Pacific Journal of Mathematics

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Volume 243 No. 2 December 2009

PACIFIC JOURNAL OF MATHEMATICS
Vol. 243, No. 2, 2009

THE LANDAU–LIFSHITZ–MAXWELL EQUATION IN DIMENSION THREE

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In dimension three, we show the existence of weak solutions (u, H, E) to the Landau–Lifshitz equation coupled with the time-dependent Maxwell equation such that u is Hölder continuous away from a closed set Σ that has locally finite 3-dimensional parabolic Hausdorff measure. For two reduced Maxwell equations, Hölder continuity of ∇u away from Σ is also established.

1. Introduction

For a bounded, smooth domain $\Omega \subseteq \mathbb{R}^3$, we consider the Landau–Lifshitz–Maxwell equation:

$$\frac{\partial u}{\partial t} = \beta_1 u \times (\Delta u + H) - \beta_2 u \times (u \times (\Delta u + H)) \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1-1}$$

$$\nabla \times H = \varepsilon_0 \frac{\partial E}{\partial t} + \sigma E \qquad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \tag{1-2}$$

$$\nabla \times E = -\frac{\partial}{\partial t}(H + \beta \bar{u}) \qquad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \tag{1-3}$$

where $u : \Omega \times \mathbb{R}_+ \to S^2$ is the magnetization field, $H : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$ is the magnetic field, $E : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$ is the electric field, $H^e \equiv \triangle Z + H$ is the effective magnetic field, β_1 is the gyromagnetic coefficient, $\beta_2 \geq 0$ is the Gilbert damping coefficient, $\varepsilon_0 \geq 0$, $\sigma \geq 0$ is the conductivity constant, β is the magnetic permeability of free space, and $\bar{u}$ is an extension of u such that $\bar{u} = 0$ outside Ω. The system (1-1)–(1-3) was originally proposed by Landau and Lifshitz [1935] to model the dynamics of magnetization, magnetic field and electric field for the ferromagnetic materials.

The coupled Maxwell equations (1-2) and (1-3) can be written as

$$\frac{\partial B}{\partial t} = -\nabla \times E \quad \text{and} \quad \frac{\partial D}{\partial t} + \sigma E = \nabla \times H \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \tag{1-4}$$

MSC2000: primary 35K55; secondary 35B65.
Keywords: Landau–Lifshitz–Gilbert equation, time-dependent Maxwell equation.
Ding is partially supported by the NSFC Grant 10471050, National 973 Program of China (No. 2006CB805902), Guangdong Provincial Natural Science Foundation (No. 7005795), and by University Special Research Foundation for Ph.D. Program (No. 20060574002). Liu is partially supported by NSFC 10631020. Wang is partially supported by NSF 0601162.

where D and B are the electric and magnetic displacements, given by

$$D = \varepsilon_0 E, \quad B = H + \beta \bar{u} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+. \tag{1-5}$$

Note that when $H = E = 0$ and $\beta = 0$, the system (1-1)–(1-3) reduces to the Landau–Lifshitz–Gilbert equation for $Z : \Omega \times \mathbb{R}_+ \to S^2$:

$$\frac{\partial Z}{\partial t} = \beta_1 Z \times \triangle Z - \beta_2 Z \times (Z \times \triangle Z). \tag{1-6}$$

Equation (1-6) is a hybrid between the Schrödinger flow into S^2 ($\partial u/\partial t = u \times \Delta u$ for $\beta_2 = 0$) and the heat flow of a harmonic map into S^2 ($\partial u/\partial t = \Delta u + |\nabla u|^2 u$ for $\beta_1 = 0$). There have been many works on both the existence and regularity of weak solutions to (1-6) in recent years. Zhou and Guo [1987] proved the existence of global weak solutions of (1-6) under suitable initial-boundary conditions. The unique smooth solution of (1-6) in dimension one was established in [Zhou et al. 1991]. Alouges and Soyeur [1992] proved that given $0 < \beta_2$ and the initial data $u_0 : \mathbb{R}^3 \to S^2$ with $\nabla u_0 \in L^2(\mathbb{R}^3)$, there exists a global weak solution of (1-6) in $\mathbb{R}^3$. Moreover, if $u_0 \in H^1(\Omega)$ and $\beta_2 > 0$, the Neumann boundary value problem of (1-6) in a bounded domain $\Omega \subset \mathbb{R}^3$ may admit infinitely many weak solutions. For regularity of weak solutions to (1-6), Guo and Hong [1993] established the existence of a global, weak solution with finitely many singular points in dimension two, and Chen, Ding and Guo [1998] proved the uniqueness of weak solutions whose energies are nonincreasing in time at dimension two. In dimension three, Melcher [2005] proved the existence of global weak solutions to (1-6) for $\Omega = \mathbb{R}^3$, which are smooth away from a closed set of locally finite 3-dimensional parabolic Hausdorff measure. Later, Wang [2006] established the existence of partially smooth weak solutions to (1-6) in any bounded domain Ω of dimension at most 4. It is unknown whether these results of Melcher and Wang can be extended to higher dimensions. It is also an interesting question to study the regularity of *suitable* weak solutions to (1-6). Moser [2002] proved, in dimensions $n \le 4$, a partial regularity theorem of weak solutions to (1-6) that are *stationary*, a notion analogous to that of heat flow of harmonic maps introduced in [Feldman 1994; Chen et al. 1995; Chen and Wang 1996] (see also [2004]). More recently, Ding and Wang [2007] proved that short-time, smooth solutions to (1-6) may develop a finite-time singularity in dimensions 3 and 4 for suitable initial-boundary data.

Motivated by these studies of (1-6), we are interested in the Landau–Lifshitz system coupled with the time-dependent Maxwell equations (1-1)–(1-3).

There has been some work on the system (1-1)–(1-3). Guo and Su [1997] used Galerkin's method to establish the existence of global, weak solutions with periodic initial conditions in dimension three. Carbou and Fabrie [1998] used the Ginzburg–Landau approximation scheme to show the existence of global, weak solutions to the system (1-1)–(1-3) under the Neumann boundary condition in dimension three, and studied the long-time behavior of the weak solution by the method of time average. See also [Joly et al. 2000; Ding et al. 2007] for related results.

The regularity issue of the system (1-1)–(1-3) is a challenging problem. There are very few results in the literature. Ding and Guo [2004] proved a partial regularity theorem for stationary solutions to the Landau–Lifshitz equation (1-1) coupled with the quasistationary Maxwell equation

$$\operatorname{div}(H+\beta\bar{u})=0 \quad \text{and} \quad \nabla\times H=0 \ \text{ in } \mathscr{D}'(\mathbb{R}^3). \tag{1-7}$$

By modifying the techniques by [Wang 2006], Ding and Guo [2008] proved the existence of partially smooth weak solutions to (1-1)+(1-7) in dimension three.

There is an essential difference between (1-7) and (1-2)–(1-3): the former is elliptic and $H\in\bigcap_{p>1}L^p(\mathbb{R}^3,\mathbb{R}^3)$, while (1-2)–(1-3) is a hyperbolic system and the regularity for $H(\cdot,t)$ and $E(\cdot,t)$ is no better than that of $H(\cdot,0)$ and $E(\cdot,0)$. The hyperbolicity of (1-2) and (1-3) imposes serious difficulties to studying the regularity of (1-1).

In this paper, we establish the existence of partially regular, weak solutions of the Landau–Lifshitz–Maxwell system (1-1)–(1-3) with respect to the following initial-boundary conditions:

$$\frac{\partial u}{\partial\nu}=0 \qquad \text{on } \ \Omega\times\mathbb{R}_+, \tag{1-8}$$

$$u(x,0)=u_0(x) \qquad \text{in } \Omega, \tag{1-9}$$

$$H(x,0)=H_0(x) \qquad \text{in } \mathbb{R}^3, \tag{1-10}$$

$$E(x,0)=E_0(x) \qquad \text{in } \mathbb{R}^3. \tag{1-11}$$

We assume throughout the paper that

$$|u_0|=1 \ \text{ a.e. in } \Omega, \qquad H_0\in L^2(\mathbb{R}^3,\mathbb{R}^3), \qquad E_0\in L^2(\mathbb{R}^3,\mathbb{R}^3). \tag{1-12}$$

For convenience, we study an equivalent form of the Landau–Lifshitz–Maxwell equation (1-1) (see [Guo and Hong 1992; 1993]):

$$\alpha_1\frac{\partial u}{\partial t}+\alpha_2 u\times\frac{\partial u}{\partial t}=\big(\Delta u+|\nabla u|^2u\big)+(H-\langle H,u\rangle\, u) \quad \text{in } \Omega\times\mathbb{R}_+, \tag{1-13}$$

where $\alpha_1,\alpha_2\in\mathbb{R}$ represent suitable normalizations of β_2 and β_1 such that

$$0<\alpha_1<1, \quad \alpha_1^2+\alpha_2^2=1.$$

Definition 1.1. The triple (u,H,E) is a weak solution of the system (1-13)+(1-2)–(1-3) with the initial-boundary conditions (1-8)–(1-11) if

(i) $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+,H^1(\Omega,S^2))$, $\partial u/\partial t\in L^2_{\mathrm{loc}}(\Omega\times\mathbb{R}_+)$ and $H,E\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+,L^2(\mathbb{R}^3))$;

(ii) u satisfies (1-13) in the distribution sense, that is, for any $\Phi \in C^\infty(\Omega\times\mathbb{R}_+, \mathbb{R}^3)$ with $\Phi(\cdot\,,0) = \Phi(\cdot\,,+\infty) = 0$,

$$\int_{\Omega\times\mathbb{R}_+}\left(\alpha_1\frac{\partial u}{\partial t}+\alpha_2 u\times\frac{\partial u}{\partial t}\right)\cdot\Phi = \int_{\Omega\times\mathbb{R}_+}\left(-\nabla u\cdot\nabla\Phi+|\nabla u|^2u\cdot\Phi\right) + \int_{\Omega\times\mathbb{R}_+}(H-\langle H,u\rangle\, u)\cdot\Phi \tag{1-14}$$

and $u(\cdot\,,0) = u_0$ in the sense of trace;

(iii) for any $\Phi \in C^\infty(\mathbb{R}^3\times\mathbb{R}_+, \mathbb{R}^3)$ with $\Phi(\cdot\,,+\infty) = 0$,

$$-\int_{\mathbb{R}^3\times\mathbb{R}_+}\left(\varepsilon_0 E\cdot\frac{\partial\Phi}{\partial t}+H\cdot\nabla\times\Phi\right)+\sigma\int_{\mathbb{R}^3\times\mathbb{R}_+}E\cdot\Phi = \varepsilon_0\int_{\mathbb{R}^3}E_0(x)\cdot\Phi(x,0); \tag{1-15}$$

(iv) for any $\Phi \in C^\infty(\mathbb{R}^3\times\mathbb{R}_+, \mathbb{R}^3)$ with $\Phi(\cdot\,,+\infty) = 0$,

$$-\int_{\mathbb{R}^3\times\mathbb{R}_+}(H+\beta\bar{u})\cdot\frac{\partial\Phi}{\partial t}+\int_{\mathbb{R}^3\times\mathbb{R}_+}E\cdot\nabla\times\Phi = \beta\int_\Omega u_0(x)\cdot\Phi(x,0)+\int_{\mathbb{R}^3}H_0(x)\cdot\Phi(x,0). \tag{1-16}$$

To state our results, we also need some notation. For $z_0 = (x_0, t_0) \in \mathbb{R}^3\times\mathbb{R}$ and $r > 0$, set

$$B_r(x_0) = \left\{x\in\mathbb{R}^3 : |x-x_0| < r_0\right\} \quad\text{and}\quad P_r(z_0) = B_r(x_0)\times(t_0-r^2, t_0).$$

For any subset $D\subset\mathbb{R}^4$, the three-dimensional parabolic Hausdorff measure, $\mathcal{P}^3(D)$, is defined by

$$\mathcal{P}^3(D) = \lim_{\delta\downarrow 0}\left(\inf\left\{\sum_{i=1}^\infty r_i^3 :\ D\subset\bigcup_{i=1}^\infty P_{r_i}(z_i),\ 0<r_i\le\delta\right\}\right).$$

We say a subset $D\subset\mathbb{R}^4$ has locally finite three-dimensional parabolic Hausdorff measure if

$$\mathcal{P}^3\left(D\cap P_R(0)\right) < +\infty \quad\text{for all } R > 0.$$

Theorem 1.2. *For any $u_0\in H^1(\Omega, S^2)$, $H_0\in L^2(\mathbb{R}^3,\mathbb{R}^3)$ and $E_0\in L^2(\mathbb{R}^3,\mathbb{R}^3)$, there exists a global weak solution (u, H, E) to the Landau–Lifshitz–Maxwell system (1-13)+(1-2)–(1-3) under the initial-boundary conditions (1-8)–(1-11) such that there exists a closed subset $\Sigma\subset\Omega\times\mathbb{R}_+$ having locally finite 3-dimensional parabolic Hausdorff measure and satisfying $u\in C^{1/2}(\Omega\times\mathbb{R}_+\setminus\Sigma, S^2)$.*

To study the higher-order regularity of weak solutions to (1-13)+(1-2)–(1-3) obtained by Theorem 1.2, we restrict to two special cases:

(i) The constant ε_0 vanishes in (1-2), and (1-2)–(1-3) become

$$\nabla \times (\nabla \times H) = -\sigma \frac{\partial}{\partial t}(H + \beta \bar{u}) \quad \text{in } \mathbb{R}^3. \tag{1-17}$$

(ii) The constant β vanishes in (1-3), and (1-2)–(1-3) become

$$\nabla \times H = \varepsilon_0 \frac{\partial E}{\partial t} + \sigma E, \quad \nabla \times E = -\frac{\partial H}{\partial t} \quad \text{in } \mathbb{R}^3. \tag{1-18}$$

Theorem 1.3. *Assume that $\delta > 0$ is a constant. For any $u_0 \in H^1(\Omega, S^2)$ and $H_0 \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ satisfying $\nabla \cdot (H_0 + \beta \bar{u}_0) = 0$ in $\mathscr{D}'(\mathbb{R}^3)$, there exists a weak solution (u, H) of the Landau–Lifshitz system (1-13) coupled with (1-17) under the initial-boundary conditions (1-8)–(1-10) such that $H \in \bigcap_{T>0} H^1(\mathbb{R}^3 \times [0, T], \mathbb{R}^3)$ and there exists a closed subset $\Sigma \subset \Omega \times \mathbb{R}_+$ of locally finite 3-dimensional parabolic Hausdorff measure and satisfying $\nabla u \in C^\alpha(\Omega \times \mathbb{R}_+ \setminus \Sigma)$ for some $0 < \alpha < 1$, and $\nabla^2 u, \partial u/\partial t \in L^6_{\text{loc}}(\Omega \times \mathbb{R}_+ \setminus \Sigma)$.*

Theorem 1.4. *For any $u_0 \in H^1(\Omega, S^2)$ and $H_0, E_0 \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ that satisfy $\nabla \cdot H_0 = \nabla \cdot E_0 = 0$ in $\mathscr{D}'(\mathbb{R}^3)$, there exists a weak solution (u, H, E) of the Landau–Lifshitz system (1-13) coupled with (1-18) under the initial-boundary conditions (1-8)–(1-11) such that $\partial H/\partial t, \partial E/\partial t \in L^\infty_{\text{loc}}(\mathbb{R}_+, L^2(\mathbb{R}^3))$, and there exists a closed subset $\Sigma \subset \Omega \times \mathbb{R}_+$ having locally finite 3-dimensional parabolic Hausdorff measure and satisfying $\nabla u \in C^\alpha(\Omega \times \mathbb{R}_+ \setminus \Sigma)$ for some $0 < \alpha < 1$, and $\nabla^2 u, \partial u/\partial t \in L^6_{\text{loc}}(\Omega \times \mathbb{R}_+ \setminus \Sigma)$.*

The ideas to approach these theorems are based on an analysis of the Ginzburg–Landau approximate equation: for $\varepsilon > 0$,

$$\alpha_1 \frac{\partial u^\varepsilon}{\partial t} + \alpha_2 u^\varepsilon \times \frac{\partial u^\varepsilon}{\partial t} = \Delta u^\varepsilon + \frac{1}{\varepsilon^2}(1 - |u^\varepsilon|^2) u^\varepsilon + u^\varepsilon \times (H^\varepsilon \times u^\varepsilon) \ \text{ in } \Omega \times \mathbb{R}_+. \tag{1-19}$$

By using an argument similar to the one in [Ding and Wang 2007], it is not hard to see that the corresponding partial regularity property at the boundary also holds for the weak solution obtained in Theorems 1.2, 1.3, and 1.4. For example, Theorem 1.2 can be extended so that there exists a closed subset $\Sigma_1 \subseteq \partial\Omega$, with $\mathscr{P}^3(\Sigma_1) < +\infty$, such that $u \in C^{1/2}(\overline{\Omega} \setminus (\Sigma \cup \Sigma_1), S^2)$.

