Pacific Journal of Mathematics

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Volume 244 No. 2 February 2010

PACIFIC JOURNAL OF MATHEMATICS
Vol. 244, No. 2, 2010

TOPOLOGICAL INVARIANTS OF PUTATIVE $\mathbb{C}^*$-SYMMETRIC EXOTIC COMPLEX PROJECTIVE SPACES

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A classical problem in complex geometry is to determine the conditions under which two manifolds with the same differentiable structure admit different complex structures. We call a complex manifold X an exotic complex projective space if it is diffeomorphic to $\mathbb{C}\mathrm{P}^n$ but not biholomorphic to $\mathbb{C}\mathrm{P}^n$. It is unknown whether such exotic structures exist, but Emery Thomas has given necessary and sufficient conditions for an element of the cohomology ring to occur as the total Chern class of an almost-complex structure in low dimensions, thus establishing the existence of almost-complex structures with exotic Chern classes. We show that most of these elements cannot occur as the total Chern class of a complex structure with $\mathbb{C}^*$ symmetry. We include an overview of the equivariant index theory used in the proof.

1. Introduction

Let X be a complex manifold diffeomorphic to $\mathbb{C}\mathrm{P}^n$, regarded as a smooth manifold. A classical problem in differential geometry and complex analysis is to determine under what conditions X is biholomorphically equivalent to $\mathbb{C}\mathrm{P}^n$, that is, when X has not only the same differentiable structure as $\mathbb{C}\mathrm{P}^n$ but also the same complex manifold structure as $\mathbb{C}\mathrm{P}^n$. We call such a manifold X an *exotic complex projective space* if X is not biholomorphically equivalent to $\mathbb{C}\mathrm{P}^n$. It is well known in the $n = 1$ case that X has the unique complex structure of a genus 0 Riemann surface. Additionally, Yau [1977] showed that any complex surface homotopic to $\mathbb{C}\mathrm{P}^2$ is biholomorphic to $\mathbb{C}\mathrm{P}^2$. Hirzebruch and Kodaira showed similar results when X is Kähler. If $n \geq 3$ and X is not assumed to be Kähler, results are scant. Indeed, little is known about the existence of exotic complex projective spaces when $n \geq 3$.

In complex dimension 3, this is related to the long-standing problem of determining whether the sphere S^6 admits a complex structure. If we take S^6 with any of its almost-complex structures and blow it up at a point, we obtain a new almost-complex manifold X. This manifold is diffeomorphic to the three-dimensional

MSC2000: 32Q55, 53C15, 53C56, 58D19.

Keywords: exotic projective space, holomorphic symmetries, almost-complex structure, complex structure.

complex projective space $\mathbb{C}P^3$, but the total Chern class of the almost-complex structure on X is different from the total Chern class of the standard (integrable) almost-complex structure on $\mathbb{C}P^3$. The blown-up S^6 is one example of an *exotic almost-complex structure* on $\mathbb{C}P^3$.

Here we focus on manifolds X admitting complex structures invariant under holomorphic $\mathbb{C}^*$ actions. We begin with a brief introduction to almost-complex and complex geometry in Section 2, concluding with a statement about almost-complex structures on complex projective spaces due to E. Thomas. In Section 3, we summarize some of the main topological tools we use in our analysis, including a particularly useful reformulation of the Atiyah–Singer index theorem due to C. Kosniowski. In Section 4, we prove in complex dimensions three and four that if X is a complex manifold diffeomorphic to $\mathbb{C}P^n$ whose complex structure is $\mathbb{C}^*$-symmetric, then the Todd genus $\mathrm{Td}(X)$ of X is 0 or 1. We show this using an index calculation. An immediate corollary restricts the possible Chern classes of putative $\mathbb{C}^*$-symmetric exotic complex projective spaces.

