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G. Robinson introduced the group invariant known as the p-local rank to study Dade's conjecture and Alperin's conjecture. It is known that, for a finite p-solvable group with trivial maximal normal p-subgroup, the p-local rank is greater than or equal to the p-rank. Along those lines, we study the p-local rank of finite simple groups, giving a group-theoretic characterization of finite simple groups having p-local rank two. These results are also necessary for the investigation of such conjectures for finite groups of p-local rank two.

1. Introduction

There are several reformulations and strengthenings of Alperin's weight conjecture [1987]. Robinson [1996] introduced an inductive invariant known as the p-local rank to give an inductive proof of his variant of the conjecture.

A p-subgroup R of a finite group G is called *radical* if $R = O_p(N_G(R))$, where p is a prime divisor of the order $|G|$ of G and $O_p(H)$ is the unique maximal normal p-subgroup of H. The normalizer $N_G(R)$ is called a *parabolic* subgroup of G. We use the symbol $\mathbb{B}_p(G)$ to denote the set of radical p-subgroups of G excluding the unique maximal normal p-subgroup $O_p(G)$ and put $\mathbb{N}_p(G) = \{N = N_G(R) \mid R \in \mathbb{B}_p(G)\}$. The sets $\mathbb{B}_p(G)$ and $\mathbb{N}_p(G)$ play important roles in various fields.

Given a chain of p-subgroups

$$\sigma : Q_0 < Q_1 < \cdots < Q_n \tag{1}$$

of G, define the length $|\sigma| = n$, the final subgroup $V^\sigma = Q_n$, the initial subgroup $V_\sigma = Q_0$, the k-th initial subchain

$$\sigma_k : Q_0 < Q_1 < \cdots < Q_k,$$

and the normalizer

$$N_G(\sigma) := N_G(Q_0) \cap N_G(Q_1) \cap \cdots \cap N_G(Q_n). \tag{2}$$

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We say that the p-chain σ is *radical* if $Q_i = O_p(N_G(\sigma_i))$ for each i, that is, if Q_0 is a radical p-subgroup of G and Q_i is a radical p-subgroup of $N_G(\sigma_{i-1})$ for each $i \neq 0$. Write $\mathcal{R} = \mathcal{R}(G)$ for the set of radical p-chains of G and write $\mathcal{R}(G|Q) = \{\sigma \in \mathcal{R}(G) : V_\sigma = Q\}$. Write $\mathcal{R}(G|Q)/G$ for a set of orbit representatives under the action of G.

Following [Robinson 1996], the p-local rank $p\mathrm{lr}(G)$ of G is the length of a longest chain in $\mathcal{R}(G)$. We say that a subgroup H of G is a *trivial intersection* (t.i.) if $H^g \cap H = 1$ for every $g \in G \setminus N_G(H)$. By [Robinson 1996, 7.1], if $p\mathrm{lr}(G) > 0$, then $p\mathrm{lr}(G) = 1$ if and only if $G/O_p(G)$ has t.i. Sylow p-subgroups.

To describe our results, we first recall some standard terms and notation. If A is an abelian p-group, let $m(A)$ denote the minimal number of generators of A. Then, if P is an arbitrary p-group, define the *rank*

$$m(P) = \max\{m(A) \mid A \text{ is an abelian subgroup of } P\}.$$

of P; if G is a finite group, then the *p-rank* $m_p(G)$ of G is the rank of a Sylow p-subgroup of G. In [Wang and Zhang 2003], we show that for a finite p-solvable group with trivial maximal normal p-subgroup, its p-local rank is greater than or equal to its p-rank. To go further, we will consider finite simple groups in this paper. We use the classification theorem on finite simple groups of p-local rank one.

Theorem 1.1 [Gorenstein and Lyons 1983, (24-1)]. *Let G be a nonabelian simple group with a noncyclic t.i. Sylow p-subgroup P. Then G is isomorphic to one of*:

(1) $\mathrm{PSL}_2(q)$, *where $q = p^n$ and $n \geq 2$*;

(2) $\mathrm{PSU}_3(q)$, *where $q = p^n$*;

(3) ${}^2B_2(2^{2m+1})$ (*and $p = 2$*);

(4) ${}^2G_2(3^{2m+1})$ *for some $m \geq 1$* (*and $p = 3$*);

(5) $\mathrm{PSL}_3(4)$ *or M_{11}* (*and $p = 3$*);

(6) ${}^2F_4(2)'$ *or* McL (*and $p = 5$*);

(7) J_4 (*and $p = 11$*).

Thus there exist many finite groups whose p-ranks are greater than their p-local ranks. Also, there does not exist a function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $m_p(G) \leq f(p\mathrm{lr}(G))$ for any finite group G.

Remark. Suppose that G is a nonabelian finite simple group and p is a prime divisor of its order $|G|$. If $m_p(G) \geq 2$ and G does not appear in the list of Theorem 1.1, then $p\mathrm{lr}(G) \geq 2$. This easy consequence of the theorem will be used again and again in our proofs.

In this paper, we will focus on what we can say about finite simple groups G with $p\mathrm{lr}(G) = 2$. Though there is a necessary and sufficient condition for a finite group to have p-local rank two (see [Wang 2005] for details), it is hard to use this characterization to determine all finite simple groups having this property. So we give another description for such groups.

Main Result. *Let p be a prime and G a finite simple group. If $p\mathrm{lr}(G) = 2$, then $m_p(G) = 2$, except for (G, p) given in the list of Theorem 6.1.*

This can be treated as a generalization of Theorem 1.1 from one viewpoint. From another viewpoint, this is the necessary group-theoretic work to verify Robinson's conjecture for finite groups of p-local rank two. (The proof of the p-local rank one case has be achieved in [Eaton 2001].)

Our results heavily depend on the classification theorem of finite simple groups. For convenience, we give an outline of the proof. For alternating groups and sporadic groups, we determine their p-local ranks directly in Section 3. In Sections 4 and 5 they are treated by being divided into several cases. For most cases, we will prove that if the p-rank of such a group G is at least 3, then the p-local rank is at least 3 except for some determined (G, p). This implies that if G is not an exception and $p\mathrm{lr}(G) = 2$ then $m_p(G) \leq 2$. Notice that for a simple group G, $m_p(G) = 1$ means that G has cyclic Sylow p-subgroups, hence those are t.i. sets. So for a nonexception G, $m_p(G) = 2$. Finally combining with Theorem 1.1, we obtain the main result.

All groups in this paper are assumed to be finite. Our notation is standard and follows [Huppert 1967; Conway et al. 1985]. Also see [Gorenstein et al. 1994] for more references.

2. Some lemmas

We begin with an easy observation that will be important later. If R is a radical p-subgroup of a finite group G properly containing $O_p(G)$ and $\bar{1} < \bar{R}_1 < \cdots < \bar{R}_m$ is a radical p-chain of $N_G(R)/R$ of length m then $O_p(G) < R < R_1 < \cdots < R_m$ is a radical chain of length $m+1$, where R_i $(i = 1, \ldots, m)$ denotes the preimage of $\bar{R}_i$ in G. In particular, we have:

Lemma 2.1. *Let G be a finite group and p a prime divisor of $|G|$. Suppose that R is a radical p-subgroup of G which properly contains $O_p(G)$. Then $p\mathrm{lr}(G) > p\mathrm{lr}(N_G(R)/R) = p\mathrm{lr}(N_G(R))$.*

The next three lemmas, from [Robinson 1996], will be used often in this paper.

Lemma 2.2. *Let U be a radical p-subgroup of a finite group G and N a normal subgroup of G. Then $U \cap N$ is a radical p-subgroup of N.*

Lemma 2.3. *Let p be a prime and G a finite group. Then $p\mathrm{lr}(H) \le p\mathrm{lr}(G)$ for each subgroup H of G. If $O^{p'}(G) \le H$, then $p\mathrm{lr}(H) = p\mathrm{lr}(G)$.*

Lemma 2.4. *Let G, p be as above. Then, whenever $N \lhd G$, we have $p\mathrm{lr}(G/N) \le p\mathrm{lr}(G)$. If $p\mathrm{lr}(N) > 0$, then $p\mathrm{lr}(G/N) < p\mathrm{lr}(G)$.*

Note that any p-local subgroup, that is, a normalizer properly in G of a p-subgroup, is contained in a maximal p-local subgroup, which is a parabolic subgroup. So by Lemma 2.1 and Lemma 2.3, a p-local subgroup of G should have a smaller p-local rank. Precisely we have the following.

Corollary 2.5. *Let G be a finite group with $O_p(G) = 1$, where p is a prime divisor of $|G|$. Suppose that P is a nontrivial p-subgroup of G. Then $p\mathrm{lr}(G) \ge 1 + p\mathrm{lr}(N_G(P))$.*

The following results generalize Lemma 7.4 in [Robinson 1996].

Lemma 2.6. *Let G be a finite group and $H \le G$. If $H \le Z(G)$, then $p\mathrm{lr}(G/H) = p\mathrm{lr}(G)$.*

Proof. By the definition of p-local rank, we can assume that H is a p'-subgroup of G. Let $\sigma : Q_0 < Q_1 < \cdots < Q_n$ be any radical p-chain of G. We denote by $\bar{Q}_i$ the quotient group $Q_i H/H$, for $i = 0, 1, \ldots, n$. Clearly, $\bar{\sigma} : \bar{Q}_0 < \bar{Q}_1 < \cdots < \bar{Q}_n$ is a p-chain of $\bar{G} = G/H$. We claim that $\bar{\sigma}$ is also radical. In fact, since $H \le Z(G)$, we have that $N_{\bar{G}}(\bar{Q}_0) = N_G(Q_0)/H = \overline{N_G(Q_0)}$. Note that H is a p'-subgroup. Then $O_p(\overline{N_G(Q_0)}) = \bar{Q}_0$, in other words, $\bar{Q}_0$ is radical. By induction, $\bar{\sigma}$ is radical, hence $p\mathrm{lr}(\bar{G}) \ge p\mathrm{lr}(G)$. By Lemma 2.4, we have $p\mathrm{lr}(\bar{G}) = p\mathrm{lr}(G)$. □

Corollary 2.7. *Let $G = M *_D N$ be a central product of two finite groups M and N with respect to D. Then*

$$p\mathrm{lr}(G) = p\mathrm{lr}(M) + p\mathrm{lr}(N).$$

Proof. Consider the quotient group G/D. Since $M \cap N = D \le Z(G)$, we have that $p\mathrm{lr}(G)$=$p\mathrm{lr}(G/D)$. Note that $G/D \cong M/D \times N/D$. By [Robinson 1996, 7.4],

$$p\mathrm{lr}(G) = p\mathrm{lr}(G/D) = p\mathrm{lr}(M/D) + p\mathrm{lr}(N/D) = p\mathrm{lr}(M) + p\mathrm{lr}(N). \qquad \square$$

3. Alternating groups and sporadic groups

In this section, we will study alternating groups and sporadic groups whose p-local ranks are at least two. Alperin and Fong [1990] determined the radical subgroups in symmetric groups. Any such radical subgroup has the form

$$P_1 = \prod_{i=1}^{s} (A_{\mathbf{c}_i})^{m_i}, \tag{3}$$

where each $\mathbf{c}_i = (c_{i1}, \dots, c_{it_i})$ is a sequence of positive integers and $A_{\mathbf{c}_i}$ is a wreath product

$$A_{\mathbf{c}_i} = A_{c_{i1}} \wr A_{c_{i2}} \wr \cdots \wr A_{c_{it_i}}.$$

Here A_c means a regular elementary abelian p-subgroup of order p^c. We call such a group $A_{\mathbf{c}_i}$ a *basic* subgroup. Though a basic subgroup is not always radical, if p is odd, a p-subgroup is radical if and only if it has the form (3), and in the case $p = 3$, in addition, only if the number of fixed points is not three [Olsson and Uno 1995].

