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We consider the exact asymptotic behavior of smooth solutions to boundary blow-up problems for the k-Hessian equation on Ω, where $\partial\Omega$ is strictly $(k-1)$-convex. Similar results were obtained by Cîrstea and Trombetti when $k=n$ (the Monge–Ampère equation) and by Bandle and Marcus for a semilinear equation.

1. Introduction and main results

We investigate the qualitative properties of solutions to the boundary blow-up problem for the k-Hessian equation of the form

$$\begin{cases} H_k[D^2u] = \sigma_k(\lambda_1, \dots, \lambda_n) = b(x) f(u), & x \in \Omega, \\ u(x) = \infty, & x \in \partial\Omega, \end{cases} \tag{1-1}$$

where $b(x)$ is a continuous weight function, $\lambda_1, \dots, \lambda_n$ are eigenvalues of D^2u, the Hessian matrix of a C^2-function u defined over Ω, and Ω is a bounded domain in $\mathbb{R}^n$. The boundary condition means $u(x) \to +\infty$ as $d(x) \triangleq \operatorname{dist}(x, \partial\Omega) \to 0_+$.

Following [Caffarelli et al. 1985; Trudinger 1995], σ_k is defined by

$$\sigma_k(\lambda_1, \dots, \lambda_n) = \sum_{1 \le i_1 < \dots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{1-2}$$

One can solve (1-1) in a class of k-convex functions by [Caffarelli et al. 1985; Jian 2006]. Recall that a function $u \in C^2(\Omega)$ is called k-convex (or strictly k-convex) if $(\lambda_1, \dots, \lambda_n) \in \overline{\Gamma}_k$ (or $(\lambda_1, \dots, \lambda_n) \in \Gamma_k$) for every $x \in \Omega$, where Γ_k is the convex cone with vertex at the origin given by

$$\Gamma_k = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0,\ j = 1, \dots, k\}.$$

Obviously,

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_j > 0,\ j = 1, \dots, k\},$$

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where Γ_n is the positive cone, and $\sigma_k(\lambda_1, \dots, \lambda_n)$ is elliptic in the class of k-convex functions.

For an open bounded subset Ω of $\mathbb{R}^n$ with boundary of class C^2 and for every $x \in \partial\Omega$, we denote by $\rho_1(x), \dots, \rho_{n-1}(x)$ the principal curvatures of $\partial\Omega$ (relative to the interior normal). Recall that Ω is said to be l-convex if $(\rho_1(x), \dots, \rho_{n-1}(x)) \in \overline{\Gamma}_l$, and it is called strictly l-convex if $(\rho_1(x), \dots, \rho_{n-1}(x)) \in \Gamma_l$, for every $x \in \partial\Omega$. In particular, strictly $(n-1)$-convex is just strictly convex.

Using radial function methods and techniques of ordinary differential inequality, Jian [2006] constructed various barriers functions, then proved existence and nonexistence theorems using those barriers. Furthermore, generic boundary blow-up rates for the solution are derived for the k-Hessian equation with boundary blow-up problem. In this paper, we derive accurately the blow-up rate of solutions to boundary blow-up problems for Hessian equations.

Let $\mathfrak{K}_\ell$ denote the set of all positive nondecreasing C^1-functions m defined on $(0, \nu)$, for some $\nu > 0$, for which there exists

$$\lim_{t\to 0^+} \frac{\int_0^t m(s)\,ds}{m(t)} = 0 \quad \text{and} \quad \lim_{t\to 0^+} \frac{d}{dt}\left(\frac{\int_0^t m(s)\,ds}{m(t)}\right) = \ell. \tag{1-3}$$

A complete characterization of $\mathfrak{K}_\ell$ (according to $\ell \neq 0$ or $\ell = 0$) is provided by [Cîrstea and Rădulescu 2006].

One has the following examples for special ℓ, where $p > 0$ is arbitrary:

(a) $m(t) = (-1/\ln t)^p$ with $\ell = 1$,

(b) $m(t) = t^p$ with $\ell = 1/(p+1)$,

(c) $m(t) = e^{-1/t^p}$ with $\ell = 0$.

Definition 1.1. A positive measurable function f defined on $[a, \infty)$, for some $a > 0$, is called regularly varying at infinity with index q, written $f \in \mathbb{RV}_q$, if for each $\lambda > 0$ and some $q \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^q. \tag{1-4}$$

The real number q is called the index of regular variation.