2. Complex structures on vector bundles

A *complex structure on a real vector space* V is an $\mathbb{R}$-linear map $J : V \to V$ with the property $J^2 = -\mathrm{I}$, where $\mathrm{I} : V \to V$ is the identity map. A complex structure J makes the real vector space V into a complex vector space, with complex scalar multiplication defined by $(\alpha + i\beta)v = \alpha v + \beta J(v)$ for $\alpha, \beta \in \mathbb{R}$ and $v \in V$.

Conversely, suppose V is a complex vector space of complex dimension n and $V_{\mathbb{R}}$ is the underlying real vector space of dimension $2n$. Then scalar multiplication by i produces a real-linear map $J : V_{\mathbb{R}} \to V_{\mathbb{R}}$, with $J^2 = -I$.

Suppose $\pi : E \to X$ is a real vector bundle, so for every point $p \in X$, the fiber $E_p = \pi^{-1}(p)$ is a real vector space. If $\pi_1 : E \to X$ and $\pi_2 : F \to X$ are real vector bundles, a real vector bundle morphism $\rho : E \to F$ is a fiber-preserving map such that the restriction $\rho_p : E_p \to F_p$ to each fiber is real-linear. A *complex structure on a real vector bundle* $\pi : E \to X$ is a real vector bundle morphism $J : E \to E$ with the property $J^2 = -\mathrm{I}$, where $\mathrm{I} : E \to E$ is the identity morphism.

A complex structure J makes the real vector bundle $\pi : E \to X$ into a complex vector bundle. Indeed, for every point $p \in X$, the restriction $J_p : E_p \to E_p$ is a complex structure on the fiber, making E_p into a complex vector space. Conversely, if $\pi : E \to X$ is a complex vector bundle, then scalar multiplication by i is an real-linear map $J_p : E_p \to E_p$ satisfying $J_p^2 = -I_p$.

We focus on the important special case of a real vector bundle that is the tangent bundle $\pi : TX \to X$ of a smooth manifold X. An *almost-complex structure on a smooth manifold* X is a complex structure J on the tangent bundle $\pi : TX \to X$.

Let X be a compact oriented smooth manifold, and let $\pi : E \to X$ be a complex vector bundle of rank k. The fundamental invariants of the complex vector bundle are the Chern classes $c_j(E) \in H^{2j}(X, \mathbb{Z})$ for $j = 1, 2, \ldots, k$, which detect nontriviality of a complex vector bundle. All Chern classes of a trivial (or product) bundle are zero. See Milnor and Stasheff [1974] for the axiomatic definitions and also Bott and Tu [1982], who use slightly different conventions. Alternately Chern classes may be defined using the curvature of a connection; see for example Kobayashi and Nomizu [1969].

We rapidly review some of the fundamental properties of Chern classes.

Proposition 2.1. *Let $\pi : E \to X$ be a complex bundle of rank k. The top Chern class $c_k(E) \in H^{2k}(X, \mathbb{Z})$ is equal to the Euler class $e(E_{\mathbb{R}}) \in H^{2k}(X, \mathbb{Z})$ of the underlying oriented real vector bundle $E_{\mathbb{R}}$.*

A complex structure on the tangent bundle $\pi : TX \to X$ of a smooth compact manifold X computes the Euler characteristic $\chi(X) = \sum_{k=0}^{2n}(-1)^k \dim H^k(X, \mathbb{Z})$ via the relation $\chi(X) = \int_X e(TX) = \int_X c_n(TX)$. Here $2n$ is the real dimension of the manifold X.

The *total Chern class* of a rank k complex vector bundle $\pi : E \to X$ is the mixed-degree element $c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_k(E) \in H^*(X, \mathbb{Z})$. The total Chern class is multiplicative for direct sums of vector bundles:

Proposition 2.2. *Let E and F be complex vector bundles over X, and let $E \oplus F$ denote the direct sum. Then $c(E \oplus F) = c(E)c(F) \in H^*(X, \mathbb{Z})$.*

The *conjugate* of a complex vector bundle $\pi : E \to X$ is the complex vector bundle $\pi : \overline{E} \to X$ with the same underlying real bundle $E_{\mathbb{R}}$, but with the "opposite" complex structure $-J$.

