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A QUOTIENT OF THE BRAID GROUP RELATED TO PSEUDOSYMMETRIC BRAIDED CATEGORIES

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Motivated by the recent concept of a pseudosymmetric braided monoidal category, we define the pseudosymmetric group PS_n to be the quotient of the braid group B_n by the relations $\sigma_i\sigma_{i+1}^{-1}\sigma_i = \sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}$ with $1 \le i \le n-2$. It turns out that PS_n is isomorphic to the quotient of B_n by the commutator subgroup $[P_n, P_n]$ of the pure braid group P_n (which amounts to saying that $[P_n, P_n]$ coincides with the normal subgroup of B_n generated by the elements $[\sigma_i^2, \sigma_{i+1}^2]$ with $1 \le i \le n-2$), and that PS_n is a linear group.

Introduction

A symmetric category consists of a monoidal category $\mathscr{C}$ equipped with a family of natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying natural "bilinearity" conditions together with the symmetry relation $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y}$ for all $X, Y \in \mathscr{C}$. This concept was generalized by Joyal and Street [1993] by dropping this symmetry relation from the axioms and arriving thus at the concept of braided category, of central importance in quantum group theory; see [Kassel 1995; Majid 1995].

Inspired by recently introduced categorical concepts of pure-braided structures [Staic 2004] and twines [Bruguières 2006], Panaite, Staic and Van Oystaeyen [Panaite et al. 2009] defined the concept of pseudosymmetric braiding to generalize symmetric braidings. A braiding c on a strict monoidal category $\mathscr{C}$ is pseudosymmetric if it satisfies the modified braid relation

$$(c_{Y,Z} \otimes \mathrm{id}_X) \circ (\mathrm{id}_Y \otimes c_{Z,X}^{-1}) \circ (c_{X,Y} \otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes c_{X,Y}) \circ (c_{Z,X}^{-1} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,Z})$$

for all $X, Y, Z \in \mathscr{C}$. The main result in [Panaite et al. 2009] asserts that, if H is a Hopf algebra with bijective antipode, then the canonical braiding of the Yetter–Drinfeld category ${}_H\mathscr{YD}^H$ is pseudosymmetric if and only if H is commutative and cocommutative.

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It is well known that, at several levels, braided categories correspond to the braid groups B_n, while symmetric categories correspond to the symmetric groups S_n. It is natural to expect that there exist some groups corresponding, in the same way, to pseudosymmetric braided categories. Indeed, it is clear that these groups, denoted by PS_n and called (naturally) the pseudosymmetric groups, should be the quotients of the braid groups B_n by the relations $\sigma_i \sigma_{i+1}^{-1} \sigma_i = \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}$. Our aim is to study and find more explicitly the structure of these groups. We prove first that the kernel of the canonical group morphism $\mathrm{PS}_n \to S_n$ is abelian, and consequently PS_n is isomorphic to the quotient of B_n by the commutator subgroup $[P_n, P_n]$ of the pure braid group P_n. (This amounts to saying that $[P_n, P_n]$ coincides with the normal subgroup of B_n generated by the elements $[\sigma_i^2, \sigma_{i+1}^2]$ with $1 \le i \le n-2$.)

There exist similarities, but also differences, between braid groups and pseudosymmetric groups. Bigelow [2001] and Krammer [2002] proved that braid groups are linear, and we show that so are pseudosymmetric groups. More precisely, we prove that the Lawrence–Krammer representation of B_n induces a representation of PS_n if the parameter q is chosen to be 1, and that this representation of PS_n is faithful over $\mathbb{R}[t^{\pm 1}]$. On the other hand, although PS_n is an infinite group, like B_n, it does have nontrivial elements of finite order, unlike B_n.

1. Preliminaries

Definition 1.1 [Panaite et al. 2007]. Let $\mathscr{C}$ be a strict monoidal category and let $T_{X,Y} : X \otimes Y \to X \otimes Y$ be a family of natural isomorphisms in $\mathscr{C}$. We call T a *strong twine* if, for all $X, Y, Z \in \mathscr{C}$,

$$T_{I,I} = \mathrm{id}_I, \qquad (T_{X,Y} \otimes \mathrm{id}_Z) \circ T_{X \otimes Y, Z} = (\mathrm{id}_X \otimes T_{Y,Z}) \circ T_{X, Y \otimes Z},$$
$$(T_{X,Y} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_X \otimes T_{Y,Z}) = (\mathrm{id}_X \otimes T_{Y,Z}) \circ (T_{X,Y} \otimes \mathrm{id}_Z).$$

Definition 1.2 [Panaite et al. 2009]. Let $\mathscr{C}$ be a strict monoidal category and c a braiding on $\mathscr{C}$. We say that c is *pseudosymmetric* if, for all $X, Y, Z \in \mathscr{C}$,

$$(1) \quad (c_{Y,Z} \otimes \mathrm{id}_X) \circ (\mathrm{id}_Y \otimes c_{Z,X}^{-1}) \circ (c_{X,Y} \otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes c_{X,Y}) \circ (c_{Z,X}^{-1} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,Z}).$$

In this case we say that $\mathscr{C}$ is a *pseudosymmetric braided category*.