The paper is written as follows. In Section 2, we establish a uniform energy estimate for (1-19). In Section 3, we sketch the time slice monotonicity. In Section 4 we establish a lower bound estimate of solutions to (1-19). In Section 5, we obtain the decay estimate of solutions to (1-19) under the smallness condition and prove Theorem 1.2. In Section 6, we establish a partial C^α-regularity of ∇u and prove both Theorems 1.3 and 1.4.

2. Estimate of the energy in (1-19)

In this section, we sketch the existence of global weak solutions to (1-19)+(1-2)–(1-3), associated with (1-8)–(1-11) by Galerkin's method and their corresponding energy estimates. Here we modify an argument from [Carbou and Fabrie 1998] to handle (1-19). Note the difference between (1-19) and the approximate equation employed by that paper: we approximate $H - \langle H, u\rangle u$ in (1-13) by $u^\varepsilon \times (H^\varepsilon \times u^\varepsilon)$ in (1-19), while Carbou and Fabrie approximate the same term in (1-13) by H^ε. An advantage of our approximation is that we have the upper bound $|u^\varepsilon| \le 1$, which plays a crucial role in establishing *a priori* continuity estimates for u^ε, and hence the existence of partially smooth solutions, while theirs yields an optimal energy inequality [Carbou and Fabrie 1998, page 387, (2.12)], which is important in their study of long-time behaviors by the method of time average.

We begin with a general L^∞-estimate of weak solutions u^ε to (1-19).

Lemma 2.1. *For $\varepsilon > 0$, assume $u_0 \in H^1(\Omega, S^2)$, $H_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ and $E_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$. Let $(u^\varepsilon, H^\varepsilon, E^\varepsilon)$ be any weak solution of (1-19)+(1-2)–(1-3) under conditions (1-8)–(1-11). Then $|u^\varepsilon|(x,t) \le 1$ for any $(x,t) \in \Omega \times \mathbb{R}_+$.*

Proof. Multiplying (1-19) by u^ε and using the equalities $u^\varepsilon \cdot u^\varepsilon \times (\partial u^\varepsilon/\partial t) = 0$ and $u^\varepsilon \cdot u^\varepsilon \times (H^\varepsilon \times u^\varepsilon) = 0$, we have

$$\left(\alpha_1 \frac{\partial}{\partial t} - \Delta\right)(|u^\varepsilon|^2 - 1) = -2\left(|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon^2}(|u^\varepsilon|^2 - 1)|u^\varepsilon|^2\right); \tag{2-1}$$

hence

$$\left(\alpha_1 \frac{\partial}{\partial t} - \Delta\right)(|u^\varepsilon|^2 - 1)_+ \le 0 \quad \text{in } \Omega \times \mathbb{R}_+,$$

where $(|u^\varepsilon|^2 - 1)_+$ is the positive part of $(|u^\varepsilon|^2 - 1)$. The conclusion now holds by the weak maximum principle of the heat equation [Lieberman 1996]. □

Now we sketch the existence of weak solutions to (1-19) that enjoy energy estimates by Galerkin's method. We borrow some notation from [Carbou and Fabrie 1998, pages 388–395]. Let $\{\phi_k\}_k \subseteq H^2(\Omega)$ be eigenfunctions of Δ with zero Neumann boundary condition that form an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $H^1(\Omega)$ and $H^2(\Omega)$. For $1 \le N < +\infty$, set $V_N = \operatorname{span}\{\phi_1, \dots, \phi_N\}$. Define

$$\mathbb{H}_{\text{curl}}(\mathbb{R}^3) = \left\{\psi \in L^2(\mathbb{R}^3, \mathbb{R}^3),\ \nabla \times \psi \in L^2(\mathbb{R}^3, \mathbb{R}^3)\right\}.$$

Let $\{\psi_k\}_k$ be an orthogonal basis of $\mathbb{H}_{\text{curl}}(\mathbb{R}^3)$ that is orthonormal in $L^2(\mathbb{R}^3)$ and $W_N = \operatorname{span}\{\psi_1, \dots, \psi_N\}$. Denote by $\Pi_{V_N} : L^2(\Omega) \to V_N$, and $\Pi_{W_N} : L^2(\Omega) \to W_N$ the orthogonal projections. Define the retraction map $\Pi : \mathbb{R}^3 \to B_1$ by setting

$$\Pi(p) = \begin{cases} p & \text{if } |p| \le 1, \\ p/|p| & \text{if } |p| > 1. \end{cases}$$

Define functions $u_N \in V_N$ and $H_N, E_N \in W_N$ by

(2-2)

$$u_N(x,t) = \sum_{k=1}^{N} v_k(t)\phi_k(x), \;\; H_N(x,t) = \sum_{k=1}^{N} h_k(t)\psi_k(x), \;\; E_N(x,t) = \sum_{k=1}^{N} e_k(t)\psi_k(x),$$

satisfying, for all $\Phi \in V_N$ and $\Psi \in W_N$,

$$\begin{aligned}\int_\Omega \left(\alpha_1 \frac{\partial u_N}{\partial t} + \alpha_2 u_N \times \frac{\partial u_N}{\partial t}\right)\cdot \Phi &= \int_\Omega \left(-\nabla u_N \cdot \nabla \Phi + \frac{1}{\varepsilon^2}(1-|u_N|^2)u_N\cdot\Phi\right) \\ &\quad + \int_\Omega \Pi(u_N)\times(H_N \times \Pi(u_N))\cdot \Phi,\end{aligned} \tag{2-3}$$

$$\int_{\mathbb{R}^3}\left(\varepsilon_0 \frac{\partial E_N}{\partial t} + \sigma E_N\right)\cdot \Psi = \int_{\mathbb{R}^3} H_N \cdot (\nabla\times\Psi), \tag{2-4}$$

$$\int_{\mathbb{R}^3} \frac{\partial}{\partial t}(H_N + \beta \bar{u}_N)\cdot\Psi = -\int_{\mathbb{R}^3} E_N\cdot(\nabla\times\Psi), \tag{2-5}$$

under the initial conditions

$$u_N\big|_{t=0} = \Pi_{V_N}(u_0), \quad H_N\big|_{t=0} = \Pi_{W_N}(H_0), \quad E_N\big|_{t=0} = \Pi_{W_N}(E_0). \tag{2-6}$$

Throughout this section, we will use the following fact:

$$\lim_{N\to\infty} \int_\Omega e_\varepsilon(u_N(0)) = \int_\Omega \frac{1}{2}|\nabla u_0|^2. \tag{2-7}$$

Note that (2-3)–(2-6) reduces to a system of first order ODEs for $(v_k, h_k, e_k)_k$. Moreover, since $P(u_N)(v) = \alpha_1 v + \alpha_2 u_N \times v : \mathbb{R}^3 \to \mathbb{R}^3$ is one-to-one, we can solve (2-3) for the derivative in time. Hence there exists a local solution (u_N, H_N, E_N) of (2-3)–(2-6). The following uniform estimate shows that (u_N, H_N, E_N) is also global in time and converges to a global weak solution of (1-19)+(1-2)–(1-3).

Lemma 2.2. *For $\varepsilon > 0$, assume $u_0 \in H^1(\Omega, S^2)$, $H_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ and $E_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$. Then there exists a global weak solution $(u^\varepsilon, H^\varepsilon, E^\varepsilon)$ to (1-19)+(1-2)–(1-3) under conditions (1-8)–(1-11), such that for any $0 < T < +\infty$ we have*

$$\sigma \int_0^T \int_{\mathbb{R}^3} |E^\varepsilon|^2 + \alpha_1 \int_0^T \int_\Omega \left|\frac{\partial u^\varepsilon}{\partial t}\right|^2 + \mathscr{E}_\varepsilon(T) \le e^{CT}\mathscr{E}_0, \tag{2-8}$$

where $C > 0$ depends only on β and α_1, and

$$\mathscr{E}_0 = \int_\Omega \tfrac{1}{2}|\nabla u_0|^2 + \int_{\mathbb{R}^3}\left(\tfrac{1}{2}\varepsilon_0|E_0|^2 + \tfrac{1}{2}|H_0|^2\right),$$

$$\mathscr{E}_\varepsilon(t) = \int_\Omega e_\varepsilon(u^\varepsilon(t)) + \int_{\mathbb{R}^3}\left(\tfrac{1}{2}\varepsilon_0|E^\varepsilon(t)|^2 + \tfrac{1}{2}|H^\varepsilon(t)|^2\right) \tag{2-9}$$

with $e_\varepsilon(u^\varepsilon(t)) = \frac{1}{2}|\nabla u^\varepsilon(t)|^2 + \frac{1}{4\varepsilon^2}\big(1-|u^\varepsilon(t)|^2\big)^2$.

Proof. We first establish the estimate (2-8) for Galerkin's approximate solutions (u_N, H_N, E_N). Then we employ this estimate to extract a subsequence that converges to a global weak solution $(u^\varepsilon, H^\varepsilon, E^\varepsilon)$ to (1-19)+(1-2)–(1-3).

Testing (2-3) with $\Phi = \partial u_N/\partial t$ and integrating over Ω gives

$$\int_\Omega \alpha_1 \Big|\frac{\partial u_N}{\partial t}\Big|^2 + \frac{d}{dt}\int_\Omega e_\varepsilon(u_N) = \int_\Omega \Pi(u_N)\times(H_N\times\Pi(u_N))\cdot\frac{\partial u_N}{\partial t} \le \int_\Omega |H_N|\Big|\frac{\partial u_N}{\partial t}\Big|, \tag{2-10}$$

where we used the inequalities $|\Pi(u_N)|\le 1$ and $\big|\Pi(u_N)\times(H_N\times\Pi(u_N))\big|\le|H_N|$.

Testing (2-4) with $\Psi = E_N$ and integrating over $\mathbb{R}^3$ gives

$$\int_{\mathbb{R}^3} \nabla\times H_N\cdot E_N = \frac{d}{dt}\int_{\mathbb{R}^3}\tfrac{1}{2}\varepsilon_0|E_N|^2 + \int_{\mathbb{R}^3}\sigma|E_N|^2. \tag{2-11}$$

Testing (2-5) with $\Psi = H_N$ and integrating over $\mathbb{R}^3$ gives

$$-\int_{\mathbb{R}^3} \nabla\times E_N\cdot H_N = \frac{d}{dt}\int_{\mathbb{R}^3}\tfrac{1}{2}|H_N|^2 + \beta\int_\Omega H_N\cdot\frac{\partial u_N}{\partial t}. \tag{2-12}$$

Adding together (2-11) and (2-12), and using the identity

$$\int_{\mathbb{R}^3}(\nabla\times H_N\cdot E_N - \nabla\times E_N\cdot H_N) = 0,$$

we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3}\big(\tfrac{1}{2}\varepsilon_0|E_N|^2+\tfrac{1}{2}|H_N|^2\big) + \sigma\int_{\mathbb{R}^3}|E_N|^2 = -\beta\int_\Omega H_N\cdot\frac{\partial u_N}{\partial t} \le \beta\int_\Omega |H_N|\Big|\frac{\partial u_N}{\partial t}\Big|. \tag{2-13}$$

Adding (2-10) and (2-13) together gives

$$\sigma\int_{\mathbb{R}^3}|E_N|^2 + \alpha_1\int_\Omega\Big|\frac{\partial u_N}{\partial t}\Big|^2 + \frac{d}{dt}\bigg(\int_\Omega e_\varepsilon(u_N) + \int_{\mathbb{R}^3}\big(\tfrac{1}{2}\varepsilon_0|E_N|^2+\tfrac{1}{2}|H_N|^2\big)\bigg) \le (1+\beta)\int_\Omega |H_N|\Big|\frac{\partial u_N}{\partial t}\Big| \le \frac{\alpha_1}{4}\int_\Omega\Big|\frac{\partial u_N}{\partial t}\Big|^2 + \frac{(1+\beta)^2}{\alpha_1}\int_{\mathbb{R}^3}|H_N|^2, \tag{2-14}$$

where we used the Cauchy–Schwarz inequality in the last step. Using Grönwall's inequality in (2-14) and integrating from $t = 0$ to $t = T$ gives

(2-15)

$$\sigma\int_0^T\int_{\mathbb{R}^3}|E_N|^2+\alpha_1\int_0^T\int_\Omega\Big|\frac{\partial u_N}{\partial t}\Big|^2+\Big(\int_\Omega e_\varepsilon(u_N)+\int_{\mathbb{R}^3}\big(\tfrac12\varepsilon_0|E_N|^2+\tfrac12|H_N|^2\big)\Big)(T)$$
$$\le e^{CT}\Big(\int_\Omega e_\varepsilon(u_N)+\int_{\mathbb{R}^3}\big(\tfrac12\varepsilon_0|E_N|^2+\tfrac12|H_N|^2\big)\Big)(0)$$
$$\le e^{CT}\Big(o(1)+\int_\Omega e_\varepsilon(u_0)+\int_{\mathbb{R}^3}\big(\tfrac12\varepsilon_0|E_0|^2+\tfrac12|H_0|^2\big)\Big)$$
$$=e^{CT}(\mathscr{E}_0+o(1)).$$

Here we have used (2-7) and $o(1)$ denotes a quantity that tends to 0 as N tends to ∞. It follows from the bound (2-15) that there exists a subsequence of (u_N, H_N, E_N), still written the same, such that for any $0<T<+\infty$,

$$u_N\rightharpoonup u^\varepsilon \text{ weak* in } L^\infty([0,T],H^1(\Omega)),\qquad \frac{\partial u_N}{\partial t}\rightharpoonup\frac{\partial u^\varepsilon}{\partial t} \text{ in } L^2(\Omega\times[0,T]),$$
$$E_N\rightharpoonup E^\varepsilon,\ H_N\rightharpoonup H^\varepsilon \text{ weak* in } L^\infty([0,T],L^2(\mathbb{R}^3)).$$