Proposition 2.3. *Let $\overline{E}$ be the conjugate of a complex vector bundle E. Then*

$$c_j(\overline{E}) = (-1)^j c_j(E).$$

A choice of Hermitian metric corresponds to a choice of complex vector bundle isomorphism between the conjugate bundle $\overline{E}$ and the dual bundle E'.

Chern classes are invariants of a *complex* vector bundle, not merely of the underlying real vector bundle. Two different complex structures on the same underlying real vector bundle may have different Chern classes. This is not the case for the Pontrjagin classes, which are specific combinations of Chern classes, and depend only on the underlying real vector bundle, that is, the Pontrjagin classes do not change if we change the complex structure on the underlying real vector bundle. In fact, Pontrjagin classes can be defined for any real vector bundle, even those without complex structure.

Definition 2.4. The *Pontrjagin classes* of a complex vector bundle $\pi : E \to X$ of rank k are defined by $p_j(E) = (-1)^j c_{2j}(E \oplus \bar{E}) \in H^{4j}(X, \mathbb{Z})$ for $j = 1, 2, \ldots, k$.

The complex bundle $E \oplus \bar{E}$ is isomorphic to its conjugate $\bar{E} \oplus E$, so the odd Chern classes $c_{2j+1}(E \oplus \bar{E})$ vanish by Proposition 2.3 and do not give nontrivial invariants.

Using the Propositions 2.2 and 2.3, we immediately obtain a relation between the Pontrjagin classes and Chern classes of a complex vector bundle:

Theorem 2.5. *Let $\pi : E \to X$ be complex vector bundle of rank k. Then*

$$1 - p_1 + p_2 - \cdots \pm p_k = (1 + c_1 + c_2 + \cdots + c_k)(1 - c_1 + c_2 - \cdots \pm c_k).$$

Theorem 2.5 expresses each Pontrjagin class in terms of the Chern classes. The first two such expressions are

$$p_1 = c_1^2 - 2c_2 \in H^4(X, \mathbb{Z}) \quad \text{and} \quad p_2 = c_2^2 - 2c_1c_3 + 2c_4 \in H^8(X, \mathbb{Z}).$$

There is a simple reason why the Pontrjagin classes of a complex vector bundle are independent of the complex structure, whereas an arbitrary combination of Chern classes is not. Let W be a complex vector space, and $W_{\mathbb{R}}$ the underlying real vector space obtained by forgetting the complex structure. It is easy to construct a natural isomorphism of complex vector spaces $W \oplus \overline{W} \simeq W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Carrying this construction over to vector bundles, we observe that the complex vector bundle $E \oplus \bar{E} \simeq E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ depends only on the underlying real bundle $E_{\mathbb{R}}$. Thus the definition of Pontrjagin classes can be extended to arbitrary real vector bundles $\pi : V \to X$ via $p_j(V) = (-1)^j c_{2j}(V \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4j}(X, \mathbb{Z})$.

Theorem 2.5 gives a clear necessary condition on the Chern classes of an almost-complex structure on $\mathbb{C}\mathrm{P}^n$. The next theorem shows that this is also a sufficient condition for $1 \le n \le 4$. Let $x \in H^2(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$ denote the class corresponding to the first Chern class of the canonical complex line bundle over $\mathbb{C}\mathrm{P}^n$. Then x^i is a generator of $H^{2i}(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$ for $1 \le i \le n$.