Proposition 3.1. *Let p be a prime and $G = A_n$ $(n \geq 5)$ an alternating group such that p divides $|G|$. If $p\mathrm{lr}(G) = 2$, then $m_p(G) = 2$, except for $(G, p) = (A_9, 3)$. In the exceptional case, $m_3(A_9) = 3$.*

Proof. Firstly, we assume that $p \geq 5$. Since $p\mathrm{lr}(S_n) = p\mathrm{lr}(A_n)$, we only consider the symmetric groups. By the discussion at the beginning of this section, we have that $Q_1 = \langle (1, 2, \dots, p) \rangle$ is a radical p-subgroup of S_n. We can determine the p-local rank by looking at the normalizer quotients, which are also determined in [Alperin and Fong 1990]. By Theorem 2B of the same reference, the normalizer quotient of Q_1 is

$$N_{S_n}(Q_1)/Q_1 = S_{n-p} \times \mathrm{GL}(1, p).$$

Since $\mathrm{GL}(1, p)$ is a cyclic group of order $p-1$ and $p\mathrm{lr}(S_n) = 2$, we have that $2 = p\mathrm{lr}(S_n) \geq 1 + p\mathrm{lr}(S_{n-p})$, hence $p\mathrm{lr}(S_{n-p}) \leq 1$. In fact, $p\mathrm{lr}(S_{n-p}) = 0$ implies that $n - p < p$ and S_n has a cyclic Sylow p-subgroup P of order p, hence $p\mathrm{lr}(S_n) = 1$. This is a contradiction. So $p\mathrm{lr}(S_{n-p}) = 1$ which implies that $p \leq n - p < 2p$. Therefore, $2p \leq n < 3p$ and $m_p(S_n) = 2$.

Secondly, when $p = 2$, it is easy to compute that $2\mathrm{lr}(A_5) = 1$, $2\mathrm{lr}(A_6) = 2\mathrm{lr}(A_7) = 2$ and $2\mathrm{lr}(A_8) = 3$. When $n \geq 8$, by the trivial embedding $A_8 \leq A_n$, we have $2\mathrm{lr}(A_n) \geq 3$. We see that both A_6 and A_7 have dihedral Sylow 2-subgroups of order 8. Our assertion holds in this case.

Lastly, for $p = 3$, A_5 and $A_6 \cong \mathrm{PSL}_2(9)$ have t.i. Sylow 3-subgroups and $3\mathrm{lr}(A_{10}) = 3$. If $n \geq 11$, since A_{10} is a subgroup of A_n, by Lemma 2.3, we have $3\mathrm{lr}(A_n) \geq 3\mathrm{lr}(A_{10}) = 3$. An easy calculation shows that $3\mathrm{lr}(A_7) = m_3(A_7) = 2$, $3\mathrm{lr}(A_8) = m_3(A_8) = 2$, and exceptionally $3\mathrm{lr}(A_9) = 2$, $m_3(A_9) = 3$. □

Now we study sporadic groups. Fundamental properties of sporadic groups are presented in several references and these groups have been investigated extensively and in great detail. Most of structure information of sporadic groups used in our proof is taken from [Conway et al. 1985; Gorenstein and Lyons 1983; Gorenstein et al. 1994]. We refer to the *Atlas of finite groups* [Conway et al. 1985] for more extensive bibliographies. Suppose that G is a finite group and p is a prime. Let $|G|_p$ denote the p-part of G, that is, $|G|_p$ is the maximal power of p

which divides the order $|G|$. Note that if $|G|_p \le p$, then $p\mathrm{lr}(G) \le 1$. So we say G in this case (that is, $|G|_p \le p$) has *trivial* p-part.

Proposition 3.2. *Let G be one of the* 26 *sporadic groups*, *and* $p\mathrm{lr}(G) = 2$, *where p is a prime divisor of $|G|$. Then we are in one of the following cases*:

(1) $p = 2$ *and* $G \cong M_{11}$, J_1, J_2 *or* J_3;

(2) $p = 3$ *and* $G \cong M_{12}$, M_{22}, M_{23}, M_{24}, J_2, J_3, HS, McL, Co_3, Ru *or* O'N;

(3) $p = 5$ *and* $G \cong J_2$, Co_3, Co_2, HS, Suz, He, Ru, Fi_{22}, Fi_{23}, Fi'_{24}, HN *or* Th;

(4) $p = 7$ *and* $G \cong \mathrm{Co}_1$, He, O'N, Fi'_{24}, Th *or* $\mathbb{B}$.

(5) $p = 11$ *and* $G \cong \mathbb{M}$;

(6) $p = 13$ *and* $G \cong \mathbb{M}$.

Proof. By [Conway et al. 1985], if $p > 13$ then G has trivial p-part. This implies that $p\mathrm{lr}(G) \le 1$ when $p > 13$. So we can assume that $p \le 13$. Now we consider the possible primes one by one. What we need is by and large contained in the references [Gorenstein and Lyons 1983; Gorenstein et al. 1994] for the case when p is odd. For the structure of normalizer of p-ordered subgroup, we refer to [Gorenstein and Lyons 1983].

Case 1 $p = 13$. In this case, there are only the Monster $\mathbb{M}$ which has $|\mathbb{M}|_p = p^3$ and the other sporadic groups which have trivial p-part, that is, $|G|_p \le p$. Let $G = \mathbb{M}$. By Theorem 1.1 and the definition of p-local rank, we have $2 \le p\mathrm{lr}(G) \le 3$. If $p\mathrm{lr}(G) = 3$, we suppose that $\sigma : 1 < R_1 < R_2 < R_3$ is a longest radical p-chain of G. So $|R_1| = 13$ and $R_3 \in \mathrm{Syl}_{13}(G)$. By [Gorenstein and Lyons 1983], there are only two classes of subgroups with order 13, namely $13A$ and $13B^*$, where the starred one lies in Sylow p-center. We also have $N_G(13A) \cong ((13\cdot 6) \times L_3(3))\cdot 2$ and $N_G(13B^*) \cong 13^{1+2}\cdot((\mathrm{SL}_2(3) \times 3)4)$, hence $13B^*$ is not a p-radical subgroup. So R_1 conjugates to $13A$. We identify R_1 with $13A$. By the definition, we must have $R_3 \le N_G(\sigma) \le N_G(13A)$. Note that R_3 is a Sylow 13-subgroup of G and $|N_G(13A)|_{13} = 13^2$. So $R_3 \not\le N_G(13A)$ which is a contradiction. Therefore, $p\mathrm{lr}(G) = 2$. This argument will be used frequently to determine the p-local ranks for groups G with $|G|_p = p^3$.

Case 2 $p = 11$. There are two groups, $\mathbb{M}$ and the Janko group J_4, which have nontrivial p-part. Note that $|\mathbb{M}|_p = p^2$ and $m_p(\mathbb{M}) = 2$. By Theorem 1.1, we have $p\mathrm{lr}(J_4) = 1$ and $p\mathrm{lr}(\mathbb{M}) = 2$.

Case 3 $p = 7$. By [Gorenstein and Lyons 1983], we can determine the full list of groups with nontrivial p-part. That is $G = \mathrm{Co}_1$, Th, $\mathbb{B}$, He, O'N, Fi'_{24} and $\mathbb{M}$. The first three of these have p-part p^2 and their Sylow p-subgroups are elementary abelian, so by Theorem 1.1, they are of p-local rank two. The next three groups have p-part p^3. So we can use the same argument as in Case 1. In the Held group He, there are three classes of p-ordered subgroups and their normalizers are $N_G(\overline{7A}) \cong (7\cdot 3)\times L_3(2)$, $N_G(7C^*) \cong (7^{1+2}\cdot 3)\cdot 6$ and $N_G(\overline{7D}) \cong (7\times D_{14})\cdot 3$. So by the same argument as in Case 1, $p\mathrm{lr}(\mathrm{He}) = 2$. Note that $\mathbb{M}$ has a subgroup isomorphic to $((7\cdot 3)\times \mathrm{He})\cdot 2$ as the normalizer $N_G(7A)$ which is a p-parabolic subgroup. By Lemma 2.1, $p\mathrm{lr}(\mathbb{M}) > 2$. In the O'Nan group O'N, there are two classes of p-ordered subgroups and their normalizers are $N_G(7A^*) \cong 7^{1+2}(3\times D_8)$ and $N_G(7B) \cong (7\times D_{14})\cdot 3$. Hence $p\mathrm{lr}(\mathrm{O'N}) = 2$. In the Fischer group Fi_{24}', there are also two classes of p-ordered subgroups and their normalizers are $N_G(7A) \cong (7\cdot 6)\times A_7$ and $N_G(7B^*) \cong (7^{1+2}\cdot 6)\cdot 6$. Hence $p\mathrm{lr}(\mathrm{Fi}_{24}') = 2$.

Case 4 $p = 5$. We again get the full list of groups with nontrivial p-part from [Gorenstein and Lyons 1983]. We set

$$\mathbb{S}_{r,n} := \left\{G \mid G \text{ is one of sporadic groups and } |G|_r = r^n\right\}.$$

When $n \geq 2$, the only nonempty possibilities are

$$\begin{aligned}
\mathbb{S}_{p,2} &= \left\{J_2, \mathrm{Suz}, \mathrm{He}, \mathrm{Fi}_{22}, \mathrm{Fi}_{23}, \mathrm{Fi}_{24}'\right\},\\
\mathbb{S}_{p,3} &= \{\mathrm{Co}_3, \mathrm{Co}_2, \mathrm{HS}, \mathrm{McL}, \mathrm{Ru}, \mathrm{Th}\},\\
\mathbb{S}_{p,4} &= \{\mathrm{Co}_1\},\\
\mathbb{S}_{p,6} &= \{\mathrm{Ly}, \mathrm{HN}, \mathbb{B}\},\\
\mathbb{S}_{p,9} &= \{\mathbb{M}\}.
\end{aligned}$$

If $G \in \mathbb{S}_{p,2}$, then G has elementary abelian Sylow p-subgroups, hence $p\mathrm{lr}(G) = 2$ by Theorem 1.1.

If $G \in \mathbb{S}_{p,3}$, we can use the argument in Case 1. In the third Conway group Co_3, the normalizer of a p-ordered subgroup is isomorphic to one of the two subgroups $N_G(5A^*) \cong 5^{1+2}\cdot(24\cdot 2)$ or $N_G(5B) \cong (5\cdot 4)\times A_5$. In the second Conway group Co_2 there are two classes of p-ordered subgroups in G whose normalizers are isomorphic to $N_G(5A^*) \cong 5^{1+2}\cdot((4 * \mathrm{SL}_2(3))\,\#\,2)$ and $N_G(5B) \cong (5\cdot 4)\times S_5$, respectively. For the Higman–Sims group HS, a normalizer of a p-ordered subgroup is isomorphic to one of following: $5^{1+2}\cdot(8\cdot 2)$, $(5\cdot 4)\times A_5$ or $E_{5^2}\cdot 4$. Also, the normalizer of a p-ordered subgroup in the Rudvalis group Ru is isomorphic to $(5^{1+2}\cdot Q_8)\cdot 4$ or $(5\cdot 4)\times A_5$. There is a single class of p-ordered subgroup in the Thompson group Th with normalizer $5^{1+2}\cdot((4 * \mathrm{SL}_2(3))\,\#\,2$. So by the same argument as in Case 1, for those groups, we have $p\mathrm{lr}(G) = 2$. By Theorem 1.1, $p\mathrm{lr}(\mathrm{McL}) = 1$.

Now we consider the first Conway group Co_1. Note that $(D_{10}\times J_2)\,\#\,2$ is a p-parabolic group of Co_1. By Lemma 2.1, we have $p\mathrm{lr}(\mathrm{Co}_1) > 2$.

The Lyons group Ly has a maximal group isomorphic to $5^3L_3(5)$, hence a parabolic subgroup. So $p\mathrm{lr}(\mathrm{Ly}) > p\mathrm{lr}(5^3L_3(5)) = 2$.

By [Wilson 1987], there is also a subgroup $N(5B^3)$ isomorphic to $5^3 \cdot L_3(5)$ as a normalizer of a radical p-subgroup with order 5^3 in the Baby Monster $\mathbb{B}$. So $N(5B^3)$ is parabolic. Therefore, by Lemma 2.1, $p\mathrm{lr}(\mathbb{B}) > p\mathrm{lr}(L_3(5)) = 2$.