When $q = 0$, we have:

Definition 1.2. A positive measurable function L defined on $[a, \infty)$, for some $a > 0$, is called regularly varying at infinity, if for each $\lambda > 0$ and some $q \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1. \tag{1-5}$$

By Definitions 1.1 and 1.2, if $f \in \mathbb{RV}_q$, it can be represented in the form

$$f(t) = u^q L(t). \tag{1-6}$$

Notation. If H is a nondecreasing function on $\mathbb{R}$, then we denote by $H^{\leftarrow}$ the (left-continuous) inverse of H [Resnick 1987], that is,

$$H^{\leftarrow}(y) = \inf\{s : H(s) \geq y\}.$$

If $\alpha > 0$ is sufficiently large, we define

$$\mathcal{P}(u) = \sup\left\{\frac{f(y)}{y^k} : \alpha \leq y \leq u\right\}, \quad \text{for } u \geq \alpha. \tag{1-7}$$

Problem (1-1) is the Laplace operator when $k = 1$. There are many papers resolving existence, uniqueness and asymptotic behavior issues for blow-up solutions of semilinear/quasilinear elliptic equations: for instance [Osserman 1957; Resnick 1987; Véron 1992; Bandle and Marcus 1992; 1995; García-Melián et al. 2001; Chuaqui et al. 2004; Cîrstea and Rădulescu 2006; García-Melián 2006].

When $k = n$, problem (1-1) is the Monge–Ampère equation, for which Cîrstea and Trombetti [2008] obtained existence, uniqueness and asymptotic behavior; see also [Guan and Jian 2004; Mohammed 2007].

The boundary blow-up problem of the k-Hessian equation was considered in [Salani 1998; Colesanti et al. 2000; Jian 2006]. See also [Takimoto 2006] for recent results on boundary blow-up problems for k-curvature equations, where there is a considerable difference between the cases $1 \leq k \leq n-1$ and $k = n$. However, we can unify them by using techniques from [Colesanti et al. 2000; Cîrstea and Trombetti 2008] for k-Hessian equations.

Our asymptotic results are obtained in the case when $\partial\Omega$ is strictly $(k-1)$-convex, but for k-curvature equations in [Cîrstea and Trombetti 2008], the condition that $\partial\Omega$ is strictly convex is needed.

Theorem 1.3. *Let $n \geq 2$ and Ω be a smooth, strictly $(k-1)$-convex bounded domain in $\mathbb{R}^n$. Assume that $f \in \mathbb{RV}_q$ with $q > k$ and there exists $m \in \mathfrak{K}_\ell$ such that*

$$0 < \beta^- = \liminf_{d(x)\to 0} \frac{b(x)}{m^{k+1}(d(x))} \quad \textit{and} \quad \limsup_{d(x)\to 0} \frac{b(x)}{m^{k+1}(d(x))} = \beta^+ < \infty. \tag{1-8}$$

Then, every k-convex blow-up solution u_∞ of (1-1) satisfies

$$\xi^- \leq \liminf_{d(x)\to 0} \frac{u}{\phi(d(x))} \quad \textit{and} \quad \limsup_{d(x)\to 0} \frac{u}{\phi(d(x))} \leq \xi^+, \tag{1-9}$$

where ϕ is defined by

$$\phi(t) = \mathcal{P}^{\leftarrow}\left(\left(\int_0^t m(s)\,ds\right)^{-k-1}\right), \quad \textit{for } t > 0 \textit{ small}, \tag{1-10}$$

and $\xi^{\pm}$ are positive constants given by

$$\frac{(\xi^+)^{k-q}}{\beta^-}\max_{\partial\Omega}\sigma_{k-1}=\frac{(\xi^-)^{k-q}}{\beta^+}\min_{\partial\Omega}\sigma_{k-1}=\frac{\big((q-k)/(n+1)\big)^{k+1}}{1+\ell(q-k)/(k+1)}. \tag{1-11}$$

On the other hand, Colesanti et al. [2000] established asymptotic estimates for the behavior of the smallest viscosity solution near the boundary of Ω for the Hessian equation

$$\begin{cases} H_k[D^2u]=f(u), & x\in\Omega,\\ u(x)=\infty, & x\in\partial\Omega.\end{cases} \tag{1-12}$$

Theorem 1.3 may also been seen as a generalization of the asymptotic behavior for the viscosity solution in [Colesanti et al. 2000].

Remark 1.4. In the setting of Theorem 1.3, $\lim_{d(x)\to 0} u/\phi(d(x))$ exists provided that Ω is a ball and (1-8) holds with $\beta^-=\beta^+\in(0,\infty)$. The latter condition is equivalent to saying that

$$b(x)\sim(m(d(x)))^{k+1}\quad\text{as } d(x)\to 0,\quad \text{for some } m\in\mathfrak{K}_\ell. \tag{1-13}$$

More exactly, when Ω is a ball of radius $R>0$, Theorem 1.3 reads as follows.

Corollary 1.5. *Let $\Omega=B_R$ be a ball of radius $R>0$ and $f\in\mathbb{RV}_q$ with $q>k$. If (1-13) holds, then every strictly k-convex blow up solution u of (1-1) satisfies*

$$u(x)\sim\xi\phi(d(x))\quad \textit{as } d(x)\to 0, \tag{1-14}$$

where ϕ is defined by (1-10) and ξ is given by

$$\xi=\left(\frac{\big((q-k)/(k+1)\big)^{k+1}R^{k-1}}{1+\ell(q-k)/(k+1)}\right)^{1/(k-q)}. \tag{1-15}$$

Under slightly more restrictive conditions than those in Theorem 1.3, there is at most one strictly k-convex blow-up solution of (1-1).