By Aubin's Lemma (see also [Carbou and Fabrie 1998]),

$$u_N\to u^\varepsilon \ \text{ strongly in } L^4(\Omega\times[0,T]).$$

Since $|\Pi(u_N)|\le|u_N|$ and $\int_\Omega|\nabla(\Pi(u_N))|^2\le\int_\Omega|\nabla u_N|^2$, we also have

$$\Pi(u_N)\to\Pi(u^\varepsilon) \ \text{ strongly in } L^4(\Omega\times[0,T]).$$

It is readily seen that (2-15) implies that $(u^\varepsilon, H^\varepsilon, E^\varepsilon)$ satisfies (2-8) and the initial conditions (1-8)–(1-11). It is also not hard to see that $(H^\varepsilon, E^\varepsilon)$ are weak solutions to (1-2)–(1-3). Similarly to [Carbou and Fabrie 1998, page 392], we can check that

$$\alpha_1\frac{\partial u^\varepsilon}{\partial t}+\alpha_2u^\varepsilon\times\frac{\partial u^\varepsilon}{\partial t}=\Delta u^\varepsilon+\frac{1}{\varepsilon^2}(1-|u^\varepsilon|^2)u^\varepsilon+\Pi(u^\varepsilon)\times(H^\varepsilon\times\Pi(u^\varepsilon)).\tag{2-16}$$

Multiplying (2-16) by u^ε and observing that $\Pi(u^\varepsilon)\times(H^\varepsilon\times\Pi(u^\varepsilon))\cdot u^\varepsilon=0$, we see that u^ε satisfies (2-1). Hence Lemma 2.1 implies that $|u^\varepsilon|\le 1$. Thus $\Pi(u^\varepsilon)=u^\varepsilon$ and (2-16) yields (1-19). □

In order to establish a partial C^α-regularity of ∇u for weak solutions u to (1-13) coupled with the Maxwell equations (1-17) or (1-18), we need uniform estimates of H^ε, E^ε in $H^1_{\mathrm{loc}}(\mathbb{R}^3\times\mathbb{R}_+)$. More precisely:

Lemma 2.3. *Suppose that $u_0\in H^1(\Omega,S^2)$ and $H_0\in H^1(\mathbb{R}^3,\mathbb{R}^3)$ satisfy the condition $\nabla\cdot(H_0+\beta\bar u_0)=0$ in $\mathscr{D}'(\mathbb{R}^3)$. Then there exists a global weak solution $(u^\varepsilon, H^\varepsilon)$ to (1-19)+(1-17) under the initial-boundary conditions (1-8)–(1-10) such* *that for any* $0 < T < +\infty$,

$$\text{(2-17)}\quad \alpha_1 \int_0^T \int_\Omega \left|\frac{\partial u^\varepsilon}{\partial t}\right|^2 + \int_0^T \int_{\mathbb{R}^3} \left(\left|\frac{\partial H^\varepsilon}{\partial t}\right|^2 + |\nabla H^\varepsilon|^2 \right) + \left(\int_\Omega e_\varepsilon(u^\varepsilon) + \int_{\mathbb{R}^3} \left(\tfrac{1}{2}\sigma |H^\varepsilon|^2 + |\nabla H^\varepsilon|^2\right) \right)(T) \le \int_{\mathbb{R}^3} |\nabla H_0|^2 + e^{CT} \left(\int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3} |H_0|^2 \right),$$

for some $C = C(\beta, \alpha_1) > 0$.

Proof. For $N \ge 1$, let $(u_N, H_N) \in V_N \times W_N$ be given by (2-2) such that u_N solves (2-3) and H_N solves

$$\text{(2-18)}\quad \int_{\mathbb{R}^3} (\nabla \times H_N) \cdot (\nabla \times \Psi) = -\sigma \int_{\mathbb{R}^3} \frac{\partial}{\partial t} (H_N + \beta \bar{u}_N) \cdot \Psi \quad \text{for all } \Psi \in W_N$$

subject to the initial condition $(u_N, H_N)|_{t=0} = (\Pi_{V_N}(u_0), \Pi_{W_N}(H_0))$.

Testing (2-18) with $\Psi = H_N$ and integrating over $\mathbb{R}^3$ gives

$$\text{(2-19)}\quad \frac{d}{dt} \int_{\mathbb{R}^3} \frac{\sigma}{2} |H_N|^2 + \int_{\mathbb{R}^3} |\nabla \times H_N|^2 = -\beta\sigma \int_\Omega H_N \cdot \frac{\partial u_N}{\partial t} \le \beta\sigma \int_\Omega |H_N| \left|\frac{\partial u_N}{\partial t}\right|.$$

Combining (2-19) with (2-10) and applying the Cauchy–Schwarz inequality yields

$$\text{(2-20)}\quad \int_\Omega \alpha_1 \left|\frac{\partial u_N}{\partial t}\right|^2 + \frac{d}{dt} \left(\int_\Omega e_\varepsilon(u_N) + \int_{\mathbb{R}^3} \tfrac{1}{2}\sigma |H_N|^2 \right) + \int_{\mathbb{R}^3} |\nabla \times H_N|^2 \le C(\alpha_1, \beta) \int_{\mathbb{R}^3} |H_N|^2.$$

This, combined with Grönwall's inequality, yields that for any $0 < T < +\infty$,

$$\text{(2-21)}\quad \alpha_1 \int_0^T \int_\Omega \left|\frac{\partial u_N}{\partial t}\right|^2 + \int_0^T \int_{\mathbb{R}^3} |\nabla \times H_N|^2 + \left(\int_\Omega e_\varepsilon(u_N) + \int_{\mathbb{R}^3} \tfrac{1}{2}\sigma |H_N|^2 \right)(T) \le e^{CT} \left(o(1) + \int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3} \tfrac{1}{2}\sigma |H_0|^2 \right)$$

for some $C = C(\beta, \alpha_1) > 0$, where we have used (2-7).

Now testing (2-18) with $\Psi = \partial H_N / \partial t$ and integrating over $\mathbb{R}^3$, we have

$$\text{(2-22)}\quad \frac{d}{dt} \int_{\mathbb{R}^3} \tfrac{1}{2} |\nabla \times H_N|^2 + \sigma \int_{\mathbb{R}^3} \left|\frac{\partial H_N}{\partial t}\right|^2 = -\beta\sigma \int_\Omega \frac{\partial H_N}{\partial t} \cdot \frac{\partial u_N}{\partial t}.$$

Thus, by the Cauchy–Schwarz inequality,

$$\text{(2-23)}\quad \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla \times H_N|^2 + \sigma \int_{\mathbb{R}^3} \left|\frac{\partial H_N}{\partial t}\right|^2 \le 16\beta^2\sigma \int_{\mathbb{R}^3} \left|\frac{\partial u_N}{\partial t}\right|^2.$$

Integrating for $0 \le t \le T$ and applying (2-21), this implies

$$(2\text{-}24)\quad \int_{\mathbb{R}^3} |\nabla \times H_N|^2(T) + \sigma \int_0^T \int_{\mathbb{R}^3} \left|\frac{\partial H_N}{\partial t}\right|^2 \le \int_{\mathbb{R}^3} |\nabla H_0|^2 + 16\beta^2\sigma \int_0^T \int_\Omega \left|\frac{\partial u_N}{\partial t}\right|^2 \le \int_{\mathbb{R}^3} |\nabla H_0|^2 + e^{CT}\left(o(1) + \int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3} |H_0|^2\right).$$

Adding (2-21) and (2-24) together, we obtain

$$(2\text{-}25)\quad \alpha_1 \int_0^T \int_\Omega \left|\frac{\partial u_N}{\partial t}\right|^2 + \int_0^T \int_{\mathbb{R}^3} \left(\left|\frac{\partial H_N}{\partial t}\right|^2 + |\nabla \times H_N|^2\right) + \left(\int_\Omega e_\varepsilon(u_N) + \int_{\mathbb{R}^3} \left(\tfrac{1}{2}\sigma |H_N|^2 + |\nabla \times H_N|^2\right)\right)(T) \le \int_{\mathbb{R}^3} |\nabla H_0|^2 + e^{CT}\left(o(1) + \int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3} \tfrac{1}{2}\sigma |H_0|^2\right).$$

From (2-25) we may assume, after taking subsequences, that for any $0 < T < +\infty$,

$$u_N \rightharpoonup u^\varepsilon \text{ weak* in } L^\infty([0,T], H^1(\Omega)), \quad \frac{\partial u_N}{\partial t} \rightharpoonup \frac{\partial u^\varepsilon}{\partial t} \text{ in } L^2(\Omega \times [0,T]),$$
$$H_N \rightharpoonup H^\varepsilon, \ \frac{\partial H_N}{\partial t} \rightharpoonup \frac{\partial H^\varepsilon}{\partial t}, \ \nabla \times H_N \rightharpoonup \nabla \times H^\varepsilon \text{ in } L^2(\mathbb{R}^3 \times [0,T]).$$

As in Lemma 2.2, we can show that $(u^\varepsilon, H^\varepsilon)$ are weak solutions to (1-19)+(1-17), under the initial condition (1-8)–(1-10). By lower semicontinuity, we also see that (2-25) holds with (u_N, H_N) replaced by $(u^\varepsilon, H^\varepsilon)$. To obtain the L^2-norm bound for ∇H, we need to use the condition $\nabla \cdot (H_0 + \beta \bar{u}_0) = 0$ in $\mathcal{D}'(\mathbb{R}^3)$. Note that

$$\int_{\mathbb{R}^3} (\nabla \times H^\varepsilon) \cdot (\nabla \times \Psi) = -\sigma \int_{\mathbb{R}^3} \frac{\partial}{\partial t}\left(H^\varepsilon + \beta \bar{u}^\varepsilon\right) \cdot \Psi, \quad \forall\, \Psi \in H^1(\mathbb{R}^3).$$

Since $\delta > 0$, by choosing $\Psi = \nabla \psi$ for $\psi \in C_0^\infty(\mathbb{R}^3)$ and observing $\nabla \times (\nabla \psi) = 0$ in $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} \frac{\partial}{\partial t}\left(H^\varepsilon + \beta \bar{u}^\varepsilon\right) \cdot \nabla \psi = 0$$

so that for a.e. $t > 0$,

$$\int_{\mathbb{R}^3} \nabla \cdot (H^\varepsilon + \beta \bar{u}^\varepsilon)\psi = \int_{\mathbb{R}^3} \nabla \cdot (H_0 + \beta \bar{u}_0)\psi = 0.$$

Thus

$$\nabla \cdot (H^\varepsilon + \beta \bar{u}^\varepsilon) = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}^3) \ \text{ for a.e. } t > 0.$$

To proceed, we claim that

$$\int_{\mathbb{R}^3} |\nabla H|^2 = \int_{\mathbb{R}^3} (|\nabla \times H|^2 + |\nabla \cdot H|^2) \quad \text{for all } H \in H^1(\mathbb{R}^3, \mathbb{R}^3).$$

Since $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3, \mathbb{R}^3)$, it suffices to verify this inequality for $H \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$. This can be seen as follows:

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla H|^2 &= \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} (H_j^i)^2 = \int_{\mathbb{R}^3} \sum_{i \neq j} (H_i^j)^2 + \int_{\mathbb{R}^3} \sum_{i=1}^{3} (H_i^i)^2 \\
&= \int_{\mathbb{R}^3} \Big(\sum_{1 \le i < j \le 3} (H_i^j - H_j^i)^2 + 2 \sum_{1 \le i < j \le 3} H_i^j H_j^i \Big) \\
&\qquad\qquad + \int_{\mathbb{R}^3} \bigg(\Big(\sum_{i=1}^{3} H_i^i \Big)^2 - 2 \sum_{1 \le i < j \le 3} H_i^i H_j^j \bigg) \\
&= \int_{\mathbb{R}^3} \big(|\nabla \times H|^2 + |\nabla \cdot H|^2 \big) + 2 \int_{\mathbb{R}^3} \sum_{1 \le i < j \le 3} (H_i^j H_j^i - H_i^i H_j^j) \\
&= \int_{\mathbb{R}^3} \big(|\nabla \times H|^2 + |\nabla \cdot H|^2 \big),
\end{aligned}$$

where the vanishing of $\int_{\mathbb{R}^3} \sum_{1 \le i < j \le 3} (H_i^j H_j^i - H_i^i H_j^j) = 0$ in the last step comes from integrating by parts twice. Thus

$$\int_{\mathbb{R}^3} |\nabla H^\varepsilon|^2 = \int_{\mathbb{R}^3} (|\nabla \times H^\varepsilon|^2 + |\nabla \cdot H^\varepsilon|^2) \le C(\beta) \Big(\int_{\mathbb{R}^3} |\nabla \times H^\varepsilon|^2 + \int_\Omega |\nabla u^\varepsilon|^2 \Big)$$

and hence (2-25), with $(u_N, H_N) = (u^\varepsilon, H^\varepsilon)$, yields (2-17). □

For the system (1-19)+(1-18), we have:

Lemma 2.4. *For any* $u_0 \in H^1(\Omega, S^2)$, $H_0 \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ *and* $E_0 \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ *with* $\nabla \cdot E_0 = \nabla \cdot H_0 = 0$ *in* $\mathcal{D}'(\mathbb{R}^3)$, *there exists a global weak solution* $(u^\varepsilon, H^\varepsilon, E^\varepsilon)$ *to* (1-19)+(1-18) *under the initial-boundary conditions* (1-8)–(1-11) *such that, for any* $0 < T < +\infty$,

$$\begin{aligned}
&\int_0^T \int_\Omega \Big| \frac{\partial u^\varepsilon}{\partial t} \Big|^2 + \mathscr{E}_\varepsilon(T) \\
&\quad + \int_{\mathbb{R}^3} \Big(|H^\varepsilon|^2 + |E^\varepsilon|^2 + \Big| \frac{\partial H^\varepsilon}{\partial t} \Big|^2 + \Big| \frac{\partial E^\varepsilon}{\partial t} \Big|^2 + |\nabla H^\varepsilon|^2 + |\nabla E^\varepsilon|^2 \Big)(T) \\
&\quad \le C(\varepsilon_0, \sigma, T) \Big(\int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3} \big(|H_0|^2 + |E_0|^2 + |\nabla H_0|^2 + |\nabla E_0|^2 \big) \Big).
\end{aligned} \tag{2-26}$$

Proof. For $N \ge 1$, let $(u_N, H_N, E_N) \in V_N \times W_N \times W_N$ of the form (2-2) be a solution to (2-3)–(2-6). Since $\beta = 0$ in this case, testing (2-4) with $\Psi = E_N$ and

(2-5) with $\Psi = H_N$ and adding the resulting identities together gives

$$\frac{d}{dt}\int_{\mathbb{R}^3}\big(|H_N|^2+\varepsilon_0|E_N|^2\big)+2\sigma\int_{\mathbb{R}^3}|E_N|^2=0. \tag{2-27}$$