The next proposition, a key result in [Panaite et al. 2009], led to the introduction of the concept of pseudosymmetric braiding. Here, it will serve as a source of inspiration for a certain key result for braids, Proposition 2.1.

Proposition 1.3 [Panaite et al. 2009]. *Let $\mathscr{C}$ be a strict monoidal category and c a braiding on $\mathscr{C}$. Then the double braiding $T_{X,Y} := c_{Y,X} \circ c_{X,Y}$ is a strong twine if and only if c is pseudosymmetric.*

2. Defining relations for PS_n

Let $n \geq 3$ be a natural number. We denote by B_n the braid group on n strands, with its usual presentation by generators σ_i with $1 \leq i \leq n-1$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i-j| \geq 2, \tag{2}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{if } 1 \leq i \leq n-2. \tag{3}$$

We begin with the analogue for braids of Proposition 1.3:

Proposition 2.1. *For all* $1 \leq i \leq n-2$, *the relations*

$$\sigma_i \sigma_{i+1}^{-1} \sigma_i = \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}, \tag{4}$$

$$\sigma_i^2 \sigma_{i+1}^2 = \sigma_{i+1}^2 \sigma_i^2 \tag{5}$$

are equivalent in B_n.

Proof. We show first that (4) implies (5):

$$\begin{aligned} \sigma_i^2 \sigma_{i+1}^2 &= \sigma_i \sigma_{i+1}^{-1} \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_{i+1} \\ &\overset{(3)}{=} \sigma_i \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1} \overset{(3),(4)}{=} \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_i \\ &\qquad\qquad\qquad\qquad\qquad\quad \overset{(3)}{=} \sigma_{i+1} \sigma_i^{-1} \sigma_i \sigma_{i+1} \sigma_i \sigma_i = \sigma_{i+1}^2 \sigma_i^2. \end{aligned}$$

Conversely, we prove that (5) implies (4):

$$\begin{aligned} \sigma_i \sigma_{i+1}^{-1} \sigma_i &= \sigma_i \sigma_{i+1}^{-2} \sigma_i^{-1} \sigma_i \sigma_{i+1} \sigma_i \\ &\overset{(3)}{=} \sigma_i \sigma_{i+1}^{-2} \sigma_i^{-1} \sigma_{i+1} \sigma_i \sigma_{i+1} \\ &= \sigma_i \sigma_{i+1}^{-2} \sigma_i^{-2} \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1} \overset{(3),(5)}{=} \sigma_i \sigma_i^{-2} \sigma_{i+1}^{-2} \sigma_{i+1} \sigma_i \sigma_{i+1}^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1}^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \sigma_{i+1} \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1}^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \overset{(3)}{=} \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_i \sigma_{i+1}^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}. \end{aligned}$$ □

Definition 2.2. For a natural number $n \geq 3$, we define the *pseudosymmetric group* PS_n as the group with generators σ_i for $1 \leq i \leq n-1$, and relations (2), (3) and (4), or equivalently (2), (3) and (5).

Proposition 2.3. *For* $1 \leq i \leq n-2$, *consider the elements*

$$p_i := \sigma_i \sigma_{i+1}^{-1} \quad \textit{and} \quad q_i := \sigma_i^{-1} \sigma_{i+1} \tag{6}$$

in PS_n. *Then, in* PS_n, *we have*

$$p_i^3 = q_i^3 = (p_i q_i)^3 = 1 \quad \textit{for all } 1 \leq i \leq n-2. \tag{7}$$

Proof. The relations $p_i^3 = 1$ and $q_i^3 = 1$ follow immediately from (4); actually each of them is equivalent to (4). Now we compute

$$\begin{aligned}(p_iq_i)^2 &= (\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1})^2\\ &= \sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}\\ &= \sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_{i+1}^{-2}\sigma_i^{-1}\sigma_{i+1}\\ &\overset{(3)}{=} \sigma_i^2\sigma_{i+1}^{-2}\sigma_i^{-1}\sigma_{i+1} \overset{(5)}{=} \sigma_{i+1}^{-2}\sigma_i\sigma_{i+1}\\ &\qquad\qquad\qquad\qquad\quad = \sigma_{i+1}^{-2}\sigma_i\sigma_{i+1}\sigma_i\sigma_i^{-1}\\ &\qquad\qquad\qquad\qquad\quad \overset{(3)}{=} \sigma_{i+1}^{-1}\sigma_i\sigma_{i+1}\sigma_i^{-1} = (p_iq_i)^{-1},\end{aligned}$$