Theorem 1.6. *Let Ω be a smooth, strictly $(k-1)$-convex, bounded domain in $\mathbb{R}^n$. Suppose $f\in\mathbb{RV}_q$ with $q>k$, and $f(u)/u^k$ is increasing on $(0,\infty)$. Then, (1-1) has at most one strictly k-convex blow-up solution, provided that either*

(i) *b is positive on $\overline{\Omega}$, or*

(ii) *b is zero on $\partial\Omega$, Ω is a ball of radius $R>0$ and (1-13) holds.*

Remark 1.7. When $k=n$ (the Monge–Ampère equation), Theorems 1.3 and 1.6 were obtained in [Cîrstea and Trombetti 2008].

2. Preliminaries

Proposition 2.1. *Let Ω be an open subset of $\mathbb{R}^n$ with $n \geq 2$. If $h \in C^2(\mathbb{R})$ and $g \in C^2(\Omega)$ then*

$$\sigma_k(D^2 h(g(x))) = \big(h'(g(x))\big)^{k-1} h''(x)\sigma_{k-1}(D^2 g|_{i,j}) g_i g_j + \big(h'(g(x))\big)^k \sigma_k(D^2 g), \quad \text{for all } x \in \Omega, \tag{2-1}$$

where $D^2 g|_{i,j}$ is the cofactor of the (i,j)-th entry of the symmetric matrix $D^2 g(x)$.

For $\mu > 0$, we set $\Gamma_\mu = \{x \in \overline{\Omega} : d(x, \partial\Omega) < \mu\}$.

Remark 2.2. If Ω is bounded and $\partial\Omega \in C^l$ for $l \geq 2$, then there exists a positive constant μ depending on Ω such that $d \in C^l(\Gamma_\mu)$. (See also Lemma 14.16 in [Gilbarg and Trudinger 1998].)

Corollary 2.3. *Let Ω be bounded with $\partial\Omega \in C^l$ for $l \geq 2$. Assume that $\mu > 0$ is small such that $d \in C^2(\Gamma_\mu)$ and h is a C^2-function on $(0, \mu)$. Let $x_0 \in \Gamma_\mu \setminus \partial\Omega$ and $y_0 \in \partial\Omega$ be such that $|x_0 - y_0| = d(x_0)$. Then, we have*

$$\sigma_k\big(D^2 h(d(x_0))\big) = \big(-h'(d(x_0))\big)^{k-1} h''(d(x_0))\sigma_{k-1}(\varepsilon_1, \dots, \varepsilon_{n-1}) + \big(-h'(d(x_0))\big)^k \sigma_k(\varepsilon_1, \dots, \varepsilon_{n-1}), \tag{2-2}$$

where $\rho_1(y_0), \dots, \rho_{n-1}(y_0)$ are the principal curvatures of $\partial\Omega$ at y_0 and $\varepsilon_i = \rho_i(y_0)/(1 - \rho_i(y_0)d(x_0))$, $i = 1, \dots, n-1$.

Proof. It is easy to calculate that the expression of the Hessian matrix of d at x_0 in terms of a principal coordinate system at y_0 (see also Lemma 14.17 in [Gilbarg and Trudinger 1998]), namely

$$D^2 d(x_0) = \operatorname{diag}\left(\frac{-\rho_1(y_0)}{1 - \rho_1(y_0)d(x_0)}, \dots, \frac{-\rho_{n-1}(y_0)}{1 - \rho_{n-1}(y_0)d(x_0)}, 0\right),$$
$$Dd(x_0) = (0, \dots, 0, 1).$$

Thus by Proposition 2.1, we obtain

$$\sigma_k\left(D^2 h(d(x_0))\right) = \big(-h'(d(x_0))\big)^{k-1} h''(d(x_0))\sigma_{k-1}\left(\begin{bmatrix} \frac{\rho_1(y_0)}{1-\rho_1(y_0)d(x_0)} & & \\ & \ddots & \\ & & \frac{\rho_{n-1}(y_0)}{1-\rho_{n-1}(y_0)d(x_0)} \end{bmatrix}\right) + \big(-h'(d(x_0))\big)^k \sigma_k\left(\begin{bmatrix} \frac{\rho_1(y_0)}{1-\rho_1(y_0)d(x_0)} & & \\ & \ddots & \\ & & \frac{\rho_{n-1}(y_0)}{1-\rho_{n-1}(y_0)d(x_0)} \end{bmatrix}\right). \qquad \square$$

We now give a brief account of the definitions and properties of regularly varying functions; see also [Resnick 1987; Cîrstea and Trombetti 2008].

Proposition 2.4 (Uniform convergence theorem). *If L is slowly varying, $\frac{L(\lambda u)}{L(u)}$ tends to 1 as $u \to \infty$, uniformly on each compact λ-set in $(0, \infty)$.*

Proposition 2.5. (See also Proposition 4.9 in [Cîrstea and Trombetti 2008].)

