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We establish first order gradient estimates for positive solutions of the heat equations on complete noncompact or closed Riemannian manifolds under Ricci flows. These estimates improve Guenther's results by weakening the curvature constraints. We also obtain a result for arbitrary solutions on closed manifolds under Ricci flows. As applications, we derive Harnack-type inequalities and second order gradient estimates for positive solutions of the heat equations under Ricci flow. The results in this paper can be considered as generalizing the estimates of Li–Yau and J. Y. Li to the Ricci flow setting.

1. Introduction

In this paper, we mainly generalize Li and Yau's [1986] and Li's [1991] gradient estimates to positive solutions of the heat equation under Ricci flow. The Ricci flow,

$$\partial_t g_{ij} = -2 \operatorname{Ric}_{ij}, \tag{1-1}$$

was introduced by Hamilton [1982] to study the Poincaré conjecture on compact three manifolds with positive Ricci curvature. Since then, in the series [1995; 1997; 1999], Hamilton created a well-developed theory of Ricci flow as an approach to the Poincaré conjecture and the geometrization conjecture. In [2002; 2003], Perelman brought in new ideas and completed the so-called Hamilton program.

Gradient estimates for solutions of the heat equation are very powerful tools in analysis, as shown for example in [Li 1991; Li and Yau 1986]. Perelman [2002] actually showed a gradient estimate for the fundamental solution of the conjugate heat equation,

$$\Delta u - Ru + \partial_t u = 0,$$

under Ricci flow on a closed Riemannian manifold M, where R is the scalar curvature. Namely, let u be the fundamental solution of the equation above in $M \times [0, T)$,

and let f be the function such that $u = (4\pi\tau)^{-n/2}e^{-f}$ with $\tau = T - t$. Then

$$\big(\tau(2\Delta f - |\nabla f|^2 + R) + f - n\big)u \le 0 \quad \text{in } M \times [0, T).$$

Equivalently,

$$\frac{|\nabla u|^2}{u^2} - \frac{u_\tau}{u} - R \le \frac{n}{\tau} + \frac{\ln u}{\tau} + \frac{n}{2}\frac{\ln(4\pi\tau)}{\tau}.$$

This estimate is important in the proof of Perelman. Recently, Kuang and Zhang [2008] established a gradient estimate that works for all positive solutions of the conjugate heat equation under Ricci flow on a closed manifold. (Here and throughout we say that a Riemannian manifold is closed if it is compact without boundary.) As an immediate consequence, they get a Harnack-type inequality. By supposing a lower bound on the Ricci curvature, Zhang [2006] established local gradient estimates for positive solutions of the heat equation under the backward Ricci flow $\partial_t g_{ij} = 2\,\mathrm{Ric}_{ij}$ on a closed Riemannian manifold. Under stronger curvature constraints, Guenther [2002] had already established gradient estimates for positive solutions of the heat equation under Ricci flow on a closed manifold. Using this result, she derived a Harnack-type inequality and found a lower bound for the heat kernel under Ricci flow. Here heat kernel means the fundamental solution of the heat equation under Ricci flow, whose existence and basic properties Guenther also proved. We weaken her curvature constraints in Section 2 using the method of Li and Yau [1986]. We also get corresponding estimates for complete noncompact manifolds under Ricci flows. All of these results are generalizations of Li and Yau's gradient estimates.

Interesting in their own right, higher order gradient estimates for heat kernels on complete noncompact Riemannian manifolds under Ricci flows are also closely related with the boundedness of the Riesz transform and the Sobolev inequality. Zhang [2007] found that the noncollapsing result, which is critical in Perelman's proof of the Poincaré conjecture, follows immediately from the Sobolev inequality under Ricci flow. Li [1994] used the boundedness of the Riesz transform to prove the Sobolev inequality on Riemannian manifold with some constraints, so it is natural to try to prove with a similar method the Sobolev inequality under Ricci flow. In that method, an important step, completed [Li 1991], is to prove an estimate for $\nabla_X \nabla_Y H(x, y, t)$, where $H(x, y, t)$ is the heat kernel and ∇_X and ∇_Y are the gradient operator in the variables x and y. However, difficulties arise in using this method to get the generalization of this estimate under Ricci flow, since in this case the heat kernel has properties different from what it had in the fixed metric case. As a consequence of the first order results in Section 2, in Section 3 we get the generalization of Li's [1991] second order gradient estimate for the positive solution $u(x, t)$ of the heat equation under Ricci flow. Let M be a complete noncompact Riemannian manifold with initial metric $g(0)$. Assume that $g(t)$ evolves

by Equation (1-1) and its first order covariant derivatives are bounded by k_1 and k_2. Then we have (the notation is defined in later sections)

$$\frac{|\nabla^2 u(x,t)|}{u(x,t)} + \alpha \frac{|\nabla u(x,t)|^2}{u^2(x,t)} - 5\alpha \frac{u_t(x,t)}{u(x,t)} \leq C(k_1 + k_2^{2/3} + 1/t).$$

In fact, the estimate for $\nabla_X \nabla_Y H(x,y,t)$ and the second order gradient estimate for $u(x,t)$ are proved similarly in [Li 1991]. The main difference is that the latter doesn't depend on the special properties of the heat kernel.

For closed Riemannian manifolds under Ricci flows, we get a gradient estimate for arbitrary solutions of the heat equation at the end of Section 2.

We will use the following notations: We denote by ∇ and Δ the gradient and Laplacian–Beltrami operator under the metric $g(t)$; by C a positive constant that may change from line to line; by $d(x,y,t)$ the geodesic distance between $x,y \in M$ under $g(t)$; and by $\psi(r)$ a C^2 function on $[0, +\infty)$, such that

$$(1\text{-}2) \qquad \psi(r) = \begin{cases} 1 & \text{if } r \in [0,1], \\ 0 & \text{if } r \in [2, +\infty), \end{cases}$$

and

$$(1\text{-}3) \qquad 0 \leq \psi(r) \leq 1, \quad \psi'(r) \leq 0, \quad \psi''(r) \geq -C, \quad \frac{|\psi'(r)|^2}{\psi(r)} \leq C,$$

where C is an absolute constant. When we say that $u(x,t)$ is a solution to the heat equation, we mean u is a solution that is C^2 in x and C^1 in t.