Differentiating (2-4) and (2-5) with respect to t and testing the resulting equations with $\Psi = \partial E_N/\partial t$ and $\Psi = \partial H_N/\partial t$ respectively, we have

$$\int_{\mathbb{R}^3}\Big(\frac{\partial E_N}{\partial t}\cdot\nabla\times\frac{\partial H_N}{\partial t}-\frac{\partial H_N}{\partial t}\cdot\nabla\times\frac{\partial E_N}{\partial t}\Big) = \int_{\mathbb{R}^3}\Big(\frac{\partial E_N}{\partial t}\cdot\Big(\varepsilon_0\frac{\partial^2 E_N}{\partial t^2}+\sigma\frac{\partial E_N}{\partial t}\Big)+\frac{\partial H_N}{\partial t}\cdot\frac{\partial^2 H_N}{\partial t^2}\Big).$$

Since

$$\int_{\mathbb{R}^3}\Big(\frac{\partial E_N}{\partial t}\cdot\nabla\times\frac{\partial H_N}{\partial t}-\frac{\partial H_N}{\partial t}\cdot\nabla\times\frac{\partial E_N}{\partial t}\Big)=0,$$

we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\varepsilon_0\Big|\frac{\partial E_N}{\partial t}\Big|^2+\Big|\frac{\partial H_N}{\partial t}\Big|^2\Big)+2\sigma\int_{\mathbb{R}^3}\Big|\frac{\partial E_N}{\partial t}\Big|^2=0. \tag{2-28}$$

Combining (2-27) with (2-28), we get

$$\frac{d}{dt}\int_{\mathbb{R}^3}\Big(|H_N|^2+\varepsilon_0\Big(|E_N|^2+\Big|\frac{\partial E_N}{\partial t}\Big|^2\Big)+\Big|\frac{\partial H_N}{\partial t}\Big|^2\Big) = -2\sigma\int_{\mathbb{R}^3}\Big(|E_N|^2+\Big|\frac{\partial E_N}{\partial t}\Big|^2\Big). \tag{2-29}$$

Since

$$\frac{\partial H_N}{\partial t}\Big|_{t=0}=-\nabla\times(\Pi_{W_N}(E_0)),\qquad \varepsilon_0\frac{\partial E}{\partial t}\Big|_{t=0}=\nabla\times(\Pi_{W_N}(H_0))-\sigma\,\Pi_{W_N}(E_0),$$

integrating (2-29) for $0\le t\le T$ yields

$$\begin{aligned}\int_{\mathbb{R}^3}\Big(|H_N|^2+\varepsilon_0\Big(|E_N|^2+\Big|\frac{\partial E_N}{\partial t}\Big|^2\Big)+\Big|\frac{\partial H_N}{\partial t}\Big|^2\Big)(T) &+2\sigma\int_0^T\int_{\mathbb{R}^3}\Big(|E_N|^2+\Big|\frac{\partial E_N}{\partial t}\Big|^2\Big)\\ &\le\int_{\mathbb{R}^3}\big(|\Pi_{W_N}(H_0)|^2+\varepsilon_0|\Pi_{W_N}(E_0)|^2+|\nabla\times(\Pi_{W_N}(E_0))|^2\\ &\qquad+\varepsilon_0^{-1}|\nabla\times(\Pi_{W_N}(H_0))-\sigma\,\Pi_{W_N}(E_0)|^2\big)\\ &\le C(\varepsilon_0,\sigma)\int_{\mathbb{R}^3}\big(|H_0|^2+|E_0|^2+|\nabla H_0|^2+|\nabla E_0|^2\big).\end{aligned} \tag{2-30}$$

For u_N, by testing (2-3) with $\Phi = \partial u_N/\partial t$ as in (2-10) of Lemma 2.2, we have

$$\alpha_1 \int_\Omega \left|\frac{\partial u_N}{\partial t}\right|^2 + \frac{d}{dt}\int_\Omega e_\varepsilon(u_N) \le C\int_{\mathbb{R}^3} |H_N|^2. \tag{2-31}$$

This, with the help of (2-30) and (2-7), implies that for any $0 < T < +\infty$,

$$\begin{aligned}\alpha_1 \int_0^T\int_\Omega \left|\frac{\partial u_N}{\partial t}\right|^2 + \int_\Omega e_\varepsilon(u_N(T)) \\ \le CT\int_{\mathbb{R}^3}\big(|H_0|^2+|E_0|^2+|\nabla H_0|^2+|\nabla E_0|^2\big) + \int_\Omega |\nabla u_0|^2 + o(1).\end{aligned} \tag{2-32}$$

It follows from (2-30), (2-4), and (2-5) with $\beta = 0$ that

$$\begin{aligned}\int_{\mathbb{R}^3}\big(|\nabla\times H_N|^2+|\nabla\times E_N|^2\big)(T) &\le C\int_{\mathbb{R}^3}\left(\left|\frac{\partial E_N}{\partial t}\right|^2+|E_N|^2+\left|\frac{\partial H_N}{\partial t}\right|^2\right)(T) \\ &\le C\int_{\mathbb{R}^3}\big(|H_0|^2+|E_0|^2+|\nabla H_0|^2+|\nabla E_0|^2\big).\end{aligned} \tag{2-33}$$

From (2-30), (2-32), and (2-33) we may assume, after taking subsequences, that for any $0 < T < +\infty$,

$$\begin{gathered} u_N \rightharpoonup u^\varepsilon \text{ weak* in } L^\infty([0,T],H^1(\Omega)), \quad \frac{\partial u_N}{\partial t} \rightharpoonup \frac{\partial u^\varepsilon}{\partial t} \text{ in } L^2(\Omega\times[0,T]), \\ H_N \rightharpoonup H^\varepsilon,\ \frac{\partial H_N}{\partial t} \rightharpoonup \frac{\partial H^\varepsilon}{\partial t},\ \nabla\times H_N \rightharpoonup \nabla\times H^\varepsilon \text{ in } L^2(\mathbb{R}^3\times[0,T]), \\ E_N \rightharpoonup E^\varepsilon,\ \frac{\partial E_N}{\partial t} \rightharpoonup \frac{\partial E^\varepsilon}{\partial t},\ \nabla\times E_N \rightharpoonup \nabla\times E^\varepsilon \text{ in } L^2(\mathbb{R}^3\times[0,T]).\end{gathered}$$

As in the previous lemmas, it is a standard exercise to check that $(u^\varepsilon, H^\varepsilon, E^\varepsilon)$ solves (1-19)+(1-18) and the initial-boundary conditions (1-8)–(1-11). Moreover, by lower semicontinuity, we have, for $0 < T < +\infty$,

$$\begin{aligned}&\int_0^T\int_\Omega \left|\frac{\partial u^\varepsilon}{\partial t}\right|^2 + \mathscr{E}_\varepsilon(T) \\ &+\int_{\mathbb{R}^3}\left(|H^\varepsilon|^2+|E^\varepsilon|^2+\left|\frac{\partial H^\varepsilon}{\partial t}\right|^2+\left|\frac{\partial E^\varepsilon}{\partial t}\right|^2+|\nabla\times H^\varepsilon|^2+|\nabla\times E^\varepsilon|^2\right) \\ &\quad \le C(\varepsilon_0,\sigma,T)\left(\int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3}\big(|H_0|^2+|E_0|^2+|\nabla H_0|^2+|\nabla E_0|^2\big)\right).\end{aligned} \tag{2-34}$$

As in the previous lemma, we can check that $\nabla\cdot H_0 = \nabla\cdot E_0 = 0$ is preserved under (1-18), that is,

$$\nabla\cdot H^\varepsilon(t) = \nabla\cdot E^\varepsilon(t) = 0 \ \text{ a.e. } t > 0. \tag{2-35}$$

Finally, it is not hard to see that (2-34) and (2-35) yield (2-26). □

Remark 2.5. Lemmas 2.3 and 2.4 show that, for any $0 < T < +\infty$, H^ε is uniformly bounded in $L^\infty([0, T], H^1(\mathbb{R}^3))$. Hence by the Sobolev embedding inequality, H^ε is uniformly bounded in $L^\infty([0, T], L^6(\mathbb{R}^3))$. This property plays an important role in the proof of C^α-regularity of ∇u claimed in Theorems 1.3 and 1.4.

We end this section with a local energy inequality.

Lemma 2.6. *There exists $C > 0$ such that for any $\varepsilon > 0$, $u_0 \in H^1(\Omega, S^2)$, $H_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ and $E_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, if $(u^\varepsilon, H^\varepsilon, E^\varepsilon)$ is the global weak solution of (1-19)+(1-2)–(1-3) with conditions (1-8)+(1-9)–(1-11) obtained in Lemma 2.2, then for any $x_0 \in \Omega$, $t_0 > 0$, and $0 < r < \min\{\operatorname{dist}(x_0, \partial\Omega), \sqrt{t}_0/2\}$, we have*

$$r^{-1}\int_{P_{r/2}(z_0)} \left|\frac{\partial u^\varepsilon}{\partial t}\right|^2 + r^{-1} \max_{t\in[t_0-r^2/4,t_0]} \int_{B_{r/2}(x_0)} e_\varepsilon(u^\varepsilon) \le Cr^{-3}\int_{P_r(z_0)} e_\varepsilon(u^\varepsilon) + Cr^{-1}\int_{P_r(z_0)} |H^\varepsilon|^2. \tag{2-36}$$

Proof. Write (u, H) for $(u^\varepsilon, H^\varepsilon)$. For $x_0 \in \Omega$ and $0 < r < \min\{\operatorname{dist}(x_0, \partial\Omega), \sqrt{t_0}\}$, by Fubini's theorem there is $\alpha \in (\frac{1}{2}, \frac{7}{8})$ such that

$$\int_{B_r(x_0)} e_\varepsilon(u)(t_0 - \alpha^2 r^2) \le 8r^{-2}\int_{P_r(z_0)} e_\varepsilon(u). \tag{2-37}$$

Let $\phi(x) \in C_0^\infty(B_r(x_0))$ be such that $0 \le \phi \le 1$, $\phi \equiv 1$ on $B_{r/2}(x_0)$. Multiplying (1-19) by $\phi^2(\partial u/\partial t)$ and integrating over $B_r(x_0)$, we get

$$\begin{aligned}\alpha_1 \int_{B_r(x_0)} \left|\frac{\partial u}{\partial t}\right|^2 \phi^2 + \frac{d}{dt}\int_{B_r(x_0)} e_\varepsilon(u)\phi^2 &= -2\int_{B_r(x_0)} \phi\nabla\phi\nabla u \cdot \frac{\partial u}{\partial t} - \int_{B_r(x_0)} \phi^2 u \times (H \times u)\cdot\frac{\partial u}{\partial t} \\ &\le \frac{\alpha_1}{2}\int_{B_r(x_0)} \left|\frac{\partial u}{\partial t}\right|^2\phi^2 + C(\alpha_1)\int_{B_r(x_0)} \big(|\nabla\phi|^2|\nabla u|^2 + \phi^2|H|^2\big).\end{aligned} \tag{2-38}$$

Integrating (2-38) from $t_0 - \alpha^2 r^2$ to $t \in [t_0 - r^2/4, t_0]$ and applying (2-37), we obtain (2-36). □

3. Energy monotonicity on time slices

An energy monotonicity property analogous to that of [Struwe 1988] (see also [Chen and Struwe 1989; Chen and Lin 1993]) is unknown for Landau–Lifshitz type equations. In order to derive an prior estimate for $(u^\varepsilon, E^\varepsilon, H^\varepsilon)$ under the small energy condition, we need an energy monotonicity of u^ε on time slices, which can be derived by a Pohozaev-type argument as in [Wang 2006].

Lemma 3.1. *For* $\varepsilon > 0$, *let* $(u^\varepsilon, H^\varepsilon)$ *be a weak solution to* (1-19). *For a.e.* $t > 0$, *any* $x_0 \in \Omega$, *and* $0 < r \le R < \min\{1, \operatorname{dist}(x_0, \partial\Omega)\}$, *we have*

$$r^{-1}E_\varepsilon(u^\varepsilon, B_r(x_0)) \le 2R^{-1}E_\varepsilon(u^\varepsilon, B_R(x_0)) + C_0 R\int_{B_R(x_0)}\Big(\Big|\frac{\partial u^\varepsilon}{\partial t}\Big|^2 + |H^\varepsilon|^2\Big), \tag{3-1}$$

$$\int_{B_R} |x-x_0|^{-1}\frac{(1-|u^\varepsilon|^2)^2}{\varepsilon^2} \le 2R^{-1}E_\varepsilon\big(u^\varepsilon, B_R(x_0)\big) + C_0 R\int_{B_R(x_0)}\Big(\Big|\frac{\partial u^\varepsilon}{\partial t}\Big|^2 + |H^\varepsilon|^2\Big) \tag{3-2}$$

for some $C_0 = C_0(\alpha_1) > 0$, *where*

$$E_\varepsilon(u^\varepsilon, A) = \int_A\Big(\tfrac12|\nabla u^\varepsilon|^2 + \frac{(1-|u^\varepsilon|^2)^2}{2\varepsilon^2}\Big), \quad A \subseteq \mathbb{R}^3.$$

Proof. The proof is a modification of [Wang 2006] (see also [Ding and Guo 2004; Melcher 2005]). We sketch it here. First observe that for a.e. $t > 0$, we have $\Delta u \in L^2(\Omega)$ and hence $\nabla^2 u \in L^2(\Omega)$. For $p \in \mathbb{R}^3$, define $R(p) : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$R(p)(v) = \alpha_1 v - \alpha_2 p \times v \quad \text{for all } v \in \mathbb{R}^3.$$

Assume $x_0 = 0 \in \Omega$. Write $(u, H) = (u^\varepsilon, H^\varepsilon)$ and $B_r = B_r(0)$. Multiplying (1-19) by $x \cdot \nabla u$ and integrating over B_r yields

$$\begin{aligned}\int_{B_r}\Big\langle R(u)\Big(\frac{\partial u}{\partial t}\Big) + u\times(u\times H),\ x\cdot\nabla u\Big\rangle &= \int_{B_r}\Big\langle \Delta u + \frac{1}{\varepsilon^2}(1-|u|^2)u,\ x\cdot\nabla u\Big\rangle \\ &= r\int_{\partial B_r}\Big(\Big|\frac{\partial u}{\partial r}\Big|^2 - \tfrac12|\nabla u|^2 - \frac{(1-|u|^2)^2}{4\varepsilon^2}\Big) + \int_{B_r}\Big(\tfrac12|\nabla u|^2 + \frac{3(1-|u|^2)^2}{4\varepsilon^2}\Big) \\ &\ge r\int_{\partial B_r}\Big(\Big|\frac{\partial u}{\partial r}\Big|^2 - \tfrac12|\nabla u|^2 - \frac{(1-|u|^2)^2}{4\varepsilon^2}\Big) + E_\varepsilon(u, B_r).\end{aligned} \tag{3-3}$$

Hence we have

$$\begin{aligned}\frac{d}{dr}\Big(r^{-1}E_\varepsilon(u, B_r) - r^{-1}\int_{B_r}\Big\langle R(u)\Big(\frac{\partial u}{\partial t}\Big) + u\times(u\times H),\ x\cdot\nabla u\Big\rangle\Big) \\ \ge r^{-1}\int_{\partial B_r}\Big(\Big|\frac{\partial u}{\partial r}\Big|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2}\Big) - r^{-1}\int_{\partial B_r}\Big\langle R(u)\frac{\partial u}{\partial t} + u\times(u\times H),\ x\cdot\nabla u\Big\rangle.\end{aligned} \tag{3-4}$$