(i) *If $R \in \mathbb{RV}_q$, then $\lim_{u\to\infty} \log R(u)/\log u = q$.*

(ii) *If $R_1 \in \mathbb{RV}_{q_1}$ and $R_2 \in \mathbb{RV}_{q_2}$ with $\lim_{u\to\infty} R_2(u) = \infty$, then*

$$R_1 \circ R_2 \in \mathbb{RV}_{q_1 q_2}.$$

(iii) *Suppose R is nondecreasing and $R \in \mathbb{RV}_q$, $0 < q < \infty$. Then*

$$R^{\leftarrow} \in \mathbb{RV}_{q^{-1}}.$$

(iv) *Suppose R_1, R_2 are nondecreasing and q-varying with $q \in (0, \infty)$. Then, for $c \in (0, \infty)$, we have*

$$\lim_{u\to\infty} \frac{R_1(u)}{R_2(u)} = c \quad \text{if and only if} \quad \lim_{u\to\infty} \frac{R_1^{\leftarrow}(u)}{R_2^{\leftarrow}(u)} = c^{-1/q}.$$

Proposition 2.6. (See also Proposition 4.10 in [Cîrstea and Trombetti 2008]). *Let $R \in \mathbb{RV}_q$ and choose $B \geq 0$ so that R is locally bounded on $[B, \infty)$. If $q > 0$, then*

(a) $\sup\{R(y) : B \leq y \leq u\} \sim R(u)$ *as $u \to \infty$,*

(b) $\inf\{R(y) : y \geq u\} \sim R(u)$ *as $u \to \infty$.*

If $q < 0$, then

(c) $\inf\{R(y) : y \geq u\} \sim R(u)$ *as $u \to \infty$,*

(d) $\inf\{R(y) : B \leq y \leq u\} \sim R(u)$ *as $u \to \infty$.*

3. Asymptotic properties of ϕ

Using Karamata's theory of regular variation and its extensions, we now consider the asymptotic properties of the function ϕ defined in (1-10).

Lemma 3.1. *Let $m \in \mathfrak{K}_\ell$ and $f \in \mathbb{RV}_q$ with $q > k$. If ϕ is defined by (1-10), then there exists a function $\psi \in C^2(0, \tau)$ with $\tau > 0$ which satisfies $\lim_{t\to 0} \psi(t)/\phi(t) = 1$ and*

$$\lim_{t\to 0} \frac{\psi(t)\psi''(t)}{(\psi'(t))^2} = 1 + \frac{(q-k)\ell}{k+1}, \tag{3-1}$$

$$\lim_{t\to 0} \frac{(-\psi'(t))^{k-1}\psi''(t)}{m^{k+1}(t) f(\psi(t))} = \left(\frac{k+1}{q-k}\right)^{k+1} \left(1 + \frac{(q-k)\ell}{k+1}\right). \tag{3-2}$$

where ℓ appears in (1-3).

Proof. To prove (3-1), denote $g(u) = f(u)/u^k$. Since $g \in \mathbb{RV}_{q-k}$ and $q > k$, by Proposition 2.6 we have $\lim_{u\to\infty} g(u)/\mathcal{P}(u) = 1$. By Remark 4.8 in [Cîrstea and Trombetti 2008] we infer that there exists a function $\hat{g} \in C^2(0, \tau)$ such that $\lim_{u\to\infty} \hat{g}(u)/g(u) = 1$ and

$$\text{(3-3)} \qquad \lim_{u\to\infty} \frac{u\hat{g}'(u)}{\hat{g}(u)} = q-k, \qquad \lim_{u\to\infty} \frac{u\hat{g}''(u)}{\hat{g}'(u)} = q-k-1,$$

where we have used $g \in \mathbb{RV}_{q-k}$.

We define ψ by

$$\text{(3-4)} \qquad \hat{g}(\psi(t)) = \left(\int_0^t m(s)\,ds\right)^{-k-1}, \quad \text{for } t > 0 \text{ small.}$$

Notice that

$$\text{(3-5)} \qquad \phi(t) = \mathcal{P}^{\leftarrow}\left(\left(\int_0^t m(s)\,ds\right)^{-k-1}\right), \quad \text{for } t > 0 \text{ small.}$$

Thus Proposition 2.5 gives

$$\lim_{t\to 0} \frac{\hat{g}^{\leftarrow}\left(\left(\int_0^t m(s)\,ds\right)^{-k-1}\right)}{\mathcal{P}^{\leftarrow}\left(\left(\int_0^t m(s)\,ds\right)^{-k-1}\right)} = \lim_{t\to 0} \frac{\hat{g}\left(\left(\int_0^t m(s)\,ds\right)^{-k-1}\right)}{\mathcal{P}\left(\left(\int_0^t m(s)\,ds\right)^{-k-1}\right)} = 1,$$

where we have used $\lim_{u\to\infty} g(u)/\mathcal{P}(u) = 1$ and $\lim_{u\to\infty} \hat{g}(u)/g(u) = 1$ in the last equality.

