Pacific Journal of Mathematics

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Volume 242 No. 2 October 2009

PACIFIC JOURNAL OF MATHEMATICS
Vol. 242, No. 2, 2009

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To the memory of Professor Jeong-Seon Baek.

We provide two characterizations of helicoids in $\mathbb{S}^2 \times \mathbb{R}$ and in $\mathbb{H}^2 \times \mathbb{R}$. First, we show that any nontrivial ruled minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ and in $\mathbb{H}^2 \times \mathbb{R}$ is a part of a helicoid. Second, we also show that these surfaces can be characterized as the only surface with zero mean curvature with respect to both the Riemannian product metric and the Lorentzian product metric on $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$.

1. Introduction

The papers [Abresch and Rosenberg 2004] and [Rosenberg 2002] generated recent study of constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$, including minimal ones. The latter were studied in general 3-dimensional product manifolds $M^2 \times \mathbb{R}$ in [Elbert and Rosenberg 2008; Meeks and Rosenberg 2004; 2005], while minimal or constant mean curvature surfaces in the manifolds $\mathbb{S}^2 \times \mathbb{R}$ and in $\mathbb{H}^2 \times \mathbb{R}$ were given a detailed treatment in [Masal'tsev 2004; Nelli and Rosenberg 2002; Sa Earp and Toubiana 2005].

Here, we study ruled minimal surfaces (RMSs) in $\mathbb{S}^2 \times \mathbb{R}$ and in $\mathbb{H}^2 \times \mathbb{R}$. RMSs in three-dimensional space forms are well studied. The classical Catalan theorem states that any nontrivial RMS in $\mathbb{E}^3$ is a part of a complete helicoid; Lawson [1970] showed that any nontrivial RMS in $\mathbb{S}^3$ is a part of a spherical helicoid of the form $(\cos kx \cos y, \sin kx \cos y, \cos x \sin y, \sin x \sin y)$ for some $k > 0$, while do Carmo and Dajczer [1983] showed that any nontrivial RMS in $\mathbb{H}^3$ is a part of a helicoid of the form $(\cosh kx \cosh y, \sinh kx \cosh y, \cos x \sinh y, \sin x \sinh y)$.

For RMSs in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$, Masal'tsev [2004] showed that any nontrivial *complete* RMS is a helicoid (see Section 4.2 for a discussion on the definition of helicoids of various types and Remark 4.6 for case of helicoids that was not given in [Masal'tsev 2004]). The completeness assumption was used in an essential way

MSC2000: 53A35.

Keywords: helicoid, ruled surface, minimal surface, spacelike maximal surface.

Kim and Yang were supported in part by Korea University grants. Koh was supported by KRF-2008-313-C00069.

in the proof in [Masal'tsev 2004], in order to simplify the differential equations. Without the completeness assumption, we show in the first part of this paper that any RMS is a part of a complete helicoid. Our proof is intrinsic in that — unlike the proof in [Masal'tsev 2004] — we do not assume $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ is embedded in $\mathbb{R}^4$. Instead we use a Jacobi field argument, which we think is new in studying RMSs. Hence our computations and results are local.

A less well-known aspect of the helicoid in $\mathbb{E}^3$ is that it is the only nontrivial surface whose mean curvature is everywhere zero with respect to both the standard Riemannian and the standard Lorentzian metric [Kim et al. 2009b; Kobayashi 1983]. We show that the same is true for the helicoid in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$. The idea of considering both the Riemannian and the Lorentzian metric on the same space is not new; see for example [Albujer and Alías 2009; Alías and Palmer 2001]. The idea of this paper applies as well to RMSs in the Riemannian 3-dimensional Heisenberg group Nil_3 to give similar results [Kim et al. 2009a].

2. Ruled surfaces in $M \times \mathbb{R}$

Let M be a 2-dimensional Riemannian manifold. We consider the Riemannian product manifold $M \times \mathbb{R}$, with Levi-Civita connection ∇.

2.1. ***A parametrization of ruled surfaces in $M \times \mathbb{R}$.*** Let $\Sigma \subset M \times \mathbb{R}$ be a surface ruled by geodesics, and let $p \in \Sigma$ be a point such that $T_p\Sigma$ is transverse to the height direction. Since a curve $u(t) = (u_1(t), u_2(t))$ in $M \times \mathbb{R}$ is a geodesic if and only if $u_1(t)$ is a geodesic in M and $u_2(t)$ is a geodesic in $\mathbb{R}$, Σ has a parametrization in a neighborhood of p given by

$$\begin{aligned} X(s,t) &= (\varphi(s,t), g(s)t + h(s)) \subset M \times \mathbb{R}, \\ \varphi(s,t) &= \exp_{\alpha(s)}(t\mathbf{x}(s)) \end{aligned} \tag{1}$$

for some functions $g(s)$ and $h(s)$, where $\alpha(s)$ is a unit speed curve in M, and $\mathbf{x}(s)$ is a vector field along α consisting of unit vectors tangent to M, with $\langle \mathbf{x}(s), \alpha'(s)\rangle \equiv 0$. We say that a ruled surface Σ is *horizontally ruled* if $g(s) = 0$ in this parametrization, that is, if the ruling geodesics are orthogonal to the height direction.

The definition of the curvature function implies this:

Lemma 2.2. *Let $\kappa(s)$ be the curvature function of the curve $\alpha(s)$. Then*

$$\mathbf{x}'(s) = \kappa(s)\alpha'(s).$$

For the parametrization $\varphi(s,t)$ on M, let $\mathbf{e}_1(s,t) = d\varphi\left(\frac{\partial}{\partial t}\right)$, and let $\mathbf{e}_2(s,t)$ be the parallel translation of $\alpha'(s)$ along the geodesic $\gamma_s(t) := \varphi(s,t)$. Then $\mathbf{e}_1, \mathbf{e}_2$ is an orthonormal frame field tangent to M along $\varphi(s,t)$, and one has

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1 = 0 \quad \text{and} \quad \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0.$$

By abuse of the notation we will consider $\mathbf{e}_1$ and $\mathbf{e}_2$ as vector fields in $M\times\mathbb{R}$.

2.3. ***Jacobi fields.*** Set $V(s,t) := d\varphi(\partial/\partial s)|_{(s,t)}$. Then $V(s,\cdot)$ is a Jacobi field along the geodesic $\gamma_s(t)$ with

$$V(s,0)=\alpha'(s)=\mathbf{e}_2(s,0). \tag{2}$$

Since $\nabla_{d\varphi(\partial/\partial t)}d\varphi(\partial/\partial s)=\nabla_{d\varphi(\partial/\partial s)}d\varphi(\partial/\partial t)$, one has by Lemma 2.2

$$\begin{aligned}\nabla_{\mathbf{e}_1}V(s,0)&=\nabla_{d\varphi(\partial/\partial t)}d\varphi\left(\tfrac{\partial}{\partial s}\right)\Big|_{(s,0)}\\&=\nabla_{d\varphi(\partial/\partial s)}d\varphi\left(\tfrac{\partial}{\partial t}\right)\Big|_{(s,0)}=\mathbf{x}'(s)=\kappa(s)\mathbf{e}_2(s,0).\end{aligned}\tag{3}$$

3. Ruled minimal surfaces in $\mathbb{S}^2\times\mathbb{R}$

Let $\Sigma\subset\mathbb{S}^2\times\mathbb{R}$ be a ruled surface with the parametrization (1).