Since $|u| \le 1$, we have $|u\times(u\times H)| \le |H|$ and $|R(u)(\partial u/\partial t)| \le |\partial u/\partial t|$. The second term of the right hand side of (3-4) can be estimated by

$$(3\text{-}5)\quad -r^{-1}\int_{\partial B_r}\Big\langle R(u)\Big(\frac{\partial u}{\partial t}\Big)+u\times(u\times H),\ x\cdot\nabla u\Big\rangle \geq -\frac{1}{4}r^{-1}\int_{\partial B_r}\Big|\frac{\partial u}{\partial r}\Big|^2 - r\int_{\partial B_r}\Big(\Big|\frac{\partial u}{\partial t}\Big|^2+|H|^2\Big).$$

The second term of the left hand side of (3-4) can be estimated by

$$(3\text{-}6)\quad \Big|r^{-1}\int_{B_r}\Big\langle R(u)\Big(\frac{\partial u}{\partial t}\Big)+u\times(u\times H),\ x\cdot\nabla u\Big\rangle\Big| \leq \tfrac{1}{4}E_\varepsilon(u,B_r)+r\int_{B_r}\Big(\Big|\frac{\partial u}{\partial t}\Big|^2+|H|^2\Big).$$

Putting (3-5) and (3-6) into (3-4) and integrating from r to R gives

$$(3\text{-}7)\quad 2R^{-1}E_\varepsilon(u,B_R)+R\int_{B_R}\Big(\Big|\frac{\partial u}{\partial t}\Big|^2+|H|^2\Big) \geq \tfrac{1}{2}r^{-1}E_\varepsilon(u,B_r)-r\int_{B_r}\Big(\Big|\frac{\partial u}{\partial t}\Big|^2+|H|^2\Big) + \int_{B_R\setminus B_r}\frac{1}{|x|}\Big(\Big|\frac{\partial u}{\partial r}\Big|^2+\frac{(1-|u|^2)^2}{4\varepsilon^2}\Big)-\int_0^R s\int_{\partial B_s}\Big(\Big|\frac{\partial u}{\partial t}\Big|^2+|H|^2\Big).$$

Since

$$r\int_{B_r}\Big(\Big|\frac{\partial u}{\partial t}\Big|^2+|H|^2\Big)\leq R\int_{B_R}\Big(\Big|\frac{\partial u}{\partial t}\Big|^2+|H|^2\Big)$$

and

$$\int_0^R s\int_{\partial B_s}\Big(\Big|\frac{\partial u}{\partial t}\Big|^2+|H|^2\Big)\leq R\int_{\partial B_R}\Big(\Big|\frac{\partial u}{\partial t}\Big|^2+|H|^2\Big),$$

(3-7) clearly implies both (3-1) and (3-2). □

4. On the lower bound of $|u^\varepsilon|$

We will now establish a lower bound estimate of $|u^\varepsilon|$ on generic time slices, under the smallness condition $r^{-3}\int_{P_r} e_\varepsilon(u^\varepsilon)$.

Definition 4.1. For any $\varepsilon\in(0,\frac{1}{2})$, $x_0\in\Omega$, $t_0>0$, $0<r<\min\{\text{dist}(x_0,\partial\Omega),\sqrt{t_0}\}$, and $\Lambda>0$, we define the set of *good* time slices by

$$(4\text{-}1)\qquad G^\Lambda_{z_0,r}=\Big\{t\in[t_0-r^2,t_0):\int_{B_r(x_0)}\Big|\frac{\partial u^\varepsilon}{\partial t}\Big|^2\leq\frac{\Lambda^2}{r^2}\int_{P_r(z_0)}\Big|\frac{\partial u^\varepsilon}{\partial t}\Big|^2\Big\},$$

and the set of *bad* time slices by

$$(4\text{-}2)\qquad B^\Lambda_{z_0,r}=[t_0-r^2,t_0)\setminus G^\Lambda_{z_0,r}.$$

By Fubini's theorem,

$$\left|B^{\Lambda}_{z_0,r}\right| \le \frac{r^2}{\Lambda^2}. \tag{4-3}$$

Similarly to [Melcher 2005; Wang 2006], we have:

Lemma 4.2. *For $\varepsilon > 0$, let $(u^\varepsilon, H^\varepsilon)$ be the weak solution of (1-19) in Lemma 2.2. Denote*

$$\|H^\varepsilon\|_{L^\infty_t L^2_x(\mathbb{R}^3\times[0,t_0])} = C_0.$$

Then for any $\Lambda > 0$, there exist $\eta_0 > 0$ and $r_0 > 0$ depending on Λ and C_0 such that for any $z_0 = (x_0, t_0) \in \Omega \times (0, +\infty)$ and $0 < r < \min\{\mathrm{dist}(x_0, \partial\Omega), \sqrt{t_0}, r_0\}$ if

$$r^{-3}\int_{P_r(z_0)} e_\varepsilon(u^\varepsilon) \le \eta_0^2, \tag{4-4}$$

then

$$|u^\varepsilon|(x,t) \ge \frac{1}{2} \quad \textit{for all } x \in B_{r/4}(x_0) \textit{ and } t \in G^{\Lambda}_{z_0,r/2}. \tag{4-5}$$

Proof. The proof is a modification of [Melcher 2005; Wang 2006]. We prove a $C^{1/2}$-estimate of $u^\varepsilon(\cdot\,, s)$ for $s \in G^{\Lambda}_{z_0,r/2}$ (see also [Melcher 2005, page 577, Lemma 5]). Define $v^\varepsilon(x,t) = u^\varepsilon(x_0+\varepsilon x, s+\varepsilon^2 t) : B_2 \times [-4,4] \to \mathbb{R}^3$. Then $w^\varepsilon(x) \equiv v^\varepsilon(x,0)$ satisfies

$$\Delta w^\varepsilon = R(w^\varepsilon)\Big(\frac{\partial v^\varepsilon}{\partial t}(0)\Big) - (1-|w^\varepsilon|^2)w^\varepsilon - w^\varepsilon \times (\widetilde{H}^\varepsilon \times w^\varepsilon), \tag{4-6}$$

where $\widetilde{H}^\varepsilon(x) = \varepsilon^2 H^\varepsilon(\varepsilon x, s)$. By the standard $W^{2,2}$ estimate, we have

$$\begin{aligned} \|\nabla^2 w^\varepsilon\|^2_{L^2(B_1)} &\le C\bigg(1 + \Big\|\frac{\partial w^\varepsilon}{\partial t}\Big\|^2_{L^2(B_2)} + \|\widetilde{H}^\varepsilon\|^2_{L^2(B_2)}\bigg) \\ &\le C\bigg(1 + \varepsilon\int_{B_{2\varepsilon}(x_0)}\Big(\Big|\frac{\partial u^\varepsilon}{\partial t}\Big|^2 + |H^\varepsilon|^2\Big)(s)\bigg) \\ &\le C\bigg(1 + C_0^2 + r\int_{B_{r/2}(x_0)}\Big|\frac{\partial u^\varepsilon}{\partial t}\Big|^2(s)\bigg) \le C(1+C_0^2+\Lambda^2\varepsilon_0^2), \end{aligned} \tag{4-7}$$

where we have used both (4-1) and Lemma 2.6 in the last step. Therefore, by the Sobolev embedding theorem, $w^\varepsilon \in C^{1/2}(B_1)$. Moreover, by rescaling and (4-7),

$$(u^\varepsilon(s))_{C^{1/2}(B_{r/2}(x_0))} \le C(\Lambda, \eta_0, C_0)\varepsilon^{-1/2} \quad \text{for all } s \in G^{\Lambda}_{z_0,r/2}. \tag{4-8}$$

Suppose that (4-5) were false. Then there exists $z_1 = (x_1, t_1) \in B_{r/4}(x_0) \times G^{\Lambda}_{z_0,r/2}$ such that $|u^\varepsilon(z_1)| < \frac{1}{2}$. Hence for sufficiently small $\theta_0 > 0$, if $y \in B_{\theta_0^2\varepsilon}(x_1)$, we would have

$$|u^\varepsilon|(y,t_1) \le |u^\varepsilon|(x_1,t_1) + (u^\varepsilon(t_1))_{C^{1/2}}|y-x_1|^{1/2} \le \tfrac{1}{2} + C(\Lambda, \eta_0, C_0)\theta_0 \le \tfrac{3}{4},$$

so that

$$\int_{B_{\theta_0^2\varepsilon}(x_1)} |x-x_1|^{-1} \frac{(1-|u^\varepsilon|^2)^2(x,t_1)}{\varepsilon^2} \geq C_1. \tag{4-9}$$

At the same time, the bound (4-4) gives $\sup_{x\in B_{r/2}(x_0)} \left(\frac{r}{2}\right)^{-3} \int_{P_{r/2}(x,t_0)} e_\varepsilon(u^\varepsilon) \leq 8\eta_0^2$. This, combined with Lemma 2.6, implies

$$\sup_{t\in[t_0-r^2/16,\, t_0)} \sup_{x\in B_{r/4}(x_0)} \left(\frac{r}{4}\right)^{-1} \int_{B_{r/4}(x)} e_\varepsilon(u^\varepsilon) \leq C(\eta_0^2 + C_0^2 r). \tag{4-10}$$

By the definition of $G^\Lambda_{z_0,\, r/2}$ and Lemma 2.6, we have

$$\begin{aligned}
\sup_{t\in G^\Lambda_{z_0,r/2}} \sup_{x\in B_{r/4}(x_0)} r \int_{B_{r/4}(x)} \left|\frac{\partial u^\varepsilon}{\partial t}\right|^2(t) &\leq \sup_{t\in G^\Lambda_{z_0,\, r/2}} r \int_{B_{r/2}(x_0)} \left|\frac{\partial u^\varepsilon}{\partial t}\right|^2(t) \\
&\leq C\left(\frac{\Lambda^2}{r^3}\int_{P_r(z_0)} e_\varepsilon(u^\varepsilon) + \Lambda^2 r \|H^\varepsilon\|^2_{L^\infty_t L^2_x}\right) \leq C\Lambda^2(\eta_0^2 + C_0^2 r).
\end{aligned} \tag{4-11}$$

With (4-10), (4-11), and the monotonicity inequality (3-2), we obtain

$$\begin{aligned}
\int_{B_{\theta_0^2\varepsilon}(x_1)} |x-x_1|^{-1} \frac{(1-|u^\varepsilon|^2)^2}{\varepsilon_0^2}(t_1) &\leq C\left(r^{-1}\int_{B_{r/4}(x_1)} e_\varepsilon(u^\varepsilon)(t_1) + r\int_{B_{r/4}(x_1)} \left(\left|\frac{\partial u^\varepsilon}{\partial t}\right|^2 + |H^\varepsilon|^2\right)(t_1)\right) \\
&\leq C(\Lambda^2\eta_0^2 + C_0^2 r_0).
\end{aligned} \tag{4-12}$$

This contradicts (4-9) provided $r_0 > 0$ and $\eta_0 > 0$ are chosen sufficiently small. □

5. Energy decay estimates and proof of Theorem 1.2

In this section, we first establish the decay estimate of the normalized energy $r^{-3}\int_{P_r(z)} e_\varepsilon(u^\varepsilon)$, provided that it is sufficiently small. Then we give a proof of Theorem 1.2. The techniques employed in the proof are suitable modifications of that by Hélein [1990] and Evans [1991] in the context of harmonic maps.

Lemma 5.1. *For any $L > 0$ and $\delta > 0$, there exist $C(\delta) > 0$, $\eta(\delta) > 0$, and $\varepsilon_1(\delta) > 0$, such that if $(u^\varepsilon, H^\varepsilon)$ is the weak solution of (1-19) in Lemma 2.2 and we take $z_0 = (x_0, t_0) \in \Omega \times \mathbb{R}^+$, $0 < r < \min\{\mathrm{dist}(x_0, \partial\Omega), \sqrt{t_0}, \varepsilon_1^2(\delta)/L^2\}$, and $0 < \varepsilon \leq \eta(\delta) r$ satisfying*

$$\|H^\varepsilon\|_{L^\infty_t L^2_x(P_r(z_0))} \leq L \quad \text{and} \quad r^{-3}\int_{P_r(z_0)} e_\varepsilon(u^\varepsilon) \leq \varepsilon_1^2(\delta), \tag{5-1}$$

then we have

$$\left(\frac{r}{8}\right)^{-3}\int_{P_{r/8}(z_0)} e_\varepsilon(u^\varepsilon) \le \delta\left(r^{-3}\int_{P_r(z_0)} e_\varepsilon(u^\varepsilon) + r\|H^\varepsilon\|^2_{L_t^\infty L_x^2(P_r(z_0))}\right) + \frac{C(\delta)}{\delta} r^{-5}\int_{P_r(z_0)} |u^\varepsilon - u^\varepsilon_{P_r(z_0)}|^2, \tag{5-2}$$

where $u^\varepsilon_{P_r(z_0)}$ is the average of u^ε over $P_r(z_0)$:

$$u^\varepsilon_{P_r(z_0)} = \frac{1}{|P_r(z_0)|}\int_{P_r(z_0)} u^\varepsilon, \quad r > 0.$$

Proof. We follow [Wang 2006, page 1631, Proposition 5.1] with suitable modifications, and outline the key steps here. For simplicity, write $(u, H) = (u^\varepsilon, H^\varepsilon)$ and assume $z_0 = (x_0, t_0) = (0, 1) \in \Omega \times \mathbb{R}_+$. For $r > 0$, let $u_r(x, t) = u(rx, 1 + r^2 t)$ and $H_r(x, t) = r^2 H(rx, 1 + r^2 t)$ for $(x, t) \in P_1$. Then it follows from (1-19) that (u_r, H_r) satisfies

$$R(u_r)\left(\frac{\partial u_r}{\partial t}\right) = \Delta u_r + \frac{(1 - |u_r|^2)}{\hat\varepsilon^2} u_r + u_r \times (H_r \times u_r) \quad \text{in } P_1,$$

where $\hat\varepsilon = r^{-1}\varepsilon$. Moreover,

$$\int_{P_1} e_{\hat\varepsilon}(u_r) = r^{-3}\int_{P_r(0,1)} e_\varepsilon(u) \le \varepsilon_1^2(\delta),$$
$$\|H_r\|^2_{L_t^\infty L_x^2(P_1)} = r\|H\|^2_{L_t^\infty L_x^2(P_r(0,1))} \le L^2 r \le \varepsilon_1^2(\delta),$$

as $r \le \dfrac{\varepsilon_1^2(\delta)}{L^2}$. From this scaling argument, we may further assume that $r = 1$ and

$$\|H\|_{L_t^\infty L_x^2(P_1(0,1))} \le \varepsilon_1(\delta). \tag{5-3}$$

Now we write

$$\int_{P_{1/8}(0,1)} e_\varepsilon(u) = \mathrm{I} + \mathrm{II},$$

with

$$\mathrm{I} = \int_{(1-(1/8)^2,1)\cap G^\Lambda_{(0,1),1/2}} \int_{B_{1/8}} e_\varepsilon(u), \quad \mathrm{II} = \int_{(1-(1/8)^2,1)\cap B^\Lambda_{(0,1),1/2}} \int_{B_{1/8}} e_\varepsilon(u). \tag{5-4}$$

By (4-3) and Lemma 2.6, we have the estimate

$$\mathrm{II} \le |B^\Lambda_{(0,1),1/2}| \sup_{\substack{t\in B^\Lambda_{(0,1),1/2} \\ t\in[1-(1/8)^2,1]}} \int_{B_{1/8}} e_\varepsilon(u) \le \frac{1}{\Lambda^2}\int_{P_1(0,1)} (e_\varepsilon(u) + |H|^2). \tag{5-5}$$