2. The first order gradient estimates

In this section, we prove the first order gradient estimates. We will denote

$$f_t(x,t) = \partial_t f(x,t) = \frac{\partial f(x,t)}{\partial t}$$

for a function f on $M \times [0,T]$, where T is a positive constant. We give a local version gradient estimate first.

Theorem 1. *Let $g(t)$ be a solution to the Ricci flow on a Riemannian manifold M^n with $n \geq 2$ for t in some time interval $[0,T]$, and suppose $-K_0 \leq \text{Ric} \leq K_1$ for some positive constants K_0 and K_1 and all $t \in [0,T]$. Let M be complete under the initial metric $g(0)$. Given $x_0 \in M$ and $R > 0$, let u be a positive solution to the equation $(\Delta - \partial_t)u(x,t) = 0$ in the cube $Q_{2R,T} := \{(x,t) \mid d(x, x_0, t) \leq 2R, \ 0 \leq t \leq T\}$. Then for $(x,t) \in Q_{R,T}$, we have*

$$(2\text{-}1) \qquad \frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha \frac{u_t(x,t)}{u(x,t)} \leq C\left(K_1 + K_0 + \frac{1}{t} + \frac{1}{R^2}\right)$$

for any $\alpha > 1$, where C depends on n and α only.

More explicitly, we have

$$(2\text{-}2) \quad \frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha \frac{u_t(x,t)}{u(x,t)} \leq \frac{n\alpha^2}{t} + \frac{C\alpha^2}{R^2}\left(R\sqrt{K_0} + \frac{\alpha^2}{\alpha-1}\right)$$
$$+ \frac{n\alpha^3}{\alpha-1}K_0 + n^{3/2}\alpha^2(K_0 + K_1) + C\alpha^2 K_1,$$

for any $\alpha > 1$, where C depends on n only.

As in the proof in [Li and Yau 1986], let $f = \log u$, and let

$$F = t\left(\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha \frac{u_t(x,t)}{u(x,t)}\right) = t(|\nabla f|^2 - \alpha f_t).$$

Lemma 1. *Suppose $(M, g(t))$ satisfies the hypotheses of Theorem 1. We have*

$$(2\text{-}3) \quad (\Delta - \partial_t)F \geq -2\nabla f \cdot \nabla F + \frac{t}{n}(|\nabla f|^2 - f_t)^2 - (|\nabla f|^2 - \alpha f_t)$$
$$- 2\alpha K_0 t |\nabla f|^2 - t\alpha^2 n^2 (K_0 + K_1)^2.$$

Proof. For a given time t, choose $\{x_1, x_2, \ldots, x_n\}$ to be a normal coordinate system at a fixed point. The subscripts i and j will denote covariant derivatives in the x_i and x_j directions. We will compute at the fixed point.

By a direct computation, we obtain

$$\Delta F = t\left(2\sum_{i,j} f_{ij}^2 + 2\sum_{i,j} f_i f_{jji} + 2\sum_{i,j} \text{Ric}_{ij}\, f_i f_j - \alpha(\Delta f)_t + 2\alpha \sum_{i,j} \text{Ric}_{ij}\, f_{ij}\right),$$

where we have used the Ricci identity and the formula

$$(2\text{-}4) \qquad\qquad \Delta(f_t) = (\Delta f)_t - 2\langle \text{Ric}, \text{Hess}(f)\rangle.$$

On the other hand, we have

$$F_t = (|\nabla f|^2 - \alpha f_t) + t\left(2\sum_{i,j} \text{Ric}_{ij}\, f_i f_j + 2\sum_i f_{ti} f_i - \alpha f_{tt}\right).$$

Then noting that $(\Delta - \partial_t)f = -|\nabla f|^2$, we arrive at

$$(2\text{-}5) \quad (\Delta - \partial_t)F = -2\nabla f \cdot \nabla F + 2t\left(\sum f_{ij}^2 + \alpha \sum \text{Ric}_{ij}\, f_{ij}\right)$$
$$+ 2\alpha t\, \text{Ric}(\nabla f, \nabla f) - (|\nabla f|^2 - \alpha f_t).$$

Because $(\text{Ric}_{ij})_{n\times n}$ is a real symmetric matrix, we obtain

$$(2\text{-}6) \qquad\qquad -K_0 - K_1 \leq \text{Ric}_{ij} \leq K_1 + K_0$$

from $-K_0 \leq \text{Ric} \leq K_1$. Applying those bounds and Young's inequality in the form

$$|\text{Ric}_{ij}|\,|f_{ij}| \leq \frac{\alpha}{2}\,\text{Ric}_{ij}^2 + \frac{1}{2\alpha}\,f_{ij}^2,$$

we conclude

$$(2\text{-}7) \quad (\Delta - \partial_t)F \geq -2\nabla f \cdot \nabla F + t\sum f_{ij}^2 - t\alpha^2 n^2 (K_0 + K_1)^2$$
$$- 2\alpha K_0 t |\nabla f|^2 - (|\nabla f|^2 - \alpha f_t).$$

The lemma is completed with the help of the inequality

$$\sum_{i,j} f_{ij}^2 \geq \sum_i f_{ii}^2 \geq \frac{1}{n}(\Delta f)^2 = \frac{1}{n}(|\nabla f|^2 - f_t)^2. \qquad \square$$

Proof of Theorem 1. Bounded Ricci curvature implies that $g(t)$ is uniformly equivalent to the initial metric $g(0)$ (see [Chow et al. 2006, Corollary 6.11]), that is,

$$e^{-2K_1 T} g(0) \leq g(t) \leq e^{2K_0 T} g(0).$$

By definition, we know that $(M, g(t))$ is also complete for $t \in [0, T]$.

