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RULED MINIMAL SURFACES IN $\mathbb{R}^3$ WITH DENSITY e^z

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We classify ruled minimal surfaces in $\mathbb{R}^3$ with density e^z. We show that there is a family of cylindrical ones and that there are no others. Also, all translation minimal surfaces are ruled.

1. Introduction

Manifolds with density, a new category in geometry, appear in many ways in mathematics, for example as quotients of Riemannian manifolds or as Gauss space. They are the smooth case of Gromov's mm-spaces. A density on a Riemannian manifold M^n is a positive function $e^{\varphi(x)}$ used to weight volume and hypersurface area. Gauss space G^n is Euclidean space with Gaussian probability density $(2\pi)^{-n/2}e^{-r^2/2}$, a space very interesting to probabilists. For details about manifolds with density and some first results of Morgan's goal to "generalize all of Riemannian geometry to manifolds with density", see [Morgan 2005; 2009a; 2006; 2009b; Morgan and Maurmann 2009; Rosales et al. 2008; Corwin et al. 2006; Doan and Tran 2008]. See especially [2009a, Chapter 18], in which Morgan describes general manifolds with density and their relationship to Perelman's proof of the Poincaré conjecture. Following Gromov [2003, page 213], we define the natural generalization of the mean curvature of hypersurfaces on a manifold with density e^{φ} as

$$H_\varphi = H - \frac{1}{n-1}\frac{d\varphi}{d\boldsymbol{n}}. \tag{1}$$

Therefore, the mean curvature of a surface in $\mathbb{R}^3$ with density e^{φ} is

$$H_\varphi = H - \frac{1}{2}\frac{d\varphi}{d\boldsymbol{n}}, \tag{2}$$

where H is the Euclidean mean curvature and $\boldsymbol{n}$ is the normal vector field of the surface. We call H_φ the mean curvature with density or mean φ-curvature of the surface.

The literature of minimal surfaces began with Lagrange in 1760; his eponymous PDE is satisfied by minimal graphs of a C^2-function of two variables. At that time, the only known solutions to Lagrange's equation were planes. In 1776, Meusnier

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solved the equation with the added assumption that the level curves were straight lines and obtained a ruled minimal surface, the helicoid. It is well known that the helicoid is the only ruled minimal surface besides the trivial case of planes; see [Barbosa and Colares 1986; do Carmo 1976]. In 1835, Scherk solved Lagrange's equation for translation functions, that is, those of the form $f(x, y) = g(x) + h(y)$, and discovered Scherk's minimal surfaces.

Often, a regular surface in $\mathbb{R}^3$ can be considered locally as the graph of a function $X : U \to \mathbb{R}$, where U is a domain in $\mathbb{R}^2$. In this paper, we consider ruled minimal surfaces in space with log-linear density $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}_\varphi$, where $\mathbb{R}_\varphi$ is the real line with log-linear density e^φ, which is equivalent to space with density e^z, as shown in Section 2. We classify all ruled minimal surfaces in $\mathbb{R}^3$ with density e^z. In contrast to the classical case, there are no noncylindrical ruled minimal surfaces, and there is a family of cylindrical ruled minimal surfaces. We also consider translation minimal surfaces and prove that all translation minimal surfaces are ruled.

All functions in this paper belong to the class C^2.

2. Minimal surfaces in spaces with densities

From the formula of the φ-curvature, it is clear that if we understand the geometric meaning of $d\varphi/d\boldsymbol{n}$ we can discover some simple minimal surfaces in a space with density. For example, in Gauss space, $d\varphi/d\boldsymbol{n}$ is the distance from the origin to the tangent hyperplane at the corresponding point of the surface. So it is easy to see that in Gauss space G^3 (see also [Corwin et al. 2006])

- planes have constant mean curvature, and planes passing through the origin are minimal;
- spheres about the origin have constant mean curvature and the one with radius $1/\sqrt{2}$ is minimal;
- circular cylinders whose axes pass through the origin have constant curvature, and the one with radius 1 is minimal.