Theorem 2.6 [Thomas 1967]. *Consider the complex projective space $\mathbb{C}\mathrm{P}^n$ for $n = 1, 2, 3, 4$. The following cohomology classes, and only these, occur as the total Chern class of an almost-complex structure on $\mathbb{C}\mathrm{P}^n$.*

$$\begin{aligned}
\mathbb{C}\mathrm{P}^1 &: \ 1 + 2x;\\
\mathbb{C}\mathrm{P}^2 &: \ 1 + 3x + 3x^2,\ 1 - 3x + 3x^2;\\
\mathbb{C}\mathrm{P}^3 &: \ 1 + 2jx + 2(j^2 - 1)x^2 + 4x^3, \qquad j \in \mathbb{Z};\\
\mathbb{C}\mathrm{P}^4 &: \ 1 + \epsilon x - 2x^2 + \epsilon x^3 + 5x^4,\\
&\quad\ 1 + 5\epsilon x + 10x^2 + 10\epsilon x^3 + 5x^4,\\
&\quad\ 1 + 25\epsilon x + 60x^2 + 1922\epsilon x^3 + 5x^4, \qquad \textit{where } \epsilon = +1 \textit{ or } \epsilon = -1.
\end{aligned}$$

In Section 4 we show that when $n = 3$ or $n = 4$ most of these almost-complex structures cannot be induced by a $\mathbb{C}^*$-symmetric complex structure.

3. Fixed-point theory for holomorphic $\mathbb{C}^*$ actions

Here we summarize some results to be used in Section 4. We use the fixed point results of Atiyah, Bott, Singer, and others as formulated in [Kosniowski 1970]. We begin by recalling many of the basic facts from equivariant geometry.

Definition 3.1. A closed subgroup H of a complex Lie group G is called a *real form* of G if $\mathfrak{g} = \mathfrak{h} \otimes \mathbb{C} = \mathfrak{h} \oplus i\mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively.

Definition 3.2. A connected complex Lie group G is called *reductive* if it has a compact real form.

Note that reductive is also well defined for disconnected complex Lie groups.

The group $\mathbb{C}^*$ is a reductive group; its real form is S^1. We can understand a $\mathbb{C}^*$ action on a manifold X by understanding the action of S^1 on X.

Theorem 3.3 (linearization). *Assume that a reductive complex Lie group G acts holomorphically on a complex manifold X, and assume that $x \in X$ is a G-fixed point. Let H be a maximal compact subgroup of G and let $L(g) : T_xX \to T_xX$ be the tangential map. Then there exist neighborhoods U of x and V of $0 \in T_xX$ and an isomorphism $\phi : U \to V$ such that $\phi \circ h = L(h) \circ \phi$ for all $h \in H$.*

Moreover, if W is a neighborhood of H, and U' is an open subset of U such that $WU' \subset U$, then $(L(w) \circ \phi)(x) = (\phi \circ w)(x)$ for all $x \in U'$.

We call U a linearizing neighborhood of x. This theorem yields useful information about the fixed point set of a $\mathbb{C}^*$ action on X. See [Huckleberry 1990].

We turn now to the main results of [Kosniowski 1970]. Suppose X is a compact complex manifold of complex dimension n. Define $\chi_p(X) := \sum_{q=0}^{n}(-1)^q h^{p,q}$, where $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X, \mathbb{C})$ are the Hodge numbers of X. The Hirzebruch–Riemann–Roch theorem gives the relationship

$$\chi_0(X) = \int_X \mathrm{Td}(X), \tag{1}$$

where $\mathrm{Td}(X)$ denotes the Todd class of X. If $X = \mathbb{C}\mathrm{P}^n$ and ω is an almost-complex structure on X, Hirzebruch and Kodaira [1957] give the following universal expression for the right side of (1):

$$\int_X \mathrm{Td}(\omega) = \binom{(\lambda + n - 1)/2}{n}, \tag{2}$$

where x is the class in $H^2(\mathbb{C}\mathrm{P}^n, \mathbb{Z})$ that corresponds to the first Chern class of the canonical complex line bundle over $\mathbb{C}\mathrm{P}^n$, and $c_1(\omega) = \lambda x$.