and so $(p_iq_i)^3 = 1$. □

Consider now the symmetric group S_n with its usual presentation by generators s_i with $1 \le i \le n-1$ and relations (2), (3) and $s_i^2 = 1$ for all $1 \le i \le n-1$. We denote by $\pi : B_n \to S_n$, $\beta : B_n \to \mathrm{PS}_n$ and $\alpha : \mathrm{PS}_n \to S_n$ the canonical surjective group homomorphisms given by $\pi(\sigma_i) = s_i$, $\alpha(\sigma_i) = s_i$ and $\beta(\sigma_i) = \sigma_i$ for all $1 \le i \le n-1$. Obviously we have $\pi = \alpha \circ \beta$; hence in particular we obtain $\mathrm{Ker}(\alpha) = \beta(\mathrm{Ker}(\pi))$. We denote as usual $\mathrm{Ker}(\pi) = P_n$, the pure braid group on n strands. It is well known (see [Kassel and Turaev 2008, page 21]) that P_n is generated by the elements

$$a_{ij} := \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1} \quad \text{for } 1 \le i < j \le n \tag{8}$$

that satisfy certain relations, of which we will use only one, namely, that for $1 \le i < j \le n$ and $1 \le r < s \le n$,

$$a_{ij}a_{rs} = a_{rs}a_{ij} \quad \text{if } s < i \ \text{ or } \ i < r < s < j. \tag{9}$$

Alternatively, P_n is generated by the elements

$$b_{ij} := \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1} \quad \text{for } 1 \le i < j \le n. \tag{10}$$

It is easy to see that in B_n we have

$$\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} = \sigma_i^{-1}\sigma_{i+1}^2\sigma_i \quad \text{and} \quad \sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1} = \sigma_i\sigma_{i+1}^2\sigma_i^{-1}, \tag{11}$$

and by using repeatedly these relations we obtain an equivalent description of the elements a_{ij} and b_{ij}:

$$a_{ij} = \sigma_i^{-1}\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^2\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i \quad \text{for } 1 \le i < j \le n, \tag{12}$$

$$b_{ij} = \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^{-1} \quad \text{for } 1 \le i < j \le n. \tag{13}$$

Now, for all $1 \le i < j \le n$, we define $A_{i,j}$ and $B_{i,j}$ as the elements in PS_n given by $A_{i,j} := \beta(a_{ij})$ and $B_{i,j} := \beta(b_{ij})$. From the discussion above it follows that $\mathrm{Ker}(\alpha)$ is generated by $\{A_{i,j}\}_{1\le i<j\le n}$ and also by $\{B_{i,j}\}_{1\le i<j\le n}$.

Lemma 2.4. *The following relations hold in* PS_n *for* $1 \le i < j < n$:

$$A_{i,j+1} = \sigma_j A_{i,j} \sigma_j^{-1}, \tag{14}$$

$$B_{i,j+1} = \sigma_j^{-1} B_{i,j} \sigma_j. \tag{15}$$

Proof. These relations are consequences of corresponding relations in B_n for the a_{ij} and b_{ij}, which in turn follow immediately from (8) and (10). □

Lemma 2.5. *For all* $i, j \in \{1, 2, \ldots, n\}$ *with* $i + 1 < j$, *we have in* PS_n

$$A_{i,j} = \sigma_i A_{i+1,j} \sigma_i^{-1}, \tag{16}$$

$$B_{i,j} = \sigma_i^{-1} B_{i+1,j} \sigma_i. \tag{17}$$

Proof. We prove (16), while (17) is similar and left to the reader. Note that in PS_n we have $\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1} = \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}$, which together with the second of (11) implies $\sigma_i\sigma_{i+1}^2\sigma_i^{-1} = \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}$; hence

$$\begin{aligned} A_{i,j} &= \sigma_{j-1}\sigma_{j-2}\cdots(\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1})\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1} \\ &= \sigma_{j-1}\sigma_{j-2}\cdots(\sigma_i\sigma_{i+1}^2\sigma_i^{-1})\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1} \\ &= \sigma_i\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}^2\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}\sigma_i^{-1} = \sigma_i A_{i+1,j}\sigma_i^{-1}. \end{aligned}$$ □

Proposition 2.6. *For all* $1 \le i < j \le n$, *we have* $A_{i,j} = B_{i,j}$ *in* PS_n.

Proof. We use (16) repeatedly:

$$\begin{aligned} A_{i,j} = \sigma_i A_{i+1,j}\sigma_i^{-1} &= \sigma_i\sigma_{i+1}A_{i+2,j}\sigma_{i+1}^{-1}\sigma_i^{-1} \\ &\cdots \\ &= \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}A_{j-1,j}\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^{-1} \\ &= \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^{-1} \overset{(13)}{=} B_{i,j}. \end{aligned}$$ □

Lemma 2.7. *For all* $1 \le i < j \le n$ *and* $1 \le h \le k < n$, *we have in* PS_n

$$A_{i,j}\sigma_i^2 = \sigma_i^2 A_{i,j}, \tag{18}$$

$$A_{h,k+1}\sigma_k^2 = \sigma_k^2 A_{h,k+1}. \tag{19}$$

Proof. Note first that (18) is obvious for $j = i + 1$. Assume that $i + 1 < j$; using the fact that $A_{r,s} = B_{r,s}$ for all r, s, we compute

$$A_{i,j}\sigma_i^2 \overset{(16)}{=} \sigma_i A_{i+1,j}\sigma_i = \sigma_i B_{i+1,j}\sigma_i \overset{(17)}{=} \sigma_i^2 B_{i,j} = \sigma_i^2 A_{i,j}.$$

Note also that (19) is obvious for $h=k$. Assume that $h<k$; using again $A_{r,s}=B_{r,s}$ for all r, s, we compute

$$A_{h,k+1}\sigma_k^2 \overset{(14)}{=} \sigma_k A_{h,k}\sigma_k = \sigma_k B_{h,k}\sigma_k \overset{(15)}{=} \sigma_k^2 B_{h,k+1} = \sigma_k^2 A_{h,k+1}. \qquad \square$$

3. The structure of PS_n

We denote by $\mathfrak{P}_n$ the kernel of the morphism $\alpha : \mathrm{PS}_n \to S_n$ defined above.