In [Norton and Wilson 1986], it is shown that any 5-local subgroup of Harada–Norton group HN is contained in one of the following groups up to conjugacy:

$$\begin{aligned} L_1 &:= N_G(5A) \cong (D_{10} \times U_3(5))2, \\ L_2 &:= N_G(5B) \cong 5_+^{1+4} : (2_-^{1+4} : 5 : 4), \\ L_3 &:= N_G(5B^2) \cong 5^2 \cdot 5^{1+2} : 4A_5. \end{aligned}$$

Hence each of them is parabolic. By the original definition of p-local rank [Robinson 1996], $p\mathrm{lr}(\mathrm{HN}) = 1 + \max\{p\mathrm{lr}(L_1) = 1, p\mathrm{lr}(L_2) = 1, p\mathrm{lr}(L_3) = 1\}$, which equals 2.

The Monster $\mathbb{M}$ has a subgroup isomorphic to $5^{1+6} \cdot ((4 * 2J_2) \cdot 2)$ which is a p-parabolic subgroup. By Lemma 2.1, we have $p\mathrm{lr}(\mathbb{M}) > 2$.

Case 5 $p = 3$. We use the notation in Case 4. When $n \geq 2$, we have the nonempty $\mathbb{S}_{p,n}$ as follows:

$$\begin{aligned} \mathbb{S}_{p,2} &= \{M_{11}, M_{22}, M_{23}, \mathrm{HS}\}, \\ \mathbb{S}_{p,3} &= \{M_{12}, M_{24}, J_2, J_4, \mathrm{He}, \mathrm{Ru}\}, \\ \mathbb{S}_{p,4} &= \{\mathrm{O'N}\}, \\ \mathbb{S}_{p,5} &= \{J_3\}, \\ \mathbb{S}_{p,6} &= \{\mathrm{Co}_2, \mathrm{McL}, \mathrm{HN}\}, \\ \mathbb{S}_{p,7} &= \{\mathrm{Co}_3, \mathrm{Suz}, \mathrm{Ly}\} \\ \mathbb{S}_{p,9} &= \{\mathrm{Co}_1, \mathrm{Fi}_{22}\}, \\ \mathbb{S}_{p} \; &= \left\{\mathrm{Fi}_{23}, \mathrm{Fi}'_{24}, \mathrm{Th}, \mathbb{B}, \mathbb{M}\right\}, \end{aligned}$$

where $\mathbb{S}_p = \bigcup_{n \geq 10} \mathbb{S}_{p,n}$. If $G \in \mathbb{S}_{p,2}$, by Theorem 1.1, it is sufficient to show that $p\mathrm{lr}(G) = 2$ except M_{11} which has t.i. Sylow p-subgroups. Now we suppose that $G \in \mathbb{S}_{p,3}$. The Mathieu groups M_{12}, M_{24}, the Hall–Janko group J_2 and the Rudvalis group Ru have normalizers $\left\{3^{1+3} \cdot E_{2^2}, S_3 \times A_4\right\}$, $\{3A_6 \cdot 2, S_3 \times L_3(2)\}$, $\{3A_6 \cdot 2, S_3 \times A_4\}$ and $3 \,\#\, \mathrm{Aut}(A_6)$ of p-ordered subgroups respectively. Following the argument in Case 1, such groups all have p-local rank two, since A_6 has t.i. Sylow p-subgroups and has index 2 in $\mathrm{Aut}(A_6)$. Note that $(6M_{22}) \cdot 2$ which is parabolic is a normalizer of p-ordered subgroup $3A$ in Janko group J_4, hence $p\mathrm{lr}(J_4) \geq 1 + p\mathrm{lr}(M_{22}) = 1 + 2 = 3$. Then $p\mathrm{lr}(J_4) = 3$. We also have $3A_7 \cdot 2 \leq \mathrm{He}$, so $p\mathrm{lr}(\mathrm{He}) \geq 1 + p\mathrm{lr}(A_7) = 1 + 2 = 3$, hence $p\mathrm{lr}(\mathrm{He}) = 3$.

For the O'Nan group O'N, radical 3-chains have been constructed in [An and O'Brien 2002]. From this, $p\mathrm{lr}(\mathrm{O'N}) = 2$.

The radical 3-chains of the Janko group J_3 have been determined by S Kotlica [1997], so we have $p\mathrm{lr}(J_3)=2$.

The second Conway group Co_2 has a subgroup isomorphic to $S_3\times\mathrm{Aut}(\mathrm{PSp}_4(3))$ which is a p-parabolic one. By the remark in [Robinson 1996], $p\mathrm{lr}(\mathrm{PSp}_4(3))=2$. So by Lemma 2.1, $p\mathrm{lr}(\mathrm{Co}_2)>2$.

By [Murray 1998; An 1999], we have $p\mathrm{lr}(\mathrm{McL}) = p\mathrm{lr}(\mathrm{Co}_3) = 2$. The Lyons group Ly has a maximal subgroup isomorphic to $2A_{11}$. Note that $p\mathrm{lr}(A_{11})=3$. So $p\mathrm{lr}(\mathrm{Ly})\geq p\mathrm{lr}(2A_{11})=3$.

The Harada-Norton group HN has a p-parabolic subgroup isomorphic to the group to $(3\times A_9)\cdot 2$. Note that $p\mathrm{lr}(A_9)=2$. So $p\mathrm{lr}(\mathrm{HN})>2$. It is known that HN can be viewed as a quotient of subgroups of $\mathbb{B}$ and $\mathbb{M}$. As a consequence, $p\mathrm{lr}(\mathbb{B})>2$ and $p\mathrm{lr}(\mathbb{M})>2$.

The Suzuki group Suz has a p-parabolic subgroup isomorphic to $3U_4(3)\cdot 2$. Since $p\mathrm{lr}(U_4(3))=2$, we also have $p\mathrm{lr}(\mathrm{Suz})>2$. Since we can treat Suz as a quotient of subgroups of Co_1, $p\mathrm{lr}(\mathrm{Co}_1)>2$ holds.

Since $S_3\times(U_4(3)\cdot 2)$ can be embedded into the Fischer group Fi_{22} as a quotient of subgroups, we have $p\mathrm{lr}(\mathrm{Fi}_{22})>2$. Moreover, $p\mathrm{lr}(\mathrm{Fi}_{23})>2$ and $p\mathrm{lr}(\mathrm{Fi}_{24}')>2$ because Fi_{22} can be embedded in Fi_{23} and Fi_{24}'.

Finally, the Thompson group Th has a p-parabolic subgroup isomorphic to $(3\times G_2(3))\cdot 2$ and $p\mathrm{lr}(G_2(3))=2$. So $p\mathrm{lr}(\mathrm{Th})>2$.

Case 6 $p=2$. In this case, we take most of the structure information from [Conway et al. 1985; Gorenstein et al. 1994]. First, we consider the Mathieu group M_{22}. Note that $2^4:A_6$ can be embedded into M_{22} as a maximal subgroup, hence as a p-parabolic subgroup since A_6 is simple. So by Lemma 2.1, $p\mathrm{lr}(M_{22})>p\mathrm{lr}(A_6)=2$. It is known that M_{22} is involved in M_{23}, HS, M_{24}, McL, Co_3, Co_2, Fi_{22}, HN, Ly, Fi_{23}, Co_1, J_4, Fi_{24}', $\mathbb{B}$ and $\mathbb{M}$ as a quotient of subgroups. Therefore, these groups have p-local rank greater than two. Secondly, let $G=M_{12}$. Since G has a maximal subgroup isomorphic to $2\times S_5$, hence p-parabolic, we have $p\mathrm{lr}(G)>p\mathrm{lr}(S_5)=2$. Since G is involved in Suz as a quotient of subgroups, $p\mathrm{lr}(\mathrm{Suz})\geq 3$. Thirdly, there is a maximal subgroup isomorphic to $S_4\times L_3(2)$ in He. So we have $p\mathrm{lr}(\mathrm{He})\geq p\mathrm{lr}(S_4)+p\mathrm{lr}(L_3(2))=1+2=3$. Fourthly, the Rudvalis group Ru has $2^{3+8}:L_3(2)$ as a maximal subgroup hence a p-parabolic one. For the same reason as above, $p\mathrm{lr}(\mathrm{Ru})>p\mathrm{lr}(2^{3+8}:L_3(2))=2$. Fifthly, the O'Nan group O'N has $4^3\cdot L_3(2)$ as a maximal subgroup which is also p-parabolic, so $p\mathrm{lr}(\mathrm{O'N})>2$. Sixthly, the Thompson group Th has $2^5\cdot L_5(2)$ as a maximal subgroup hence p-parabolic. Note that $L_3(2)\leq L_5(2)$. Thus $p\mathrm{lr}(\mathrm{Th})>2$.

For the Mathieu group M_{11} and J_1, radical 2-chains have been constructed in [Dade 1992], hence $p\mathrm{lr}(M_{11})=p\mathrm{lr}(J_1)=2$. We can give an easy proof that $p\mathrm{lr}(J_1)=2$ following the argument in Case 1. In fact, the group $G=J_1$ has an elementary abelian Sylow 2-subgroup E of order 8. There is a single class of involutions of G and $N_G(z) = C_G(z) \simeq Z_2 \times A_5$ for $z \in E \setminus \{1\}$. Then $C_G(F) = E$ for every subgroup F of E properly containing $\langle z\rangle$. Hence there is no radical 2-subgroup of order 4. As $O_2(C_G(z)) = \langle z\rangle$, we see that $\langle z\rangle$ is a radical 2-subgroup. Notice that the unique nontrivial radical 2-subgroups of $A_5 = \mathrm{PSL}_2(4)$ are Sylow 2-subgroups by Theorem 1.1. Thus $1 < \langle z\rangle < E$ is a radical 2-chain of length 2 which is the maximal length. So we have $2\mathrm{lr}(J_1) = 2$.

Following [Kotlica 1997], we get $p\mathrm{lr}(J_3) = 2$.

According to Uno, Dade verified his remarkable conjecture for J_2 and Ru in the final and the ordinary forms respectively, but Dade's notes seem to be unpublished. In [Yoshiara 2005], the list of radical 2-subgroups and their normalizer quotients of J_2 is given. Here we relist the normalizer quotients (see the Appendix in [Yoshiara 2005]): $3 \times A_5, 3 \times S_3, 3^2, A_5$ and the group Z_3 of order 3. So we can determine that $p\mathrm{lr}(J_2) = 2$ by the definition, since all p-local ranks of these quotients are less than or equal to 1. □

Corollary 3.3. *Let G be one of the* 26 *sporadic finite simple groups, and* $p\mathrm{lr}(G) = 2$, *where p is a prime divisor of $|G|$. Then $m_p(G) = 2$ unless G is isomorphic to one of the following groups*:

(1) $p = 2$ *and* $G \cong J_1,\ J_2$ *or* J_3 (*with* $m_2(G) = 3,\ 4$ *or* 4 *respectively*);

(2) $p = 3$ *and* $G \cong J_3$, McL, O'N *or* Co_3 (*with* $m_3(G) = 3,\ 4, 4$ *or* 5 *respectively*);

(3) $p = 5$ *and* $G \cong$ HN (*with* $m_5(G) = 3$).

Proof. In [Gorenstein et al. 1994], the p-ranks of sporadic groups are determined for all primes p. So by Proposition 3.2, we are done. □

4. For certain primes

In this section and the next, we investigate the finite groups of Lie type. A number of basic results which are used in our study of finite groups of Lie type can be found, for example, in [Carter 1972; Gorenstein et al. 1994]. In fact, the groups constructed in these two references are the same, except that Carter uses the adjoint versions. We follow the notation and terminology of [Gorenstein et al. 1994]. We can also identify the classical groups, almost simple as well as simple, in the Lie-theoretic language which will be used.