Let $\varphi(x)$ be the linear function $\varphi(x) = \sum_{i=1}^{n} a_i x_i$ on Euclidean space $\mathbb{R}^n$, and consider the log-linear density $e^{\varphi(x)}$. Any set of points in $\mathbb{R}^n$ with constant density is a hyperplane. By a suitable change in coordinates, we can assume that the density has the form e^{x_n}, and therefore view the space $\mathbb{R}^n$ with density $e^{\varphi(x)}$ as the product $\mathbb{R}^{n-1} \oplus \mathbb{R}_\varphi$, where $\mathbb{R}^{n-1}$ is Euclidean $(n-1)$-space and $\mathbb{R}_\varphi$ is the real line with density e^{x_n}.

Since $\nabla\varphi = (0, 0, \ldots, 1)$, $d\varphi/d\boldsymbol{n} = \langle \nabla\varphi, \boldsymbol{n} \rangle$ is the cosine of the angle between $\boldsymbol{n}$ and the z-axis. By the definition of the mean φ-curvature, it is easy to see that H_φ does not change under a translation or a rotation about the z-axis, and moreover

- hyperplanes in $\mathbb{R}^n$ with density e^{x_n} have constant mean curvature;

- hyperplanes in $\mathbb{R}^n$ with density e^{x_n} that are parallel to the x_n-axis have zero mean curvature;
- circular hypercylinders with rules parallel to the x_n-axis have constant mean curvature.

3. Ruled minimal surfaces in $\mathbb{R}^3$ with density e^z

Now we consider the problem of classifying all ruled minimal surfaces in $\mathbb{R}^3$ with a log-linear density. Coordinates in $\mathbb{R}^3$ are denoted by (x, y, z). Without loss of generality we can assume that the density is e^z.

Locally, a ruled surface is given by the equation

$$X(u, v) = \alpha(u) + v\beta(u) \quad \text{for } u \in (a, b) \text{ and } v \in (c, d). \tag{3}$$

We can assume that $|\alpha'| = 1$, that $|\beta| = 1$, and that $\langle \alpha', \beta \rangle = 0$.

We will focus on the cases of cylindrical ruled surfaces (β constant) and non-cylindrical ruled surfaces ($\beta' \neq 0$ for all $u \in (a, b)$).

Denote by E, F and G the coefficients of the first fundamental form and by e, f and g the coefficients of the second fundamental form. A direct computation yields

$$\begin{aligned} &E = 1 + 2v\langle \alpha', \beta' \rangle + v^2|\beta'|^2, \quad && F = 0, \quad && G = 1; \\ &e = \langle N, \alpha'' + v\beta'' \rangle, && f = \langle N, \beta' \rangle, && g = 0; \end{aligned}$$

where $N = ((\alpha' + v\beta') \wedge \beta)/|(\alpha' + v\beta') \wedge \beta)|$. Thus

$$H_\varphi = \frac{1}{2}\Big(\frac{\langle N, \alpha'' + v\beta'' \rangle}{1 + 2v\langle \alpha', \beta' \rangle + v^2|\beta'|^2} - \langle N, \nabla\varphi \rangle\Big). \tag{4}$$

Proposition 1. *$H_\varphi = 0$ if and only if*

$$\begin{aligned} \langle \alpha' \wedge \beta, \alpha'' \rangle &= \langle \alpha' \wedge \beta, \nabla\varphi \rangle, \\ \langle \alpha' \wedge \beta, \beta'' \rangle + \langle \beta' \wedge \beta, \alpha'' \rangle &= \langle \beta' \wedge \beta, \nabla\varphi \rangle + \langle \alpha' \wedge \beta, 2\langle \alpha', \beta' \rangle \nabla\varphi \rangle, \\ \langle \beta' \wedge \beta, \beta'' \rangle &= \langle \beta' \wedge \beta, 2\langle \alpha', \beta' \rangle \nabla\varphi \rangle + \langle \alpha' \wedge \beta, |\beta'|^2 \nabla\varphi \rangle, \\ \langle \beta' \wedge \beta, |\beta'|^2 \nabla\varphi \rangle &= 0. \end{aligned}$$

Proof. By (4), $H_\varphi = 0$ if and only if

$$\frac{\langle N, \alpha'' + v\beta'' \rangle}{1 + 2v\langle \alpha', \beta' \rangle + v^2|\beta'|^2} = \langle N, \nabla\varphi \rangle.$$