Proposition 3.1. *$\mathfrak{P}_n$ is an abelian group.*

Proof. It is enough to prove that any two elements $A_{i,j}$ and $A_{k,l}$ commute in PS_n. We only have to analyze the following seven cases for the numbers i, j, k, l:

(i) $i<j<k<l$. This is an obvious consequence of (9).

(ii) $i<j=k<l$. We write

$$A_{i,j} = \sigma_i^{-1}\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^2\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i,$$
$$A_{j,l} = \sigma_{l-1}\sigma_{l-2}\cdots\sigma_{j+1}\sigma_j^2\sigma_{j+1}^{-1}\cdots\sigma_{l-2}^{-1}\sigma_{l-1}^{-1},$$

and we obtain $A_{i,j}A_{j,l}=A_{j,l}A_{i,j}$ by using (2) and the fact that σ_{j-1}^2 and σ_j^2 commute in PS_n.

(iii) $i<k<j<l$. This follows since $A_{k,l}=B_{k,l}$ in PS_n (Proposition 2.6), and a_{ij} and b_{kl} commute in P_n if $i<k<j<l$, which is easily seen geometrically.

(iv) $i=k<j=l$. This is trivial.

(v) $i<k<l<j$. This is an obvious consequence of (9).

(vi) $i=k<j<l$. In case $j=i+1$, we have $A_{i,j}=\sigma_i^2$ and so we obtain $A_{i,j}A_{i,l}=A_{i,l}A_{i,j}$ by using (18); assuming now $i+1<j$, by using repeatedly (16) we can compute

$$\begin{aligned} A_{i,j}A_{i,l} &= \sigma_i A_{i+1,j}A_{i+1,l}\sigma_i^{-1} \\ &= \sigma_i\sigma_{i+1}A_{i+2,j}A_{i+2,l}\sigma_{i+1}^{-1}\sigma_i^{-1} \\ &\cdots \\ &= \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}A_{j-1,j}A_{j-1,l}\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^{-1}, \end{aligned}$$

and similarly

$$A_{i,l}A_{i,j} = \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}A_{j-1,l}A_{j-1,j}\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^{-1};$$

these are equal since $A_{j-1,j}=\sigma_{j-1}^2$ and by (18), $\sigma_{j-1}^2A_{j-1,l}=A_{j-1,l}\sigma_{j-1}^2$.

(vii) $i < k < j = l$. In case $j = k+1$, we have $A_{k,j} = \sigma_k^2$ and so we obtain $A_{i,j}A_{k,j} = A_{k,j}A_{i,j}$ by using (19); assuming now $k+1 < j$, by repeatedly using (14) we can compute

$$\begin{aligned} A_{i,j}A_{k,j} &= \sigma_{j-1}A_{i,j-1}A_{k,j-1}\sigma_{j-1}^{-1} \\ &= \sigma_{j-1}\sigma_{j-2}A_{i,j-2}A_{k,j-2}\sigma_{j-2}^{-1}\sigma_{j-1}^{-1} \\ &\cdots \\ &= \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{k+1}A_{i,k+1}A_{k,k+1}\sigma_{k+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, \end{aligned}$$

and similarly

$$A_{k,j}A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{k+1}A_{k,k+1}A_{i,k+1}\sigma_{k+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1};$$

these are equal since $A_{k,k+1} = \sigma_k^2$ and by (19), $A_{i,k+1}\sigma_k^2 = \sigma_k^2 A_{i,k+1}$. □

Let G be a group. If $x, y \in G$ we denote by $[x, y] := x^{-1}y^{-1}xy$ the commutator of x and y, and by G' the commutator subgroup of G (the subgroup of G generated by all commutators $[x, y]$), which is the smallest normal subgroup N of G with the property that G/N is abelian. Moreover, G' is a characteristic subgroup of G, that is, $\theta(G') = G'$ for all $\theta \in \mathrm{Aut}(G)$.

Proposition 3.2. $\mathfrak{P}_n \simeq P_n/P_n' \simeq \mathbb{Z}^{n(n-1)/2}$.