- The linear and unitary groups $A_m^{\pm}(q)$, $m \geq 1$: For the group $K = A_m(q)$, we obtain $K_u = \mathrm{SL}_{m+1}(q)$ and $K_a = \mathrm{PSL}_{m+1}(q) = L_{m+1}(q)$. If $K = {}^2A_m(q)$, we also have $K_u = \mathrm{SU}_{m+1}(q)$ and $K_a = \mathrm{PSU}_{m+1}(q) = U_{m+1}(q)$.
- The odd-dimensional orthogonal groups $B_m(q)$, q odd, $m \geq 1$: Here there is just one group in each dimension; although there are two equivalence classes of nonsingular quadratic forms, representatives of the two classes are similar

and thus give rise to the same group. We may identify K_a as follows: $K_a = \Omega_{2m+1}(q)$, of index 2 in $\mathrm{SO}_{2m+1}(q)$.

- The symplectic groups $C_m(q)$, $m \geq 2$: Here we may identify K_u as the symplectic group which we write as $\mathrm{Sp}_{2m}(q)$. Consequently $K_a = \mathrm{PSp}_{2m}(q)$.
- The even-dimensional orthogonal groups $D_m^{\pm}(q)$, $m \geq 2$: Here there are two finite groups in each even dimension, corresponding to the two equivalence classes of nonsingular quadratic forms. To summarize, we have

$$K = \Omega_{2m}^{\pm}(q), \quad K_u = \mathrm{Spin}_{2m}^{\pm}(q) \quad \text{and} \quad K_a = P\Omega_{2m}^{\pm}(q).$$

Remark. By Lemma 2.6, for a given root system Σ, when we consider the p-local rank of $K = \Sigma(q)$, there is no difference among any version of $\Sigma(q)$. In the following, we may use any one of them for convenience when the discussion is not sensitive to the version and specify the version only when it is of importance. The same remark applies to the symbols ${}^d\Sigma(q)$.

The defining characteristic. G. Robinson [1996] remarked that as an easy consequence of finite version of the Borel–Tits Theorem, if G is a finite group with a (B, N)-pair of rank n and characteristic p, then $p\mathrm{lr}(G) = n$. So we can determine the p-local ranks of finite groups of Lie type directly, where p is the defining characteristic of such groups. In fact, as an easy consequence of [Gorenstein et al. 1994], we have:

Proposition 4.1. *Let $K = {}^d\Sigma(p^a) \in \mathscr{L}ie(p)$, p prime. If $p\mathrm{lr}(K) = 2$, then $m_p(K) = 2$ unless K is isomorphic to one of $A_2(p^a)$ $(a \geq 2)$, $B_2(p^a) = C_2(p^a)$, $G_2(p^a)$, ${}^2A_3(p^a)$, ${}^3D_4(p^a)$ and ${}^2F_4(2^{a/2})$ for $a \geq 1$.*

Even primes.

Theorem 4.2 [Gorenstein et al. 1994]. *Let $K = {}^d\Sigma(q) \in \mathscr{L}ie(r)$, r odd. Let $P \in \mathrm{Syl}_2(K)$ and $m = m_2(K)$. Also let $P^* = \mathrm{Syl}_2(\mathrm{Inndiag}(K))$.*

(a) *$m = 1$ if and only if $K \cong \mathrm{SL}_2(q)$, in which case P is a quaternion group.*

(b) *If $K \cong \mathrm{PSL}_2(q)$, then P and P^* are dihedral groups and $m = 2$.*

(c) *If $K \cong \mathrm{PSL}_3^{\varepsilon}(q)$ or $\mathrm{SL}_3^{\varepsilon}(q)$, then $P \cong Z_{2^a} \wr Z_2$ if the 2-part of $q - \varepsilon$ is $2^a \geq 4$ while P is quasidihedral if $q \equiv -\varepsilon \bmod 4$. In either case $m = 2$.*

(d) *If $K \cong \mathrm{Sp}_4(q)$ then $P \cong Q \wr Z_2$ where Q is quaternion and $m = 2$. If $K \cong \mathrm{PSp}_4(q)$, then $P = P_1 P_2 \langle t \rangle$ where P_1 and P_2 are quaternion groups, $Z(P_1) = Z(P_2)$, $[P_1, P_2] = 1$, $t^2 = 1$, $P_1^t = P_2$ and $m(P) = 4$.*

(e) *If $K \cong G_2(q)$, ${}^2G_2(q)$ or ${}^3D_4(q)$ then $m = 3$.*

(f) *If $K \cong \mathrm{PSL}_4^{\varepsilon}(q)$, then $m = 4$. Moreover if $q \equiv -\varepsilon \bmod 4$, then $P \cong D \wr Z_2$ for some dihedral group D. If $q \equiv -\varepsilon + 4 \bmod 8$, then P is a split extension of* *an elementary abelian normal subgroup $E \cong E_{16}$ by D_8 whose isomorphism type is independent of q and ε.*

(g) *If $m \leq 3$, then K is one of the groups mentioned in* (a)–(e) *or else $K \cong \mathrm{Sp}_6(q)$ or $\mathrm{SL}_4^{\pm}(q)$ for some q.*

(h) *P is abelian if and only if $K \cong \mathrm{PSL}_2(q)$ $q \equiv \pm 3 \bmod 8$, or $K \cong {}^2G_2(q)$ for some q.*

Now we deal with the even primes.

Proposition 4.3. *Let $K = {}^d\Sigma(q) \in \mathscr{L}ie(r)$, r odd. Assume that K_a is the adjoint version of K. If $m_2(K_a) \geq 3$ then $2\mathrm{lr}(K_a) \geq 3$, except for $2\mathrm{lr}(\mathrm{PSp}_4(3)) = 2$.*

Proof. By Theorem 4.2, K_a is isomorphic to $G_2(q), {}^2G_2(q)$ or ${}^3D_4(q)$, if and only if $m_2(K_a) = 3$. In [An 1996; An 2002; An 1994], the radical 2-chains of $G_2(q), {}^2G_2(q)$ or ${}^3D_4(q)$ were constructed which implies our assertion for such groups. Now we consider the cases of $m_2(K_a) \geq 4$ which include almost all finite groups of Lie type. We will prove our assertion type by type. The main idea of the proof is try to find subgroups of each group which have 2-local ranks greater than or equal to 3. If this fails, we will compute $2\mathrm{lr}(K_a)$ directly. For the trivial embedding ${}^dX(q) \leq {}^dL(q)$, where X is a subsystem of L, if $2\mathrm{lr}({}^dX(q)) \geq 3$, then so does $2\mathrm{lr}({}^dL(q))$. Now we may only consider the minimal cases for each type.

Firstly, we consider the linear groups and unitary groups. By Theorem 4.2, we may assume $K_a = \mathrm{PSL}_4^{\varepsilon}(q)$, since $\mathrm{PSL}_3^{\varepsilon}(q)$ has Sylow 2-groups of rank 2. We see that $\mathrm{PSL}_2^{\varepsilon}(q) \times \mathrm{PSL}_2^{\varepsilon}(q) \leq \mathrm{PSL}_4^{\varepsilon}(q)$. By Theorem 1.1, we have $2\mathrm{lr}(\mathrm{PSL}_2^{\varepsilon}(q)) \geq 2$ except for $2\mathrm{lr}(\mathrm{PSL}_2(3)) = 0$. So by Corollary 2.5 and Lemma 2.3, we have $2\mathrm{lr}(\mathrm{PSL}_4^{\varepsilon}(q)) \geq 4$ except for $2\mathrm{lr}(\mathrm{PSL}_4(3)) = 3$. This implies that our assertion holds for linear and unitary groups.

Secondly, assume that K is a symplectic group. Let $K_a = \mathrm{PSp}_4(q)$. Note that $\mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$ is also a subgroup of $\mathrm{PSp}_4(q)$. By the proof for linear groups, we also have $2\mathrm{lr}(K_a) \geq 4$ when $q \neq 3$. Note that $m_2(\mathrm{PSp}_4(3)) = 4$. We get the exception $2\mathrm{lr}(\mathrm{PSp}_4(3)) = 2$ by the fact $\mathrm{PSp}_4(3) \cong {}^2A_3(2)$ and Proposition 4.1. Now we have that if $q \neq 3$, then $2\mathrm{lr}(\mathrm{PSp}_{2m}(q)) \geq 4$, where $m \geq 2$. If we can prove that $2\mathrm{lr}(\mathrm{PSp}_6(3)) \geq 3$, we will solve our problem for symplectic groups. In fact, $K = \mathrm{PSp}_6(3)$ has a subgroup H isomorphic to $\mathrm{SL}_2(3) * C_2(3)_u$ where $C_2(3)_u$ denotes the universal version of $\mathrm{PSp}_4(3)$. Note that $\mathrm{SL}_2(3) * C_2(3)$ has a normal 2-subgroup of order 8. So H must be contained in some 2-local subgroup of K_a. Since K_a is simple, by Corollary 2.5 we have $2\mathrm{lr}(K_a) \geq 1 + 2\mathrm{lr}(\mathrm{SL}_2(3) * C_2(3)_u) = 1 + 2\mathrm{lr}(\mathrm{PSp}_4(3)) = 1 + 2 = 3$.

Note that $P\Omega_5(q) \cong \mathrm{PSp}_4(q)$, $P\Omega_6^+(q) \cong \mathrm{PSL}_4(q)$, $P\Omega_6^-(q) \cong \mathrm{PSU}_4(q)$. So for classical groups, there is only one class of groups $P\Omega_{2m+1}(3)$, $m \geq 3$ for which our assertion remains to be verified. The same as for symplectic groups, it is sufficient to show $2\mathrm{lr}(P\Omega_7(3)) \geq 3$. Suppose $G = P\Omega_7(3)$. Let $S_{13} \in \mathrm{Syl}_{13}(G)$

and $N_{13} = N_G(S_{13})$. We have that $|N_{13}| = 2 \cdot 3 \cdot 13$. Now let $S_2 \in \mathrm{Syl}_2(N_{13})$. Then $N := N_G(S_2)$ has order $2^9 \cdot 3^3$ and 2-local rank two. By Corollary 2.5, we have $2\mathrm{lr}(G) \geq 3$.

Lastly, we consider the exceptional groups. Since q is odd, we may ignore ${}^2B_2(2^{2m+1})$ and ${}^2F_4(2^{2m+1})$. Notice that $E_8(q) \geq E_7(q) \geq E_6^{\varepsilon}(q) \geq F_4(q) \geq {}^3D_4(q)$. Since $2\mathrm{lr}({}^3D_4(q)) \geq 3$ has been proved, we are done. □

5. For odd nondefining characteristic

Prime p dividing d_{K_u}. As remarked above, there is no difference among the p-local ranks of any version of a finite group of Lie type. But when we consider p-ranks of such groups, the version is of importance for the investigation. Especially if p divides the order d_{K_u} of the center of K_u, a universal version, we may have $m_p(K_u) = m_p(K_u/Z(K_u)) + 1$ for some cases. We will now deal with such sensitive cases. In fact, we have the following proposition in which only adjoint versions are considered.

Proposition 5.1. *Let $K_u = {}^dX_l(q) \in \mathscr{L}ie$ be a universal version, where ${}^1X_l(q^1)$ is an untwisted version $X_l(q)$. The adjoint version K_a of K_u is simple. Assume p is an odd prime dividing $|K_u|$ and d_{K_u}. If $p\mathrm{lr}(K_a) = 2$, then $m_p(K_a) = 2$.*

Proof. By [Gorenstein et al. 1994, Table 2.2], there are only four possible cases:

$$A_l(q)\ (p \mid d_{A_l(q)}), \qquad E_6(q)\ (p = 3,\ q - 1 \equiv 0\ (3)),$$
$${}^2A_l(q)\ (p \mid d_{{}^2A_l(q)}), \qquad {}^2E_6(q)\ (p = 3,\ q + 1 \equiv 0\ (3)).$$

Case $E_6(q)$ $(p = 3)$. We see that $E_6(q)$ has $\mathrm{SL}_3(q) * \mathrm{SL}_3(q) * \mathrm{SL}_3(q)$ as a subgroup. Note that $\mathrm{PSL}_3(q)$ is simple and 3 divides $|\mathrm{PSL}_3(q)|$. So $3\mathrm{lr}(\mathrm{SL}_3(q) * \mathrm{SL}_3(q) * \mathrm{SL}_3(q)) \geq 6$ when $q \neq 4$ and $3\mathrm{lr}(\mathrm{SL}_3(4) * \mathrm{SL}_3(4) * \mathrm{SL}_3(4)) = 3$. This is not our case to be considered.