Let

$$\varphi(x, t) = \varphi(d(x, x_0, t)) = \psi\left(\frac{d(x, x_0, t)}{R}\right) = \psi\left(\frac{\rho(x, t)}{R}\right),$$

where $\rho(x, t) = d(x, x_0, t)$. For the purpose of applying the maximum principle, the argument of [Calabi 1958] allows us to assume that the function $\varphi(x, t)$, with support in $Q_{2R,T}$, is C^2 at the maximum point.

For any $0 < T_1 \leq T$, let (x_1, t_1) be the point in Q_{2R,T_1} at which φF achieves its maximum value. We can assume that this value is positive, because otherwise the proof is trivial. Then at the point (x_1, t_1), we have

$$(2\text{-}8) \qquad \nabla(\varphi F) = F\nabla\varphi + \varphi\nabla F = 0, \quad \Delta(\varphi F) \leq 0, \quad \partial_t(\varphi F) \geq 0.$$

Therefore,

$$(2\text{-}9) \qquad 0 \geq (\Delta - \partial_t)(\varphi F)$$
$$= (\Delta\varphi)F + \varphi(\Delta - \partial_t)F - \varphi_t F + 2\nabla\varphi\nabla F.$$

Using the Laplacian comparison theorem, we have

$$\Delta\varphi \geq -\frac{C}{R^2} - \frac{C}{R}\sqrt{K_0}.$$

By the evolution formula of the geodesic length under Ricci flow (see [Chow and Knopf 2004]), we calculate

$$-F\varphi_t = -F\psi'\left(\frac{\rho}{R}\right)\frac{1}{R}\frac{d\rho}{dt} = F\psi'\left(\frac{\rho}{R}\right)\frac{1}{R}\int_{\gamma_{t_1}} \text{Ric}(S, S)\, ds$$
$$\geq F\psi'\left(\frac{\rho}{R}\right)\frac{1}{R}K_1\rho \geq F\psi'\left(\frac{\rho}{R}\right)K_1 \geq -F\sqrt{C}K_1,$$

where γ_{t_1} is the geodesic connecting x and x_0 under the metric $g(t_1)$, S is the unit tangent vector to γ_{t_1}, and ds is the element of arc length. Substituting the two inequalities above into (2-9) and using (2-8), we obtain

$$(2\text{-}10) \qquad 0 \geq \left(-\frac{C}{R^2} - \frac{C}{R}\sqrt{K_0}\right)F - \sqrt{C}K_1 F + \varphi(\Delta - \partial_t)F.$$

Applying Lemma 1 to this inequality yields

$$(2\text{-}11) \quad 0 \geq \left(-\frac{C}{R^2} - \frac{C}{R}\sqrt{K_0}\right)F - \sqrt{C}K_1 F - 2\frac{\sqrt{C}}{R}\sqrt{\varphi}|\nabla f|F + \frac{t_1}{n}\varphi(|\nabla f|^2 - f_t)^2$$
$$- \varphi(|\nabla f|^2 - \alpha f_t) - 2\alpha K_0 t_1 \varphi |\nabla f|^2 - t_1 \alpha^2 n^2 \varphi(K_1 + K_0)^2.$$

The following computation is almost the same as one in [Li and Yau 1986]. Multiplying through by φt_1 and setting $y = \varphi|\nabla f|^2$ and $z = \varphi f_t$, Equation (2-11) becomes

$$(2\text{-}12) \quad 0 \geq t_1\left(-\frac{C}{R^2} - \frac{C}{R}\sqrt{K_0}\right)(\varphi F) - \sqrt{C}K_1 t_1(\varphi F) - 2\frac{\sqrt{C}}{R}t_1^2 y^{1/2}(y - \alpha z)$$
$$+ \frac{t_1^2}{n}(y - z)^2 - 2\alpha K_0 t_1^2 y - \varphi^2 F - t_1^2 \alpha^2 n^2 \varphi^2 (K_0 + K_1)^2.$$

Using the inequality $ax^2 - bx \geq -b^2/(4a)$ for $a, b > 0$, one obtains

$$\frac{t_1^2}{n}(y - z)^2 - 2\frac{\sqrt{C}}{R}t_1^2 y^{1/2}(y - \alpha z) - 2\alpha K_0 t_1^2 y$$
$$= \frac{t_1^2}{n}\left(\frac{1}{\alpha^2}(y - \alpha z)^2 + \left(\frac{\alpha - 1}{\alpha}\right)^2 y^2 - 2\alpha n K_0 y + \left(2\frac{\alpha - 1}{\alpha^2}y - \frac{2n\sqrt{C}}{R}y^{1/2}\right)(y - \alpha z)\right)$$
$$\geq \frac{t_1^2}{n}\left(\frac{1}{\alpha^2}(y - \alpha z)^2 - \frac{\alpha^4 n^2 K_0^2}{(\alpha - 1)^2} - \frac{\alpha^2 n^2 C}{2(\alpha - 1)R^2}(y - \alpha z)\right).$$

Hence (2-12) becomes

$$(2\text{-}13) \quad \frac{1}{n\alpha^2}(\varphi F)^2 - (\varphi F)\left(1 + \frac{C}{R^2}t_1 + \frac{C}{R}\sqrt{K_0}t_1 + \frac{Cn\alpha^2 t_1}{2(\alpha - 1)R^2} + \sqrt{C}K_1 t_1\right)$$
$$- \left(\frac{nK_0^2\alpha^4 t_1^2}{(\alpha - 1)^2} + t_1^2\alpha^2 n^2 \varphi^2(K_1 + K_0)^2\right) \leq 0.$$

We apply the quadratic formula and then arrive at

$$(2\text{-}14) \quad \varphi F(x_1, t_1) \leq n\alpha^2 + \frac{Cn\alpha^2}{R^2}\left(R\sqrt{K_0} + \frac{\alpha^2}{\alpha - 1}\right)t_1 + \sqrt{C}n\alpha^2 K_1 t_1$$
$$+ \frac{n\alpha^3 t_1}{\alpha - 1}K_0 + t_1 \varphi n^{3/2}(K_0 + K_1)\alpha^2.$$

This estimate for φF is also correct on $\{(x, T_1) \mid d(x, x_0, T_1) \leq 2R\}$ since $t_1 \leq T_1$. Since T_1 is arbitrary in $0 < T_1 \leq T$, we have completed the proof of Theorem 1. $\square$

The local result above implies a global one.