With N as defined, this is an equality of polynomials in v. Identifying coefficients, we obtain the claim. □

The case $\beta' \neq 0$. From the last equation of Proposition 1, we have $\langle \beta' \wedge \beta, \nabla\varphi \rangle = 0$. Since $\nabla\varphi = (0, 0, 1)$ and $\beta \perp \beta'$, we see β belongs to a plane containing the z-axis. After a rotation about the z-axis, we can assume that $\beta = (\cos t(u), 0, \sin t(u))$, with $t' \neq 0$. Therefore the third equality of Proposition 1 becomes

$$\langle \alpha' \wedge \beta, |\beta'|^2 \nabla\varphi \rangle = 0.$$

From this we conclude that α' belongs to the plane $\{y = 0\}$ and the curve α lies on a plane parallel to the xz-plane. It is clear that α and β satisfy the system of equations in Proposition 1.

Corollary 2. *If $\beta'(u) \neq 0$ for all $u \in (a, b)$, the ruled minimal surfaces determined by (3) are vertical planes.*

If $\beta'(v) \neq 0$ at some $v \in (a, b)$, then locally the surface is planar. Since minimal surfaces solve elliptic PDEs, Aronszajn's unique continuation theorems [1956; 1957] for solutions of elliptic PDEs guarantee that the surface is planar globally.

The case $\beta' = 0$. Since $\beta' = 0$, $\beta = (a, b, c) = \text{constant}$, and $a^2 + b^2 + c^2 = 1$, the system in Proposition 1 becomes

$$\langle \alpha' \wedge \beta, \alpha'' \rangle = \langle \alpha' \wedge \beta, \nabla\varphi \rangle,$$
$$\beta = (a, b, c) = \text{constant}.$$

The first of these implies $\alpha'' - \nabla\varphi = m\alpha' + n\beta$, and hence

$$\langle \alpha'' - \nabla\varphi, \alpha' \rangle = m = -\langle \nabla\varphi, \alpha' \rangle \quad \text{and} \quad \langle \alpha'' - \nabla\varphi, \beta \rangle = n = -\langle \nabla\varphi, \beta \rangle.$$

Thus, $\alpha'' - \nabla\varphi = -\langle \nabla\varphi, \alpha' \rangle \alpha' - \langle \nabla\varphi, \beta \rangle \beta$, or

$$\alpha'' + \langle \nabla\varphi, \alpha' \rangle \alpha' = \nabla\varphi - \langle \nabla\varphi, \beta \rangle \beta. \tag{5}$$

Since the mean φ-curvature does not change under a rotation about z-axis, we can assume $a = 0$. So we have $\nabla\varphi - \langle \nabla\varphi, \beta \rangle \beta = (0, -cb, 1 - c^2)$. Because $b^2 = 1 - c^2$, we have

$$\nabla\varphi - \langle \nabla\varphi, \beta \rangle \beta = (0, -cb, b^2).$$

Suppose that $\alpha = (x(u), y(u), z(u))$. Then this equation is equivalent to the system

$$x'' + x'z' = 0, \qquad y'' + y'z' = -cb, \qquad z'' + z'^2 = b^2. \tag{6}$$

We treat the special cases $\beta = (0, 0, \pm 1)$ and $\beta = (0, \pm 1, 0)$ first.

If $\beta = (0, 0, \pm 1)$, the right side of (5) equals zero. We conclude that α' and α'' are parallel, and hence $\alpha'' = 0$. Thus, α is a straight line and we have this:

Proposition 3. *If $\beta = (0, 0, \pm 1)$, the ruled minimal surfaces determined by (3) are planes parallel to the z-axis.*

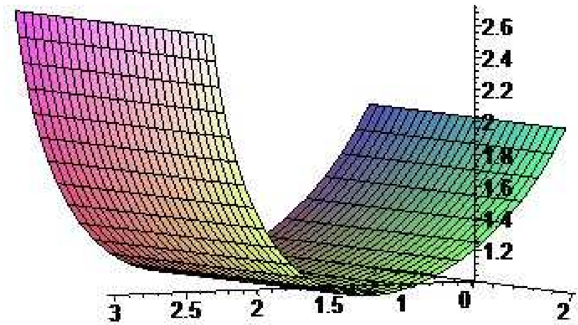

Figure 1. Ruled minimal surface with $\beta = (0, 1, 0)$.