Following [Hirzebruch 1966] we also introduce the polynomial $\chi(X, y)$,

$$\chi(X, y) = \sum_{p=0}^{n} \chi_p(X) y^p,$$

where y is an indeterminate. Note that if $y = -1$, then $\chi(X, -1) = \chi(X)$, the Euler characteristic of X.

Let A be a holomorphic vector field with simple isolated zeros. At each zero x of A, there is an induced linear endomorphism $L_x(A)$ of $T_x X$, which is nonsingular at all of the zeros of A. The eigenvalues of A are in general nonzero complex numbers. Let c be a complex number such that $\mathrm{Real}(\theta/c) \neq 0$ for all eigenvalues θ of $L(A)$. Define $s(x, c)$ to be the number of eigenvalues of $L_x(A)$ for which $\mathrm{Real}(\theta/c)$ is positive.

Theorem 3.4 [Kosniowski 1970]. *If X is a compact complex manifold and A a holomorphic vector field with simple isolated zeros, then*

$$\chi(X, y) = \sum (-y)^{s(x,c)},$$

where the sum is over all of the zeros x of A.

Theorem 3.4 can be extended to arbitrary fixed point sets by using the holomorphic Lefschetz fixed point formula of Atiyah and Segal [1968]. It is required that the one parameter group of the vector field lies in a compact group. We call such a vector field a compact vector field. Decompose the normal bundle N^A of the zero set of A as $N^A = \sum_\theta N^A(\theta)$, where $N^A(\theta)$ is the subbundle of N^A on which $\exp(A)$ acts as $\exp(i\theta)$ and θ is a real number. Then define $s(k, \pm)$ as the number of θ with sign $\pm$ at a component X_k^A of the zero set X^A.

Theorem 3.5 [Kosniowski 1970]. *Let X be a compact complex manifold and A a compact holomorphic vector field. Then*

$$\chi(X, y) = \sum_k (-y)^{s(k,+)} \chi(X_k^A, y) = \sum_k (-y)^{s(k,-)} \chi(X_k^A, y),$$

where X_k^A is a component of the zero set X^A of A.

These theorems together give necessary conditions for a vector field to be a holomorphic vector field on X. We will use these to determine necessary conditions for X to admit a holomorphic $\mathbb{C}^*$ action.

We conclude this discussion with properties of the fixed point sets of holomorphic $\mathbb{C}^*$ actions on complex projective spaces.

Theorem 3.6 [Su 1963; Bredon 1972]. *Suppose S^1 acts on a smooth manifold X diffeomorphic to $\mathbb{C}\mathrm{P}^n$, with fixed point set F. Then F has at most $n+1$ components F_i for $i=1,\dots,m$, and each F_i has the same cohomology ring of complex projective k_i-space, with $\sum_{i=1}^m k_i = n-m+1$.*

This theorem holds in the more general case where X is an integral cohomology complex projective space, that is, a topological space with the same cohomology ring as $\mathbb{C}\mathrm{P}^n$ with integer coefficients.

4. Symmetry and fixed loci

Suppose $\mathbb{C}^*$ acts holomorphically on a complex manifold X that is diffeomorphic to $\mathbb{C}\mathrm{P}^n$. In this setting we can apply Theorem 3.6 for an initial restriction of the fixed point set of this action. Moreover, a consequence of Theorem 3.3 is that each component of the fixed point set is itself a complex submanifold of X.

Lemma 4.1. *Suppose X is a complex manifold diffeomorphic to $\mathbb{C}\mathrm{P}^n$. If X admits a holomorphic $\mathbb{C}^*$ action whose fixed locus contains a hypersurface, then $\chi_0(X)=1$.*

Proof. Since the fixed point set of the action contains a hypersurface S, this set has two connected components S and x, where x is an isolated point and S is a complex submanifold with the cohomology of complex projective space.