Corollary 1. *Let $(M, g(0))$ be a complete noncompact Riemannian manifold without boundary, and suppose $g(t)$ evolves by Ricci flow in such a way that $-K_0 \leq \mathrm{Ric} \leq K_1$ for $t \in [0, T]$. Let u be a positive solution to the equation $(\Delta - \partial_t)u(x, t) = 0$. Then for $(x, t) \in M \times (0, T]$, we have*

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha \frac{u_t(x,t)}{u(x,t)} \leq \frac{n\alpha^2}{t} + C(K_1 + K_0), \tag{2-15}$$

for any $\alpha > 1$, where C depends on n and α only.

Proof. By the uniform equivalence of $g(t)$, we know that $(M, g(t))$ is complete noncompact for $t \in [0, T]$. Then let $R \to +\infty$ in (2-2). $\qquad\square$

Remark 1. When $(M, g(0))$ is a complete noncompact Riemannian manifold, Shi [1989] gives a sufficient condition for the short time existence of the Ricci flow: It suffices that the curvature tensor $\{R_{ijkl}\}$ of $g(0)$ satisfies

$$|R_{ijkl}|^2 \leq \kappa \quad \text{on M},$$

where $0 < \kappa < +\infty$ is a constant.

Using Lemma 1, we can also derive a similar gradient estimate on a closed Riemannian manifold.

Theorem 2. *Let $(M, g(t))$ be a closed Riemannian manifold, where $g(t)$ evolves by Ricci flow in such a way that $-K_0 \leq \mathrm{Ric} \leq K_1$ for $t \in [0, T]$. If u is a positive solution to the equation $(\Delta - \partial_t)u(x, t) = 0$, then for $(x, t) \in M \times (0, T]$, we have*

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha \frac{u_t(x,t)}{u(x,t)} \leq \frac{n\alpha^2}{t} + \frac{n\alpha^3 K_0}{\alpha - 1} + n^{3/2}\alpha^2(K_0 + K_1), \tag{2-16}$$

for any $\alpha > 1$, where C depends on n and α only.

Proof. Let notations F and f be as above. Set

$$\bar{F}(x, t) = F(x, t) - \frac{n\alpha^3 K_0}{\alpha - 1}t - n^{3/2}\alpha^2(K_0 + K_1)t.$$

If $\bar{F}(x, t) \leq n\alpha^2$ for any $(x, t) \in M \times (0, T]$, then the theorem is proved.

If (2-16) doesn't hold, then at the maximal point (x_0, t_0) of $\bar{F}(x, t)$, we have $\bar{F}(x_0, t_0) > n\alpha^2$. Noting $\bar{F}(x, 0) = 0$, we know here $t_0 > 0$. Then applying the maximal principle, we have at the point (x_0, t_0) that

$$\nabla \bar{F}(x_0, t_0) = 0, \quad \Delta \bar{F}(x_0, t_0) \leq 0, \quad \frac{\partial}{\partial t}\bar{F}(x_0, t_0) \geq 0. \tag{2-17}$$

Therefore we obtain

$$0 \geq (\Delta - \partial_t)\bar{F} \geq (\Delta - \partial_t)F. \tag{2-18}$$

Using Lemma 1 and the trick in calculating (2-11), we get

$$(2\text{-}19) \quad 0 \geq \frac{t_0}{n\alpha^2}\left(\frac{F}{t_0}\right)^2 - \left(\frac{F}{t_0}\right) - \frac{n\alpha^4 K_0^2}{(\alpha-1)^2}t_0 - t_0\alpha^2 n^2(K_0+K_1)^2$$
$$+ \frac{2t_0}{n}\frac{\alpha-1}{\alpha^2}|\nabla f|^2\frac{F}{t_0}.$$

Since

$$\frac{F}{t_0} = \frac{\overline{F}}{t_0} + \frac{n\alpha^3 K_0}{\alpha-1} + n^{3/2}\alpha^2(K_0+K_1) > 0,$$

we get the inequality

$$(2\text{-}20) \qquad \frac{t_0}{n\alpha^2}\left(\frac{F}{t_0}\right)^2 - \left(\frac{F}{t_0}\right) - \frac{n\alpha^4 K_0^2}{(\alpha-1)^2}t_0 - t_0\alpha^2 n^2(K_0+K_1)^2 \leq 0.$$

Again the quadratic formula gives

$$(2\text{-}21) \qquad \frac{F}{t_0} \leq \frac{n\alpha^2}{t_0} + \frac{n\alpha^3 K_0}{\alpha-1} + n^{3/2}\alpha^2(K_0+K_1).$$

This implies $\overline{F}(x_0,t_0) \leq n\alpha^2$, a contradiction. So (2-16) holds.   □

Remark 2. In Corollary 1 and Theorem 2, if $K_0 = 0$, we can let $\alpha \to 1$.

In fact, the Theorem 2 improves the gradient inequality in [Guenther 2002], which requires the boundedness of the gradient of scalar curvature in addition to the boundedness of the Ricci curvature. Beginning with this result, we can do things similar to what was done in [Guenther 2002], such as deriving Harnack-type inequalities.