Proof. For $1 \le i \le n-2$ we define $t_i \in P_n$ by $t_i := [\sigma_i^2, \sigma_{i+1}^2] = [a_{i,i+1}, a_{i+1,i+2}]$. These elements are the relators added to the ones of B_n in order to obtain PS_n; therefore, as a particular case of a general fact about groups given by generators and relations (see for instance [Coxeter and Moser 1972, page 2]), the kernel of the map $\beta : B_n \to \mathrm{PS}_n$ defined above coincides with the normal subgroup of B_n generated by $\{t_i\}_{1\le i\le n-2}$, which will be denoted by L_n. We obviously have $L_n \subseteq P_n$, and if we consider the map β restricted to P_n, we have a surjective morphism $P_n \to \mathfrak{P}_n$ with kernel L_n, so $\mathfrak{P}_n \simeq P_n/L_n$. By Proposition 3.1 we know that $\mathfrak{P}_n$ is abelian, so we obtain $P_n' \subseteq L_n$. On the other hand, since P_n' is characteristic in P_n and P_n is normal in B_n, it follows (see [Suzuki 1982, Proposition 6.14]) that P_n' is normal in B_n, and since $t_1, \dots, t_{n-2} \in P_n'$ and L_n is the normal subgroup of B_n generated by $\{t_i\}_{1\le i\le n-2}$, we obtain $L_n \subseteq P_n'$. Thus, we have obtained $L_n = P_n'$ and so $\mathfrak{P}_n \simeq P_n/P_n'$. On the other hand, it is well known that $P_n/P_n' \simeq \mathbb{Z}^{n(n-1)/2}$; see for instance [Kassel and Turaev 2008, Corollary 1.20]. □

As a consequence of the equality $L_n = P_n'$, we obtain B_n/P_n':

Corollary 3.3. $\mathrm{PS}_n \simeq B_n/P_n'$.

The extension with abelian kernel $1 \to \mathfrak{P}_n \to \mathrm{PS}_n \to S_n \to 1$ induces an action of S_n on $\mathfrak{P}_n$, given by $\sigma \cdot a = \tilde{\sigma} a \tilde{\sigma}^{-1}$ for $\sigma \in S_n$ and $a \in \mathfrak{P}_n$, where $\tilde{\sigma}$ is an element of PS_n with $\alpha(\tilde{\sigma}) = \sigma$. In particular, on generators we have $s_k \cdot A_{i,j} = \sigma_k A_{i,j} \sigma_k^{-1}$, for $1 \le k \le n-1$ and $1 \le i < j \le n$. By using some of the formulas given above, one can describe explicitly this action as

$$s_k \cdot A_{i,j} = A_{i,j} \qquad \text{if } k < i-1, \tag{20a}$$

$$s_{i-1} \cdot A_{i,j} = A_{i-1,j}, \tag{20b}$$

$$s_i \cdot A_{i,j} = A_{i+1,j} \qquad \text{if } j-i > 1 \text{ and } s_i \cdot A_{i,i+1} = A_{i,i+1}, \tag{20c}$$

$$s_k \cdot A_{i,j} = A_{i,j} \qquad \text{if } i < k < j-1, \tag{20d}$$

$$s_{j-1} \cdot A_{i,j} = A_{i,j-1} \qquad \text{if } j-i > 1 \text{ and } s_{j-1} \cdot A_{j-1,j} = A_{j-1,j}, \tag{20e}$$

$$s_j \cdot A_{ij} = A_{i,j+1} \qquad \text{for } 1 \le i < j < n, \tag{20f}$$

$$s_k \cdot A_{i,j} = A_{i,j} \qquad \text{if } j < k. \tag{20g}$$

Note that the first equality in (20c) follows by using (17) together with the fact that $A_{i,j} = B_{i,j}$ (Proposition 2.6), and the first equality in (20e) follows by an easy computation using also the fact that $A_{i,j} = B_{i,j}$. Also, one can easily see that these formulas may be expressed more compactly as follows: If $\sigma \in \{s_1, \ldots, s_{n-1}\}$ and $1 \le i < j \le n$, then $\sigma \cdot A_{i,j} = A_{\sigma(i),\sigma(j)}$, where we made the convention $A_{r,t} := A_{t,r}$ for $t < r$. Since $s_1, \ldots, s_{n-1}$ generate S_n, we have found the action of S_n on $A_{i,j}$:

Proposition 3.4. *For any $\sigma \in S_n$ and $1 \le i < j \le n$, the action of σ on $A_{i,j}$ is given by $\sigma \cdot A_{i,j} = A_{\sigma(i),\sigma(j)}$, with the convention $A_{r,t} := A_{t,r}$ for $t < r$.*

Lemma 3.5. *Let F be a free $\mathbb{Z}$-module of rank m, and let $\{X_1, \ldots, X_m\}$ be a generating system for F over $\mathbb{Z}$. Then $\{X_1, \ldots, X_m\}$ is a basis of F over $\mathbb{Z}$.*