By the definition of the inverse of $\hat{g}$ we see that

$$\text{(3-6)} \qquad \lim_{t\to 0} \frac{\psi(t)}{\phi(t)} = \lim_{t\to 0} \frac{\hat{g}^{\leftarrow}\left(\left(\int_0^t m(s)\,ds\right)^{-k-1}\right)}{\mathcal{P}^{\leftarrow}\left(\left(\int_0^t m(s)\,ds\right)^{-k-1}\right)} = 1.$$

By differentiating (3-4) we obtain

$$\text{(3-7)} \qquad \hat{g}'(\psi(t))\psi'(t) = -(k+1)\left(\int_0^t m(s)\,ds\right)^{-k-2} m(t), \quad \text{for } t > 0 \text{ small.}$$

Then, by (3-3), (3-4) and (3-7),

$$\text{(3-8)} \qquad \frac{\psi'(t)}{\psi(t)} \sim \frac{-(k+1)}{q-k} \frac{m(t)}{\int_0^t m(s)\,ds}, \quad \text{as } t \to 0.$$

We differentiate (3-7), then use (1-3) and (3-3) to deduce that as $t \to 0$

$$\text{(3-9)} \qquad \hat{g}'(\psi(t)) \frac{(\psi'(t))^2}{\psi(t)} \left(q-k-1+\frac{\psi(t)\psi''(t)}{(\psi'(t))^2}\right) \sim (k+1)(k+1+\ell)m^2(s)\left(\int_0^t m(s)\,ds\right)^{-k-3}.$$

Putting (3-7) and (3-8) into (3-9), we have

$$-(k+1)\Big(\int_0^t m(s)\,ds\Big)^{-k-2} m(t)\frac{-(k+1)}{q-k}\frac{m(t)}{\int_0^t m(s)\,ds}\Big(q-k-1+\frac{\psi(t)\psi''(t)}{(\psi'(t))^2}\Big)$$

$$(3\text{-}10)\qquad =\frac{(k+1)^2}{q-k}m^2(t)\Big(\int_0^t m(s)\,ds\Big)^{-k-3}\Big(q-k-1+\frac{\psi(t)\psi''(t)}{(\psi'(t))^2}\Big)$$

$$\sim (k+1)(k+1+\ell)m^2(s)\Big(\int_0^t m(s)\,ds\Big)^{-k-3}.$$

Thus,

$$(3\text{-}11)\qquad \frac{(k+1)}{q-k}\Big(q-k-1+\frac{\psi(t)\psi''(t)}{(\psi'(t))^2}\Big)\sim(k+1+\ell).$$

(3-1) now follows from (3-11).

From (3-4) and (3-8), we find

$$(3\text{-}12)\qquad \lim_{t\to 0}\Big(-\frac{\psi'(t)}{\psi(t)}\Big)^{k+1}\frac{1}{m^{k+1}(t)\hat{g}(\psi(t))}=\Big(\frac{k+1}{q-k}\Big)^{k+1}.$$

This, combined with (3-1), proves (3-2). □

4. Proof of Theorem 1.3

Fix $\epsilon\in(0,1/2)$ and choose $\delta>0$ small enough such that:

(a) m is nondecreasing on $(0,2\delta)$.

(b) $\beta^-(1-\epsilon)\big(m(d(x))\big)^{k+1}\le b(x)\le\beta^+(1+\epsilon)\big(m(d(x))\big)^{k+1}$, for every $x\in\Omega_{2\delta}$, where for $\lambda>0$ we set

$$\Omega_\lambda=\{x\in\Omega: d(x)<\lambda\}.$$

(c) $d(x)$ is a C^2 function on $\Gamma_{2\delta}=\{x\in\overline{\Omega}: d(x)<2\delta\}$.

(d) $0<\psi$, $\psi'<0$, and $\psi''>0$ on $(0,2\delta)$, where ψ is as in Lemma 3.1.

(e) $\sigma_{k-1}(\mathrm{diag}(1-\rho_1(y)d(x),\dots,1-\rho_{n-1}(y)d(x)))>1-\varepsilon$, for every $x\in\Omega_{2\delta}$. Recall that $\rho_i(y)$, $i=1,\dots,n-1$, denote the principal curvatures of $\partial\Omega$ at y, where $y\in\partial\Omega$ is such that $|x-y|=d(x)$.