Corollary 2. *Let $(M, g(0))$ be a complete noncompact Riemannian manifold without boundary or a closed Riemannian manifold, and suppose $g(t)$ evolves by Ricci flow for $t \in [0, T]$ in such a way that $-K_0 \leq \mathrm{Ric} \leq K_1$. If u is a positive solution to the equation $(\Delta - \partial_t)u(x,t) = 0$, then for any points $(x,t_1), (y,t_2) \in M \times (0,T]$ such that $t_1 < t_2$, we have*

$$(2\text{-}22) \quad u(x,t_1) \leq u(y,t_2)\left(\frac{t_2}{t_1}\right)^{2n\varepsilon} \exp\left(\int_0^1 \frac{\varepsilon|\gamma'(s)|_\sigma^2}{2(t_2-t_1)}ds + C\frac{t_2-t_1}{2\varepsilon}(K_1+K_0)\right),$$

for any $\varepsilon > 1/2$, where C depends on n and ε only, $\gamma(s)$ is a smooth curve connecting x and y with $\gamma(1) = x$ and $\gamma(0) = y$, and $|\gamma'(s)|_\sigma$ is the length of the vector $\gamma'(s)$ at time $\sigma = (1-s)t_2 + st_1$.

Proof. First note that the gradient estimates in Corollary 1 and Theorem 2 can both be written as

$$(2\text{-}23) \qquad \frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha\frac{u_t(x,t)}{u(x,t)} \leq \frac{n\alpha^2}{t} + C_{n,\alpha}(K_1+K_0).$$

Define $l(s) = \ln u(\gamma(s), (1-s)t_2 + st_1)$. It is easy to see that $l(0) = \ln u(y, t_2)$ and $l(1) = \ln u(x, t_1)$. Direct calculation gives

$$\frac{\partial l(s)}{\partial s} = (t_2 - t_1)\left(\frac{\nabla u}{u}\frac{\gamma'(s)}{t_2 - t_1} - \frac{u_t}{u}\right)$$

$$\leq \frac{\varepsilon|\gamma'(s)|_\sigma^2}{2(t_2 - t_1)} + \frac{t_2 - t_1}{2\varepsilon}\left(C(K_1 + K_0) + \frac{4\varepsilon^2 n}{\sigma(s)}\right).$$

Integrating this inequality over $\gamma(s)$, we have

$$\ln\frac{u(x, t_1)}{u(y, t_2)} = \int_0^1 \frac{\partial l(s)}{\partial s}ds$$

$$\leq \int_0^1 \frac{\varepsilon|\gamma'(s)|_\sigma^2}{2(t_2 - t_1)}ds + C\frac{t_2 - t_1}{2\varepsilon}(K_1 + K_0) + 2\varepsilon n\ln(t_2/t_1). \qquad \square$$

We can also get a gradient estimate for an arbitrary solution of the heat equation under Ricci flow on a closed manifold without any curvature conditions. The auxiliary function F we take in the following proof is inspired by Hamilton's [1995] proof of Shi's [1989] derivative estimates.

Theorem 3. *Let $(M, g(t))$ be a closed Riemannian manifold, where $g(t)$ solves the Ricci flow for $t \in [0, T]$. If u solves $\Delta u - \partial_t u = 0$, then*

$$(2\text{-}24) \quad |\nabla u(x, t)|^2 \leq \frac{1}{2t}\left(\max_{x \in M} u^2(x, 0) - u^2(x, t)\right) \quad \textit{for } (x, t) \in M \times [0, T].$$

Proof. Since $\partial_t u = \Delta u$, we have

$$\partial_t(|\nabla u|^2) = 2\operatorname{Ric}(\nabla u, \nabla u) + 2\langle\nabla u, \nabla(\Delta u)\rangle.$$

Using Bochner's formula, this becomes

$$(2\text{-}25) \qquad \partial_t(|\nabla u|^2) = \Delta(|\nabla u|^2) - 2|\nabla^2 u|^2.$$

On the other hand,

$$(2\text{-}26) \qquad \partial_t(u^2) = \Delta(u^2) - 2|\nabla u|^2.$$

Let $F = t|\nabla u|^2 + Au^2$, where A is a constant to be fixed. Then combining (2-25) and (2-26) gives

$$\partial_t F = |\nabla u|^2 + t(\Delta(|\nabla u|^2) - 2|\nabla^2 u|^2) + A(\Delta(u^2) - 2|\nabla u|^2)$$

$$(2\text{-}27) \qquad \leq \Delta F + (1 - 2A)|\nabla u|^2.$$

Setting $A = 1/2$ and applying maximum principle, we conclude

$$(2\text{-}28) \qquad F(x, t) \leq \max_{x \in M} F(x, 0) = \tfrac{1}{2}\max_{x \in M} u^2(x, 0).$$

This inequality implies the theorem. $\qquad \square$

Remark 3. For a positive solution of the heat equation u on closed manifolds under Ricci flow, Zhang [2006] gives a stronger estimate,

$$(2\text{-}29) \qquad \frac{|\nabla u|}{u} \leq \sqrt{\frac{1}{t}} \sqrt{\ln \frac{M}{u}},$$

where $M = \max_{x \in M} u(x, 0)$. Similar to the fact that the interpolation inequality follows from this estimate in [Zhang 2006], here we get for any $x, y \in M$ and $0 < t \leq T$ that

$$(2\text{-}30) \qquad u(x, t) \leq u(y, t) + \sqrt{\frac{C}{2}} \frac{d(x, y, t)}{\sqrt{t}},$$

where $C = \max_{x \in M} u^2(x, 0)$.

3. The second order gradient estimates

Using Corollary 1, we can generalize to the Ricci flow setting the second order gradient estimate for the positive solution of the heat equation in [Li 1991].

Theorem 4. *Let $g(t)$ be a solution to the Ricci flow on a Riemannian manifold M^n with $n \geq 2$ for t in some time interval $[0, T]$. Assume that $(M, g(0))$ is a complete noncompact manifold without boundary. Suppose the curvature tensor and its first order covariant derivatives are bounded throughout by k_1 and k_2, respectively. Let u be a positive solution to $(\Delta - \partial_t)u(x, t) = 0$. Then for $(x, t) \in M \times (0, T]$,*

$$(3\text{-}1) \qquad \frac{|\nabla^2 u(x, t)|}{u(x, t)} + \alpha \frac{|\nabla u(x, t)|^2}{u^2(x, t)} - 5\alpha \frac{u_t(x, t)}{u(x, t)} \leq C\left(k_1 + k_2^{2/3} + \frac{1}{t}\right),$$

for any $\alpha > 1$, where C depends on n and α only.