Theorem 3.5 gives the relationship

$$\chi(X,y) = (-y)^{s(x,+)} + (-y)^{s(S,+)}\chi(S,y), \tag{3}$$

where $s(x,+)\in\mathbb{Z}$, $0\le s(x,+)\le n$, and $s(S,+)=0$ or 1. We will prove Lemma 4.1 by a calculation in which we compare coefficients of y in (3).

Recall the relationship

$$h^{p,q} = h^{n-p,n-q} \tag{4}$$

among the Hodge numbers of a compact complex manifold resulting from Serre duality. We also have an expression for the Euler characteristic:

$$\chi(X) = \sum_{p=0}^{n}\sum_{q=0}^{n}(-1)^{p+q}h^{p,q} = \sum_{p=0}^{n}(-1)^p\chi_p(X) \tag{5}$$

Suppose n is even. Equation (4) implies

$$\chi(X,y) = \chi_0(X) + \chi_1(X)y + \cdots + \chi_{n/2}(X)y^{n/2} + \cdots + \chi_1(X)y^{n-1} + \chi_0(X)y^n$$

and

$$\chi(S,y) = \chi_0(S) + \chi_1(S)y + \cdots + \chi_{n/2-1}(S)y^{n/2-1} - \chi_{n/2-1}(S)y^{n/2+1} + \cdots - \chi_1(S)y^{n-2} - \chi_0(S)y^{n-1}.$$

We can assume that $s(S,+)=0$. Then (3) becomes

$$\chi_0(X)+\chi_1(X)y+\ldots+\chi_0(X)y^n=(-y)^{s(x,+)}+\chi_0(S)+\chi_1(S)y+\ldots-\chi_0(S)y^{n-1}.$$

Suppose $s(x,+)=n$. Then

$$\chi_k(X)=(-1)^k \tag{6}$$

Now suppose $s(x,+)=n-1$. Then $\chi_0(X)=0$, but (6) still holds for $1\le k\le n$. We can continue in this way and use (5) to express the Euler characteristic in terms of $s(x,+)$ as

$$\chi(X)=-n+2s(x,+)+1.$$

But $\chi(X)$ is a topological invariant of X, so $\chi(X)=\chi(\mathbb{C}\mathrm{P}^n)=n+1$. This is exactly the case when $s(x,+)=n$, so $\chi_0(X)=1$.

Now suppose n is odd. Equation (4) implies

$$\begin{aligned}\chi(X,y)=\chi_0(X)+\chi_1(X)y+\cdots+\chi_{(n-1)/2}(X)y^{(n-1)/2}\\ -\chi_{(n-1)/2}(X)y^{(n-1)/2+1}-\cdots-\chi_1(X)y^{n-1}-\chi_0(X)y^n\end{aligned}$$

and

$$\begin{aligned}\chi(S,y)=\chi_0(S)+\chi_1(S)y+\cdots+\chi_{(n-1)/2}(S)y^{(n-1)/2}\\ +\chi_{(n-1)/2}(S)y^{(n-1)/2+1}+\cdots+\chi_1(S)y^{n-2}+\chi_0(S)y^{n-1}.\end{aligned}$$

Again we assume that $s(S,+)=0$. Then (3) becomes

$$\chi_0(X)+\chi_1(X)y+\ldots+\chi_0(X)y^n=(-y)^{s(x,+)}+\chi_0(S)+\chi_1(S)y+\ldots-\chi_0(S)y^{n-1}.$$

Suppose $s(x,+)=n$. As above, (6) holds. Now suppose $s(x,+)=n-1$. Then $\chi_0(X)=0$, but (6) still holds for $1\le k\le n$. Continuing this way, we obtain the same expression for the Euler characteristic. As above this implies $s(x,+)=n$, so $\chi_0(X)=1$. □

Theorem 4.2. *Suppose X is a complex manifold diffeomorphic to $\mathbb{C}\mathrm{P}^n$ for $n=3$ or $n=4$. Suppose $\mathbb{C}^*$ acts holomorphically on X. Then $\chi_0(X)=0$ or $\chi_0(X)=1$.*

Proof. We consider the cases $n=3$ and $n=4$ separately. Suppose that $n=3$. The fixed point set F has at most four connected components F_1, F_2, F_3, F_4 with $H^*(F_i,\mathbb{Z})\cong H^*(\mathbb{C}\mathrm{P}^{n_i},\mathbb{Z})$, and each is a complex submanifold of X.