Proof. Assume $X_1, \ldots, X_m$ are linearly dependent over $\mathbb{Z}$ and take $\sum_{i=1}^m \alpha_i X_i = 0$ a nontrivial linear combination over $\mathbb{Z}$. Choose a prime number p such that $|\alpha_i| < p$ for all $1 \le i \le m$, and consider $\overline{F} := F/pF$, a linear space over the field $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, and $\overline{X}_i$, the images of the elements X_i in $\overline{F}$. These elements generate $\overline{F}$ over $\mathbb{Z}_p$, and since the dimension of $\overline{F}$ over $\mathbb{Z}_p$ is m, it follows that $\{\overline{X}_1, \ldots, \overline{X}_m\}$ is a basis of $\overline{F}$ over $\mathbb{Z}_p$. Thus, it follows that $\alpha_i \equiv 0 \pmod p$ for all $1 \le i \le m$, which is a contradiction because we have chosen p so that $|\alpha_i| < p$ for all $1 \le i \le m$. □

Proposition 3.6. *In PS_n, there is no element of order 2 whose image in S_n is the transposition $s_1 = (1, 2)$. Consequently, the extension $1 \to \mathfrak{P}_n \to \mathrm{PS}_n \to S_n \to 1$ is* not *split.*

Proof. Take $x \in \mathrm{PS}_n$ such that $\alpha(x) = s_1$. Since $\alpha(\sigma_1) = s_1$, we obtain that $x\sigma_1^{-1} \in \mathrm{Ker}(\alpha) = \mathfrak{P}_n$. By Proposition 3.2 and Lemma 3.5, it follows that the abelian group $\mathfrak{P}_n$ is freely generated by $\{A_{i,j}\}_{1 \le i < j \le n}$, so we can write uniquely

$x = \prod_{1 \le i < j \le n} A_{i,j}^{m_{ij}} \sigma_1$, with $m_{ij} \in \mathbb{Z}$. We compute

$$\begin{aligned} x^2 &= \Big(\prod_{1 \le i < j \le n} A_{i,j}^{m_{ij}} \sigma_1 \Big) \Big(\prod_{1 \le i < j \le n} A_{i,j}^{m_{ij}} \sigma_1 \Big) \\ &= \Big(\prod_{1 \le i < j \le n} A_{i,j}^{m_{ij}} \Big) \Big(\sigma_1 \prod_{1 \le i < j \le n} A_{i,j}^{m_{ij}} \sigma_1^{-1} \Big) \sigma_1^2 \\ &= \Big(\prod_{1 \le i < j \le n} A_{i,j}^{m_{ij}} \Big) \Big(\prod_{1 \le i < j \le n} \sigma_1 A_{i,j}^{m_{ij}} \sigma_1^{-1} \Big) A_{1,2} \\ &= A_{1,2}^{2m_{12}+1} \Big(\prod_{3 \le j \le n} A_{1,j}^{m_{1j}+m_{2j}} A_{2,j}^{m_{1j}+m_{2j}} \Big) \Big(\prod_{3 \le i < j \le n} A_{i,j}^{2m_{ij}} \Big), \end{aligned}$$

and this element cannot be trivial because $2m_{12} + 1$ cannot be 0. Note that for the last equality we used the commutation relations

$$\begin{aligned} \sigma_1 A_{1,2} \sigma_1^{-1} &= A_{1,2}, \\ \sigma_1 A_{1,j} \sigma_1^{-1} &= A_{2,j} \quad \text{for all } j \ge 3, \\ \sigma_1 A_{2,j} \sigma_1^{-1} &= A_{1,j} \quad \text{for all } j \ge 3, \\ \sigma_1 A_{i,j} \sigma_1^{-1} &= A_{i,j} \quad \text{for all } 3 \le i < j, \end{aligned}$$

which can be easily proved by using some of the formulas given above. □

Remark 3.7. As is well known [Brown 1982], any extension with abelian kernel corresponds to a 2-cocycle. Specifically, the extension $1 \to \mathfrak{P}_n \to \mathrm{PS}_n \to S_n \to 1$ corresponds to an element in $H^2(S_n, \mathbb{Z}^{n(n-1)/2})$. We illustrate this by computing explicitly the corresponding 2-cocycle for $n = 3$. We consider the set-theoretical section $f : S_3 \to \mathrm{PS}_3$ defined by $f(1) = 1$, $f(s_2) = \sigma_2$, $f(s_1) = \sigma_1$, $f(s_1 s_2) = \sigma_1 \sigma_2$, $f(s_2 s_1) = \sigma_2 \sigma_1$ and $f(s_2 s_1 s_2) = \sigma_2 \sigma_1 \sigma_2$. The 2-cocycle afforded by this section is defined by $u : S_3 \times S_3 \to \mathfrak{P}_3$, $(x, y) \mapsto f(x) f(y) f(xy)^{-1}$, and a direct computation gives its explicit formula as in Table 1, where we have chosen an additive notation for the abelian group $\mathfrak{P}_3 \simeq \mathbb{Z}^3$.