To prove the theorem, we set

$$F(x, y, t) = t F_1 = t\left(\frac{|\nabla^2 u(x, t)|}{u(x, t)} + \alpha \frac{|\nabla u(x, t)|^2}{u^2(x, t)} - \beta \frac{u_t(x, t)}{u(x, t)}\right),$$

where β is a constant to be fixed.

Lemma 2. *Suppose $(M, g(t))$ satisfies the hypotheses of Theorem 4. Then for sufficiently small $\delta > 0$ and $\gamma - 1 > 0$ and some $\varepsilon > 0$, we have*

$$\begin{aligned}
(\Delta - \partial_t)F \geq\ & -2\nabla F \cdot \nabla \log u + \frac{\delta\alpha}{t}F^2 + 2\delta\alpha\beta F \frac{u_t}{u} - 2\delta\alpha^2 F \frac{|\nabla u|^2}{u^2} \\
& - Ck_1 F - \frac{Ck_1^2}{4(\gamma - 1)^2}t - Ck_2^{4/3}t - 4\beta\varepsilon n^2 k_1^2 t - F/t \\
& \qquad\qquad - 2Ct\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right)\left(\frac{1}{t} + k_1\right)^2,
\end{aligned}$$

where $\beta = 5\alpha$ and C depends on n and α.

Proof. As in the proof of [Lemma 1](#), choose $\{x_1, x_2, \ldots, x_n\}$ to be a normal coordinate system at a fixed point. Subscripts i, j and k will denote covariant derivatives in the x_i, x_j and x_k directions.

We will calculate the evolution equation for F_1. The computation is a little long, so we divide it into three parts.

Part 1. We calculate the equation for $|\nabla^2 u|/u$.

It follows from [Li 1991] that

$$(3\text{-}2) \quad \Delta\left(\frac{|\nabla^2 u|}{u}\right) = \frac{\sum u_{ijk}^2 + \sum u_{ij}u_{ijkk}}{u|\nabla^2 u|} - 2\frac{\sum u_{ij}u_{ijk}u_k}{u^2|\nabla^2 u|} - \frac{\sum_k \left(\sum_{ij} u_{ij}u_{ijk}\right)^2}{u|\nabla^2 u|^3}$$
$$- \frac{|\nabla^2 u|\Delta u}{u^2} + 2\frac{|\nabla^2 u||\nabla u|^2}{u^3},$$

$$(3\text{-}3) \quad \partial_t\left(\frac{|\nabla^2 u|}{u}\right) = \frac{\partial_t(|\nabla^2 u|^2)}{2u|\nabla^2 u|} - \frac{|\nabla^2 u|u_t}{u^2}.$$

Noting the metric evolves by the Ricci flow, we have

$$\partial_t(|\nabla^2 u|^2) = 4\sum \text{Ric}(\nabla u_j, \nabla u_j) + 2\sum u_{ijt}u_{ij},$$
$$u_{ijt} = \partial_t(u_{ij} - \Gamma_{ij}^l u_l) = u_{tij} + \sum_l u_l(\nabla_i \text{Ric}_{jl} + \nabla_j \text{Ric}_{il} - \nabla_l \text{Ric}_{ij}).$$

The Ricci identity gives

$$u_{ijkk} = u_{kkij} + \sum_l \left(R_{kikl,j}u_l + R_{kikl}u_{lj} + R_{kjil}u_{lk} + R_{kjkl}u_{li} + R_{ijkl,k}u_l + R_{ijkl}u_{lk}\right).$$

Combining the above and using the Schwarz inequality, we conclude

$$(\Delta - \partial_t)\left(\frac{|\nabla^2 u|}{u}\right) \geq -2\nabla\left(\frac{|\nabla^2 u|}{u}\right) \cdot \nabla \log u - Ck_1 \frac{|\nabla^2 u|}{u} - Ck_2 \frac{|\nabla u|}{u}.$$

Part 2. We calculate the equation for $|\nabla u|^2/u^2$. A direct computation shows

$$(\Delta - \partial_t)\left(\alpha \frac{|\nabla u|^2}{u^2}\right)$$
$$= 2\alpha \frac{\sum u_{ij}^2}{u^2} + 2\alpha \frac{\sum u_i u_{jji}}{u^2} + 2\alpha \frac{\text{Ric}(\nabla u, \nabla u)}{u^2} - 8\alpha \frac{\sum u_{ij}u_i u_j}{u^3} - 2\alpha \frac{\sum u_i^2 u_{jj}}{u^3}$$
$$+ 6\alpha \frac{\sum u_i^2 u_j^2}{u^4} - 2\alpha \frac{\text{Ric}(\nabla u, \nabla u)}{u^2} - 2\alpha \frac{\sum u_i u_{it}}{u^2} + 2\alpha \frac{\sum u_i^2 u_t}{u^3}$$
$$= 2\alpha \frac{\sum u_{ij}^2}{u^2} - 8\alpha \frac{\sum u_{ij}u_i u_j}{u^3} + 6\alpha \frac{\sum u_i^2 u_j^2}{u^4}$$
$$= -2\nabla\left(\alpha \frac{|\nabla u|^2}{u^2}\right) \cdot \nabla \log u + 2\alpha \frac{\sum u_{ij}^2}{u^2} - 4\alpha \frac{\sum u_{ij}u_i u_j}{u^3} + 2\alpha \frac{\sum u_i^2 u_j^2}{u^4}.$$