To estimate I, observe that (5-3) and Lemma 4.2 imply that

$$|u|(x, t) \ge \tfrac{1}{2} \quad \text{for all } x \in B_{1/4} \text{ and } t \in G^\Lambda_{(0,1),1/2}. \tag{5-6}$$

This, combined with the fact $|u| \le 1$ in $\Omega \times \mathbb{R}_+$, implies

$$|\nabla u|^2 \le 4|u|^2|\nabla u|^2 = 4|\nabla u \times u|^2 + |\nabla |u|^2|^2 \le 4(|\nabla u \times u|^2 + |\nabla |u||^2).$$

Therefore $\int_{B_{1/8}} e_\varepsilon(u) \le \mathrm{III} + \mathrm{IV}$ for $t \in G^{\Lambda}_{(0,1),1/2}$, with

$$\mathrm{III} = 2\int_{B_{1/8}} |\nabla u \times u|^2, \quad \mathrm{IV} = +\int_{B_{1/8}} \left(2|\nabla |u||^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2}\right). \tag{5-7}$$

By the definition of $G^{\Lambda}_{(0,1),1/2}$ and Lemma 2.6, we have

$$\begin{aligned} \int_{B_{1/2}} e_\varepsilon(u) + \int_{B_{1/2}} \left|\frac{\partial u}{\partial t}\right|^2 &\le C\Lambda^2\left(\int_{P_1(0,1)} e_\varepsilon(u) + \int_{P_1(0,1)} |H|^2\right) \\ &\le C\Lambda^2\left(\int_{P_1(0,1)} e_\varepsilon(u) + \|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\right). \end{aligned} \tag{5-8}$$

Hence, for $t \in G^{\Lambda}_{(0,1),1/2}$,

$$\begin{aligned} \sup_{x \in B_{1/4}} \left\{\int_{B_{1/4}(x)} e_\varepsilon(u) + \int_{B_{1/4}(x)} \left|\frac{\partial u}{\partial t}\right|^2\right\} &\le C\Lambda^2 \int_{P_1(0,1)} e_\varepsilon(u) \\ &\quad + C\Lambda^2 \|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}. \end{aligned} \tag{5-9}$$

It follows from (5-9) and Lemma 3.1 that

$$\begin{aligned} \sup\left\{s^{-1}\int_{B_s(x)} |\nabla u|^2 : \ x \in B_{1/4},\ 0 < s < \tfrac{1}{4}\right\} \\ \le C\Lambda^2 \int_{P_1(0,1)} e_\varepsilon(u) + C\Lambda^2 \|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}. \end{aligned} \tag{5-10}$$

To estimate III, let $\phi \in C^\infty_0(B_{1/4})$ be such that $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_{1/8}$, and $|\nabla \phi| \le 128$. Then we have, by integration by parts,

$$\begin{aligned} \int_{B_{1/8}} |\nabla u \times u|^2 \le \int_{\mathbb{R}^3} \phi^2 |\nabla u \times u|^2 &= \int_{\mathbb{R}^3} \phi^2 (\nabla u \times u) \cdot (\nabla u \times u) \\ &= \int_{\mathbb{R}^3} \phi^2 (\nabla u \times u) \cdot \big(\nabla(u - c_{1/4}(t)) \times u\big) \\ &= \int_{\mathbb{R}^3} \big(\phi^2 (\nabla u \times u) \times \nabla u\big) \cdot \big(u - c_{1/4}(t)\big) \\ &\quad - \int_{\mathbb{R}^3} \nabla \cdot \big(\phi^2 (\nabla u \times u)\big) \cdot \big((u - c_{1/4}(t)) \times u\big), \end{aligned} \tag{5-11}$$

where

$$c_r(t) = \frac{1}{|B_r|}\int_{B_r} u(t) \ \text{ for } r > 0.$$

Setting

$$\lambda = \frac{\int_{\mathbb{R}^3} \phi^2 (\nabla u \times u) \times \nabla u}{\int_{\mathbb{R}^3} \phi^2},$$

we can rewrite the expression on the last two lines of (5-11) to obtain

$$\int_{B_{1/8}} |\nabla u \times u|^2 = \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3,$$

where

$$\mathrm{III}_1 = \int_{\mathbb{R}^3} \phi^2 \big((\nabla u \times u) \times \nabla u - \lambda\big) \cdot (u - c_{1/4}(t)), \quad \mathrm{III}_2 = \lambda \int_{\mathbb{R}^3} \phi^2 (u - c_{1/4}(t)),$$
$$\mathrm{III}_3 = -\int_{\mathbb{R}^3} \nabla \cdot \big(\phi^2 (\nabla u \times u)\big) \cdot \big((u - c_{1/4}(t)) \times u\big).$$

It follows from Lemma 2.6 that

$$|\lambda| \le C \int_{B_{1/4}} |\nabla u|^2 \le C \bigg(\int_{P_1(0,1)} e_\varepsilon(u) + \|H\|^2_{L^\infty_t L^2_x(P_1(0,1))} \bigg) \tag{5-12}$$

so that by Hölder's inequality and Poincaré's inequality,

$$\begin{aligned} |\mathrm{III}_2| &\le |\lambda| \| u - c_{1/4}(t)\|_{L^2(B_{1/4})} \\ &\le C \bigg(\int_{P_1(0,1)} e_\varepsilon(u) + \|H\|^2_{L^\infty_t L^2_x(P_1(0,1))} \bigg) \|\nabla u\|_{L^2(B_{1/4})} \\ &\le C \bigg(\int_{P_1(0,1)} e_\varepsilon(u) + \|H\|^2_{L^\infty_t L^2_x(P_1(0,1))} \bigg)^{3/2}. \end{aligned} \tag{5-13}$$

To estimate III_3, we first note that (1-19) is equivalent to

$$\nabla \cdot (\nabla u \times u) = \bigg(R(u) \Big(\frac{\partial u}{\partial t} \Big) + u \times (u \times H) \bigg) \times u. \tag{5-14}$$

Hence, by using (5-14), (5-10), and Lemma 2.6,

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla \cdot (\phi^2 \nabla u \times u)|^2 &\le \int_{\mathbb{R}^3} \big(|\nabla \phi|^2 |\nabla u|^2 + \phi^2 \, |\nabla \cdot (\nabla u \times u)|^2 \big) \\ &\le C \int_{B_{1/4}} |\nabla u|^2 + C \int_{B_{1/4}} \bigg(\Big| \frac{\partial u}{\partial t} \Big|^2 + |H|^2 \bigg) \\ &\le C \Lambda^2 \bigg(\int_{P_1(0,1)} e_\varepsilon(u) + \|H\|^2_{L^\infty_t L^2_x(P_1(0,1))} \bigg). \end{aligned}$$

Therefore, by Hölder's inequality we have, for any $\delta > 0$,

$$|\mathrm{III}_3| \le \|\nabla\cdot(\phi^2\nabla u\times u)\|_{L^2(\mathbb{R}^3)}\|u-c_{1/4}(t)\|_{L^2(B_{1/4})} \le \frac{\delta}{4}\biggl(\int_{P_1(0,1)} e_\varepsilon(u)+\|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\biggr)+C\frac{\Lambda^2}{\delta}\|u-c_{1/4}(t)\|^2_{L^2(B_{1/4})}. \tag{5-15}$$

To estimate III_1, we utilize the duality between Hardy and BMO spaces (see also [Hélein 1990; Evans 1991; Wang 2006]). First, by the definition of the BMO norm, the Poincaré inequality, and (5-10), we have

$$(u-c_{1/4}(t))^2_{\mathrm{BMO}(B_{1/4})} \le \sup\biggl\{s^{-1}\int_{B_s(x)}|\nabla u|^2 : x\in B_{1/4},\ 0<s<\tfrac14\biggr\} \le C\Lambda^2\biggl(\int_{P_1(0,1)} e_\varepsilon(u)+\|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\biggr). \tag{5-16}$$

Therefore by (5-15), (5-16), and [Wang 2006, Propositions 5.6–5.8], we have

$$\begin{aligned}|\mathrm{III}_1| &= \biggl|\int_{\mathbb{R}^3}\phi^2\bigl((\nabla u\times u)\times\nabla u-\lambda\bigr)\cdot(u-c_{1/4}(t))\biggr| \\ &\le C\bigl\|\phi^2((\nabla u\times u)\times\nabla u-\lambda)\bigr\|_{\mathcal{H}^1(\mathbb{R}^3)} u-c_{1/4}(t)_{\mathrm{BMO}(B_{1/4})} \\ &\le C\bigl\|\phi^2(\nabla u\times u)\times\nabla u\bigr\|_{\mathcal{H}^1(B_{1/4},B_{1/2})}(u-c_{1/4}(t))_{\mathrm{BMO}(B_{1/4})} \\ &\le C(u-c_{1/4}(t))_{\mathrm{BMO}(B_{1/4})}\bigl(\|\nabla u\|^2_{L^2(B_{1/2})}+\|\nabla\cdot(\nabla u\times u)\|^2_{L^2(B_{1/2})}\bigr) \\ &\le C\Lambda^3\biggl(\int_{P_1(0,1)} e_\varepsilon(u)+\|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\biggr)^{3/2}.\end{aligned} \tag{5-17}$$

Putting the estimates (5-3), (5-13), (5-15) and (5-17) together, we get

$$\int_{B_{1/8}}|\nabla u\times u|^2 \le \Bigl(C\Lambda^3\varepsilon_1(\delta)+\frac{\delta}{4}\Bigr)\biggl(\int_{P_1(0,1)} e_\varepsilon(u)+\|H\|^2_{L^\infty_t L^2_x(P_1(0,1))}\biggr) + C\frac{\Lambda^2}{\delta}\int_{B_{1/4}}|u-c_{1/4}(t)|^2. \tag{5-18}$$

Now we estimate IV as follows. It follows from (5-6) that we can write $u=\rho\omega$, with $\rho=|u|\ge\frac12$ and $\omega=u/|u|$. Then ρ satisfies

$$\Delta\rho-\rho|\nabla\omega|^2+\frac{(1-\rho^2)\rho}{\varepsilon^2}=R(u)\Bigl(\frac{\partial u}{\partial t}\Bigr)\cdot\omega \quad \text{in } B_{1/4}. \tag{5-19}$$

Multiplying (5-19) by $\phi^2(1-\rho)$ for $\phi\in C_0^\infty(B_{1/4})$ and integrating over $B_{1/4}$, we can write

$$\int_{B_{1/4}}\phi^2\Bigl(|\nabla\rho|^2+\frac{(1-\rho)^2}{\varepsilon^2}\rho(1+\rho)\Bigr)=\mathrm{IV}_1+\mathrm{IV}_2+\mathrm{IV}_3, \tag{5-20}$$

where

$$\mathrm{IV}_1=\int_{B_{1/4}}(1-\rho)\nabla\rho\cdot\nabla\phi^2,\quad \mathrm{IV}_2=\int_{B_{1/4}}\phi^2(1-\rho)R(u)\Big(\frac{\partial u}{\partial t}\Big)\cdot\omega,$$
$$\mathrm{IV}_3=\int_{B_{1/4}}\phi^2\rho(1-\rho)|\nabla\omega|^2.$$

Since $|\nabla\rho|\le|\nabla u|$, Lemma 2.6 gives

$$|\mathrm{IV}_1|\le\int_{B_{1/4}}|\nabla u|(1-|\rho|^2)\le\varepsilon\Big(\int_{B_{1/4}}|\nabla u|^2\Big)^{1/2}\Big(\int_{B_{1/4}}\frac{(1-|u|^2)^2}{\varepsilon^2}\Big)^{1/2}$$
$$\le C\Lambda^2\varepsilon\Big(\int_{P_1(0,1)}e_\varepsilon(u)+\|H\|_{L_t^\infty L_x^2(P_1(0,1))}\Big). \tag{5-21}$$

For IV_2, we have

$$|\mathrm{IV}_2|\le\int_{B_{1/4}}\Big|\frac{\partial u}{\partial t}\Big|(1-|\rho|^2)\le\varepsilon\Big(\int_{B_{1/4}}\Big|\frac{\partial u}{\partial t}\Big|^2\Big)^{1/2}\Big(\int_{B_{1/4}}\frac{(1-|u|^2)^2}{\varepsilon^2}\Big)^{1/2}$$
$$\le C\Lambda^2\varepsilon\Big(\int_{P_1(0,1)}e_\varepsilon(u)+\|H\|^2_{L_t^\infty L_x^2(P_1^+(0,1))}\Big). \tag{5-22}$$

Since $|\omega|=1$ and $\rho\ge\frac12$, we have $|\nabla\omega|^2\le 14|\nabla u\times u|^2$. Hence

$$|\mathrm{IV}_3|\le C\int_{B_{1/4}}|\nabla u\times u|^2. \tag{5-23}$$

Therefore, for $t\in G^\Lambda_{(0,1),1/2}$,

$$|\mathrm{IV}|\le C\Lambda^2\varepsilon\Big(\int_{P_1(0,1)}e_\varepsilon(u)+\|H\|^2_{L_t^\infty L_x^2(P_1(0,1))}\Big)+C\int_{B_{1/4}}|\nabla u\times u|^2. \tag{5-24}$$

Putting the estimates for III and IV together, we obtain for any $t\in G^\Lambda_{(0,1),1/2}$,

$$\int_{B_{1/8}}e_\varepsilon(u)\le\Big(C\Lambda^2(\varepsilon+\Lambda\varepsilon_1(\delta))+\frac{\delta}{4}\Big)\Big(\int_{P_1(0,1)}e_\varepsilon(u)+\|H\|^2_{L_t^\infty L_x^2(P_1(0,1))}\Big)$$
$$+C\frac{\Lambda^2}{\delta}\int_{B_1}|u-c_1(t)|^2. \tag{5-25}$$

Integrating (5-25) over $t\in G^\Lambda_{(0,1),1/2}$ and adding (5-5), we obtain

$$\Big(\frac18\Big)^{-3}\int_{P_{1/8}(0,1)}e_\varepsilon(u)$$
$$\le\Big(C\Lambda^2(\varepsilon+\Lambda\varepsilon_1(\delta))+\frac{\delta}{4}+\frac{1}{\Lambda^2}\Big)\Big(\int_{P_1(0,1)}e_\varepsilon(u)+\|H\|^2_{L_t^\infty L_x^2(P_1(0,1))}\Big)$$
$$+\frac{C\Lambda^2}{\delta}\int_{P_1(0,1)}|u-c_1(t)|^2. \tag{5-26}$$

Lemma 5.1 will be proved if we choose, for any fixed small $\delta > 0$, a sufficiently large $\Lambda = 2/\sqrt{\delta} > 0$, a sufficiently small $\varepsilon = \delta/(16C)$ and $\varepsilon_1(\delta) = \delta^{5/2}/(32C)$. Here we have also used in the last step the fact that

$$\int_{P_1(0,1)} |u - c_1(t)|^2 \le 2\int_{P_1(0,1)} |u - u_{P_1(0,1)}|^2. \qquad \square$$

Lemma 5.2. *There exists a constant $C_0 > 0$ such that for any $L > 0$, $\theta \in (0, \frac{1}{4})$ there are $\varepsilon(\theta), \varepsilon_1(\theta) > 0$ such that if $(u^\varepsilon, H^\varepsilon)$ is the weak solution of (1-19) in Lemma 2.2 and we take $0 < r < \min\{\operatorname{dist}(x_0, \partial\Omega), \sqrt{t_0}, \varepsilon_1^2(\theta)/L^2\}$, $\varepsilon < \varepsilon(\theta)r$, and $z_0 = (x_0, t_0) \in \Omega \times \mathbb{R}_+$ satisfying*

$$\|H^\varepsilon\|_{L_t^\infty L_x^2(P_r(z_0))} \le L \quad \textit{and} \quad \int_{P_r(z_0)} e_\varepsilon(u^\varepsilon) \le \varepsilon_1^2(\theta), \tag{5-27}$$

then

$$\frac{1}{(\theta r)^5}\int_{P_{\theta r}(z_0)} |u^\varepsilon - u^\varepsilon_{P_{\theta r}(z_0)}|^2 \le C_0\theta^2 \max\left\{r^{-3}\int_{P_r(z_0)} e_\varepsilon(u^\varepsilon),\ r\|H^\varepsilon\|^2_{L_t^\infty L_x^2(P_r(z_0))}\right\}$$

where $u^\varepsilon_{P_{\theta r}(z_0)}$ is the average of u^ε over $P_{\theta r}(z_0)$.