3.1. ***Ruled surfaces in $\mathbb{S}^2\times\mathbb{R}$.*** Since the curvature of $\mathbb{S}^2$ is (assumed to be) 1, the Jacobi field along $\gamma_s(t)$ with the initial conditions (2) and (3) is computed as

$$d\varphi\left(\tfrac{\partial}{\partial s}\right)=V(s,t)=(\kappa(s)\sin t+\cos t)\mathbf{e}_2(s,t).$$

Now we compute $\nabla_{\mathbf{e}_2}\mathbf{e}_1$ and $\nabla_{\mathbf{e}_2}\mathbf{e}_2$. Since

$$\begin{aligned}\nabla_{d\varphi(\partial/\partial t)}d\varphi\left(\tfrac{\partial}{\partial s}\right)&=\nabla_{\mathbf{e}_1}(\kappa(s)\sin t+\cos t)\mathbf{e}_2\\&=(\kappa(s)\cos t-\sin t)\mathbf{e}_2+(\kappa(s)\sin t+\cos t)\nabla_{\mathbf{e}_1}\mathbf{e}_2\\&=(\kappa(s)\cos t-\sin t)\mathbf{e}_2,\\\nabla_{d\varphi(\partial/\partial s)}d\varphi\left(\tfrac{\partial}{\partial t}\right)&=\nabla_{(\kappa(s)\sin t+\cos t)\mathbf{e}_2}\mathbf{e}_1\\&=(\kappa(s)\sin t+\cos t)\nabla_{\mathbf{e}_2}\mathbf{e}_1,\\\nabla_{d\varphi(\partial/\partial t)}d\varphi\left(\tfrac{\partial}{\partial s}\right)&=\nabla_{d\varphi(\partial/\partial s)}d\varphi\left(\tfrac{\partial}{\partial t}\right),\end{aligned}$$

one has

$$\nabla_{\mathbf{e}_2}\mathbf{e}_1=\frac{\kappa(s)\cos t-\sin t}{\kappa(s)\sin t+\cos t}\mathbf{e}_2.$$

Since $\langle\nabla_{\mathbf{e}_2}\mathbf{e}_2,\mathbf{e}_1\rangle=-\langle\mathbf{e}_2,\nabla_{\mathbf{e}_2}\mathbf{e}_1\rangle$ and $\langle\nabla_{\mathbf{e}_2}\mathbf{e}_2,\mathbf{e}_2\rangle=0$, one also has

$$\nabla_{\mathbf{e}_2}\mathbf{e}_2=-\frac{\kappa(s)\cos t-\sin t}{\kappa(s)\sin t+\cos t}\mathbf{e}_1. \tag{4}$$

Let $\mathbf{e}_3$ be a unit vector field on Σ tangent to $\mathbb{R}$. Then, for the parametrization X in (1), one has

$$\begin{aligned} X_s &= d\varphi\left(\tfrac{\partial}{\partial s}\right) + (g'(s)t + h'(s))\mathbf{e}_3 \\ &= V + (g'(s)t + h'(s))\mathbf{e}_3 \\ &= (\kappa(s)\sin t + \cos t)\mathbf{e}_2 + (g'(s)t + h'(s))\mathbf{e}_3, \\ X_t &= d\varphi\left(\tfrac{\partial}{\partial t}\right) + g(s)\mathbf{e}_3 = \mathbf{e}_1 + g(s)\mathbf{e}_3. \end{aligned}$$

The coefficients of the first fundamental form are

$$\begin{aligned} E &= \langle X_s, X_s\rangle = (\kappa(s)\sin t + \cos t)^2 + (g'(s)t + h'(s))^2, \\ F &= \langle X_s, X_t\rangle = g(s)(g'(s)t + h'(s)), \\ G &= \langle X_t, X_t\rangle = 1 + g(s)^2, \end{aligned}$$

and the unit normal vector field $\mathbf{n}$ of Σ is

$$\mathbf{n} = \tfrac{1}{W}\big(g(s)(\kappa(s)\sin t + \cos t)\mathbf{e}_1 + (g'(s)t + h'(s))\mathbf{e}_2 - (\kappa(s)\sin t + \cos t)\mathbf{e}_3\big),$$

where $W = ((1 + g(s)^2)(\kappa(s)\sin t + \cos t)^2 + (g'(s)t + h'(s))^2)^{1/2}$. By abuse of notation, let ∇ be the covariant differentiation in $\mathbb{S}^2 \times \mathbb{R}$. Since $\nabla_{\mathbf{e}_1}\mathbf{e}_3 = \nabla_{\mathbf{e}_2}\mathbf{e}_3 = 0$ one has by (4)

$$\begin{aligned} \nabla_{X_s} X_s &= \kappa'(s)\sin t\,\mathbf{e}_2 + (g''(s)t + h''(s))\mathbf{e}_3 + (\kappa(s)\sin t + \cos t)^2\nabla_{\mathbf{e}_2}\mathbf{e}_2 \\ &= -(\kappa(s)\sin t + \cos t)(\kappa(s)\cos t - \sin t)\mathbf{e}_1 + \kappa'(s)\sin t\,\mathbf{e}_2 \\ &\qquad\qquad + (g''(s)t + h''(s))\mathbf{e}_3, \\ \nabla_{X_t} X_s &= (\kappa(s)\cos t - \sin t)\mathbf{e}_2 + g'(s)\mathbf{e}_3, \\ \nabla_{X_t} X_t &= 0. \end{aligned}$$

The coefficients of the second fundamental form are

$$\begin{aligned} l &= \langle \nabla_{X_s} X_s, \mathbf{n}\rangle \\ &= \tfrac{1}{W}\big(-g(s)(\kappa(s)\sin t + \cos t)^2(\kappa(s)\cos t - \sin t) \\ &\qquad + \kappa'(s)\sin t\,(g'(s)t + h'(s)) - (\kappa(s)\sin t + \cos t)(g''(s)t + h''(s))\big), \\ m &= \langle \nabla_{X_t} X_s, \mathbf{n}\rangle \\ &= \tfrac{1}{W}\big((\kappa(s)\cos t - \sin t)(g'(s)t + h'(s)) - (\kappa(s)\sin t + \cos t)g'(s)\big), \\ n &= \langle \nabla_{X_t} X_t, \mathbf{n}\rangle = 0. \end{aligned}$$

3.2. ***Ruled minimal surfaces in*** $\mathbb{S}^2 \times \mathbb{R}$**.** In this section, we are to find all not necessarily complete RMSs in $\mathbb{S}^2 \times \mathbb{R}$. We first note that the horizontal level surface $\mathbb{S}^2 \times \{z_0\}$ for $z_0 \in \mathbb{R}$ is a RMS in $\mathbb{S}^2 \times \mathbb{R}$. Other trivial RMS are those of the form $\gamma \times \mathbb{R}$, where γ is a geodesic in $\mathbb{S}^2$.