In the above, the two Ricci curvature terms generated by the Ricci identity and the evolution of the metric are canceled. Applying Young's inequality, that is,

$$4\alpha \frac{\sum u_{ij}u_iu_j}{u^3} \le 2(1-\delta)\alpha \frac{\sum u_{ij}^2}{u^2} + \frac{2\alpha}{1-\delta}\left(\frac{\sum u_i^2}{u^2}\right)^2 \quad \text{for any } 0 < \delta < 1,$$

we conclude

$$(\Delta - \partial_t)\left(\alpha \frac{|\nabla u|^2}{u^2}\right) \ge -2\nabla\left(\alpha \frac{|\nabla u|^2}{u^2}\right) \cdot \nabla \log u + 2\delta\alpha \frac{\sum u_{ij}^2}{u^2} - \frac{2\delta\alpha}{1-\delta}\left(\frac{\sum u_i^2}{u^2}\right)^2.$$

Part 3. Using the formula (2-4) and Young's inequality, we get for any $\varepsilon > 0$,

$$(\Delta - \partial_t)\left(\beta \frac{u_t}{u}\right) \ge -2\nabla\left(\beta \frac{u_t}{u}\right) \cdot \nabla \log u - \frac{\beta}{\varepsilon}\frac{\sum u_{ij}^2}{u^2} - \beta\varepsilon \sum \text{Ric}_{ij}^2.$$

Combining Parts 1–3 and using (2-6), we obtain for any $0 < \delta < 1$ and $\varepsilon > 0$

$$(3\text{-}4) \quad (\Delta - \partial_t)F_1 \ge -2\nabla F_1 \cdot \nabla \log u - Ck_1 \frac{|\nabla^2 u|}{u} + \left(\delta\alpha - \frac{\beta}{\varepsilon}\right)\frac{\sum u_{ij}^2}{u^2} + \delta\alpha \frac{\sum u_{ij}^2}{u^2}$$
$$- \frac{2\delta\alpha}{1-\delta}\left(\frac{\sum u_i^2}{u^2}\right)^2 - Ck_2 \frac{|\nabla u|}{u} - 4\beta\varepsilon n^2 k_1^2.$$

By the definition of F_1, we have

$$(3\text{-}5) \qquad\qquad \frac{|\nabla^2 u|}{u} \le F_1 + \beta \frac{u_t}{u},$$

and

$$\frac{\sum u_{ij}^2}{u^2} = \left(F_1 - \alpha \frac{|\nabla u|^2}{u^2} + \beta \frac{u_t}{u}\right)^2$$
$$(3\text{-}6) \quad = F_1^2 + \alpha^2\left(\frac{\sum u_i^2}{u^2}\right)^2 + \beta^2 \frac{u_t^2}{u^2} + 2\beta F_1 \frac{u_t}{u} - 2\alpha F_1 \frac{\sum u_i^2}{u^2} - 2\alpha\beta \frac{\sum u_i^2}{u^2}\frac{u_t}{u}.$$

Inserting (3-5) and (3-6) into (3-4) and applying Young's inequality to separate the mixed items, we arrive at

$$(3\text{-}7) \quad (\Delta - \partial_t)F_1 \ge -2\nabla F_1 \cdot \nabla \log u - Ck_1 F_1 - \tfrac{1}{4}Ck_1^2/(\gamma - 1)^2$$
$$+ \left(\tfrac{1}{2}\delta\alpha\beta^2 - C\beta^2(\gamma - 1)^2\right)\frac{u_t^2}{u^2} + \left(\delta\alpha - \frac{\beta}{\varepsilon}\right)\frac{\sum u_{ij}^2}{u^2}$$
$$- \left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right)\left(\frac{\sum u_i^2}{u^2}\right)^2 + \delta\alpha F_1^2$$
$$+ 2\delta\alpha\beta F_1 \frac{u_t}{u} - 2\delta\alpha^2 F_1 \frac{\sum u_i^2}{u^2} - C\delta^{-1/3}k_2^{4/3} - 4\beta\varepsilon n^2 k_1^2,$$

for any $\gamma - 1 > 0$.

Using the inequality

$$(3\text{-}8) \qquad \left(\frac{|\nabla u|^2}{u^2}\right)^2 \le 2\left(\frac{|\nabla u|^2}{u^2} - \gamma\frac{u_t}{u}\right)^2 + 2\gamma^2 u_t^2/u^2,$$

we calculate

$$(3\text{-}9) \quad \left(\tfrac{1}{2}\delta\alpha\beta^2 - C\beta^2(\gamma-1)^2\right)\frac{u_t^2}{u^2} - \left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right)\left(\frac{\sum u_i^2}{u^2}\right)^2$$

$$\ge \left(\tfrac{1}{2}\delta\alpha\beta^2 - 2\gamma^2\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right) - C\beta^2(\gamma-1)^2\right)\frac{u_t^2}{u^2}$$

$$- 2\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right)\left(\frac{|\nabla u|^2}{u^2} - \gamma\frac{u_t}{u}\right)^2.$$

Setting $\beta = 5\alpha$, we check that

$$\tfrac{1}{2}\delta\alpha\beta^2 - 2\gamma^2\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right) - C\beta^2(\gamma-1)^2$$

$$= 8\delta\alpha^3\left(\tfrac{25}{16} - \gamma^2\right) - \frac{\delta}{\alpha(1-\delta)^2}\gamma^2 - C\beta^2(\gamma-1)^2$$

$$\ge 0 \quad \text{when } \delta > 0 \text{ and } \gamma - 1 > 0 \text{ are sufficiently small.}$$

Then we can take $\varepsilon \ge 5/\delta$ such that $\delta\alpha - \beta/\varepsilon \ge 0$.