Case $A_l(q)$. In this case, our assumption is: p is an odd prime dividing $q - 1$ and $l + 1$, and $G = \mathrm{SL}_{l+1}(q)$ with $l \geq 1$. We assume that $p\mathrm{lr}(G) \leq 2$.

First assume that p is a proper divisor of $l+1$. There exists $1 \neq t \in \mathrm{GF}(q)^{\times}$ such that $t^p = 1$. Set $h = \mathrm{diag}(1, t, t^{-1}, 1, \ldots, 1)$ and $w = A \oplus I_{(l+1)-p}$ in G, where $I_{(l+1)-p}$ is the identity matrix of size $(l+1) - p$ and

$$A = \begin{pmatrix} 0 & 0 & & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{pmatrix} \in \mathrm{SL}_p(q)$$

is a permutation matrix of order p. Therefore the normalizer in $G = \mathrm{SL}_{l+1}(q)$ of a p-subgroup $\langle w, h\rangle$ of order p^p contains $\langle w, h\rangle \times \mathrm{SL}_{l+1-p}(q)$. Notice that

$\mathrm{SL}_{l+1-p}(q) \geq \mathrm{SL}_p(q)$ and $\mathrm{PSL}_p(q)$ contains a p-group isomorphic to $\langle D, A\rangle/Z$, where $D = \mathrm{diag}\ (t, t^{-1}, 1, \dots, 1)$ and Z is the cyclic subgroup of order p of $Z(\mathrm{SL}_p(q))$. As $\langle D, A\rangle$ is a group of order p^p generated by elements of order p, its central quotient group $\langle D, A\rangle/Z$ of order p^{p-1} is noncyclic. Then we can apply Theorem 1.1 to the nonabelian simple group $\mathrm{PSL}_p(q)$ with noncyclic Sylow p-subgroup to conclude that either we have $p\mathrm{lr}(\mathrm{SL}_{l+1-p}(q)) \geq p\mathrm{lr}(\mathrm{SL}_p(q)) = p\mathrm{lr}(\mathrm{PSL}_p(q)) \geq 2$ by Lemmas 2.3 and 2.6 or $(p, q) = (3, 4)$. In the former case, we have $p\mathrm{lr}(G) \geq p\mathrm{lr}(\mathrm{SL}_p(q)) + p\mathrm{lr}(\mathrm{SL}_{l+1-p}(q)) \geq 1+2 = 3$ by Lemma 2.3 and Corollary 2.7, which contradicts our assumption.

Thus we conclude that one of the following holds if $p\mathrm{lr}(\mathrm{SL}_{l+1}(q)) \leq 2$ for an odd prime p dividing $q-1$:

$$p = l+1 \quad \text{or} \quad (p, q) = (3, 4).$$

Now assume that $p > 3$. Then $p = l+1$ occurs. Let P be the unique Sylow p-subgroup of $\mathrm{GF}(q)^\times$ and t be a generator of P. We claim that the subgroup

$$R := \{\mathrm{diag}(a, b, z, z, \dots, z) \mid a, b, z \in P, abz^{p-2} = 1\} \cong P \times P$$

is a radical p-subgroup of G. Let V be the natural module with natural basis $(e_1, \dots, e_{l+1})$ with respect to which of G is represented as a matrix group. Observe that the subspaces $\langle e_1\rangle$, $\langle e_2\rangle$ and $\langle e_3, \dots, e_{l+1}\rangle$ are the all simultaneous eigensubspaces for R. Thus they are permuted by $N_G(R)$. As the last subspace has dimension $l-1 = p-2$, which is larger than the dimension of the former subspace, we see that $N_G(R)$ at most interchanges the former two 1-dimensional subspaces and stabilizes the last subspace. Thus if we denote by X the subgroup of $N_G(R)$ which stabilizes all three eigenspaces, then $[N_G(R) : X] \leq 2$ and $O_p(N_G(R)) = O_p(X)$ as p is odd. It is now easy to see that X coincides with the following subgroup (in fact $X = C_G(R)$ and $[N_G(R) : X] = 2$):

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & X \end{pmatrix} \;\middle|\; \begin{array}{c} a, b \in \mathrm{GF}(q)^\times,\ X \in \mathrm{GL}_{l-1}(q), \\ ab\det(X) = 1 \end{array} \right\} \cong Z_{q-1} \times \mathrm{GL}_{l-1}(q).$$

As $l-1 = p-2 \geq 3$, every normal subgroup of $\mathrm{GL}_{l-1}(q)$ is either contained in the center or a subgroup containing $\mathrm{SL}_{l-1}(q)$. Hence $O_p(X)$ should coincide with R. Thus $R = O_p(N_G(R))$ is a radical p-subgroup of G.

Furthermore, $N_G(R)/R$ contains a subgroup $X'R/R \cong X'/(X' \cap R)$, where $X' \cong \mathrm{SL}_{l-1}(q)$ and $X' \cap R$ is the Sylow p-subgroup of $Z(X')$. Thus we have $p\mathrm{lr}(N_G(R)/R) \geq p\mathrm{lr}(\mathrm{SL}_{l-1}(q))$ by Lemma 2.3 and Lemma 2.6.

Note that $\mathrm{PSL}_{l-1}(q)$ contains a noncyclic p-subgroup $\langle \bar{A}, \bar{B}\rangle$, where $A := \mathrm{diag}$ $(t, t^{-1}, 1, 1, \dots, 1)$ and $B := \mathrm{diag}(1, t, t^{-1}, 1, \dots, 1)$. This is clear if $(l-1)-3 = p-5 > 0$. In the remaining case when $p = 5$ and $l-1 = 3$, we have $\langle \bar{A}\rangle = \langle \bar{B}\rangle$

if and only if there exists some integer i and some $z \in \mathrm{GF}(q)^\times$ such that $A = zB^i$. It then follows that $z = t = t^{-i-1} = t^i$, whence $t^3 = 1$. This contradicts that t has order a proper power of 5. Thus $\mathrm{PSL}_{l-1}(q)$ is a nonabelian simple group with a noncyclic Sylow p-subgroup with $l - 1 = p - 2 \geq 3$. Notice that $q \geq 6$ as $q - 1$ is a multiple of $p > 5$. Then it follows from Theorem 1.1 that $p\mathrm{lr}(\mathrm{PSL}_{l-1}(q)) \geq 2$.

Summarizing, $p\mathrm{lr}(G) \geq p\mathrm{lr}(N_G(R)) + 1$ by Lemma 2.1, $p\mathrm{lr}(N_G(R)/R) \geq p\mathrm{lr}(\mathrm{SL}_{l-1}(q)) = p\mathrm{lr}(\mathrm{PSL}_{l-1}(q))$ and $p\mathrm{lr}(\mathrm{PSL}_{l-1}(q)) \geq 2$. Hence $p\mathrm{lr}(G) \geq 1+2 = 3$, which contradicts our assumption.

Thus we established that $p > 3$ does not occur. So the unique possibility is $p = 3$. In this case, if $p = l + 1$ holds, then $G/Z(G) \cong \mathrm{PSL}_3(q)$ with 3 dividing $q - 1$; hence $G/Z(G)$ has a Sylow 3-subgroup isomorphic to $Z_3 \times Z_3$. It is obvious that $3\mathrm{lr}(\mathrm{PSL}_3(q)) = m_3(\mathrm{PSL}_3(q)) = 2$, except for $q = 4$, in which case, $G = \mathrm{SL}_3(4)$ has 3-local rank one and $m_3(\mathrm{SL}_3(q)) = 2$ by Theorem 1.1.

The unique possible exception is $\mathrm{SL}_{2p}(q) = \mathrm{SL}_6(4)$, because $\mathrm{SL}_{3p}(4) = \mathrm{SL}_9(4)$ contains $\mathrm{SL}_3(4) \times \mathrm{SL}_3(4) \times \mathrm{SL}_3(4)$ and we have $3\mathrm{lr}(\mathrm{SL}_{3k}(4)) \geq 3\mathrm{lr}(\mathrm{SL}_9(4)) \geq 3 \times 3\mathrm{lr}(\mathrm{SL}_3(4)) = 3$. Actually, we can calculate $3\mathrm{lr}(\mathrm{SL}_6(4)) = 4$ directly, so the proof of our claim in this case is complete.

Case $^2A_l(q)$. Let $G = {}^2A_l(q)$. Since $2 \neq p \mid l + 1$ and $p \mid q + 1$, we have $l + 1 \geq 3$ and there exists $1 \neq t \in \{\alpha \in \mathrm{GF}(q^2)^\times \mid 1 = \alpha\theta(\alpha) = \alpha\alpha^q = \alpha^{q+1}\} \cong Z_{q+1}$ of order p, where θ is the associated field automorphism of order 2. We see that $\mathrm{SU}_{l+1}(q) = \{X \in \mathrm{SL}_{l+1}(q^2) \mid {}^tX\theta(X) = I\}$. Therefore we can apply a similar argument as in the case of $A_l(q)$. Here we emphasize that $3\mathrm{lr}(\mathrm{PSU}_3(2)) = 0$ and $3\mathrm{lr}(\mathrm{PSU}_6(2)) = 3$. So we have verified our assertion in this case.

Case $^2E_6(q)$ $(p = 3,\ 3 \mid q + 1)$. The group $^2E_6(q)$ contains $\mathrm{SU}_6(q)$ as a Levi factor of its parabolic subgroup (see [Aschbacher 1977, Table 14.5]). $\mathrm{SU}_6(q)$ has a subgroup $H \cong \mathrm{SU}_3(q) \times \mathrm{SU}_3(q)$. Since a Sylow 3-subgroup of $\mathrm{SU}_3(q)$ is not cyclic, by Theorem 1.1, $3\mathrm{lr}(\mathrm{SU}_3(q)) \geq 2$ for any $q > 2$ and $\mathrm{SU}_6(2)$ is of 3-local rank 3 as shown in the case $^2A_l(q)$. By Lemma 2.6, we have $3\mathrm{lr}(^2E_6(q)) \geq 3$.