Lemma 3.3. *Let Σ be a ruled minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ whose parametrization is given by (1). Then $g(s) = 0$, that is, Σ is horizontally ruled and $h(s)$ is a linear function. If $h(s)$ is constant, Σ is a part of the horizontal level surface $\mathbb{S}^2 \times \{z_0\}$. If $h(s)$ is nonconstant, the curvature $\kappa(s)$ of the base curve $\alpha(s)$ is constant.*

Proof. We may assume that $\alpha(s)$ is a nonconstant curve. Recall the formula

$$H = \frac{1}{2}\frac{Gl - 2Fm + En}{EG - F^2},$$

where H is the mean curvature. Since Σ is minimal, we have $Gl - 2Fm + En = 0$. A computation gives

$$A(s)\cos^3 t + B(s)\sin^3 t + \sum_{k=0}^{2} A_k(s)t^k \cos t + \sum_{k=0}^{2} B_k(s)t^k \sin t = 0, \tag{5}$$

where

$$\begin{aligned}
A(s) &= -g(1+g^2)(\kappa^3 - 3\kappa),\\
B(s) &= -g(1+g^2)(3\kappa^2 - 1),\\
A_0(s) &= g(1+g^2)(\kappa^3 - 2\kappa) + (1+g^2)h'' + 2g\kappa h'^2 - 2gg'h',\\
B_0(s) &= g(1+g^2)(2\kappa^2 - 1) - \kappa'(1+g^2)h' + (1+g^2)h''\kappa - 2gh'^2 - 2gg'\kappa h',\\
A_1(s) &= (1+g^2)g'' + 4gg'h'\kappa - 2gg'^2,\\
B_1(s) &= -\kappa'(1+g^2)g' + (1+g^2)g''\kappa - 4gg'h' - 2gg'^2\kappa,\\
A_2(s) &= 2g\kappa g'^2,\\
B_2(s) &= -2gg'^2.
\end{aligned}$$

Since the functions $\cos^3 t$, $\sin^3 t$, $t^2\cos t$, $t^2 \sin t$, $t\cos t$, $t\sin t$, $\cos t$, $\sin t$ are linearly independent, all the coefficients above are zero. From

$$\begin{aligned}
A(s) &= -g(s)(1+g(s)^2)(\kappa(s)^3 - 3\kappa(s)) = 0,\\
B(s) &= -g(s)(1+g(s)^2)(3\kappa(s)^2 - 1) = 0,
\end{aligned}$$

one has $g(s) = 0$, and (5) reduces to

$$h''(s)\cos t + (h''(s)\kappa(s) - \kappa'(s)h'(s))\sin t = 0.$$

Finally this equation gives $h''(s) = 0$ and $\kappa'(s)h'(s) = 0$. Therefore $h(s) = as + b$ for some constant a and b. If $a = 0$, it is clear that $\Sigma \subset \mathbb{S}^2 \times \{z_0\}$ for some $z_0 \in \mathbb{R}$. If $a \neq 0$, one has $\kappa'(s) = 0$ from $\kappa'(s)h'(s) = 0$. □

If a ruled minimal surface Σ coincides with a helicoid or $\mathbb{S}^2 \times \{z_0\}$ in an open set $U \subset \Sigma$, then Σ coincides with the helicoid or $\mathbb{S}^2 \times \{z_0\}$ along all the ruling

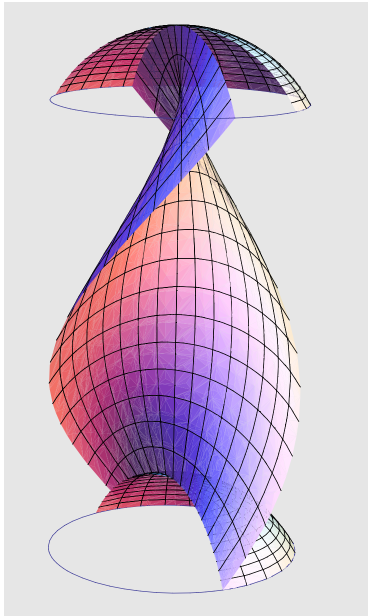

Figure 1. Helicoid in $S^2 \times \mathbb{R}$. The lines of latitude in the surface represent the rulings by radial geodesics on S^2.

geodesics through U. By Lemma 3.3, the condition that a ruled minimal surface Σ coincides with a helicoid along the base curve is an open and closed condition.

Theorem 3.4. *A ruled minimal surface in* $\mathbb{S}^2 \times \mathbb{R}$ *is either a part of a cylinder* $\gamma \times \mathbb{R}$ *over a geodesic* γ, *a horizontal level surface* $\mathbb{S}^2 \times \{z_0\}$ *or a helicoid.*

Proof. It suffices to consider the case when the curvature $\kappa(s)$ of the base curve $\alpha(s)$ is constant. Any constant curvature curve in $\mathbb{S}^2$ is a circle, so let p be its center. Then Σ is a helicoid in $\mathbb{S}^2 \times \mathbb{R}$ with axis $\{p\} \times \mathbb{R}$. □

Since a totally geodesic surface is an RMS, we have the following:

Corollary 3.5. *A totally geodesic surface in* $\mathbb{S}^2 \times \mathbb{R}$ *is either a part of a cylinder* $\gamma \times \mathbb{R}$ *over a geodesic* γ *or a horizontal level surface* $\mathbb{S}^2 \times \{z_0\}$.

Remark 3.6. Take a parameter s such that $h(s) = bs$ for a constant b. Let $p \in \mathbb{S}^2$ be the center of the circle $\alpha(s)$ in Theorem 3.4. Considering $\mathbb{S}^2 \times \mathbb{R}$ as the hypersurface in $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4) given by the equation $x_1^2 + x_2^2 + x_3^2 = 1$ and chosen so that $p = (0, 0, \pm 1, 0)$, one obtains the parametrization

$$x_1(s,t) = \cos s \cos t, \quad x_2(s,t) = \sin s \cos t, \quad x_3(s,t) = \sin t, \quad x_4(s,t) = bs.$$

of the ruled minimal surface $\Sigma \subset \mathbb{S}^2 \times \mathbb{R}$ given in [Masal'tsev 2004].

4. Ruled minimal surfaces in $\mathbb{H}^2\times\mathbb{R}$

We consider in this section RMSs in $\mathbb{H}^2\times\mathbb{R}$. The computations are almost the same as those of the previous section, but we include them here for the sake of completeness. Let $\Sigma\subset\mathbb{H}^2\times\mathbb{R}$ be a ruled surface.