Consequently, (3-7) becomes

$$(3\text{-}10) \quad (\Delta - \partial_t)F_1 \ge -2\nabla F_1 \cdot \nabla \log u - Ck_1 F_1 - \frac{Ck_1^2}{4(\gamma-1)^2} + \delta\alpha F_1^2$$

$$- 2\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right)\left(\frac{|\nabla u|^2}{u^2} - \gamma\frac{u_t}{u}\right)$$

$$+ 2\delta\alpha\beta F_1 u_t/u - 2\delta\alpha^2 F_1\frac{\sum u_i^2}{u^2} - Ck_2^{4/3} - 4\beta\varepsilon n^2 k_1^2.$$

We complete the lemma by applying Corollary 1 and noting that

$$(\Delta - \partial_t)F = t(\Delta - \partial_t)F_1 - F_1. \qquad \square$$

Proof of Theorem 4. As in the proof of Theorem 1, $(M, g(t))$ is complete for $t \in [0, T]$. Let

$$\rho(x, t) = d(x, x_0, t) \quad \text{and} \quad \varphi(x, t) = \psi(\rho(x, t)/R).$$

Let $\varphi F(x, t) := \psi(\rho(x, t)/R)F(x, t)$, where $(x, t) \in Q_{2R,T}$. Suppose (x_1, t_1) is the point where φF achieves its maximum in Q_{2R,T_1}, where $0 < T_1 \le T$.

If $|\nabla^2 u(x_1, t_1)| = 0$, then by Corollary 1, we have

$$(3\text{-}11) \qquad (\varphi F)(x_1, t_1) = \varphi t_1 \left(\alpha \frac{|\nabla u|^2}{u^2} - \beta \frac{u_t}{u} \right) \le C_{n,\alpha,\beta}\,(k_1 t_1 + 1),$$

which implies (3-1).

Using the arguments of [Calabi 1958] and [Li 1991], we can assume that φF is smooth at (x_1, t_1) and that $\varphi F(x_1, t_1) > 0$.

As in the proof of Theorem 1, at the point (x_1, t_1), we have

$$(3\text{-}12) \qquad \begin{aligned} 0 &\ge (\Delta - \partial_t)(\varphi F) \\ &\ge \left(-\frac{C}{R^2} - \frac{C}{R}\sqrt{k_1} \right) F - Ck_1 F + \varphi(\Delta_X - \partial_t)F. \end{aligned}$$

Applying Lemma 2 and Young's inequality, we obtain for $s > 0$ that

$$(3\text{-}13) \quad \begin{aligned} 0 \ge{} & \left(-\frac{C}{R^2} - \frac{C}{R}\sqrt{k_1} \right) F - Ck_1 F - \frac{F|\nabla\varphi|^2}{2s\varphi} - 2Fs\varphi\frac{|\nabla u|^2}{u^2} \\ & - Ck_1\varphi F + \frac{\delta\alpha}{t_1}\varphi F^2 - 2C\varphi t_1\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2} \right)(1/t_1 + k_1)^2 \\ & + 2\delta\alpha\beta\varphi Fu_t/u - 2\delta\alpha^2\varphi F\frac{\sum u_i^2}{u^2} - \frac{Ck_1^2}{4(\gamma-1)^2}\varphi t_1 \\ & - Ck_2^{4/3}\varphi t_1 - 4\beta\varepsilon n^2\varphi t_1 k_1^2 - \varphi F/t_1. \end{aligned}$$

Using Corollary 1, we have

$$(3\text{-}14) \quad \begin{aligned} 2\delta\alpha\beta\varphi Fu_t/u &- 2\delta\alpha^2\varphi F\frac{\sum u_i^2}{u^2} - 2Fs\varphi\frac{|\nabla u|^2}{u^2} \\ &\ge (2\delta\alpha\beta - 2s\gamma - 2\delta\alpha^2\gamma)\varphi Fu_t/u - C(2\delta\alpha^2 + 2s)\varphi F(1/t_1 + K_1). \end{aligned}$$

Observe that $2\delta\alpha\beta - 2s\gamma - 2\delta\alpha^2\gamma = 0$ when we set $s = \delta\alpha^2(4/\gamma - 1)$. Then (3-13) becomes

$$(3\text{-}15) \quad \begin{aligned} 0 \ge{} & \frac{\delta\alpha}{t_1}\varphi F^2 - \varphi FC(2s + 2\delta\alpha^2)\,(1/t_1 + k_1) - \frac{CF}{2sR^2} + \left(-\frac{C}{R^2} - \frac{C}{R}\sqrt{k_1} \right)F \\ & - Ck_1 F - Ck_1\varphi F - \varphi F/t_1 - \frac{Ck_1^2}{4(\gamma-1)^2}\varphi t_1 - 4\beta\varepsilon n^2\varphi t_1 k_1^2 \\ & - Ck_2^{4/3}\varphi t_1 - 2C\varphi t_1\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2} \right)(1/t_1 + k_1)^2. \end{aligned}$$

Multiplying through by φt_1 and using $0 \leq \varphi \leq 1$, we have

$$
\begin{aligned}
(3\text{-}16) \qquad 0 \geq {} & \delta\alpha(\varphi F)^2 - (\varphi F)\left(\left(\frac{C}{R^2} + \frac{C}{R}\sqrt{k_1}\right)t_1 + (8C\delta\alpha^2/\gamma)(1/t_1 + k_1)t_1\right) \\
& - (\varphi F)\left(\frac{Ct_1}{2\delta\alpha^2(4/\gamma - 1)R^2} + (1 + Ck_1t_1)\right) - \frac{Ck_1^2}{4(\gamma-1)^2}t_1^2 \\
& - 2C\left(4\delta\alpha^3 + \frac{\delta}{2\alpha(1-\delta)^2}\right)(1/t_1 + k_1)^2 t_1^2 - 4\beta\varepsilon n^2 k_1^2 t_1^2 - Ck_2^{4/3}t_1^2.
\end{aligned}
$$

Applying the quadratic formula, we get

$$
(3\text{-}17) \qquad \varphi F(x_1, t_1) \leq C(1 + k_1 t_1 + k_2^{2/3}t_1 + (C/R^2)t_1).
$$

By an argument similar to one in the proof of Theorem 1, we conclude

$$
(3\text{-}18) \qquad F_1(x, t) \leq C(k_1 + k_2^{2/3} + 1/t + 1/R^2) \quad \text{in } Q_{R,T},
$$

where C depends on n and α. Because M is noncompact, we can let $R \to +\infty$. This completes the proof of Theorem 4. $\qquad\square$

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