Proof. Write (u, H) for $(u^\varepsilon, H^\varepsilon)$. Assume that $z_0 = (0, 1)$, $r = 1$, and

$$\|H\|_{L_t^\infty L_x^2(P_1(0,1))} \le \varepsilon_1(\theta).$$

Now we argue by contradiction. Suppose that Lemma 5.2 is false. Then there are $\theta_0 \in (0, \frac{1}{4})$, $\varepsilon_k \downarrow 0$, and a sequence of weak solutions (u^k, H^k) of (1-19) corresponding to $\varepsilon = \varepsilon_k$ such that

$$\int_{P_1(0,1)} e_{\varepsilon_k}(u^k) = \delta_k^2 \downarrow 0, \quad \|H^k\|^2_{L_t^\infty L^2(P_1(0,1))} \le \delta_k^2, \tag{5-28}$$

but

$$\theta_0^{-5}\int_{P_{\theta_0}(0,1)} |u^k - u^k_{P_{\theta_0}(0,1)}|^2 \ge k\theta_0^2 \max\left\{\int_{P_1(0,1)} e_{\varepsilon_k}(u^k),\ \|H^k\|^2_{L_t^\infty L^2(P_1(0,1))}\right\}. \tag{5-29}$$

Define

$$v^k = \frac{u^k - u^k_{P_1(0)}}{\delta_k}.$$

By Lemma 2.6, $\{v^k\}$ is uniformly bounded in $H^1(P_{1/2}(0, 1))$ and $(v^k)_{P_1(0,1)} = 0$. Assume that $v^k \to v$ weakly in $H^1(P_{1/2}(0, 1), \mathbb{R}^3)$, strongly in $L^2(P_{1/2}(0, 1), \mathbb{R}^3)$, and $u^k \to p$ for some $p \in S^2$. It is not hard to show that $v \in T_pS^2$ and hence we have $R(p)(\partial v/\partial t) - \Delta v \in T_pS^2$. Observe that

$$\left(R(u^k)\left(\frac{\partial v^k}{\partial t}\right) - \Delta v^k - \delta_k^{-1}(u^k \times (H^k \times u^k))\right) \times u^k = 0,$$

and (5-29) implies

$$\left|\delta_k^{-1}(u^k\times(H^k\times u^k))\times u^k\right|\le\frac{|H^k|}{\delta_k}\to 0\quad\text{in } L^2(P_1(0,1))\text{ as } k\to\infty.$$

By sending k to ∞, we conclude that v solves

$$\Bigl(R(p)\frac{\partial v}{\partial t}-\Delta v\Bigr)\times p=0.$$

Therefore

$$R(p)\frac{\partial v}{\partial t}-\Delta v=0\text{ in } P_{1/2}(0,1).\tag{5-30}$$

Standard parabolic theory [Lieberman 1996] implies

$$\theta_0^{-5}\int_{P_{\theta_0}(0,1)}|v|^2\le C\theta_0^2\int_{P_1}|\nabla v|^2,$$

which contradicts (5-29). □

Combining Lemma 5.1 and Lemma 5.2, we can prove:

Lemma 5.3. *For any $\gamma\in(0,1)$, there are $\theta\in(0,\frac14)$, $C_1>0$, $k_0>0$, $\varepsilon_2>0$ such that if $(u^\varepsilon,H^\varepsilon)$ is the weak solution of (1-19) in Lemma 2.2 and we take $z_0=(x_0,t_0)\in\Omega\times\mathbb{R}_+$, $L>0$, $0<r<\min\{\operatorname{dist}(x_0,\partial\Omega),\sqrt{t_0},\varepsilon_2^2/L^2\}$, and $0<\varepsilon\le k_0r$ satisfying*

$$\|H^\varepsilon\|_{L_t^\infty L_x^2(P_r(z_0))}\le L\quad\textit{and}\quad r^{-3}\int_{P_r(z_0)}e_\varepsilon(u^\varepsilon)\le\varepsilon_2^2,\tag{5-31}$$

then

$$(\theta r)^{-3}\int_{P_{\theta r}(z_0)}e_\varepsilon(u^\varepsilon)\le C_1\Bigl(\theta^{2\gamma}r^{-3}\int_{P_r(z_0)}e_\varepsilon(u^\varepsilon)+\theta r\|H^\varepsilon\|^2_{L_t^\infty L_x^2(P_r(z_0))}\Bigr).\tag{5-32}$$

Proof. Again we write (u,H) for $(u^\varepsilon,H^\varepsilon)$. As in the proof of Lemmas 5.1 and 5.2, we may assume that $z_0=(0,1)$, $r=1$, and

$$\|H\|_{L_t^\infty L_x^2(P_1(0,1))}\le\varepsilon_2.\tag{5-33}$$

Set $\delta=8^{-3}$ and

$$\theta=\theta(\gamma)\le\Bigl(\frac{\delta^2}{2C_0C(\delta)}\Bigr)^{1/(2-2\gamma)}.$$

Here $C_0>0$ and $C(\delta)>0$ are given by Lemma 5.2 and Lemma 5.1 respectively. Let $k\ge1$ be such that $8^k\theta=1$. For $0<\rho<1$, set

$$E(u,\rho)=\rho^{-3}\int_{P_\rho(0,1)}e_\varepsilon(u),\qquad F(H,\rho)=\rho\|H\|^2_{L_t^\infty L_x^2(P_\rho(0,1))}.$$

For $0 \le i \le k-1$, if $E(u, 8^{i+1}\theta) \le \varepsilon_1^2(\delta)$ and $E(u,1) \le \varepsilon_1^2(8^{i+1}\theta)$, then Lemmas 5.1 and 5.2 imply

$$E(u, 8^i\theta) \le \delta \max\{E(u, 8^{i+1}\theta), F(H, 8^{i+1}\theta)\} + \frac{C_0C(\delta)}{\delta}\max\{E(u,1), F(H,1)\}. \tag{5-34}$$

Now we choose

$$\varepsilon_2 \equiv \frac{\delta}{2C_0C(\delta)}\min\{\varepsilon_1(8\theta), \dots, \varepsilon_1(8^k\theta), \varepsilon_1(\delta)\}.$$

Since $F(H,\rho) \le \rho F(H,1) \le F(H,1) \le \varepsilon_2^2$, (5-34) implies that

$$E(u, 8^i\theta) \le \min\{\varepsilon_1^2(8\theta), \dots, \varepsilon_1^2(8^k\theta), \varepsilon_1^2(\delta)\} \quad \text{for all } 0 \le i \le k.$$

Hence by iteration, (5-34) implies

$$\begin{aligned} E(u,\theta) &\le \delta^k E(u,1) + F(H,1)\sum_{i=1}^{k}(8\delta\theta)^i + \frac{C_0C(\delta)}{1-64\delta}\left(\frac{\theta}{\delta}\right)^2 \max\{E(u,1), F(H,1)\} \\ &\le \delta^k E(u,1) + \frac{8\delta\theta}{1-8\delta\theta}F(H,1) + \frac{C_0C(\delta)}{1-64\delta}\left(\frac{\theta}{\delta}\right)^2 \max\{E(u,1), F(H,1)\}. \end{aligned} \tag{5-35}$$

According to the definition, $\delta^k = \theta^3$ and $2C_0C(\delta)/\delta^2 \le \theta^{2-2\gamma}$. So (5-35) gives $E(u,\theta) \le \max\{C_1\theta^{2\gamma}E(u,1), C_1\theta F(H,1)\}$, which clearly implies (5-32). □

The following proposition plays a crucial role in the proof of Theorem 1.2:

Proposition 5.4. *For any $u_0 \in H^1(\Omega, S^2)$, $H_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, $E_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, $\varepsilon > 0$ and $0 < T < +\infty$, let*

$$(u^\varepsilon, H^\varepsilon, E^\varepsilon) \in H^1(\Omega \times [0,T], \mathbb{R}^3) \times L^2(\mathbb{R}^3 \times [0,T], \mathbb{R}^3) \times L^2(\mathbb{R}^3 \times [0,T], \mathbb{R}^3)$$

be the weak solution of (1-19)+(1-2)–(1-3) under conditions (1-8)–(1-11) obtained in Lemma 2.2. Then there exist universal constants $k_0 > 0$, $\varepsilon_3 > 0$, $C_2 > 0$, such that for any $z_0 = (x_0, t_0) \in \Omega \times \mathbb{R}_+$, $0 < r < \min\{\mathrm{dist}(x_0, \partial\Omega), \sqrt{t_0}, \varepsilon_3^2/C_2\}$, if

$$\mathscr{E}(u^\varepsilon, z_0, r) \equiv r^{-3}\int_{P_r(z_0)} e_\varepsilon(u^\varepsilon) \le \varepsilon_3^2, \tag{5-36}$$

then for any $z \in P_{r/2}(z_0)$, $\varepsilon/k_0 \le \rho \le \frac{1}{4}r$,

$$\rho^{-3}\int_{P_\rho(z)}\left(e_\varepsilon(u^\varepsilon) + \rho^2\left|\frac{\partial u^\varepsilon}{\partial t}\right|^2\right) \le C_2\frac{\rho}{r}\max\{\mathscr{E}(u^\varepsilon, z_0, r), r\|H^\varepsilon\|^2_{L_t^\infty L_x^2(P_r(z_0))}\}. \tag{5-37}$$

Proof. By (2-8) of Lemma 2.2, we have $H^\varepsilon \in L^\infty([0,T], L^2(\mathbb{R}^3))$ and

$$\|H^\varepsilon\|_{L_t^\infty L_x^2(\mathbb{R}^3\times[0,T])} \le e^{CT}\left(\int_\Omega |\nabla u_0|^2 + \int_{\mathbb{R}^3} (\varepsilon_0|E_0|^2 + |H_0|^2)\right) \equiv C_2. \tag{5-38}$$

This implies that for any $0<\rho\le r$ and $z\in P_{r/2}(z_0)$

$$\rho\|H^\varepsilon\|_{L_t^\infty L_x^2(P_\rho(z))} \le r\|H^\varepsilon\|_{L_t^\infty L_x^2(\mathbb{R}^3\times[0,T])} \le rC_2 \le \varepsilon_3^2.$$

Choose $\varepsilon_3 \le \varepsilon_2$, where ε_2 is given by Lemma 5.3. Then the condition (5-31) of Lemma 5.3 is satisfied for $P_{r/2}(z)$ with $z\in P_{r/2}(z_0)$. Hence we can repeatedly apply Lemma 5.3 with $\gamma=\frac{1}{2}$ to obtain that for $0<\rho<\frac{r}{4}$, $\varepsilon\le k_0\rho$,

$$\mathscr{E}(u^\varepsilon, z, \rho) \le C_1\frac{\rho}{r}\max\bigl\{\mathscr{E}(u^\varepsilon, z_0, r), r\|H^\varepsilon\|^2_{L_t^\infty L_x^2(P_r(z_0))}\bigr\}. \tag{5-39}$$

This, combined with Lemma 2.6, implies (5-37). □

Proof of Theorem 1.2. For $\varepsilon>0$, let $(u^\varepsilon, H^\varepsilon, E^\varepsilon)$ be the weak solution of the system (1-19)+(1-2)–(1-3) with conditions (1-8)–(1-11) obtained in Lemma 2.2. It follows from (2-8) that we may assume that $u^\varepsilon \to u$ weakly in $H^1_{\text{loc}}(\Omega\times\mathbb{R}_+, \mathbb{R}^3)$, $(H^\varepsilon, E^\varepsilon)\to(H,E)$ weakly in $L^2_{\text{loc}}(\mathbb{R}^3\times\mathbb{R}_+, \mathbb{R}^3)$. By the argument in [Carbou and Fabrie 1998], we know that (u, H, E) is a weak solution of the Landau–Lifshitz–Maxwell system (1-13)+(1-2)–(1-3) under the initial-boundary conditions (1-8)–(1-11).

Now we want to show partial regularity of u as follows. Let ε_3 be given by Proposition 5.4, and define the concentrate set of u^ε by

$$\Sigma = \bigcap_{r>0}\left\{z\in\Omega\times\mathbb{R}^+ : \liminf_{\varepsilon\to 0} r^{-3}\int_{P_r(z)} e_\varepsilon(u^\varepsilon) \ge \varepsilon_3^2\right\}. \tag{5-40}$$

Then a standard covering argument (see [Chen and Struwe 1989]) shows that $\mathscr{P}^3(\Sigma\cap K)<\infty$ for any compact subset of $\Omega\times\mathbb{R}^+$. Since u is a weak limit in $H^1_{\text{loc}}(\Omega\times\mathbb{R}_+,\mathbb{R}^3)$ of u_ε as $\varepsilon\downarrow 0$, we conclude that for any $z_0\in\Omega\times\mathbb{R}_+\setminus\Sigma$, the lower semicontinuity, (5-40), and Proposition 5.4 imply that there exists $r_0>0$ such that for any $z\in P_{r/2}(z_0)$ and $0<\rho\le\frac{1}{4}r$,

$$\rho^{-3}\int_{P_\rho(z)}\left(|\nabla u|^2 + \rho^2\left|\frac{\partial u}{\partial t}\right|^2\right) \le C_3\frac{\rho}{r} \tag{5-41}$$

for some universal constant $C_3>0$. This implies that $u\in C^{1/2}(\Omega\times\mathbb{R}_+\setminus\Sigma, S^2)$, by the parabolic version of Morrey's Lemma [Chen et al. 1995]. This completes the proof of Theorem 1.2. □

6. C^α-regularity of ∇u and proofs of Theorems 1.3 and 1.4

This section is devoted to the discussion of partial C^α-regularity of ∇u, when (u, H, E) is a weak solution of (1-13)+(1-2)–(1-3) obtained as in Theorem 1.2 in two special cases: (i) either $\varepsilon_0 = 0$ in (1-2) or (ii) $\beta = 0$ in (1-3). For case (i), we assume that the initial data $(u_0, H_0) \in H^1(\Omega, S^2) \times H^1(\mathbb{R}^3, \mathbb{R}^3)$ and H_0 satisfies $\nabla \cdot (H_0 + \beta u_0) = 0$. For case (ii), we assume that the initial data $(u_0, H_0, E_0) \in H^1(\Omega, S^2) \times H^1(\mathbb{R}^3, \mathbb{R}^3) \times H^1(\mathbb{R}^3, \mathbb{R}^3)$ and H_0, E_0 satisfy $\nabla \cdot H_0 = \nabla \cdot E_0 = 0$.

