz inequality, we have

$$\int_{B_\rho(2R)\setminus B_\rho(R)} \rho g^2 \leq \Big(\int_{B_\rho(2R)\setminus B_\rho(R)} \rho \exp(2r_\rho)|\omega|^2\Big)^{m-p-1/(m-p)} \cdot \Big(\int_{B_\rho(2R)\setminus B_\rho(R)} \rho \exp(-2(m-p-1)r_\rho)\Big)^{1/(m-p)}. \tag{10}$$

By the volume growth condition, the second term on the right side satisfies

$$\int_{B_\rho(2R)\setminus B_\rho(R)} \rho \exp(-2(m-p-1)r_\rho) \leq c\,R.$$

On the other hand, for $a = d/(d-2)$, we have

$$\begin{aligned}\int_{B_\rho(2R)\setminus B_\rho(R)} \rho \exp(-2r_\rho)|\omega|^2 &\leq \Big(\int_{B_\rho(2R)\setminus B_\rho(R)} \rho^a \exp(-2\,a\,r_\rho)\Big)^{1/a} \cdot \Big(\int_{B_\rho(2R)\setminus B_\rho(R)} |\omega|^d\Big)^{2/d} \\ &\leq C\Big(\int_{B_\rho(2R)\setminus B_\rho(R)} \rho \exp(-2(m-p-1)r_\rho)\Big)^{1/a} \\ &\leq CR^{1/a},\end{aligned}$$

since $w$ is in $L^d$. Now according to Lemma 2.1, one has

$$\int_{B_\rho(2R)\setminus B_\rho(R)} \rho \exp(2r_\rho)|w|^2 \le CR.$$

Then (10) can be written as $\int_{B_\rho(2R)\setminus B_\rho(R)} \rho g^2 \le CR$.

To finish the proof of the theorem, note that for a cutoff function $\phi$, we have

$$\int_M \rho_0\phi^2 g^2 \le \int_M |\nabla(\phi g)|^2 = \int_M |\nabla\phi|^2 g^2 + 2\phi g\langle\nabla\phi, \nabla g\rangle + |\phi|^2|\nabla g|^2.$$

Also,

$$\int_M 2\phi g\langle\nabla\phi, \nabla g\rangle = \int_M g\langle\nabla\phi^2, \nabla g\rangle = -\int_M \phi^2|\nabla g|^2 - \int_M \phi^2 g\Delta g.$$

Therefore

$$\int_M \rho_0\phi^2 g^2 \le \int_M |\nabla\phi|^2 g^2 - \int_M \phi^2 g\Delta g,$$

or in other words,

$$\int_M \phi^2 g(\rho_0 g + \Delta g) \le \int_M |\nabla\phi|^2 g^2. \tag{11}$$

Let us now choose $\phi = \phi(r_\rho)$ to satisfy the properties that

$$\phi = \begin{cases} 1 & \text{on } B_\rho(R), \\ 0 & \text{on } M\setminus B_\rho(2R), \end{cases}$$

and

$$|\phi'(t)| \le 2\,R^{-1} \quad \text{for } R \le t \le 2R.$$

Then

$$\int_{B_\rho(2R)} \phi^2 g^2\Big(\rho_0 + \frac{m-p-1}{m-p}K_p\Big) \le CR^{-2}\int_{B_\rho(2R)\setminus B_\rho(R)} \rho g^2.$$

The right side of this tends to zero as $R\to\infty$. Since $K_p \ge -(m-p)\rho_0/(m-p-1)$, we conclude that $g$ must be identically zero, or

$$\Delta g = \frac{m-p-1}{m-p}K_p g. \tag{12}$$

This in particular implies that $w$ must satisfy (8). □

Next we prove the finiteness theorem for the space of harmonic $p$-forms if the curvature lower bound only holds on $M\setminus B_q(R_0)$, where $B_q(R_0)$ is a geodesic ball in $M$.

**Theorem 3.2.** *For $m \geq 3$, let $M^m$ be a complete noncompact Riemannian manifold with properties $(\mathcal{P}_{\rho_0})$ and $(\mathcal{P}_{p,\rho})$. Suppose the volume growth of $M$ satisfies*

$$\int_{B_\rho(2R)\setminus B_\rho(R)} \rho e^{-2(m-p-1)r_\rho} \leq c\,R$$

*and the curvature operator $K_p$ of $M^m$ has the lower bound*

$$K_p \geq -\Big(\frac{m-p}{m-p-1} - \epsilon\Big)\rho_0 \quad on\ M \setminus B_q(R_0).$$

*Then* $\dim H^p_d(M) \leq C(m, p, \epsilon, B_q(R_0))$ *with* $2 \leq d \leq 2(m-p-1)/(m-p-2)$.

*Proof.* According to the proof of the vanishing Theorem 3.1, for each $w \in H^p_d$, function $g = |w|^{2(m-p-1)/(m-p)}$ satisfies the estimate

$$\int_{B_\rho(2R)\setminus B_\rho(R)} g^2 \leq C\,R.$$

Also, the Bochner formula together with the curvature assumption implies that the function $g$ satisfies the differential inequality

$$\Delta g \geq \Big(\frac{m-p-1}{m-p}\epsilon - \rho_0\Big) g \quad \text{on } M\setminus B_q(R_0).$$

Let $\phi$ be a cutoff function satisfying

$$\phi = \begin{cases} 0 & \text{on } B_q(R_0), \\ 1 & \text{on } B_\rho(R)\setminus B_q(2R_0), \\ 0 & \text{on } M\setminus B_\rho(2R), \end{cases}$$
$$|\nabla\phi| \leq C R_0^{-1} \qquad \text{on } B_q(2R_0)\setminus B_q(R_0),$$
$$|\nabla\phi| \leq C\sqrt{\rho}\,R^{-1} \quad \text{on } B_\rho(2R)\setminus B_\rho(R)$$

for some constant $C > 0$.

Since

$$\int_M \rho_0\phi^2 g^2 \leq \int_M |\nabla(\phi g)|^2 = \int_M |\nabla\phi|^2 g^2 + 2\phi g\langle\nabla\phi, \nabla g\rangle + |\phi|^2|\nabla g|^2$$

and

$$\int_M 2\phi g\langle\nabla\phi, \nabla g\rangle = \int_M g\langle\nabla\phi^2, \nabla g\rangle = -\int_M \phi^2|\nabla g|^2 - \int_M \phi^2 g\,\Delta g,$$

we conclude

$$\int_M \rho_0\phi^2 g^2 \leq \int_M |\nabla\phi|^2 g^2 - \int_M \phi^2 g\,\Delta g.$$

Hence, we have

$$\int_{M\setminus B_q(R_0)} \phi^2 g(\rho_0 g + \Delta g) \le C R_0^{-2} \int_{B_q(2R_0)\setminus B_q(R_0)} g^2 + C R^{-2} \int_{B_p(2R)\setminus B_p(R)} \rho g^2.$$

Let $R \to \infty$. Then

$$\frac{m-p-1}{m-p}\epsilon \int_{M\setminus B_q(2R_0)} g^2 \le C R_0^{-2} \int_{B_q(2R_0)\setminus B_q(R_0)} g^2.$$

In particular,

$$\int_{B_q(3R_0)} g^2 \le \Big(1 + \frac{C}{\epsilon R_0^2}\Big) \int_{B_q(2R_0)} g^2. \tag{13}$$

It is now standard to conclude [Li 1980] that $\dim H_d^p \le C$. □

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