	1	s_2	s_1	$s_1 s_2$	$s_2 s_1$	$s_2 s_1 s_2$
1	0	0	0	0	0	0
s_2	0	$A_{2,3}$	0	0	$A_{2,3}$	$A_{2,3}$
s_1	0	0	$A_{1,2}$	$A_{1,2}$	0	$A_{1,2}$
$s_1 s_2$	0	$A_{1,3}$	0	$A_{1,2}$	$A_{1,2} + A_{1,3}$	$A_{1,2} + A_{1,3}$
$s_2 s_1$	0	0	$A_{1,3}$	$A_{1,3} + A_{2,3}$	$A_{2,3}$	$A_{1,3} + A_{2,3}$
$s_2 s_1 s_2$	0	$A_{1,2}$	$A_{2,3}$	$A_{1,3} + A_{2,3}$	$A_{1,2} + A_{1,3}$	$A_{1,2} + A_{1,3} + A_{2,3}$

Table 1. The 2-cocycle for $n = 3$ associated to the section f.

4. PS_n is linear

Bigelow [2001] and Krammer [2002] proved that the braid group B_n is linear. More precisely, let R be a commutative ring, let q and t be two invertible elements in R, and let V be a free R-module of rank $n(n-1)/2$ with a basis $\{x_{i,j}\}_{1\le i<j\le n}$. Then the map $\rho : B_n \to \mathrm{GL}(V)$, defined by

$$\begin{aligned}
\sigma_k x_{k,k+1} &= tq^2 x_{k,k+1}, && \\
\sigma_k x_{i,k} &= (1-q)x_{i,k} + qx_{i,k+1} && \text{for } i<k,\\
\sigma_k x_{i,k+1} &= x_{i,k} + tq^{k-i+1}(q-1)x_{k,k+1} && \text{for } i<k,\\
\sigma_k x_{k,j} &= tq(q-1)x_{k,k+1} + qx_{k+1,j} && \text{for } k+1<j,\\
\sigma_k x_{k+1,j} &= x_{k,j} + (1-q)x_{k+1,j} && \text{for } k+1<j,\\
\sigma_k x_{i,j} &= x_{i,j} && \text{for } i<j<k \text{ or } k+1<i<j,\\
\sigma_k x_{i,j} &= x_{i,j} + tq^{k-i}(q-1)^2 x_{k,k+1} && \text{for } i<k<k+1<j,
\end{aligned}$$

and $\rho(x)(v) = xv$ for $x \in B_n$ and $v \in V$, gives a representation of B_n, and if also $R = \mathbb{R}[t^{\pm 1}]$ and $q \in \mathbb{R} \subseteq R$ with $0<q<1$, then the representation is faithful; see [Krammer 2002].

We consider now the general formula for ρ, in which we take $q=1$:

$$\begin{aligned}
\sigma_k x_{k,k+1} &= tx_{k,k+1}, && \\
\sigma_k x_{i,k} &= x_{i,k+1} && \text{for } i<k,\\
\sigma_k x_{i,k+1} &= x_{i,k} && \text{for } i<k,\\
\sigma_k x_{k,j} &= x_{k+1,j} && \text{for } k+1<j,\\
\sigma_k x_{k+1,j} &= x_{k,j} && \text{for } k+1<j,\\
\sigma_k x_{i,j} &= x_{i,j} && \text{for } i<j<k \text{ or } k+1<i<j,\\
\sigma_k x_{i,j} &= x_{i,j} && \text{for } i<k<k+1<j.
\end{aligned}$$

One can easily see that these formulas imply

$$\sigma_k^2 x_{k,k+1} = t^2 x_{k,k+1} \qquad \text{and} \qquad \sigma_k^2 x_{i,j} = x_{i,j} \quad \text{if } (i,j) \neq (k,k+1).$$

One can then check that $\rho(\sigma_k^2)$ commutes with $\rho(\sigma_{k+1}^2)$ for all $1 \le k \le n-2$, and so for $q=1$ it turns out that ρ is a representation of PS_n.

Theorem 4.1. *This representation of* PS_n *is faithful if* $R = \mathbb{R}[t^{\pm 1}]$. *Therefore*, PS_n *is linear.*

Proof. We first prove that $A_{i,j}x_{i,j} = t^2 x_{i,j}$, and $A_{i,j}x_{k,l} = x_{k,l}$ if $(i,j) \neq (k,l)$. We do it by induction over $|j-i|$. If $|j-i| = 1$, the relations follow from the fact that $A_{i,i+1} = \sigma_i^2$. Assume the relations hold for $|j-i| = s-1$. We want to prove them for $|j-i|=s$. We recall that $A_{i,j}=\sigma_{j-1}A_{i,j-1}\sigma_{j-1}^{-1}$; see (14). We compute

$$\begin{aligned} A_{i,j}x_{i,j}=\sigma_{j-1}A_{i,j-1}\sigma_{j-1}^{-1}x_{i,j} &= \sigma_{j-1}A_{i,j-1}x_{i,j-1} \\ &= \sigma_{j-1}t^2x_{i,j-1} \quad \text{(by induction)} \\ &= t^2x_{i,j}. \end{aligned}$$

On the other hand, if $(i,j)\neq(k,l)$ then $\sigma_{j-1}^{-1}x_{k,l}=x_{u,v}$ with $(i,j-1)\neq(u,v)$, and so

$$\begin{aligned} A_{i,j}x_{k,l}=\sigma_{j-1}A_{i,j-1}\sigma_{j-1}^{-1}x_{k,l} &= \sigma_{j-1}A_{i,j-1}x_{u,v} \\ &= \sigma_{j-1}x_{u,v} \quad \text{(by induction)} \\ &= \sigma_{j-1}\sigma_{j-1}^{-1}x_{k,l}=x_{k,l}, \end{aligned}$$

as desired.