There are two steps to proving C^α-regularity of ∇u in $\Omega \times \mathbb{R}_+ \setminus \Sigma$, where Σ is the concentration set (5-40). The first step is to use $H \in L_t^\infty L_x^6(\mathbb{R}^3 \times [0, T])$ for any $0 < T < +\infty$ to show that $u \in C^\gamma(\Omega \times \mathbb{R}_+ \setminus \Sigma, S^2)$ for any $\gamma \in (0, 1)$. The second step is to employ a parabolic hole-filling technique similar to [Giaquinta and Hildebrandt 1982; Giaquinta and Struwe 1982] to show that for $z \in \Omega \times \mathbb{R}_+ \setminus \Sigma$,

$$\rho^{-5} \int_{P_\rho(z)} |\nabla u - (\nabla u)_{P_\rho(z)}|^2 \le C\rho^{2\alpha} \quad \text{for some } \alpha \in (0, 1).$$

This can be summarized as follows.

Lemma 6.1. *For any $u_0 \in H^1(\Omega, S^2)$, $H_0 \in H^1(\mathbb{R}^3, \mathbb{R}^3)$, and $0 < T < +\infty$, let $(u, H) \in H^1(\Omega \times [0, T], S^2) \times L_t^\infty L_x^2(\mathbb{R}^3 \times [0, T], \mathbb{R}^3)$ be a weak solution to (1-13) coupled with (1-17) under the initial-boundary conditions (1-8)–(1-10) obtained as the weak limit of $(u^\varepsilon, H^\varepsilon)$ given by Lemma 2.3. Let $\Sigma \subset \Omega \times \mathbb{R}_+$ be defined by (5-40). For any $z_0 \in \Omega \times \mathbb{R}_+ \setminus \Sigma$, there exists $r_0 > 0$ such that $\nabla u \in C^\alpha(P_{r_0}(z_0))$ for some $\alpha \in (0, 1)$.*

Proof. By (2-17) of Lemma 2.3,

$$\text{(6-1)} \quad \sup_{\varepsilon>0}\big(\|H^\varepsilon\|_{L_t^\infty L_x^2(\mathbb{R}^3\times[0,T])} + \|\nabla H^\varepsilon\|_{L_t^\infty L_x^2(\mathbb{R}^3\times[0,T])}\big) \le e^{CT}\big(\|\nabla u_0\|^2_{L^2(\Omega)} + \|H_0\|^2_{L^2(\mathbb{R}^3)} + \|\nabla H_0\|^2_{L^2(\mathbb{R}^3)}\big).$$

By the Sobolev embedding theorem, (6-1) implies $H^\varepsilon \in L_t^\infty L_x^6(\mathbb{R}^3 \times [0, T])$ and

$$\text{(6-2)} \quad \sup_{\varepsilon>0} \|H^\varepsilon\|_{L_t^\infty L_x^6(\mathbb{R}^3\times[0,T])} \le C_3 \equiv C\left(T, \|u_0\|_{H^1(\Omega)}, \|H_0\|_{H^1(\mathbb{R}^3)}\right).$$

Since $z_0 \in \Omega \times \mathbb{R}_+ \setminus \Sigma$, by (5-40), there exists $0 < r_0 \le \varepsilon_3^2/C_3^2$ such that

$$\mathscr{E}(u^\varepsilon, z_0, r_0) \equiv r_0^{-3} \int_{P_{r_0}(z_0)} e_\varepsilon(u^\varepsilon) \le \varepsilon_3^2, \quad \mathscr{F}(H^\varepsilon, z_0, r_0) \equiv r_0 \|H^\varepsilon\|^2_{L_t^\infty L_x^2(P_{r_0}(z_0))} \le \varepsilon_3^2.$$

Hence we can apply Lemma 5.3 to conclude that for any $\theta \in (0, \frac{1}{2})$, $\gamma \in (0, 1)$, $z \in P_{r_0/2}(z_0)$ and $0 < r < r_0/2$, there is $C_4 > 0$ such that

$$\text{(6-3)} \quad \mathscr{E}(u^\varepsilon, z, \theta r) \le C_4 \theta^{2\gamma} \mathscr{E}(u^\varepsilon, z, r) + C_4 \theta \mathscr{F}(H^\varepsilon, z, r).$$

By Hölder's inequality we have

$$\mathcal{F}(H^\varepsilon, z, r) \le r^3 \|H^\varepsilon\|_{L_t^\infty L_x^2(P_r(z))} \le C_3 r^3 \quad \text{for all } 0 < r \le r_0.$$

Therefore (6-3) yields, for $z \in P_{r_0/2}(z_0)$ and $0 < r < r_0/2$,

$$\mathcal{E}(u^\varepsilon, z, \theta r) \le C_5 \left(\theta^{2\gamma} \mathcal{E}(u^\varepsilon, z, r) + \theta r^3\right). \tag{6-4}$$

Iterating (6-4) k times, we obtain

$$\begin{aligned} \mathcal{E}(u^\varepsilon, z, \theta^k r) &\le (C_5\theta^{2\gamma})^k \mathcal{E}(u^\varepsilon, z, r) + \left(\sum_{i=0}^{k-1} (C_5\theta^{2\gamma})^{k-1-i} (\theta^3)^i\right) r^3 \\ &\le (C_5\theta^{2\gamma})^k \left(\mathcal{E}(u^\varepsilon, z, r) + \frac{r^3}{C_5\theta^{2\gamma} - \theta^3}\right). \end{aligned} \tag{6-5}$$

In particular,

$$\mathcal{E}(u^\varepsilon, z, s) \le \left(\frac{s}{r_0}\right)^{2\gamma} \left(\mathcal{E}\left(u^\varepsilon, z, \frac{r_0}{2}\right) + C_6 r_0^3\right) \quad \text{for all } z \in P_{r_0/2}(z_0),\ 0 < s \le \frac{r_0}{2}.$$

In view of Lemma 2.6 and taking $\varepsilon \downarrow 0$, this implies that for $z \in P_{r_0/2}(z_0)$ and $0 < s \le r_0/2$,

$$s^{-3} \int_{P_s(z)} \left(|\nabla u|^2 + s^2 \left|\frac{\partial u}{\partial t}\right|^2\right) \le \left(\frac{s}{r_0}\right)^{2\gamma} (\varepsilon_3^2 + C_6 r_0^3). \tag{6-6}$$

Hence the parabolic version of Morrey's Lemma implies that $u \in C^\gamma\,(P_{r_0/2}(z_0), S^2)$ for any $0 < \gamma < 1$, and

$$\operatorname{osc}_{P_r(z_0)} u \le C \left(\frac{r}{r_0}\right)^\gamma (\varepsilon_3^2 + C_6 r_0^3), \quad 0 < r \le \frac{r_0}{2}. \tag{6-7}$$

Next we will use a parabolic hole-filling argument to show $\nabla u \in C^\alpha(P_{r_0/2}(z_0))$ for some $\alpha \in (0, 1)$. The linear map $R(u)\xi = \alpha_1\xi + \alpha_2 u \times \xi : \mathbb{R}^3 \to \mathbb{R}^3$ can be represented by

$$R(u) = \begin{pmatrix} \alpha_1 & -\alpha_2 u_3 & \alpha_2 u_2 \\ \alpha_2 u_3 & \alpha_1 & -\alpha_2 u_1 \\ -\alpha_2 u_2 & \alpha_2 u_1 & \alpha_1 \end{pmatrix}.$$

It is easy to check that $R(u)$ has an inverse $M(u)$, and that this inverse a uniformly elliptic matrix. Now we can rewrite the equation of u as

$$\frac{\partial u}{\partial t} - \nabla \cdot (M(u)\nabla u) = M(u) \left(|\nabla u|^2 u + (H - \langle H, u\rangle u)\right) - \nabla(M(u)) \cdot \nabla u. \tag{6-8}$$

For any $z_1 \in P_{r_0/2}(z_0)$ and $0 < r < r_0/2$, consider an auxiliary equation for $v : P_r(z_1) \to \mathbb{R}^3$:

$$\frac{\partial v}{\partial t} - \nabla \cdot (M(u(z_1))\nabla v) = 0 \text{ in } P_r(z_1), \quad v = u \text{ on } \partial_p P_r(z_1), \tag{6-9}$$

where $\partial_p P_r(z_1)$ denotes the parabolic boundary of $P_r(z_1)$. It follows from the maximum principle, (6-7), and (6-6) that

$$\text{(6-10)} \qquad \operatorname{osc}_{P_r(z_1)} v \le C_7 r^\gamma, \quad \int_{P_r(z_1)} |\nabla v|^2 \le \int_{P_r(z_1)} |\nabla u|^2 \le C_7 r^{3+2\gamma}.$$

Multiplying (6-8) and (6-9) by $w \equiv u - v$ and integrating over $P_r(z_1)$, we obtain

$$\text{(6-11)} \qquad \int_{P_r(z_1)} \langle M(u(z_1))\nabla w, \nabla w\rangle \le \mathrm{I} + \mathrm{II},$$

with $\mathrm{I} = C_8 \int_{P_r(z_1)} \big(|\nabla u|^2 + |H|\big)|w|$ and $\mathrm{II} = C_8 \int_{P_r(z_1)} \big|M(u) - M(u(z_1))\big|\,|\nabla u|\,|\nabla w|$. By the ellipticity of $M(u(z_1))$, we have

$$\int_{P_r(z_1)} \langle M(u(z_1))\nabla w, \nabla w\rangle \ge \alpha_1 \int_{P_r(z_1)} |\nabla w|^2.$$

By Hölder's inequality, (6-7) and (6-10), we have $I \le C_9 \Big(\dfrac{r}{r_0}\Big)^{3+3\gamma} (\varepsilon_0^2 + r_0^3)$, and

$$\text{(6-12)} \qquad \begin{aligned} \mathrm{II} &\le \frac{\alpha_1}{2} \int_{P_r(z_1)} |\nabla u|^2 + C_{10} (\operatorname{osc}_{P_r(z_1)} u)^2 \int_{P_1(z_1)} |\nabla w|^2 \\ &\le \frac{\alpha_1}{2} \int_{P_r(z_1)} |\nabla u|^2 + C_{10} \Big(\frac{r}{r_0}\Big)^{3+4\gamma}. \end{aligned}$$

Putting these estimates into (6-11), we obtain

$$\text{(6-13)} \qquad \int_{P_r(z_1)} |\nabla w|^2 \le C_{11} r^{3+3\gamma}.$$

Since v solves (6-9), standard parabolic theory implies that for any $0 < \rho < r$,

$$\text{(6-14)} \qquad \int_{P_\rho(z_1)} \big|\nabla v - (\nabla v)_{P_\rho(z_1)}\big|^2 \le C_{12} \Big(\frac{\rho}{r}\Big)^7 \int_{P_r(z_1)} \big|\nabla v - (\nabla v)_{P_r(z_1)}\big|^2.$$

Combining (6-13) with (6-14), we obtain that

$$\text{(6-15)} \qquad \begin{aligned} \int_{P_\rho(z_1)} \big|\nabla u - (\nabla u)_{P_\rho(z_1)}\big|^2 &\le \int_{P_\rho(z_1)} \big|\nabla v - (\nabla v)_{P_\rho(z_1)}\big|^2 + \int_{P_r(z_1)} |\nabla w|^2 \\ &\le C_{12} \Big(\frac{\rho}{r}\Big)^7 \int_{P_r(z_1)} |\nabla u|^2 + C_{12} r^{3+3\gamma}. \end{aligned}$$

We now choose some $\gamma \in (\frac{2}{3}, 1)$, whence $3 + 3\gamma > 5$. Applying the algebraic Lemma 2.1 in [Giaquinta 1983, Chapter III], we conclude that

$$\text{(6-16)} \qquad \rho^{-5} \int_{P_\rho(z_1)} \big|\nabla u - (\nabla u)_{P_\rho(z_1)}\big|^2 \le C_{13} \rho^{3\gamma-2} \Big(1 + r^{-(3+3\gamma)} \int_{P_r(z_1)} |\nabla u|^2\Big)$$

for any $z_1 \in P_{r_0/2}(z_0)$ and $0 < \rho \le r \le r_0/2$.

A well known characterization of Hölder continuous functions due to Campanato [1965] yields that $\nabla u \in C^{(3\gamma-2)/2}(P_{r_0/2}(z_0))$. This completes the proof of Lemma 6.1. □

Completion of proof of Theorem 1.3. It follows immediately from Lemma 6.1 that $\nabla u \in C^{\alpha}(\Omega \times \mathbb{R}_+ \setminus \Sigma)$ for some $\alpha \in (0, 1)$. It remains to show that $\nabla^2 u, \partial u/\partial t \in L^6_{\text{loc}}(\Omega \times \mathbb{R}_+ \setminus \Sigma)$. To see this, observe that

$$\begin{aligned}\left|\frac{\partial u}{\partial t} - \nabla\cdot(M(u)\nabla u)\right| &= \left|M(u)|\nabla u|^2 u - \nabla(M(u))\cdot\nabla u + M(u)\,(H - \langle H, u\rangle u)\right| \\ &\le C_{14}(|\nabla u|^2 + |H|) \in L^6(P_R),\end{aligned}$$

for any $P_R \Subset \Omega \times \mathbb{R}_+ \setminus \Sigma$. Since $M(u)$ is Hölder continuous and uniformly elliptic, from the $W^{2,1}_p$-estimate for the linear parabolic equation [Lieberman 1996], we can conclude that $\nabla^2 u, \partial u/\partial t \in L^6(P_{R/2})$. This implies the second conclusion of Theorem 1.3. □

Proof of Theorem 1.4. By applying Lemma 2.4, we can conclude H^{ε} is bounded in $L^{\infty}_t L^6_x(\mathbb{R}^3 \times [0, T])$ for any $0 < T < +\infty$, uniformly in ε. Hence applying the same argument of Lemma 6.1 shows $\nabla u \in C^{\alpha}(\Omega \times \mathbb{R}_+ \setminus \Sigma)$ for some $\alpha \in (0, 1)$, and $\nabla^2 u, \partial u/\partial t \in L^6_{\text{loc}}(\Omega \times \mathbb{R}_+ \setminus \Sigma)$. We leave the details to interested readers. □

Acknowledgements

The first two authors would like to thank Department of Mathematics at the University of Kentucky for its hospitality where most parts of this joint work were conducted. We thank the referee for several useful comments.

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Received December 4, 2008. Revised February 9, 2009.

SHIJIN DING
DEPARTMENT OF MATHEMATICS
SOUTH CHINA NORMAL UNIVERSITY
GUANGZHOU, GUANGDONG 510631
CHINA

dingsj@scnu.edu.cn

XIANGAO LIU
SCHOOL OF MATHEMATICS
FUDAN UNIVERSITY
SHANGHAI, 200433
CHINA

xgliu@fudan.edu.cn

CHANGYOU WANG
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KENTUCKY
LEXINGTON, KY 40506
UNITED STATES

cywang@ms.uky.edu
http://www.ms.uky.edu/~cywang