We can also treat this case by solving (6). In this case (6) becomes

$$x'' + x'z' = 0, \qquad y'' + y'z' = 0, \qquad z'' + z'^2 = 0. \tag{7}$$

Since $\beta \perp \alpha'$, we get $z' = 0$, and hence $x'' = y'' = 0$. We conclude that $\alpha = (x, y, z)$ is a straight line lying on the plane $z =$ constant. Hence the ruled surface is a plane parallel to the z-axis.

If $\beta = (0, \pm 1, 0)$, then (6) becomes

$$x'' + x'z' = 0, \qquad y'' + y'z' = 0, \qquad z'' + z'^2 = 1.$$

Since $\beta \perp \alpha'$, we get $y' = 0$ and conclude that α lies on the plane $y =$ constant. The last equation above has the solution

$$z' = 1 - \frac{2}{1 + Ae^{2u}} = \frac{Ae^{2u} - 1}{Ae^{2u} + 1},$$
$$z = \log(1 + Ae^{2u}) - u = \log(e^{-u} + Ae^{u}),$$

where $A > 0$, whereas the first equation gives

$$x' = Be^{-z} \frac{B}{e^{-u} + Ae^{u}} = \frac{Be^{u}}{1 + Ae^{2u}};$$

hence

$$x = \frac{B}{\sqrt{A}} \arctan \sqrt{A} e^{u} + C.$$

Since the mean φ-curvature does not change under a translation, we can put $C = 0$.

Since $x'^2 + y'^2 + z'^2 = 1$, we have $4A = B^2$. This proves:

Proposition 4. *If* $\beta = (0, \pm 1, 0)$, *a ruled minimal surface determined by* (3) *has a parametric equation of the form*

$$x = 2 \arctan \sqrt{A} e^{u}, \qquad y = \pm v, \qquad z = \log(e^{-u} + Ae^{u}). \tag{8}$$

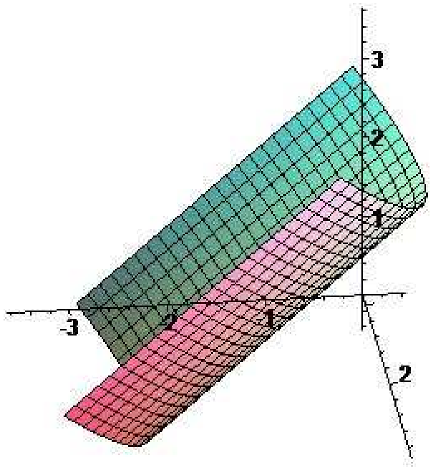

Figure 2. Ruled minimal surface with $\beta = (0, 1/\sqrt{2}, 1/\sqrt{2})$.

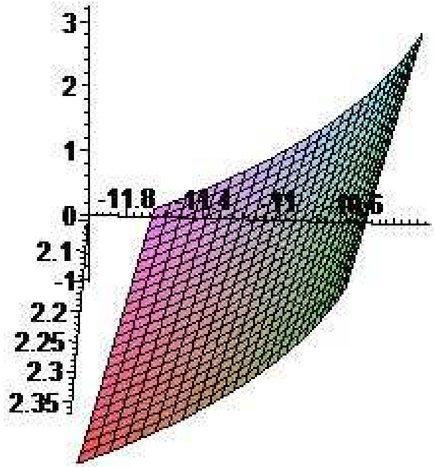

Figure 3. Ruled minimal surface with $\beta = (0, 0.1, 0.99)$.