4.1. ***Ruled surfaces in*** $\mathbb{H}^2\times\mathbb{R}$**.** Since the curvature of $\mathbb{H}^2$ is (assumed to be) -1, the Jacobi field along $\gamma_s(t)$ with the initial conditions (2) and (3) is computed as

$$d\varphi\big(\tfrac{\partial}{\partial s}\big)=V(s,t)=(\kappa(s)\sinh t+\cosh t)\mathbf{e}_2(s,t).$$

Now we compute $\nabla_{\mathbf{e}_2}\mathbf{e}_1$ and $\nabla_{\mathbf{e}_2}\mathbf{e}_2$. Since

$$\begin{aligned}
&\nabla_{d\varphi(\partial/\partial t)}d\varphi\big(\tfrac{\partial}{\partial s}\big)=\nabla_{\mathbf{e}_1}(\kappa(s)\sinh t+\cosh t)\mathbf{e}_2=(\kappa(s)\cosh t+\sinh t)\mathbf{e}_2,\\
&\nabla_{d\varphi(\partial/\partial s)}d\varphi\big(\tfrac{\partial}{\partial t}\big)=\nabla_{(\kappa(s)\sinh t+\cosh t)\mathbf{e}_2}\mathbf{e}_1=(\kappa(s)\sinh t+\cosh t)\nabla_{\mathbf{e}_2}\mathbf{e}_1,\\
&\nabla_{d\varphi(\partial/\partial t)}d\varphi\big(\tfrac{\partial}{\partial s}\big)=\nabla_{d\varphi(\partial/\partial s)}d\varphi\big(\tfrac{\partial}{\partial t}\big),
\end{aligned}$$

one has

$$\nabla_{\mathbf{e}_2}\mathbf{e}_1=\frac{\kappa(s)\cosh t+\sinh t}{\kappa(s)\sinh t+\cosh t}\mathbf{e}_2.$$

Since $\langle\nabla_{\mathbf{e}_2}\mathbf{e}_2,\mathbf{e}_1\rangle=-\langle\mathbf{e}_2,\nabla_{\mathbf{e}_2}\mathbf{e}_1\rangle$ and $\langle\nabla_{\mathbf{e}_2}\mathbf{e}_2,\mathbf{e}_2\rangle=0$, one also has

$$\nabla_{\mathbf{e}_2}\mathbf{e}_2=-\frac{\kappa(s)\cosh t+\sinh t}{\kappa(s)\sinh t+\cosh t}\mathbf{e}_1. \tag{6}$$

Let $\mathbf{e}_3$ be a unit vector field on Σ tangent to $\mathbb{R}$. Then, for the parametrization X in (1), one has

$$\begin{aligned}
X_s&=d\varphi\big(\tfrac{\partial}{\partial s}\big)+(g'(s)t+h'(s))\mathbf{e}_3=V+(g'(s)t+h'(s))\mathbf{e}_3\\
&=(\kappa(s)\sinh t+\cosh t)\mathbf{e}_2+(g'(s)t+h'(s))\mathbf{e}_3,\\
X_t&=d\varphi\big(\tfrac{\partial}{\partial t}\big)+g(s)\mathbf{e}_3=\mathbf{e}_1+g(s)\mathbf{e}_3.
\end{aligned}$$

The coefficients of the first fundamental form are

$$\begin{aligned}
E&=\langle X_s,X_s\rangle=(\kappa(s)\sinh t+\cosh t)^2+(g'(s)t+h'(s))^2,\\
F&=\langle X_s,X_t\rangle=g(s)(g'(s)t+h'(s)),\\
G&=\langle X_t,X_t\rangle=1+g(s)^2,
\end{aligned}$$

and the unit normal vector field $\mathbf{n}$ of Σ is

$$\mathbf{n}=\tfrac{1}{W}\big(g(s)(\kappa(s)\sinh t+\cosh t)\mathbf{e}_1+(g'(s)t+h'(s))\mathbf{e}_2-(\kappa(s)\sinh t+\cosh t)\mathbf{e}_3\big),$$

where $W=((1+g(s)^2)(\kappa(s)\sinh t+\cosh t)^2+(g'(s)t+h'(s))^2)^{1/2}$.

By abuse of notation once again let ∇ be the covariant differentiation in $\mathbb{H}^2 \times \mathbb{R}$. Since $\nabla_{\mathbf{e}_1}\mathbf{e}_3 = \nabla_{\mathbf{e}_2}\mathbf{e}_3 = 0$, one has by (6)

$$\begin{aligned}
\nabla_{X_s} X_s &= \kappa'(s)\sinh t\,\mathbf{e}_2 + (g''(s)t + h''(s))\mathbf{e}_3 + (\kappa(s)\sinh t + \cosh t)^2 \nabla_{\mathbf{e}_2}\mathbf{e}_2 \\
&= -(\kappa(s)\sinh t + \cosh t)(\kappa(s)\cosh t + \sinh t)\mathbf{e}_1 + \kappa'(s)\sinh t\,\mathbf{e}_2 \\
&\qquad\qquad + (g''(s)t + h''(s))\mathbf{e}_3, \\
\nabla_{X_t} X_s &= (\kappa(s)\cosh t + \sinh t)\mathbf{e}_2 + g'(s)\mathbf{e}_3, \\
\nabla_{X_t} X_t &= 0.
\end{aligned}$$

The coefficients of the second fundamental form are

$$\begin{aligned}
l &= \langle \nabla_{X_s} X_s, \mathbf{n} \rangle \\
&= \tfrac{1}{W}\big(-g(s)(\kappa(s)\sinh t + \cosh t)^2(\kappa(s)\cosh t + \sinh t) \\
&\qquad + \kappa'(s)\sinh t\,(g'(s)t + h'(s)) - (\kappa(s)\sinh t + \cosh t)(g''(s)t + h''(s))\big), \\
m &= \langle \nabla_{X_t} X_s, \mathbf{n} \rangle \\
&= \tfrac{1}{W}\big((\kappa(s)\cosh t + \sinh t)(g'(s)t + h'(s)) - (\kappa(s)\sinh t + \cosh t)g'(s)\big), \\
n &= \langle \nabla_{X_t} X_t, \mathbf{n} \rangle = 0.
\end{aligned}$$

4.2. *Ruled minimal surfaces in* $\mathbb{H}^2 \times \mathbb{R}$. Here we will find all not necessarily complete RMSs in $\mathbb{H}^2 \times \mathbb{R}$. The horizontal level surface $\mathbb{H}^2 \times \{z_0\}$ for $z_0 \in \mathbb{R}$ is an RMS in $\mathbb{H}^2 \times \mathbb{R}$. Other obvious RMSs are those of the form $\gamma \times \mathbb{R}$, where γ is a geodesic in $\mathbb{H}^2$.