This completes the proof of Proposition 5.1. □

Notation and further lemmas. To describe the Sylow structure of Chevalley groups for primes distinct from the characteristic, we use the following notation. Let $K = {}^dX_l(q)$ be a universal Chevalley group over $\mathrm{GF}(q^d)$. If ${}^dX_l = {}^2B_2$, ${}^2G_2$, or ${}^2F_4$ then we set $K = {}^2B_2(2^{2m+1})$, ${}^2G_2(3^{2m+1})$, or ${}^2F_4(2^{2m+1})$, respectively. There is an integer N, a set $\mathbb{O}({}^dX_l)$ of positive integers, and for each $m \in \mathbb{O}({}^dX_l)$ a "multiplicity" $r_m \in \mathbb{Z}_+$ such that

$$|K| = q^N \prod_{m \in \mathbb{O}({}^dX_l)} \Phi_m(q)^{r_m},$$

where $\Phi_m(q)$ is the cyclotomic polynomial for the m-th roots of unity, N is the number of positive roots in the root system corresponding to X. Table 1 and Table 2 [Gorenstein and Lyons 1983]. show the multiplicities r_m for the classical and ex-

A_l	$\left[\frac{l+1}{m}\right]$	if $m > 1$;	l if $m = 1$		
B_l, C_l	$\left[\frac{2l}{\operatorname{lcm}(2,m)}\right]$				
D_l	$\frac{2l}{m} - 1$	if $m \mid 2l$ and $m \nmid l$;		$\left[\frac{2l}{\operatorname{lcm}(2,m)}\right]$	otherwise
${}^2A_l$	$\left[\frac{l+1}{\operatorname{lcm}(2,m)}\right]$	if $m \not\equiv 2\ (4)$;	l if $m = 2$;	$\left[\frac{2(l+1)}{m}\right]$	otherwise
${}^2D_l$	$\left[\frac{2l}{\operatorname{lcm}(2,m)}\right]$	if $m \nmid l$;		$\left[\frac{2l}{\operatorname{lcm}(2,m)}\right] - 1$ if $m \mid l$	

Table 1. Values of r_m for classical groups [Gorenstein and Lyons 1983].

ceptional groups. Let e be the smallest positive integer such that p divides $\Phi_e(q)$

X_l	$\prod \Phi_m^{r_m}$ (product over $\mathbb{O}(X_l)$)
${}^2B_2$	$\Phi_1\Phi_4$
${}^3D_4$	$\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_{12}$
G_2	$\Phi_1^2\Phi_2^2\Phi_3\Phi_6$
${}^2G_2$	$\Phi_1\Phi_2\Phi_6$
F_4	$\Phi_1^4\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$
${}^2F_4$	$\Phi_1^2\Phi_2^2\Phi_4^2\Phi_6\Phi_{12}$
E_6	$\Phi_1^6\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$
${}^2E_6$	$\Phi_1^4\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$
E_7	$\Phi_1^7\Phi_2^7\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
E_8	$\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$

Table 2. Values of r_m for exceptional groups [Gorenstein and Lyons 1983].

and

$$\pi := \left\{ p \in \pi(K) \mid p \neq 2,\ p \nmid q,\ p \notin \pi(Z(K)) \right\}.$$

By [Gorenstein and Lyons 1983, (10-2)], with the above notation, if $p \in \pi$, then $m_p(K) = m_p(K/Z(K)) = r_e$. Here we restate the important theorems in [Gorenstein and Lyons 1983].

Lemma 5.2. *Suppose $K = {}^dX_l(q)$ is a universal Chevalley group or a twisted variation. Let p be an odd prime not dividing q. If $d = 3$, assume that $p \neq 3$, in* (1)–(3) *below. Let P be a Sylow p-subgroup of K, and assume that $P \neq 1$. Let m_0 be the smallest element of $\mathbb{O}({}^dX_l(q))$ such that p divides $\Phi_{m_0}(q)$, and let*

$$w = \{m \in \mathbb{O}({}^dX_l(q)) \mid m = p^a m_0,\ a > 0\}.$$

(1) $m_0 = \mathrm{ord}_p(q)$ (*the multiplicative order of q* (mod p)).

(2) *P has a nontrivial normal homocyclic abelian subgroup* (*denote by P_H*) *of rank r_{m_0}* (*the multiplicity of m_0*) *and exponent $|\Phi_{m_0}(q)|_p$. P is a split extension of P_H by a* (*possibly trivial*) *group P_W. P_W is isomorphic to a subgroup of the Weyl group W of type X, and have order p^σ, where $\sigma = \Sigma_{m \in w} r_m$.*

(3) *If $p \nmid |W|$, or if $pm_0 \nmid m$ for all $m \in \mathbb{O}({}^dX_l(q))$, then $P = P_H$ is homocyclic abelian.*

Lemma 5.3. *Suppose $K = {}^dX_l(q)$ is a finite group of Lie type, where ${}^1X_l(q)$ is an untwisted version $X_l(q)$ and $K^* = \mathrm{Inndiag}(K)$. Let p be an odd prime, and $P \in \mathrm{Syl}_p(K)$ and $P^* \in \mathrm{Syl}_p(K^*)$.*

(1) *If m_0 is as in the previous lemma* (*that is, $m_0 = \mathrm{ord}_p(q)$*), *then $m_p(P) = r_{m_0}$ unless p divides $|Z(K)|$, in which case $m_p(P) = r_{m_0}$ or $r_{m_0} - 1$.*

(2) *If any one the groups P, P^*, $\Omega_1(P)$ and $\Omega_1(P^*)$ is abelian, then they are all abelian.*

(3) *If P is abelian and $p \nmid q$, then P* ($\cong P^*$) *is homocyclic of rank r_{m_0} and exponent $|\Phi_{m_0}|_p$, where $m_0 = \mathrm{ord}_p(q)$ and r_{m_0} is the multiplicity of $\Phi_{m_0}(q)$ in $|{}^dX_l|$; also $\Phi_{m_0}(a)p^a \nmid |{}^dX_l|$ for any $a > 0$.*

Prime divisors p with $p \nmid d_{K_u}$. Let K be a universal Chevalley group. There is only one class of cases remaining for our problem, that is, (K_a, p) where K_a is the adjoin version of K and p is a prime divisor of the order $|K_u|$ of universal version K_u with $p \nmid |Z(K_u)|$. In such cases, all versions have the same p-ranks and p-local ranks, namely, $m_p(K_a) = m_p(K_u)$ and $p\mathrm{lr}(K_a) = p\mathrm{lr}(K_u)$. So we can consider the universal version in general. At first, we give several lemmas.

Lemma 5.4. *Let $G = \mathrm{PSL}_4(q)$, p an odd prime dividing $q - 1$, where q is a power of some prime r distinct from p. Then $p\mathrm{lr}(G) \geq 3$.*

Proof. Let $S = \mathrm{SL}_4(q)$ and $Z = Z(S)$. Note $p\mathrm{lr}(S) = p\mathrm{lr}(G)$ and they have isomorphic Sylow p-subgroups. There exists $1 \neq t \in \mathrm{GF}(q)^\times$ of order $p^k = (q-1)_p$. Set $h = \mathrm{diag}(t^{-3}, t, t, t)$. Since p is odd, $h \notin Z(S)$. It is obvious that $H := \langle h \rangle \leq \mathrm{SL}_4(q)$.

Let $N := N_S(H)$. N/Z is a proper p-local subgroup of G, since HZ/Z is a nontrivial p-subgroup of G which is simple. Therefore, $p\mathrm{lr}(N) = p\mathrm{lr}(N/Z) < p\mathrm{lr}(G) = p\mathrm{lr}(S)$. Notice that $N \geq \mathrm{SL}_3(q) \times H$ and $m_p(\mathrm{SL}_3(q)) = 2$. By Theorem 1.1 and the proof of Proposition 5.1, $p\mathrm{lr}(\mathrm{SL}_3(q)) \geq 2$ unless $(p, q) = (3, 4)$. When $(p, q) \neq (3, 4)$, we have $p\mathrm{lr}(G) \geq 1 + p\mathrm{lr}(\mathrm{SL}_3(q)) \geq 1 + 2 = 3$. By an easy calculation, we determine that the 3-local rank of $\mathrm{PSL}_4(4)$ is 3. $\square$

By using a similar argument as above, we can prove the following lemma.

Lemma 5.5. *Let $G = \mathrm{PSU}_4(q)$, p an odd prime dividing $q+1$, where q is a power of some prime r distinct from p. Then $p\mathrm{lr}(G) \geq 3$ except for $3\mathrm{lr}(\mathrm{PSU}_4(2)) = 2$.*

Proof. As in the proof of Proposition 5.1, we can take an element

$$1 \neq t \in \left\{\alpha \in \mathrm{GF}(q^2)^\times \mid 1 = \alpha\theta(\alpha) = \alpha\alpha^q = \alpha^{q+1}\right\}$$

of order $(q+1)_p$, where θ is the associated field automorphism of order 2. Note that $3\mathrm{lr}(\mathrm{PSU}_3(2)) = 0$. Then applying a similar argument as in the above lemma, we can prove that $p\mathrm{lr}(\mathrm{PSU}_4(q)) \geq 3$ where $(p, q) \neq (3, 2)$ and $3\mathrm{lr}(\mathrm{PSU}_4(2)) = 2$ as required. $\square$

Lemma 5.6. *Let $G = \Omega_8^-(q)$ and $\varepsilon = \pm 1$, where q is a power of a prime r. Assume that p is an odd prime dividing $q - \varepsilon$. Then $p\mathrm{lr}(G) \geq 3$.*

Proof. Now suppose that q is odd. Notice that $\left|\mathrm{SO}_8^-(q) : \Omega_8^-(q)\right| = 2$ and p is odd. By Lemma 2.3, we see that $p\mathrm{lr}(\mathrm{SO}_8^-(q)) = p\mathrm{lr}(G)$. Consider the subgroup $H = \mathrm{SO}_2^\varepsilon(q) \times \mathrm{SO}_6^{-\varepsilon}(q) \leq \mathrm{SO}_8^-(q)$. Note that $\mathrm{SO}_2^\varepsilon(q)$ is a cyclic group of order $q - \varepsilon$. Let a be an element of order $(q-\varepsilon)_p$ in $\mathrm{SO}_2^\varepsilon(q)$. Denote $N = N_{\mathrm{SO}_8^-(q)}(\langle a\rangle)$. So we can see that $N \geq H \geq \mathrm{SO}_6^{-\varepsilon}(q)$. Since $m_p(\mathrm{SO}_6^{-\varepsilon}(q)) = 2$, by Theorem 1.1, $p\mathrm{lr}(\mathrm{SO}_6^{-\varepsilon}(q)) \geq 2$. Since $O_p(\mathrm{SO}_8^-(q))$ is trivial, by Corollary 2.5, $p\mathrm{lr}(G) \geq 1 + p\mathrm{lr}(N) \geq 1 + p\mathrm{lr}(\mathrm{SO}_6^{-\varepsilon}(q)) \geq 1 + 2 = 3$. If q is even, we replace SO_l by O_l. Note that in this case $|O_l(V) : \Omega_l(V)| = 2$. We can get the result by a similar argument. $\square$

Proposition 5.7. *Let $K = {}^dX_l(q)$ be a universal Chevalley group or twisted variation, p is an odd prime divisor of the order K, where q is a power of a prime r distinct from p. Assume that $p \nmid d_{K_u} = |Z(K_u)|$ where K_u is the universal version of K. If $m_p(K) \geq 3$ then $p\mathrm{lr}(K) \geq 3$ except for $3\mathrm{lr}(\mathrm{PSU}_4(2)) = 2$ and $3\mathrm{lr}(\mathrm{PSp}_6(2)) = 2$.*

Proof. The strategy is rather standard and uniform: to find some subgroup of shape $X_i(q) \times X_j(q)$ inside $X_n(q)$. Suppose that p divides $|K|$ and the p-rank of K is $k = r_e$ where e is the smallest element of $\mathbb{O}({}^dX_l(q))$ such that p divides $\Phi_e(q)$. We will verify the proposition type by type.

Type A_l. In this type, by Table 1, we have $r_m = [(l+1)/m]$ for $m \geq 2$ and $r_1 = l$. Note that for this type, $d_{A_l(q)} = (l+1, \Phi_1(q))$ (see [Gorenstein et al. 1994, Table 2.2]). So there are two cases.

(1) $e \geq 2$. By assumption, we see that $k = [(l+1)/e] \geq 3$, hence $l \geq 3e-1$. By Lemma 2.3, we need only to prove our assertion in the case $l = 3e-1$. Since $A_{e-1}(q) \times A_{e-1}(q) \times A_{e-1}(q)$ is a subgroup of $A_{3e-1}(q)$, we have that $p\mathrm{lr}(A_{3e-1}(q)) \geq 3 \cdot p\mathrm{lr}(A_{e-1}(q))$. Notice that p divides the order of $A_{e-1}(q) = \mathrm{SL}_e(q)$ and $p \nmid |Z(\mathrm{SL}_e(q)|$. When $(p,q,e) \neq (3,2,2)$, since $\mathrm{SL}_e(q)$ has nonnormal Sylow p-subgroups, $p\mathrm{lr}(A_{e-1}(q)) \geq 1$ which implies that $p\mathrm{lr}(A_{3e-1}(q)) \geq 3$. For the possible exception, we have $3\mathrm{lr}(\mathrm{PSL}_6(2)) = 3$ by direct calculation.

(2) $e = 1$. By the same reason as in the above case, we only need to consider the case $K = A_3(q)$. Note that $\Phi_1(q) = q-1$ and $|Z(K)| = (4, q-1)$. By Lemma 5.4, we have verified our assertion in this case.