To show that the representation is faithful, take $b\in \mathrm{PS}_n$ such that $\rho(b)=\mathrm{id}_V$ and consider $\alpha(b)$, the image of b in S_n. From the way ρ is defined it follows that

$$bx_{i,j}=t^px_{\alpha(b)(i),\alpha(b)(j)} \quad \text{for all } 1\le i<j\le n,$$

with $p\in\mathbb{Z}$, where we made the convention $x_{r,s}:=x_{s,r}$ if $1\le s<r\le n$. Since $x_{i,j}$ is a basis in V and we assumed $\rho(b)=\mathrm{id}_V$, we find that the permutation $\alpha(b)\in S_n$ has the property that if $1\le i<j\le n$, then either $\alpha(b)(i)=i$ and $\alpha(b)(j)=j$ or $\alpha(b)(i)=j$ and $\alpha(b)(j)=i$. Since we assumed $n\ge 3$, the only such permutation is the trivial one. Thus, we have obtained that $b\in\mathrm{Ker}(\alpha)=\mathfrak{P}_n$ and so we can write $b=\prod_{1\le i<j\le n}A_{i,j}^{m_{i,j}}$, with $m_{i,j}\in\mathbb{Z}$. By using the formulas given above for the action of $A_{i,j}$ on $x_{k,l}$ we immediately obtain $bx_{k,l}=t^{2m_{k,l}}x_{k,l}$ for all $1\le k<l\le n$. Using again the assumption $\rho(b)=\mathrm{id}_V$, we obtain $t^{2m_{k,l}}=1$ and hence $m_{k,l}=0$ for all $1\le k<l\le n$, that is $b=1$, finishing the proof. □

5. Pseudosymmetric groups and pseudosymmetric braidings

We recall from [Kassel 1995, XIII.2] that to braid groups one can associate the so-called *braid category* $\mathcal{B}$, a universal braided monoidal category. Similarly, we can construct a pseudosymmetric braided category $\mathcal{PS}$ associated to pseudosymmetric groups. Namely, the objects of $\mathcal{PS}$ are natural numbers $n\in\mathbb{N}$. The set of morphisms from m to n is empty if $m\neq n$ and is PS_n if $m=n$. The monoidal structure of $\mathcal{PS}$ is defined as the one for $\mathcal{B}$, and so is the braiding, namely

$$\begin{aligned} &c_{n,m}: n\otimes m\to m\otimes n, \\ &c_{0,n}=\mathrm{id}_n=c_{n,0}, \\ &c_{n,m}=(\sigma_m\sigma_{m-1}\cdots\sigma_1)(\sigma_{m+1}\sigma_m\cdots\sigma_2)\cdots(\sigma_{m+n-1}\sigma_{m+n-2}\cdots\sigma_n) \quad \text{if } m,n>0. \end{aligned}$$

We denote by $t_{m,n} = c_{n,m} \circ c_{m,n}$ the double braiding. In view of Proposition 1.3, to prove that c is pseudosymmetric it is enough to check that, for all $m, n, p \in \mathbb{N}$,

$$(t_{m,n} \otimes \mathrm{id}_p) \circ (\mathrm{id}_m \otimes t_{n,p}) = (\mathrm{id}_m \otimes t_{n,p}) \circ (t_{m,n} \otimes \mathrm{id}_p). \tag{21}$$

Note that $t_{m,n} \otimes \mathrm{id}_p$ and $\mathrm{id}_m \otimes t_{n,p}$ are elements in $\mathfrak{P}_{m+n+p}$, which is an abelian group, and the composition $\circ$ between $t_{m,n} \otimes \mathrm{id}_p$ and $\mathrm{id}_m \otimes t_{n,p}$ is just the multiplication in the group $\mathfrak{P}_{m+n+p}$, so (21) is obviously true.

Let $\mathscr{C}$ be a strict braided monoidal category with braiding c, let n be a natural number and let $V \in \mathscr{C}$. Consider the automorphisms $c_1, \ldots, c_{n-1}$ of $V^{\otimes n}$ defined by $c_i = \mathrm{id}_{V^{\otimes(i-1)}} \otimes c_{V,V} \otimes \mathrm{id}_{V^{\otimes(n-i-1)}}$. It is well known (see [Kassel 1995, XV.4]) that there exists a unique group morphism $\rho_n^c : B_n \to \mathrm{Aut}(V^{\otimes n})$ such that $\rho_n^c(\sigma_i) = c_i$ for all $1 \leq i \leq n-1$. It is clear that, if c is pseudosymmetric, then ρ_n^c factorizes to a group morphism $\mathrm{PS}_n \to \mathrm{Aut}(V^{\otimes n})$. Thus, pseudosymmetric braided categories provide representations of pseudosymmetric groups.

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