Lemma 4.3. *Let Σ be a ruled minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ whose parametrization is given by (1). Then $g(s) = 0$, that is, Σ is horizontally ruled and $h(s)$ is a linear function. If $h(s)$ is constant, Σ is a part of the horizontal level surface $\mathbb{H}^2 \times \{z_0\}$. If $h(s)$ is nonconstant, the curvature $\kappa(s)$ of the base curve $\alpha(s)$ is constant.*

Proof. We may assume that $\alpha(s)$ is a nonconstant curve. Since Σ is minimal, we have $Gl - 2Fm + En = 0$. A computation gives

$$C(s)\cosh^3 t + D(s)\sinh^3 t + \sum_{k=0}^{2} C_k(s)t^k \cosh t + \sum_{k=0}^{2} D_k(s)t^k \sinh t = 0, \tag{7}$$

where

$$\begin{aligned}
C(s) &= -g(1+g^2)(\kappa^3 + 3\kappa), \\
D(s) &= -g(1+g^2)(3\kappa^2 + 1), \\
C_0(s) &= g(1+g^2)(\kappa^3 + 2\kappa) - (1+g^2)h'' - 2g\kappa h'^2 + 2gg'h', \\
D_0(s) &= -g(1+g^2)(2\kappa^2 + 1) + \kappa'(1+g^2)h' - (1+g^2)h''\kappa - 2gh'^2 + 2gg'\kappa h', \\
C_1(s) &= -(1+g^2)g'' - 4gg'h'\kappa + 2gg'^2,
\end{aligned}$$

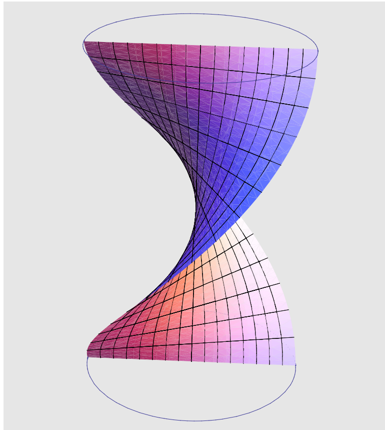

Figure 2. Helicoid in $\mathbb{H}^2 \times \mathbb{R}$ of elliptic type. $\mathbb{H}^2$ is shown in the Poincaré disk model.

$$D_1(s) = \kappa'(1+g^2)g' - (1+g^2)g''\kappa - 4gg'h' + 2gg'^2\kappa,$$
$$C_2(s) = -2g\kappa g'^2,$$
$$D_2(s) = -2gg'^2.$$

Since the functions $\cosh^3 t$, $\sinh^3 t$, $t^2 \cosh t$, $t^2 \sinh t$, $t \cosh t$, $t \sinh t$, $\cosh t$, $\sinh t$ are linearly independent, all the above coefficients are zero. From $D(s) = 0$ one has $g(s) = 0$, and (7) reduces to

$$-h''(s) \cosh t - (h''(s)\kappa(s) - \kappa'(s)h'(s)) \sinh t = 0.$$

Hence finally one has $h''(s) = 0$ and $\kappa'(s)h'(s) = 0$. Therefore $h(s) = as + b$ for some constant a and b. If $a = 0$, it is clear that $\Sigma \subset \mathbb{H}^2 \times \{z_0\}$ for some $z_0 \in \mathbb{R}$. If $a \neq 0$, one has $\kappa'(s) = 0$ from $\kappa'(s)h'(s) = 0$. □

Any nonconstant curve in $\mathbb{H}^2$ with constant curvature is a geodesic circle, a horocircle, an equidistance curve, or a geodesic. If the $\alpha(s)$ of Lemma 4.3 is a geodesic circle, that is, if $\kappa(s) > 1$, let $p \in \mathbb{H}^2$ be its center, and let $\mathrm{Rot}(p)$ denote a rotation on $\mathbb{H}^2$ with respect to p. Then Lemma 4.3 shows that these RMSs are invariant under a screw motion $\mathrm{Rot}(p) \times \mathrm{Trans} : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$, where $\mathrm{Trans} : \mathbb{R} \to \mathbb{R}$ denotes a translation, and hence are helicoids in this case.

On the other hand, if $0 \leq \kappa(s) \leq 1$, that is, if the curve $\alpha(s)$ is a geodesic, an equidistance curve or a horocircle, the surface Σ is not invariant under a screw motion. However, if we extend the classical notion of helicoid to include surfaces invariant under $\mathrm{Iso}^+ \times \mathrm{Trans} : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}$, where $\mathrm{Iso}^+ : \mathbb{H}^2 \to \mathbb{H}^2$ is an orientation preserving isometry, then Σ can be regarded as a helicoid in this generalized sense. The similarity of the parametrization (1) of these surfaces to

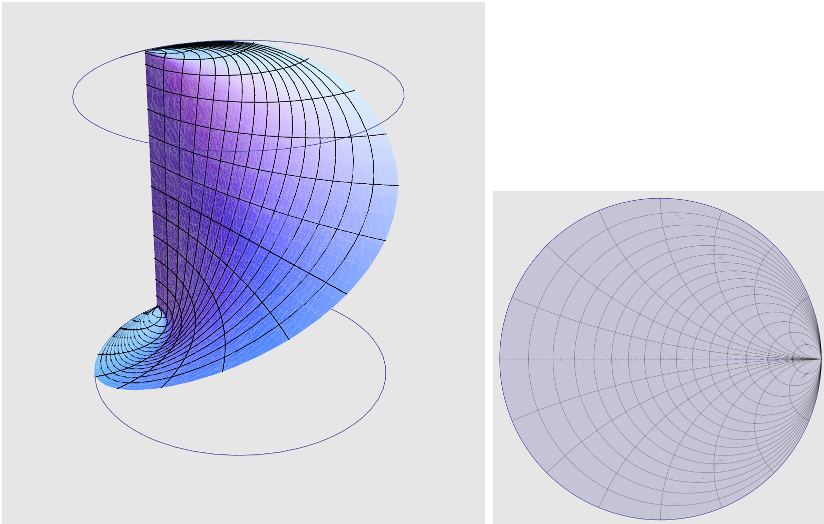

Figure 3. Helicoid in $\mathbb{H}^2 \times \mathbb{R}$ of parabolic type and its top view. $\mathbb{H}^2$ is shown in the Poincaré disk model. The height is given in the same scale as the diameter of $\mathbb{H}^2$.

the classical helicoid (that is, the case when the base curve $\alpha(s)$ is a geodesic circle) backs up this viewpoint. Following [Casson and Bleiler 1988], let us call a (complete) RMS Σ a helicoid of *elliptic type* if $\alpha(s)$ is a geodesic circle, a helicoid of *parabolic type* if $\alpha(s)$ is a horocircle, and a helicoid of *hyperbolic type* if $\alpha(s)$ is a geodesic or an equidistance curve. As in the case of $\mathbb{S}^2 \times \mathbb{R}$, we have the following.