Fix $\tau\in(0,\delta)$. With $\xi^\pm$ given by (1-11), we set

$$(4\text{-}1)\qquad \eta^\pm=\big((1\mp\varepsilon)(1\mp 2\varepsilon)\big)^{1/(k-q)}\xi^\pm.$$

Define

$$(4\text{-}2)\qquad \begin{cases} v_\tau^+=\eta^+\psi((1-e^{-T(d(x)-\tau)})/T), & x\in\Omega_{2\delta}\setminus\overline{\Omega}_\tau,\\ v_\tau^-=\eta^-\psi((1-e^{-T(d(x)+\tau)})/T), & x\in\Omega_{2\delta-\tau}.\end{cases}$$

Step 1. *We prove that, near the boundary,* v_τ^+*(respectively,* v_τ^-*) is an upper (respectively, lower) solution of* (1-1), *that is,*

(4-3) $$\begin{cases} H_k[D^2 v_\tau^+] \le b(x) f(v_\tau^+), & x \in \Omega_{2\delta} \setminus \overline{\Omega}_\tau, \\ H_k[D^2 v_\tau^-] \ge b(x) f(v_\tau^-), & x \in \Omega_{2\delta-\tau}. \end{cases}$$

We denote by

(4-4) $$M^+ = \max_{y \in \partial\Omega} \sigma_{k-1}(y) \quad \text{and} \quad M^- = \min_{y \in \partial\Omega} \sigma_{k-1}(y).$$

After some computations we obtain, for a point $x \in \Omega_{2\delta} \setminus \overline{\Omega}_\tau$,

$$[v_\tau^+]_{ij} = \eta^+ e^{-T(d(x)-\tau)} \big(\psi' d_{ij} + d_i d_j (\psi'' e^{-T(d(x)-\tau)} - T\psi')\big).$$

Since $|Dd(x)| = 1$ in $x \in \Omega_{2\delta} \setminus \overline{\Omega}_\tau$, we can choose a coordinate system such that

$$Dd(x) = (0, \dots, 0, 1),$$
$$D^2 d(x) = \operatorname{diag}\big(d_{11}(x), \dots, d_{n-1,n-1}(x), 0\big),$$

where $d_{ii}(x) = -\rho_i(y)/(1 - \rho_i(y) d(x))$, and $y \in \partial\Omega$ is such that $|x - y| = d(x)$ as in Corollary 2.3.

Hence

$$D^2 v_\tau^+ = \eta^+ e^{-T(d(x)-\tau)} \operatorname{diag}\big(\psi' d_{11}(x), \dots, \psi' d_{n-1,n-1}(x), \psi'' e^{-T(d(x)-\tau)} - T\psi'\big).$$

Using this and Corollary 2.3, we can easily compute the k-Hessian of v_τ^+:

(4-5) $$H_k[D^2 v_\tau^+] = (\eta^+)^k e^{-(k+1)T(d(x)-\tau)} [-\psi']^{k-1} \psi'' \sigma_{k-1}(-D^2 d(x)) + (\eta^+)^k e^{-kT(d(x)-\tau)} [-\psi']^k \left(T\sigma_{k-1}(-D^2 d(x)) + \sigma_k(-D^2 d(x))\right).$$

Now, if

$$T_1 \le -\frac{\max_{\Omega_{2\delta} \setminus \overline{\Omega}_\tau} |\sigma_k(D^2 d(x))|}{\min_{\Omega_{2\delta} \setminus \overline{\Omega}_\tau} \sigma_{k-1}(-D^2 d(x))},$$

then (4-5) and condition (e) yield for $T \le T_1$,

$$\begin{aligned} H_k[D^2 v_\tau^+] &\le (\eta^+)^k e^{-(k+1)T(d(x)-\tau)} [-\psi']^{k-1} \psi'' \sigma_{k-1}(-D^2 d(x)), \\ &\le \frac{(\eta^+)^k}{1-\varepsilon} M^+ e^{-(k+1)T(d(x)-\tau)} [-\psi']^{k-1} \psi'', \ x \in \Omega_{2\delta} \setminus \overline{\Omega}_\tau. \end{aligned}$$

Similarly, we have for T_2

$$T_2 \ge \frac{\max_{\Omega_{2\delta-\tau}} |\sigma_k(D^2 d(x))|}{\min_{\Omega_{2\delta-\tau}} \sigma_{k-1}(-D^2 d(x))},$$

for $T \geq T_2$,

$$H_k[D^2 v_\tau^-] \geq (\eta^-)^k e^{-(k+1)T(d(x)+\tau)}[-\psi']^{k-1}\psi''\sigma_{k-1}(-D^2 d(x)),$$
$$\geq \frac{(\eta^-)^k}{1+\varepsilon} M^- e^{-(k+1)T(d(x)+\tau)}[-\psi']^{k-1}\psi'', \quad x \in \Omega_{2\delta-\tau}.$$

Therefore, to deduce (4-3) it is enough to establish that

$$\lim_{t\to 0}(\eta^{\pm})^k \frac{M^{\pm}}{\beta^{\mp}} \frac{[-\psi'(t)]^{k-1}\psi''(t)}{m^{k+1}(t) f(\eta^{\pm}\psi(t))} = (1\mp\varepsilon)(1\mp\varepsilon). \tag{4-6}$$

Since $f \in \mathbb{RV}_q$, Lemma 3.1 and our choice of $\eta^{\pm}$ in (4-1),

$$\lim_{t\to 0}(\eta^{\pm})^k \frac{M^{\pm}}{\beta^{\mp}} \frac{[-\psi'(t)]^{k-1}\psi''(t)}{m^{k+1}(t) f(\eta^{\pm}\psi(t))} = (\eta^{\pm})^k \frac{M^{\pm}}{\beta^{\mp}}\Big(\frac{k+1}{q-k}\Big)^{k+1}\Big(1+\frac{(q-k)\ell}{k+1}\Big)(\eta^{\pm})^{-q}$$
$$= \big((1\mp\varepsilon)(1\mp 2\varepsilon)\big)\xi^{\pm(k-q)}\frac{M^{\pm}}{\beta^{\mp}}\Big(\frac{k+1}{q-k}\Big)^{k+1}\Big(1+\frac{(q-k)\ell}{k+1}\Big) = (1\mp\varepsilon)(1\mp 2\varepsilon),$$

where we have used (1-11) in the last equality.