Type C_l, $l \geq 2$. For symplectic groups, we will prove our assertion in two cases.

(1) q is even. The symplectic group $C_l(q)$ has a subgroup of shape $C_i(q) \times C_j(q)$, $i + j \leq l$. By Table 1, if e is even, $k = [2l/e] \geq 3$, hence $l \geq (3/2)e$. Suppose that $e = 2t$, $t \geq 1$. Now we only need to consider the case $l = (3/2)e = 3t$. Let $H = C_t(q) \times C_e(q) \leq C_l(q)$. We see that $m_p(C_t(q)) = 1$ and $m_p(C_e(q)) = 2$. If $(p,q,e) \neq (3,2,2)$, then $C_t(q) = \mathrm{PSp}_{2t}(q)$ is simple since $\mathrm{PSp}_2(q) \cong \mathrm{PSL}_2(q)$ and q is even. So

$$p\mathrm{lr}(C_t(q)) \geq 1 \quad \text{and} \quad p\mathrm{lr}(C_e(q)) \geq 2$$

by Theorem 1.1. Hence $p\mathrm{lr}(C_l(q)) \geq p\mathrm{lr}(H) \geq 1+2 = 3$ unless $(p,q,e) = (3,2,2)$. By direct calculation, we determine one of the exceptions, precisely, $3\mathrm{lr}(\mathrm{PSp}_6(2)) = 2$. We also get $3\mathrm{lr}(\mathrm{PSp}_8(2)) = 3$, hence $p\mathrm{lr}(\mathrm{PSp}_{2m}(2)) \geq 3$, if $m \geq 4$. Now suppose that e is odd. We have $k = [2l/(2e)] \geq 3$, hence $l \geq 3e$. Let $H = C_{2e}(q) \times C_e(q)$. Note that $\mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$, $\mathrm{Sp}_4(2) \cong \Omega_5(2) \cong S_6$, otherwise $\mathrm{PSp}_{2e}(q)$ is simple. If $e = 1$, for $\mathrm{Sp}_2(2)$, $\Phi_1(2) = 2-1 = 1$ which is not our case. So by the same argument as above, we have $p\mathrm{lr}(C_{3e}(q)) \geq p\mathrm{lr}(H) \geq 2+1 = 3$.

(2) q is odd. The proof is the same as that in the even case except for a small difference. When we investigate the p-local rank of $C_1(q)$, the nonsimple one is $\mathrm{Sp}_2(3) \cong \mathrm{SL}_2(3)$. Note that $\mathrm{SL}_2(3)$ is 2-closed, but not 3-closed. So $\mathrm{PSp}_6(3)$ is not an exception. We are done in this case.

Type D_l. Now we assume that $l \geq 4$. In this type, there also exist two cases to discuss.

(1) q is odd.

(a) Suppose that $e = 2t$ $(t \geq 1)$ is even, and e does not satisfy the conditions $e \mid 2l$, $e \nmid l$. Since $k = [2l/e] \geq 3$, we see that $l \geq (3/2)e = 3t$. We may assume that $l = 3t+1$, since if $l = 3t$, then $e \mid 2l$ and $e \nmid l$. Let $H = D_e(q) \times D_{t+1}(q) \leq D_l(q)$. We can get that $m_p(D_e(q)) = 2$ and $m_p(D_{t+1}(q)) \geq 1$. Note that when $e = 2$, $D_4(q)_a$, the adjoint version (that is, $P\Omega_8^+(q)$) has a subgroup of shape

$L = \mathrm{SL}_2(q) * \mathrm{SL}_2(q) * \mathrm{SL}_2(q) * \mathrm{SL}_2(q)$. Since q and p are both odd, $\mathrm{SL}_2(q)$ has nonnormal Sylow p-subgroups, hence $p\mathrm{lr}(\mathrm{SL}_2(q)) \geq 1$. So we have shown $p\mathrm{lr}(D_4(q)) \geq p\mathrm{lr}(L) = 4$. When $e \geq 4$, we have $p\mathrm{lr}(D_l(q)) \geq p\mathrm{lr}(H) \geq 2+1 = 3$ as required by Theorem 1.1.

(b) Now assume that e is odd. We also suppose that e does not satisfy the conditions as in Case (a). Since $k = [2l/(2e)] \geq 3$, $l \geq 3e$. Let $H = D_{2e}(q) \times D_e(q)$, and note that $m_p(D_{2e}(q)) = 2$, $m_p(D_e(q)) = 1$. Then $p\mathrm{lr}(D_l(q)) \geq p\mathrm{lr}(H) \geq 2+1 = 3$ by Theorem 1.1 for $e \geq 3$. When $e = 1$, we have $l \geq 3$. Note that $l \geq 4$. We only need to consider $D_4(q)$. Our assertion holds for the same reason as in the case $e = 2$ in (a). From another point of view, since $D_3(q) \cong A_3(q)$ and the trivial embedding $D_3(q) \leq D_4(q)$, we also proved our assertion by Type A_l when $e = 1$ as required.

(c) Lastly, we deal with the case that $e \mid 2l$ and $e \nmid l$. Since $k = 2l/e - 1 \geq 3$, we have $l \geq 2e+1$. Let $H = D_{e+1}(q) \times D_e(q)$. Note that there is an even number in $\{e, e+1\}$. We have that $p\mathrm{lr}(D_l(q)) \geq p\mathrm{lr}(H) \geq 2+1 = 3$ for $e \geq 3$ by Theorem 1.1. When $e = 1$, it does not satisfy the hypotheses that $e \mid 2l$ and $e \nmid l$. In fact we have verified our assertion for $e = 1$ in Case (b). When $e = 2$, with the assumption $e \mid 2l$ and $e \nmid l$, we have $l \geq 5$. Since $D_5(q) \geq D_4(q)$, our assertion holds (see Case (a)).

(2) q is even. Note that $D_1(q) \cong Z_{q-1}$, $D_2(q) \cong A_1(q) \times A_1(q)$ and $D_3(q) \cong A_3(q)$. By a similar argument to the p odd case, we can verify our assertion in this case. In fact, following the argument in the case that q is odd, the unique possible exception is $D_4(2)$ when $(p, q, e) = (3, 2, 2)$ (that is, $L = \mathrm{SL}_2(2) * \mathrm{SL}_2(2) * \mathrm{SL}_2(2) * \mathrm{SL}_2(2)$ for $p = 3$). But the calculation shows $3\mathrm{lr}(D_4(2)) = 3$. So we are done in this case.

Type B_l. Here q is odd and $l \geq 3$. There exists a subgroup in $B_l(q)$ which has the form $B_i(q) \times D_j(q)$, where $i, j \in \mathbb{Z}_+$ and $i + j = l$. When $e = 2t$ is even ($t \geq 1$), we have $k = [2l/e] \geq 3$, hence $l \geq (3/2)e = 3t$. Let $H = D_e(q) \times B_t(q)$. Since $e \mid 2e$ and $e \mid e$, we have that $m_p(D_e(q)) = 2$ and $m_p(B_t(q)) = 1$, hence $p\mathrm{lr}(B_l(q)) \geq p\mathrm{lr}(H) \geq 2 + 1 = 3$ for $e \geq 4$ by Theorem 1.1. When $e = 2$, we see that $L = \mathrm{SL}_2(q) * \mathrm{SL}_2(q) * B_1(q) \leq B_3(q)$. It is known that $B_1(q) \cong A_1(q)$. Note that both p and q are odd. By the same reason in the q odd case of Type D_l, $p\mathrm{lr}(B_3(q)) \geq 3$. Now we suppose that e is odd. Since $k = [2l/(2e)] \geq 3$, $l \geq 3e$. Let $H = D_{2e}(q) \times B_e(q)$. Note that $m_p(D_{2e}(q)) = 2$ and $m_p(B_e(q)) = 1$. We also have that $p\mathrm{lr}(B_l(q)) \geq 3$ for $e \geq 3$. When $e = 1$, our assertion holds by the same argument as the case $e = 2$, since we can also take $H = A_1(q) \times A_1(q) \times B_1(q)$ as a subgroup of $B_3(q)$.

Type ^2A_l. In this type, $K = \mathrm{SU}_{l+1}(q)$, where $l \geq 2$. By [Gorenstein et al. 1994, Table 2.2], $d_K = (l+1, \Phi_2(q))$.

(1) q is odd. There exists a subgroup of the form $^2A_i(q) \times {}^2A_j(q)$ in K, where $i + j = l - 1$ and $0 \leq i < l/2$. If $e \equiv \pm 1 (\mathrm{mod}\, 4)$, $k = [(l+1)/\mathrm{lcm}(2, e)] \geq 3$,

hence $l \geq 6e-1$. Let $H = {}^2A_{2e-1}(q) \times {}^2A_{4e-1}(q)$. By an easy computation, we get $m_p({}^2A_{2e-1}(q)) = 1$ and $m_p({}^2A_{4e-1}(q)) = 2$. Since p and q are both odd, ${}^2A_{2e-1}(q)$ has nonnormal Sylow p-subgroups. So by Theorem 1.1, we have that $p\mathrm{lr}(K) \geq p\mathrm{lr}(H) \geq 3$. If $4 \mid e$, then $k = [(l+1)/e] \geq 3$, hence $l \geq 3e-1$. Let $H = \mathrm{SU}_e(q) \times \mathrm{SU}_{2e}(q)$. By the same argument, $p\mathrm{lr}(K) \geq 3$. Now we suppose $e \equiv 2 (\mathrm{mod}\, 4)$ and $e > 2$. Let $e = 2t$, where $t \geq 3$ is odd. We see that $l \geq 3e/2 - 1 = 3t-1$. Let $H = \mathrm{SU}_t(q) \times \mathrm{SU}_e(q)$. We get $p\mathrm{lr}(\mathrm{SU}_l(q)) \geq p\mathrm{lr}(H) \geq 3$, since both $\mathrm{PSU}_t(q)$ and $\mathrm{PSU}_e(q)$ are simple. For $e = 2$, we may assume $K = \mathrm{SU}_4(q)$. Note $\Phi_2(q) = q+1$ and $|Z(K)| = (4, q+1)$. By Lemma 5.5, we are done in this case.

(2) q is even. K has subgroups of the form ${}^2A_i(q) \times {}^2A_j(q)$ where $i + j = l - 1$. Using the same argument as in preceding case, we obtain the result and another exception $3\mathrm{lr}(\mathrm{PSU}_4(2)) = 2$ by Lemma 5.5 and the fact that $3\mathrm{lr}(\mathrm{PSU}_5(2)) = 3$.

So our assertion holds for groups of this type.

Type ${}^2D_l$. We also have $l \geq 4$.

There exists a subgroup of K of the form ${}^2D_i(q) \times D_j(q)$ where $i + j = l$ and $2 \leq i < l/2$. When $e = 2t$ ($t \geq 1$) is even and $e \nmid l$, we have that $k = [2l/e] \geq 3$, hence $l \geq 3e/2 = 3t$. Let $H_1 = {}^2D_t(q)$, $H_2 = D_e(q)$ if t is even; $H_1 = {}^2D_{t+1}(q)$, $H_2 = D_e(q)$ if t is odd and greater than 1. Note that with each condition above on e, we have $e \geq 4$. Then $H_1 \times H_2 \leq K$. Since $m_p(H_2) = 2$ and $m_p(H_1) = 1$, we can prove that $p\mathrm{lr}(K) \geq 3$ (note that ${}^2D_2(q) \cong A_1(q^2)$ and ${}^2D_3(q) \cong {}^2A_3(q)$). If $e = 2$, then $l \geq 5$. For the trivial embedding ${}^2D_4(q) \leq {}^2D_5(q)$, we can omit this case and treat the following. Now suppose that $e \mid l$. From $k + 1 = [2l/e] \geq 4$, we see that $l \geq 2e$. If $e \geq 4$ or $l > 2e$, we can use the same argument as above and our assertion holds. The remaining case is $l = 4$ and $e = 2$, and for this our claim has been verified in Lemma 5.6. Now we suppose that e is odd. From $k = [2l/(2e)] \geq 3$, we have that $l \geq 3e$. When $e \nmid l$, since $l \geq 3e+1$ and $e > 1$, we take $H = {}^2D_{e+1}(q) \times D_{2e}(q)$. As above, we can show $p\mathrm{lr}(G) \geq 3$. If $e \mid l$, we have that $l \geq 4e$. For $e > 1$, our proof is the same as the case when e is even. When $e = 1$, our assertion is proved in Lemma 5.6.