Theorem 4.4. *A ruled minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ is either a part of a cylinder $\gamma \times \mathbb{R}$ over a geodesic γ, a horizontal level surface $\mathbb{H}^2 \times \{z_0\}$, or a helicoid.*

Corollary 4.5. *A totally geodesic surface in $\mathbb{H}^2 \times \mathbb{R}$ is either a part of a cylinder $\gamma \times \mathbb{R}$ over a geodesic γ, or a horizontal level surface $\mathbb{H}^2 \times \{z_0\}$.*

Remark 4.6. By taking the hyperboloid model of $\mathbb{H}^2$, let us consider $\mathbb{H}^2 \times \mathbb{R}$ as the hypersurface in the Lorentz–Minkowski space $\mathbb{R}^{3,1}$ given by the equation $-x_0^2 + x_1^2 + x_2^2 = -1$ with $x_0 > 0$. By for example [Dillen and Kühnel 1999], the orientation preserving isometries of $\mathbb{H}^2$ of elliptic type, of parabolic type and of hyperbolic type are given respectively as follows:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos s & \sin s \\ 0 & -\sin s & \cos s \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

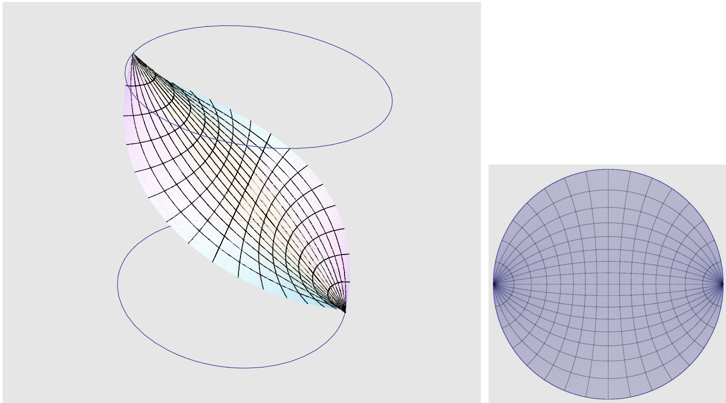

Figure 4. Helicoid in $\mathbb{H}^2 \times \mathbb{R}$ of hyperbolic type and its top view. $\mathbb{H}^2$ shown in the Poincaré disk model. The height is given in the same scale as the diameter of $\mathbb{H}^2$.

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} 1+s^2/2 & -s^2/2 & s \\ s^2/2 & 1-s^2/2 & s \\ s & -s & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \cosh s & \sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

Hence we have the following parametrization for helicoids of each type:

(1) elliptic type

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos s & \sin s & 0 \\ 0 & -\sin s & \cos s & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh t \\ \sinh t \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ bs \end{pmatrix} = \begin{pmatrix} \cosh t \\ \sinh t \cos s \\ -\sinh t \sin s \\ bs \end{pmatrix}$$

(2) parabolic type

$$\begin{pmatrix} 1+s^2/2 & -s^2/2 & s & 0 \\ s^2/2 & 1-s^2/2 & s & 0 \\ s & -s & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh t \\ \sinh t \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ bs \end{pmatrix} = \begin{pmatrix} \cosh t + s^2 e^{-t}/2 \\ \sinh t + s^2 e^{-t}/2 \\ s e^{-t} \\ bs \end{pmatrix}$$

(3) hyperbolic type

$$\begin{pmatrix} \cosh s & \sinh s & 0 & 0 \\ \sinh s & \cosh s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cosh t \\ 0 \\ \sinh t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ bs \end{pmatrix} = \begin{pmatrix} \cosh s \cosh t \\ \sinh s \cosh t \\ \sinh t \\ bs \end{pmatrix}$$

These give the parametrizations of helicoids in [Masal'tsev 2004]. In fact, that of parabolic type is missed there, as the author omits the case when the rotation axis vector is lightlike.

5. Minimal and maximal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

We show in this section that the helicoid in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ is the only nontrivial surface whose mean curvature is everywhere zero with respect to both the standard Riemannian product metric and the standard Lorentzian product metric. For both cases, we write the inner product as $\langle \cdot , \cdot \rangle$ and the Levi-Civita connection as ∇.

5.1. ***Riemannian and Lorentzian metrics on $M \times \mathbb{R}$.*** Suppose (M, ds^2) is a 2-dimensional Riemannian manifold. Consider the Riemannian metric $g = ds^2 + dz^2$ and the Lorentzian metric $h = ds^2 - dz^2$ on the product space $M \times \mathbb{R}$, where z is a canonical coordinate and dz^2 is the canonical metric on $\mathbb{R}$. A surface in the Lorentzian product space is called *spacelike* if its induced metric is Riemannian. A spacelike surface is called *maximal* if its mean curvature is zero everywhere.

5.2. ***A frame field on $M \times \mathbb{R}$.*** Take a frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \partial/\partial z$ on $M \times \mathbb{R}$ so that $\nabla_{\mathbf{e}_3}\mathbf{e}_1 = \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0$ and so that $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthonormal vectors tangent to $M \times \{c\}$ for every $c \in \mathbb{R}$. Then $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form an orthonormal frame with respect to both g and h. A computation gives

$$\begin{aligned} &\nabla_{\mathbf{e}_1}\mathbf{e}_1 = a\mathbf{e}_2, && \nabla_{\mathbf{e}_1}\mathbf{e}_2 = -a\mathbf{e}_1, && \nabla_{\mathbf{e}_1}\mathbf{e}_3 = 0, \\ &\nabla_{\mathbf{e}_2}\mathbf{e}_1 = -b\mathbf{e}_2, && \nabla_{\mathbf{e}_2}\mathbf{e}_2 = b\mathbf{e}_1, && \nabla_{\mathbf{e}_2}\mathbf{e}_3 = 0, \\ &\nabla_{\mathbf{e}_3}\mathbf{e}_1 = \nabla_{\mathbf{e}_3}\mathbf{e}_2 = \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0, \end{aligned} \tag{8}$$

where $a = \langle \nabla_{\mathbf{e}_1}\mathbf{e}_1, \mathbf{e}_2 \rangle$ and $b = \langle \nabla_{\mathbf{e}_2}\mathbf{e}_2, \mathbf{e}_1 \rangle$.

5.3. ***Graphs in $M \times \mathbb{R}$.*** Let Σ be the graph of a function $f : M \to \mathbb{R}$. We assume when considering the metric h that $\mathbf{e}_1(f)^2 + \mathbf{e}_2(f)^2 < 1$. In other words, we consider the spacelike graph in the Lorentzian product space. At a point of Σ, both $T_1 = \mathbf{e}_1 + \mathbf{e}_1(f)\mathbf{e}_3$ and $T_2 = \mathbf{e}_2 + \mathbf{e}_2(f)\mathbf{e}_3$ are tangent to Σ with respect to either g or h. Then

$$\mathbf{n}_g=\frac{1}{W_g}\big(-\mathbf{e}_1(f)\mathbf{e}_1-\mathbf{e}_2(f)\mathbf{e}_2+\mathbf{e}_3\big),\quad \text{where } W_g=\sqrt{\mathbf{e}_1(f)^2+\mathbf{e}_2(f)^2+1},$$
$$\mathbf{n}_h=\frac{1}{W_h}(-\mathbf{e}_1(f)\mathbf{e}_1-\mathbf{e}_2(f)\mathbf{e}_2-\mathbf{e}_3),\quad \text{where } W_h=\sqrt{1-\mathbf{e}_1(f)^2-\mathbf{e}_2(f)^2}.$$

are the unit normal vector fields of Σ with respect to g and h. By differentiating $\langle\mathbf{n}_g,\mathbf{n}_g\rangle_g=1$ and $\langle\mathbf{n}_h,\mathbf{n}_h\rangle_h=-1$, one can easily check the following:

Lemma 5.4. *With W_g and W_h given above, we have*

$$\mathbf{e}_1\Big(\frac{\mathbf{e}_1(f)}{W_g}\Big)\mathbf{e}_1(f)+\mathbf{e}_1\Big(\frac{\mathbf{e}_2(f)}{W_g}\Big)\mathbf{e}_2(f)+\mathbf{e}_1\Big(\frac{1}{W_g}\Big)=0,$$
$$\mathbf{e}_2\Big(\frac{\mathbf{e}_1(f)}{W_g}\Big)\mathbf{e}_1(f)+\mathbf{e}_2\Big(\frac{\mathbf{e}_2(f)}{W_g}\Big)\mathbf{e}_2(f)+\mathbf{e}_2\Big(\frac{1}{W_g}\Big)=0,$$
$$\mathbf{e}_1\Big(\frac{\mathbf{e}_1(f)}{W_h}\Big)\mathbf{e}_1(f)+\mathbf{e}_1\Big(\frac{\mathbf{e}_2(f)}{W_h}\Big)\mathbf{e}_2(f)-\mathbf{e}_1\Big(\frac{1}{W_h}\Big)=0,$$
$$\mathbf{e}_2\Big(\frac{\mathbf{e}_1(f)}{W_h}\Big)\mathbf{e}_1(f)+\mathbf{e}_2\Big(\frac{\mathbf{e}_2(f)}{W_h}\Big)\mathbf{e}_2(f)-\mathbf{e}_2\Big(\frac{1}{W_h}\Big)=0.$$

5.5. ***Mean curvature of a graph in the Riemannian product $M\times\mathbb{R}$.*** Now let S_g be the shape operator of the graph Σ in the Riemannian product space $(M\times\mathbb{R},g)$. By (8) and Lemma 5.4, one computes

$$\begin{aligned}
S_g(T_1)&=-\nabla_{T_1}\mathbf{n}_g=-\nabla_{\mathbf{e}_1}\mathbf{n}_g-\mathbf{e}_1(f)\nabla_{\mathbf{e}_3}\mathbf{n}_g=-\nabla_{\mathbf{e}_1}\mathbf{n}_g\\
&=\mathbf{e}_1\Big(\frac{\mathbf{e}_1(f)}{W_g}\Big)\mathbf{e}_1+\mathbf{e}_1\Big(\frac{\mathbf{e}_2(f)}{W_g}\Big)\mathbf{e}_2-\mathbf{e}_1\Big(\frac{1}{W_g}\Big)\mathbf{e}_3\\
&\qquad\qquad+\frac{\mathbf{e}_1(f)}{W_g}\nabla_{\mathbf{e}_1}\mathbf{e}_1+\frac{\mathbf{e}_2(f)}{W_g}\nabla_{\mathbf{e}_1}\mathbf{e}_2-\frac{1}{W_g}\nabla_{\mathbf{e}_1}\mathbf{e}_3\\
&=\Big(\mathbf{e}_1\Big(\frac{\mathbf{e}_1(f)}{W_g}\Big)-a\frac{\mathbf{e}_2(f)}{W_g}\Big)\mathbf{e}_1+\Big(\mathbf{e}_1\Big(\frac{\mathbf{e}_2(f)}{W_g}\Big)+a\frac{\mathbf{e}_1(f)}{W_g}\Big)\mathbf{e}_2-\mathbf{e}_1\Big(\frac{1}{W_g}\Big)\mathbf{e}_3\\
&=\Big(\mathbf{e}_1\Big(\frac{\mathbf{e}_1(f)}{W_g}\Big)-a\frac{\mathbf{e}_2(f)}{W_g}\Big)T_1+\Big(\mathbf{e}_1\Big(\frac{\mathbf{e}_2(f)}{W_g}\Big)+a\frac{\mathbf{e}_1(f)}{W_g}\Big)T_2,
\end{aligned}$$

$$\begin{aligned}
S_g(T_2)&=-\nabla_{T_2}\mathbf{n}_g=-\nabla_{\mathbf{e}_2}\mathbf{n}_g\\
&=\Big(\mathbf{e}_2\Big(\frac{\mathbf{e}_1(f)}{W_g}\Big)+b\frac{\mathbf{e}_2(f)}{W_g}\Big)\mathbf{e}_1+\Big(\mathbf{e}_2\Big(\frac{\mathbf{e}_2(f)}{W_g}\Big)-b\frac{\mathbf{e}_1(f)}{W_g}\Big)\mathbf{e}_2-\mathbf{e}_2\Big(\frac{1}{W_g}\Big)\mathbf{e}_3\\
&=\Big(\mathbf{e}_2\Big(\frac{\mathbf{e}_1(f)}{W_g}\Big)+b\frac{\mathbf{e}_2(f)}{W_g}\Big)T_1+\Big(\mathbf{e}_2\Big(\frac{\mathbf{e}_2(f)}{W_g}\Big)-b\frac{\mathbf{e}_1(f)}{W_g}\Big)T_2.
\end{aligned}$$

Now set $f_i=\mathbf{e}_i(f)$ and $f_{ij}=\mathbf{e}_j(\mathbf{e}_i(f))$ for $i,j=1,2$. Note then that $f_{ij}\neq f_{ji}$ in general. Then the mean curvature H_g of Σ as a surface in the Riemannian product space $(M \times \mathbb{R}, g)$ is computed as

$$\begin{aligned} H_g = \tfrac{1}{2}\operatorname{tr} S_g &= \frac{1}{2}\left(\mathbf{e}_1\left(\frac{\mathbf{e}_1(f)}{W_g}\right) + \mathbf{e}_2\left(\frac{\mathbf{e}_2(f)}{W_g}\right) - a\frac{\mathbf{e}_2(f)}{W_g} - b\frac{\mathbf{e}_1(f)}{W_g}\right) \\ &= \frac{1}{2W_g^3}\big(f_{11}W_g^2 - f_1(f_1 f_{11} + f_2 f_{21}) + f_{22}W_g^2 \\ &\qquad\qquad - f_2(f_1 f_{12} + f_2 f_{22}) - af_2 W_g^2 - bf_1 W_g^2\big) \\ &= \frac{1}{2W_g^3}\big(f_{11}(1+f_2^2) - (f_{21}+f_{12})f_1 f_2 + f_{22}(1+f_1^2) - (af_2 + bf_1)W_g^2\big). \end{aligned}$$