Step 2. *Every strictly k-convex blow-up solution u of (1-1) satisfies (1-9).*

Let $C = \max_{d(x)=\delta} u(x)$. Notice that

$$\begin{cases} v_\tau^+ + C = \infty > u(x), & x \in \Omega \text{ with } d(x) = \tau, \\ v_\tau^+ + C \geq u(x), & x \in \Omega \text{ with } d(x) = \delta. \end{cases} \tag{4-7}$$

Using (4-3) we deduce that for every $x \in \Omega_\delta \setminus \overline{\Omega}_\tau$,

$$H_k[D^2(v_\tau^+ + C)] = H_k[D^2 v_\tau^+] \leq b(x) f(v_\tau^+) \leq b(x) f(v_\tau^+ + C).$$

Since u is a solution to (1-1), by the comparison principle for k-Hessians [Jian 2006, Lemma 2.1] we find

$$v_\tau^+ + C \geq u(x), \quad \text{for all } x \in \Omega_\delta \setminus \overline{\Omega}_\tau. \tag{4-8}$$

We set $C' = \xi^- \psi(\delta)$. Hence, we have $C' \geq v_\tau^-(x)$ for every $x \in \Omega$ with $d(x) = \delta - \tau$. It follows that

$$u(x) + C' \geq v_\tau^-(x), \quad \text{for all } x \in \partial\Omega_{\delta-\tau}. \tag{4-9}$$

We see that, for every $x \in \Omega_{\delta-\tau}$,

$$H_k[u(x) + C'] = H_k[D^2 u(x)] \leq b(x) f(u(x)) \leq b(x) f(u(x) + C'),$$

while by (4-3) we have

$$H_k[D^2 v_\tau^-] \geq b(x) f(v_\tau^-), \quad x \in \Omega_{\delta-\tau}. \tag{4-10}$$

Using again the comparison principle for k-Hessian equations, we infer that

$$u(x)+C' \geq v_\tau^-(x), \quad \text{for all } x \in \Omega_{\delta-\tau}. \tag{4-11}$$

By (4-8) and (4-11), letting $\tau \to 0$ we obtain

$$\begin{cases} \big((1+\epsilon)(1+2\epsilon)\big)^{1/(k-q)}\xi^- \psi((1-e^{-T_2 d(x)})/T_2)-C' \leq u(x), \;\; x \in \Omega_\delta, \\ u(x) \leq \big((1-\epsilon)(1-2\epsilon)\big)^{1/(k-q)}\xi^+ \psi((1-e^{-T_1 d(x)})/T_1)+C. \end{cases} \tag{4-12}$$

Dividing by $\psi((1-e^{-T_i d(x)})/T_i)$ for $i=1,2$ and noticing that $\lim_{t\to 0}\psi(t)/\phi(t)=1$, letting $d(x)\to 0$, we obtain

$$\begin{cases} \liminf_{d(x)\to 0} u/\phi(d(x)) \geq \big((1+\epsilon)(1+2\epsilon)\big)^{1/(k-q)}\xi^-, \\ \liminf_{d(x)\to 0} u/\phi(d(x)) \leq \big((1-\epsilon)(1-2\epsilon)\big)^{1/(k-q)}\xi^+. \end{cases} \tag{4-13}$$

Since $\epsilon > 0$ is arbitrary, we let $\epsilon \to 0$ and obtain (1-9). This completes the proof of Theorem 1.3.

5. Proof of Theorem 1.6

We follow the methods in [Cîrstea and Trombetti 2008] and divide the proof into two steps:

Step 1. *For all strictly k-convex blow-up solutions u_1, u_2 of (1-1),*

$$\lim_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} = 1. \tag{5-1}$$

Step 2. *There is at most one strictly convex blow-up solution of (1-1).*

Proof of Step 1. The argument breaks into two cases.

Case (i): $b > 0$ on $\overline{\Omega}$. Since u_1 and u_2 are arbitrary, it suffices to show that

$$\liminf_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} \geq 1. \tag{5-2}$$

Without loss of generality, we can assume that 0 belongs to Ω. Let $\varepsilon \in (0,1)$ be fixed and let $\lambda > 1$ be close to 1.