Combining (4) and (5) yields

$$\chi(X,y)=\chi_0(X)+(2-\chi_0(X))y-(2-\chi_0(X))y^2-\chi_0(X)y^3. \tag{7}$$

Suppose that F has two connected components F_1 and F_2. The relationship $n_1+n_2=2$ gives two cases: first, $F_1=x$ and $F_2=S$, where x is an isolated point and S is a hypersurface; or second $F_1=\mathbb{C}\mathrm{P}^1$ and $F_2=\mathbb{C}\mathrm{P}^1$. In the first case we know from Lemma 4.1 that $\chi_0(X) = 1$. Suppose then that $F_1 = \mathbb{C}\mathrm{P}^1$ and $F_2 = \mathbb{C}\mathrm{P}^1$. This gives

$$\begin{aligned}\chi(X, y) &= (-y)^{s(1,+)}(1-y) + (-y)^{s(2,+)}(1-y)\\ &= (-y)^{s(1,+)} + (-y)^{s(1,+)+1} + (-y)^{s(2,+)} + (-y)^{s(2,+)+1}.\end{aligned}$$

This expression for $\chi(X, y)$ is consistent with (7) if and only if $\chi_0(X) = 1$ or $\chi_0(X) = 0$.

Consider now the case F has three connected components: two isolated points and $\mathbb{C}\mathrm{P}^1$. Theorem 3.5 gives

$$\chi(X, y) = (-y)^{s(1,+)} + (-y)^{s(2,+)} + (-y)^{s(3,+)}(1-y).$$

Suppose $\chi_0(X) \neq 0$ in (7). Performing the same calculations as above gives that $\chi_0(X) = 1$. We see then that this expression is consistent with (7) if and only if $\chi_0(X) = 0$ or $\chi_0(X) = 1$.

Finally, if F has four connected components, then each component F_i is a point x_i, and we have by Theorem 3.4

$$\chi(X, y) = (-y)^{a_1} + (-y)^{a_2} + (-y)^{a_3} + (-y)^{a_4},$$

and again this is consistent with (7) if and only if $\chi_0(X) = 0$ or $\chi_0(X) = 1$.

Now suppose $n = 4$. The argument is the same as above but now simpler, even though there are more cases to check. This is because of one of the consequences of Theorem 2.6 is that $\chi_0(X) = 0$, 1, or 1001. A quick check of the same relationships as above shows that $\chi_0(X) \neq 1001$ if X admits a holomorphic $\mathbb{C}^*$ action. □

Corollary 4.3. *Suppose X is a complex manifold diffeomorphic to $\mathbb{C}\mathrm{P}^n$ for $n = 3$ or $n = 4$. Suppose further that $\mathbb{C}^*$ acts holomorphically on X. Then the total Chern class of X is one of the following.*

$$\begin{aligned}n = 3:\ & 1 + 2x^2 + 4x^3,\\ & 1 + 2x + 4x^3,\\ & 1 - 2x + 4x^3,\\ & 1 + 4x + 6x^2 + 4x^3;\\ n = 4:\ & 1 + \epsilon x - 2x^2 + \epsilon x^3 + 5x^4,\\ & 1 + 5\epsilon x + 10x^2 + 10\epsilon x^3 + 5x^4, \quad \textit{where } \epsilon = +1 \textit{ or } \epsilon = -1.\end{aligned}$$

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Received March 24, 2009.

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