5.6. ***Mean curvature of a spacelike graph in the Lorentzian product $M \times \mathbb{R}$.*** Let S_h be the shape operator of the graph Σ in the Lorentzian product space $(M \times \mathbb{R}, h)$. By (8) and Lemma 5.4, one computes

$$\begin{aligned} S_h(T_1) &= -\nabla_{T_1}\mathbf{n}_h = -\nabla_{\mathbf{e}_1}\mathbf{n}_h \\ &= \mathbf{e}_1\left(\frac{\mathbf{e}_1(f)}{W_h}\right)\mathbf{e}_1 + \mathbf{e}_1\left(\frac{\mathbf{e}_2(f)}{W_h}\right)\mathbf{e}_2 + \mathbf{e}_1\left(\frac{1}{W_h}\right)\mathbf{e}_3 \\ &\qquad + \frac{\mathbf{e}_1(f)}{W_h}\nabla_{\mathbf{e}_1}\mathbf{e}_1 + \frac{\mathbf{e}_2(f)}{W_h}\nabla_{\mathbf{e}_1}\mathbf{e}_2 + \frac{1}{W_h}\nabla_{\mathbf{e}_1}\mathbf{e}_3 \\ &= \left(\mathbf{e}_1\left(\frac{\mathbf{e}_1(f)}{W_h}\right) - a\frac{\mathbf{e}_2(f)}{W_h}\right)T_1 + \left(\mathbf{e}_1\left(\frac{\mathbf{e}_2(f)}{W_h}\right) + a\frac{\mathbf{e}_1(f)}{W_h}\right)T_2, \\ S_h(T_2) &= \left(\mathbf{e}_2\left(\frac{\mathbf{e}_1(f)}{W_h}\right) + b\frac{\mathbf{e}_2(f)}{W_h}\right)T_1 + \left(\mathbf{e}_2\left(\frac{\mathbf{e}_2(f)}{W_h}\right) - b\frac{\mathbf{e}_1(f)}{W_h}\right)T_2. \end{aligned}$$

The mean curvature H_h of Σ as a surface in $(M \times \mathbb{R}, h)$ is computed as

$$\begin{aligned} H_h = \tfrac{1}{2}\operatorname{tr} S_h &= \frac{1}{2}\left(\mathbf{e}_1\left(\frac{\mathbf{e}_1(f)}{W_h}\right) + \mathbf{e}_2\left(\frac{\mathbf{e}_2(f)}{W_h}\right) - a\frac{\mathbf{e}_2(f)}{W_h} - b\frac{\mathbf{e}_1(f)}{W_h}\right) \\ &= \frac{1}{2W_h^3}\big(f_{11}W_h^2 + f_1(f_1 f_{11} + f_2 f_{21}) + f_{22}W_h^2 \\ &\qquad\qquad + f_2(f_1 f_{12} + f_2 f_{22}) - af_2 W_h^2 - bf_1 W_h^2\big) \\ &= \frac{1}{2W_h^3}\big(f_{11}(1-f_2^2) + (f_{21}+f_{12})f_1 f_2 + f_{22}(1-f_1^2) - (af_2 + bf_1)W_h^2\big). \end{aligned}$$

5.7. ***Another characterization of (generalized) helicoids in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.*** In this section we show that — except for the trivial level surfaces $\mathbb{S}^2 \times \{z_0\}$ and $\mathbb{H}^2 \times \{z_0\}$ — helicoids are the only surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ that are minimal with respect to the Riemannian product metric g and at the same time maximal with respect to the Lorentzian product metric h.

Lemma 5.8. *Let Σ be a surface in $M \times \mathbb{R}$ that is spacelike with respect to the Lorentzian metric h. If $H_g = H_h = 0$, then Σ is horizontally ruled by geodesics.*

Proof. To prove the theorem, it is enough to show that $\Sigma \cap \{z = c\}$ is a geodesic in $M \times \mathbb{R}$. Since Σ is spacelike with respect to h, Σ is locally a graph of a function $f : M \to \mathbb{R}$. The vector field $X = f_2\mathbf{e}_1 - f_1\mathbf{e}_2$ is tangent to the curve $\Sigma \cap \{z = c\}$. Since $H_g = 0$ and $H_h = 0$, one has respectively

$$\begin{aligned} &f_{11}(1+f_2^2)-(f_{21}+f_{12})f_1f_2+f_{22}(1+f_1^2)-(af_2+bf_1)(1+f_1^2+f_2^2) = 0,\\ &f_{11}(1-f_2^2)+(f_{21}+f_{12})f_1f_2+f_{22}(1-f_1^2)-(af_2+bf_1)(1-f_1^2-f_2^2) = 0. \end{aligned} \tag{9}$$

Subtracting the first of these from the second, one has

$$0 = f_{11}f_2^2 - (f_{12} + f_{21})f_1f_2 + f_{22}f_1^2 - (af_2 + bf_1)(f_1^2 + f_2^2). \tag{10}$$

Now, one has

$$\begin{aligned} \nabla_X X &= f_2\nabla_{\mathbf{e}_1}(f_2\mathbf{e}_1 - f_1\mathbf{e}_2) - f_1\nabla_{\mathbf{e}_2}(f_2\mathbf{e}_1 - f_1\mathbf{e}_2)\\ &= (f_2f_{21} + af_1f_2 - f_1f_{22} + bf_1^2)\mathbf{e}_1 + (-f_2f_{11} + af_2^2 + f_1f_{12} + bf_1f_2)\mathbf{e}_2. \end{aligned}$$

Then, the vector $X \wedge \nabla_X X$, which is orthogonal to X and to $\nabla_X X$, is computed as

$$\begin{aligned} X \wedge \nabla_X X &= \big(f_2(-f_2f_{11} + af_2^2 + f_1f_{12} + bf_1f_2)\\ &\qquad\qquad + f_1(f_2f_{21} + af_1f_2 - f_1f_{22} + bf_1^2)\big)\mathbf{e}_3\\ &= -\big(f_{11}f_2^2 - (f_{12} + f_{21})f_1f_2 + f_{22}f_1^2 - (af_2 + bf_1)(f_1^2 + f_2^2)\big)\mathbf{e}_3, \end{aligned}$$

which vanishes by (10). This implies that the vector field $\nabla_X X$ is parallel to X. Thus $\Sigma \cap \{z = c\}$ is a geodesic in $M \times \mathbb{R}$ since X is a vector field tangent to it. □

Let Σ be a surface in $\mathbb{S}^2 \times \mathbb{R}$ with $H_g = H_h = 0$. Then Σ is a horizontally ruled surface in the Riemannian product space by Lemma 5.8, and then Σ is $\mathbb{S}^2 \times \{z_0\}$ or a helicoid by Theorem 3.4. Note that the surface $\gamma \times \mathbb{R}$ is excluded since it is not spacelike. Now let Σ be a surface in $\mathbb{H}^2 \times \mathbb{R}$ with $H_g = H_h = 0$. Then Σ is a horizontally ruled surface in the Riemannian product space by Lemma 5.8, and then Σ is a $\mathbb{H}^2 \times \{z_0\}$ or a generalized helicoid by Theorem 4.4. Again the surface $\gamma \times \mathbb{R}$ is excluded since it is not spacelike. We state this result as follows:

Theorem 5.9. (1) *A surface in $\mathbb{S}^2 \times \mathbb{R}$ with $H_g = H_h = 0$ is $\mathbb{S}^2 \times \{z_0\}$ or a helicoid.*

(2) *A surface in $\mathbb{H}^2 \times \mathbb{R}$ with $H_g = H_h = 0$ is $\mathbb{H}^2 \times \{z_0\}$ or a helicoid.*

Acknowledgment

We thank the referee for pointing out computational mistakes and providing valuable suggestions which helped us to improve this paper.

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Received January 14, 2009. Revised April 8, 2009.

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