Types ${}^2B_2$, ${}^2G_2$, G_2, ${}^3D_4$ and ${}^2F_4$. In these types, we see that

$$\begin{aligned}
\left|{}^2B_2(q^2)\right| &= q\Phi_1(q)\Phi_4(q),\\
\left|{}^2G_2(q^2)\right| &= q^3\Phi_1(q)\Phi_2(q)\Phi_6(q),\\
\left|G_2(q)\right| &= q^6\Phi_1^2(q)\Phi_2^2(q)\Phi_3(q)\Phi_6(q),\\
\left|{}^3D_4(q)\right| &= q^{13}\Phi_1^2(q)\Phi_2^2(q)\Phi_3^2(q)\Phi_6^2(q)\Phi_{12}(q),\\
\left|{}^2F_4(q^2)\right| &= q^{12}\Phi_1^2(q)\Phi_2^2(q)\Phi_4^2(q)\Phi_6(q)\Phi_{12}(q).
\end{aligned}$$

For each type, there does not exist odd prime p distinct from the defining characteristic such that the p-ranks of those Sylow p-subgroups are greater than two. So there is nothing to prove.

Type F_4. In the remainding types, we use $\Phi_n := \Phi_n(q)$ for short. We see that $|F_4(q)| = q^{24}\Phi_1^4\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$. So $m_p(K) \geq 3$ if and only if $e = 1$ or 2. If $p \mid \Phi_1$, notice that $F_4(q)$ has $\mathrm{SL}_2(q) * C_3(q)_u$ as a subgroup, where $C_3(q)_u$ is the universal version. By Table 1, we have $m_p(C_3(q)) = 3$, hence $p\mathrm{lr}(C_3(q)_u) \geq 3$ by Type C_l. So $3\mathrm{lr}(F_4(q)) \geq 3$. If $p \mid \Phi_2$, we also have that $B_4(q)_u$, the universal version, is a subgroup of $F_4(q)$. When q is even, $B_l(q) \cong C_l(q)$. Since $m_p(B_4(q)) = 4$, so we are done by Types B_l and C_l.

Type E_6. Since $|E_6(q)| = q^{36}\Phi_1^6\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$, we only need to consider when p is a prime divisor of Φ_1, Φ_2 or Φ_3. When p divides Φ_1 or Φ_2, since we can treat $F_4(q)$ as a subgroup of $E_6(q)$, $p\mathrm{lr}(E_6(q)) \geq p\mathrm{lr}(F_4(q)) \geq 3$. Note that $\mathrm{SL}_3(q) * \mathrm{SL}_3(q) * \mathrm{SL}_3(q)$ can be embedded in $E_6(q)$ as a subgroup. We see that $\mathrm{SL}_3(q)$ has the trivial multiplicity $r_3 = 1$ and $\mathrm{PSL}_3(q)$ is simple. So if p divides Φ_3 and $e = 3$, $p\mathrm{lr}(E_6(q)) \geq 3$.

Type ${}^2E_6$. We know $F_4(q) \leq {}^2E_6(q)$. Since

$$\left|{}^2E_6(q)\right| = q^{36}\Phi_1^4\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18},$$

we only consider that $p \mid \Phi_6$ and $e = 6$ is the smallest number i such that p divides Φ_i. Note that $e = 6$ implies $p \nmid \Phi_2$. Let $H = \mathrm{SU}_3(q) * \mathrm{SU}_3(q) * \mathrm{SU}_3(q)$ which is a subgroup of ${}^2E_6(q)$. Note that the multiplicity r_6 of $\mathrm{SU}_3(q)$ is 1. Since $\Phi_6(2) = \Phi_2(2) = 3$, $\mathrm{SU}_3(2)$ is not involved in H (that is, the case $(G, p) = ({}^2E_6(2), 3)$ is not a case to be considered) and otherwise $\mathrm{PSU}_3(q)$ is simple. So $p\mathrm{lr}({}^2E_6(q)) \geq 3$.

Type E_7. In this type, $E_6(q)$ and ${}^2E_6(q)$ are both subgroups of $E_7(q)$. Note that $|E_7(q)| = q^{63}\Phi_1^7\Phi_2^7\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$. We are done by above two types.

Type E_8. In this type, since $E_7(q) \leq E_8(q)$ and

$$|E_8(q)| = q^{120}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30},$$

we just consider the case that p divides Φ_4 and $e = 4$ is the smallest subscript i such that p divides Φ_i. We see that $D_8(q)$ is a subgroup of $E_8(q)$ and the multiplicity r_4 of $D_8(q)$ is 4, hence $m_p(D_8(q)) = 4$, so we are done by Type D_l.

This completes the proof of Proposition 5.7. □

6. Main result

In Section 4 and 5, we have determined all finite simple groups of Lie type satisfying the conditions $m_p(G) \geq 3$ and $p\mathrm{lr}(G) = 2$. Note that $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$.

In Proposition 4.1, we have $2\mathrm{lr}({}^2F_4(2)) = 2$. Since $|{}^2F_4(2) : {}^2F_4(2)'| = 2$, by Theorem 1.1, we have $2\mathrm{lr}({}^2F_4(2)') = 2$. Combining with the results on alternating and sporadic groups, we have:

Theorem 6.1. *Let p be a prime and G a finite simple group with $p\mathrm{lr}(G) = 2$. Then $m_p(G) = 2$, where $m_p(G)$ is the p-rank of G, unless we are in one of the following cases*::

(1) $G \cong \mathrm{PSL}_3(q)$, *where* $q = p^n$ *and* $n \geq 2$;

(2) $G \cong P\Omega_5(q), \mathrm{PSU}_4(q), G_2(q)$ *or* ${}^3D_4(q)$ *where* $q = p^n$ *and* $n \geq 1$;

(3) $p = 2$ *and* $G \cong {}^2F_4(2^{2m+1})$ *for* $m \geq 1$;

(4) $p = 2$ *and* $G \cong {}^2F_4(2)', J_1, J_2$ *or* J_3;

(5) $p = 3$ *and* $G \cong A_9, \mathrm{PSU}_4(2), \mathrm{PSp}_6(2), J_3$, McL, O'N *or* Co_3;

(6) $p = 5$ *and* $G \cong \mathrm{HN}$.

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References

[Alperin 1987] J. L. Alperin, "Weights for finite groups", pp. 369–379 in *The Arcata Conference on Representations of Finite Groups* (Arcata, CA, 1986), edited by P. Fong, Proc. Sympos. Pure Math. **47**, Amer. Math. Soc., Providence, RI, 1987. MR 89h:20015 Zbl 0657.20013

[Alperin and Fong 1990] J. L. Alperin and P. Fong, "Weights for symmetric and general linear groups", *J. Algebra* **131**:1 (1990), 2–22. MR 91h:20014 Zbl 0714.20007

[An 1994] J. An, "Dade's conjecture for the simple Ree groups ${}^2G_2(q^2)$ in non-defining characteristics", *Indian J. Math.* **36**:1 (1994), 7–27. MR 96c:20029 Zbl 0864.20009

[An 1996] J. An, "Dade's conjecture for Chevalley groups $G_2(q)$ in non-defining characteristics", *Canad. J. Math.* **48**:4 (1996), 673–691. MR 97i:20016 Zbl 0894.20018

[An 1999] J. An, "The Alperin and Dade conjectures for the simple Conway's third group", *Israel J. Math.* **112** (1999), 109–134. MR 2000f:20019 Zbl 0938.20011

[An 2002] J. An, "Dade's conjecture for Steinberg triality groups ${}^3D_4(q)$ in non-defining characteristics", *Math. Z.* **241**:3 (2002), 445–469. MR 2003k:20014 Zbl 1045.20009

[An and O'Brien 2002] J. An and E. A. O'Brien, "The Alperin and Dade conjectures for the O'Nan and Rudivalis simple groups", *Comm. Algebra* **30**:3 (2002), 1305–1348. MR 2003a:20020 Zbl 1009.20016

[Aschbacher 1977] M. Aschbacher, "A characterization of Chevalley groups over fields of odd order", *Ann. of Math.* (2) **106**:2 (1977), 353–398. MR 58 #16865a Zbl 0393.20011

[Carter 1972] R. W. Carter, *Simple groups of Lie type*, Pure and Applied Math. **28**, Wiley, London-New York-Sydney, 1972. MR 53 #10946 Zbl 0248.20015

[Conway et al. 1985] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups: Maximal subgroups and ordinary characters for simple groups*, Oxford Univ. Press, 1985. MR 88g:20025 Zbl 0568.20001

[Dade 1992] E. C. Dade, "Counting characters in blocks. I", *Invent. Math.* **109**:1 (1992), 187–210. MR 93g:20021 Zbl 0738.20011

[Eaton 2001] C. W. Eaton, "On finite groups of p-local rank one and conjectures of Dade and Robinson", *J. Algebra* **238**:2 (2001), 623–642. MR 2002b:20012 Zbl 0988.20005

[Gorenstein and Lyons 1983] D. Gorenstein and R. Lyons, *The local structure of finite groups of characteristic* 2 *type*, vol. 42, Mem. Amer. Math. Soc. **276**, Amer. Math. Soc., Providence, 1983. MR 84g:20025 Zbl 0519.20014

[Gorenstein et al. 1994] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups*, Math. Surveys and Monogr. **40**, Amer. Math. Soc., Providence, RI, 1994. MR 95m:20014 Zbl 0816.20016

[Huppert 1967] B. Huppert, *Endliche Gruppen. I*, Grund. der Math. Wissenschaften **134**, Springer, Berlin, 1967. MR 37 #302 Zbl 0217.07201

[Kotlica 1997] S. Kotlica, "Verification of Dade's conjecture for Janko group J_3", *J. Algebra* **187**:2 (1997), 579–619. MR 97k:20026 Zbl 0871.20008

[Murray 1998] J. Murray, *Dade's conjecture for the McLaughlin simple group*, Ph.D. thesis, University of Illinois at Urbana–Champaign, 1998, Available at http://homepage.tinet.ie/~jcmurray/thesis.ps.

[Norton and Wilson 1986] S. P. Norton and R. A. Wilson, "Maximal subgroups of the Harada–Norton group", *J. Algebra* **103**:1 (1986), 362–376. MR 88b:20029 Zbl 0595.20017

[Olsson and Uno 1995] J. B. Olsson and K. Uno, "Dade's conjecture for symmetric groups", *J. Algebra* **176**:2 (1995), 534–560. MR 96h:20025 Zbl 0839.20018

[Robinson 1996] G. R. Robinson, "Local structure, vertices and Alperin's conjecture", *Proc. London Math. Soc.* (3) **72**:2 (1996), 312–330. MR 97c:20015 Zbl 0854.20010

[Wang 2005] B. Wang, "A note on the p-local rank", *Arch. Math.* (*Basel*) **84**:3 (2005), 198–204. MR 2005m:20029 Zbl 1069.20009

[Wang and Zhang 2003] B. Wang and J. P. Zhang, "On the p-local rank of finite groups", *Acta Math. Sin.* (*Engl. Ser.*) **19**:1 (2003), 29–34. MR 2003m:20021 Zbl 1035.20017

[Wilson 1987] R. A. Wilson, "Some subgroups of the Baby Monster", *Invent. Math.* **89**:1 (1987), 197–218. MR 88d:20030 Zbl 0622.20007

[Yoshiara 2005] S. Yoshiara, "Radical 2-subgroups of the Monster and the Baby Monster", *J. Algebra* **287**:1 (2005), 123–139. MR 2006a:20029 Zbl 1071.20024

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