We set

$$C_\lambda = \Big((1+\varepsilon)\lambda^{2k} \max_{x\in(1/\lambda)\overline{\Omega}} \frac{b(\lambda x)}{b(x)}\Big)^{1/(q-k)}, \tag{5-3}$$

where $(1/\lambda)\overline{\Omega} = \big\{(1/\lambda)x : x \in \overline{\Omega}\big\}$. Notice that $C_\lambda \to (1+\varepsilon)^{1/(q-k)}$ as $\lambda \to 1$.

Hence, by Proposition 2.4 and $\lim_{d(x)\to 0} u_1(x) = \infty$, we deduce that there exists $\delta = \delta(\varepsilon) > 0$, independent of λ, such that

$$C_\lambda^q \frac{f(u_1(x))}{f(C_\lambda u_1(x))} \leq 1+\varepsilon, \quad \text{for all } x \in \Omega_\delta \text{ and } \lambda \in (1, 1+\eta) \text{ for some } \eta. \tag{5-4}$$

We now define U_λ as

$$U_\lambda(x) = C_\lambda u_1(\lambda x), \quad \text{for all } x \in (1/\lambda)\Omega_\delta. \tag{5-5}$$

Notice by (5-3)–(5-5),

$$\begin{aligned} H_k[D^2 U_\lambda(x)] &= \lambda^{2k} C_\lambda^k b(\lambda x) f(u_1(\lambda x)) \\ &\le \lambda^{2k} C_\lambda^{k-q}(1+\varepsilon) b(\lambda x) f(C_\lambda u_1(\lambda x)) \\ &\le b(x) f(C_\lambda u_1(\lambda x)) = b(x) f(U_\lambda(x)), \quad x \in (1/\lambda)\Omega_\delta, \end{aligned} \tag{5-6}$$

which says that $U_\lambda(x)$ is a supersolution of (1-1) with domain $(1/\lambda)\Omega_\delta$.

Since f is increasing on $(0, \infty)$ and (5-6), for each constant $M > 0$,

$$\begin{aligned} H_k[D^2(U_\lambda(x) + M)] &= H_k[D^2 U_\lambda(x)] \le b(x) f(U_\lambda(x)) \\ &\le b(x) f(U_\lambda(x) + M), \quad \text{for all } x \in (1/\lambda)\Omega_\delta. \end{aligned} \tag{5-7}$$

Notice also that $U_\lambda(x) = \infty > u_2(x)$, for every $x \in (1/\lambda)\partial\Omega$. Moreover, $x \in (1/\lambda)\partial\Omega$ implies that $d(x) < \delta$ (as $\lambda > 1$ is close to 1).

Thus, if we choose $M > 0$ large enough (for example, $M = \max_{d(x)=\delta} u_2(x)$), then by the comparison principle for k-Hessian equations we obtain

$$U_\lambda(x) + M \ge u_2(x), \quad \text{for all } x \in \Omega_\delta \cap (1/\lambda)\Omega_\delta. \tag{5-8}$$

Letting $\lambda \to 1$ in (5-8), we find

$$(1+\varepsilon)^{1/(q-k)} u_1(x) + M \ge u_2(x), \quad \text{for all } x \in \Omega_\delta, \tag{5-9}$$

which implies that

$$\liminf_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} \ge (1+\varepsilon)^{1/(k-q)}, \tag{5-10}$$

and then letting $\varepsilon \to 0$ we obtain (5-2).

Case (ii): $b \equiv 0$ on $\partial\Omega$, Ω is a ball of radius $R > 0$, and (1-13) holds. By Corollary 1.5, every strictly k-convex blow-up solution u of (1-1) satisfies

$$\lim_{d(x)\to 0} \frac{u}{\phi(d(x))} = \left(\frac{\big((q-k)/(k+1)\big)^{k+1} R^{k-1}}{1 + \ell(q-k)/(k+1)} \right)^{1/(k-q)}, \tag{5-11}$$

where ϕ is defined by (1-10) and ℓ appears in (1-3). □

Proof of Step 2. If u_1, u_2 are arbitrary strictly k-convex blow-up solutions of (1-1), it suffices to show that $u_1 \le u_2$ in Ω. Fix $\varepsilon > 0$. By Step 1 we infer that

$$\lim_{d(x)\to 0} \big(u_1(x) - (1+\varepsilon)u_2(x)\big) = -\infty. \tag{5-12}$$

Since $f(u)/u^k$ is increasing on $(0,\infty)$, we deduce that

$$H_k[D^2(1+\varepsilon)u_2(x)] = (1+\varepsilon)^k H_k[D^2u_2(x)] \le (1+\varepsilon)^k b(x) f(u_2(x)) \le b(x) f((1+\varepsilon)u_2(x)), \quad \text{for all } x \in \Omega. \tag{5-13}$$

By (5-12), (5-13) and the comparison principle for k-Hessian equations,

$$u_1 \le (1+\varepsilon)u_2, \quad \text{for all } x \in \Omega. \tag{5-14}$$

Letting $\varepsilon \to 0$, thus $u_1 \le u_2$ in Ω. This completes the proof of Step 2 and hence of Theorem 1.6. □

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