If $\beta = (0, b, c)$ with $b, c \neq 0$, the system (6) becomes

$$x'' + x'z' = 0, \qquad y'' + y'z' = -cb, \qquad z'' + z'^2 = b^2.$$

Since $\beta \perp \alpha'$, we get $by' = -cz'$ and conclude that α lies on the plane $by+cz+d=0$. The last equation above has the solution

$$z' = b - \frac{2b}{1+Ae^{2bu}} = \frac{bAe^{2bu}-b}{Ae^{2bu}+1},$$
$$z = \log(1 + Ae^{2bu}) - bu = \log(e^{-bu} + Ae^{bu}),$$

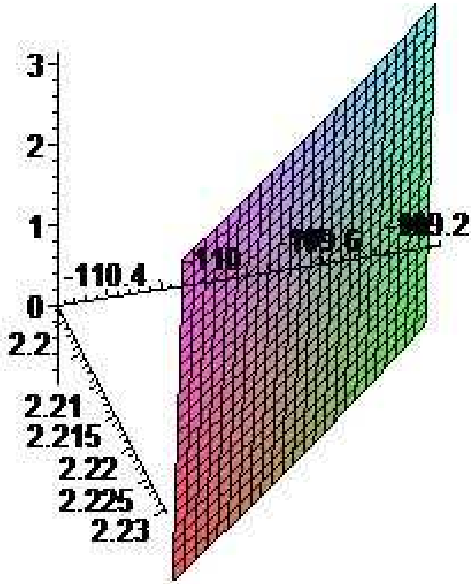

Figure 4. Ruled minimal surface with $\beta = (0, 0.01, 0.999)$.

where $A > 0$, whereas the first gives $x' = Be^{-z} = B/(e^{-bu} + Ae^{bu})$, and hence

$$x = \frac{B}{b\sqrt{A}} \arctan(\sqrt{A}e^{bu}) + C.$$

Since $x'^2 + y'^2 + z'^2 = 1$, we have $4A = B^2$ and another result:

Proposition 5. *If $\beta = (0, b, c)$ with $b, c \neq 0$, the ruled minimal surface determined by (3) has a parametric equation of the form*

$$x = 2\arctan\sqrt{A}e^{bu}, \quad y = -\frac{c}{b}\log(e^{-bu} + Ae^{bu}) + bv, \quad z = \log(e^{-bu} + Ae^{bu}) + cv.$$

We now combine the results above:

Theorem 6. *Besides planes parallel to the z-axis, the ruled minimal surfaces in $\mathbb{R}^3$ with density e^z are the cylindrical ones given by Proposition 5.*

4. Translation minimal surfaces in space with log-linear density e^z.

In this section we study translation minimal surfaces in $\mathbb{R}^3$ with density e^z. We prove that all translation surfaces that are minimal must be ruled.

Theorem 7. *A translation surface given by*

$$X(u, v) = (u, v, g(u) + h(v))$$

is minimal if either $g(u) = au + b$ or $h(v) = cv + d$.

Proof. A straightforward computation shows that $H_\varphi = 0$ if and only if

$$g''(1+h'^2) + h''(1+g'^2) = 1 + g'^2 + h'^2. \tag{9}$$

We fix $v = v_0$, and set $A = 1 - h''(v_0)$, $B = 1 + h'^2(v_0)$, $C = 1 + h'^2(v_0) - h''(v_0)$. Note that $B > 0$ and $C = B - A - 1$. Thus, f satisfies $Ag'^2 + Bg'' = C$, and hence $g'' = (C - Ag'^2)/B$.

Substituting g'' into (9), we get

$$g'^2(h'' - A(1+h'^2)/B - 1) = 1 - h'' + h'^2 - C(1+h'^2)/B.$$

This implies that unless $g' =$ constant, we must have

$$h'' - A(1+h'^2)/B - 1 = 0,$$
$$1 - h'' + h'^2 - C(1+h'^2)/B = 0.$$

Substituting h'' from the first of these into the second, we obtain

$$(1+h'^2)(1 - C/B - A/B) = 1.$$

Noting that $C = B - A - 1$, we get $h'^2 = B - 1$. Thus, $h' =$ constant. □

Since the roles of g and h are the same, we only need to consider translation minimal surfaces of the form

$$X(u,v) = (u, v, g(u) + cv + d).$$

A straightforward computation shows that g must be of the form

$$g(u) = -(1+c^2)\log\left|\cos\frac{u+D}{\sqrt{1+c^2}}\right|.$$

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