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**AREA-MINIMIZING REGIONS WITH SMALL VOLUME
IN RIEMANNIAN MANIFOLDS WITH BOUNDARY**

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Given a domain Ω of a Riemannian manifold, we prove that regions minimizing the area (relative to Ω) are nearly the maxima of the mean curvature of $\partial\Omega$ when their volume tends to zero. We deduce some sharp local relative isoperimetric inequalities involving mean curvature comparisons.

1. Introduction

Let Ω be a bounded smooth domain of a Riemannian manifold (\mathcal{M}^{n+1}, g) . Recall that the *partitioning problem* in Ω consists in finding, for a given $v < |\Omega|_g$, a critical point of the perimeter functional $\mathcal{P}_g(\cdot, \Omega)$ in the class of Borel sets in Ω that enclose a volume v . A set that minimizes the perimeter will be called an *isoperimetric region*. It is clear that the boundary of a smooth solution to the partitioning problem in Ω have constant mean curvature and, if it touches $\partial\Omega$, it will intersect it orthogonally; see for example [Ros and Vergasta 1995]. In light of standard results in geometric measure theory, minimizers do exist for any given volume and may have various topologies; see the survey [Ros 2005].

Up to now the complete description of minimizers has been achieved only in special cases; see for example [Bürger and Kuwert 2008; Ros and Vergasta 1995; Ritoré and Rosales 2004; Sternberg and Zumbrun 1998]. However, the study of existence and geometric and topological properties of stationary surfaces (not necessarily minimizers) is far from complete. Grüter and Jost [1986] have proved the existence of minimal discs in convex bodies, while Jost [1986] proved the existence of embedded minimal surfaces of higher genus. In the particular case of the free boundary Plateau problem, some global existence results were obtained by M. Struwe [1984; 1988]. In [Fall 2007] we proved the existence of surfaces similar to half spheres surrounding a small volume near nondegenerate critical points of the mean curvature of $\partial\Omega$; in the same paper it was shown that the boundary mean

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curvature determines the main terms when studying the problem via a Lyapunov–Schmidt reduction. In [Appendix B](#) we complement this last result as follows:

Proposition B.1. *There exist $r_0 > 0$ and a smooth function $f : (0, r_0) \times \partial\Omega \rightarrow \mathbb{R}$ such that for every $r \in (0, r_0)$, if p is a critical point of $f(r, \cdot)$, the geodesic ball centered at p with radius r can be perturbed smoothly to a set which is a solution to the partitioning problem in Ω . Furthermore*

$$\|f(r, \cdot) - H_{\partial\Omega}(\cdot)\|_{\mathcal{C}^1(\partial\Omega)} \leq cr,$$

where $H_{\partial\Omega}(p)$ is the mean curvature of $\partial\Omega$ at p .

In the body of the paper our main goal is the location of minimal area separating hypersurfaces of Ω enclosing a small volume:

Theorem 1.2. *Isoperimetric regions with small volume in Ω are near global maxima of the mean curvature of $\partial\Omega$.*

Results of this kind were recently obtained in [[Druet 2002](#); [Nardulli 2007](#)]. These authors showed that isoperimetric regions with small volume are nearly maxima of the scalar curvature.

To prove the above theorem, we first show a regularity result that generalizes Theorem 2.2 in [[Morgan and Johnson 2000](#)] (see [Lemma 4.2](#)). We notice that the original proof of that theorem highlights that the diameter of an isoperimetric region E_v tends to zero as the volume v tends to zero. Moreover as pointed out by Bayle and Rosales [[2005](#)], this set must touch the boundary $\partial\Omega$ if v is small enough (E_v is not compactly contained in Ω). From this one sees that E_v is contained in a geodesic sphere centered at some point $p \in \partial\Omega$ for v small. Hence, using the result of Morgan and Johnson just mentioned, one concludes that the hypersurface $\Sigma_v = \partial E_v \cap \Omega$ can be written, after suitable scaling, as a graph over a round hemisphere, and the function defining the graph tends to zero. This also shows that $\partial\Sigma_v \subset \partial\Omega$. But, according to our argument we need convergence up to the free boundary. We achieve this, following [[Grüter 1987](#)], by proving a monotonicity result for the area of Σ_v in a tubular neighborhood of $\partial\Omega$. This allows us to get a bound for the area of $\partial\Sigma_v$ and hence, by compactness, weak convergence up to the free boundary and smoothly by [[Grüter and Jost 1986](#)].

The second step is to reduce the isoperimetric problem to a finite-dimensional variational one (see [Lemma 4.7](#)) by adopting a variant of the method in [[Nardulli 2007](#)]. To this end, by means of the implicit function theorem we construct, for any fixed v sufficiently small, a manifold \mathfrak{C}_v of sets having volume v and diffeomorphic to $\partial\Omega$ (see [Lemma 4.6](#)). Any set $E \in \mathfrak{C}_v$ is a *pseudo-half-ball* (see [Definition 4.5](#)) which is uniquely determined by its center of mass $p \in \partial\Omega$ while its boundary, $\partial E = \Sigma_{p, \omega^{p,v}}$, is a normal graph over a geodesic sphere centered at p with $\omega^{p,v}$ (defining the graph) tending to zero as $v \rightarrow 0$. Finally we show that an

isoperimetric region with small volume v must belong to \mathfrak{C}_v , so looking for the minimum of the perimeter among sets in Ω with volume v is equivalent to taking the minimum among sets in \mathfrak{C}_v . Taking advantage of the role of the mean curvature in the expansion of the area of normal graphs centered at the free boundary $\partial\Omega$ (see [Appendix A](#)), the theorem then follows.

It is well known that much of the information concerning the partitioning problem is contained in the isoperimetric profile relative to Ω , namely the mapping

$$v \mapsto I_\Omega(v) = \min_{\substack{E \subset \Omega \\ |E|_g = v}} \mathcal{P}_g(E, \Omega).$$

Explicit lower bounds for the profile I_Ω are very important in applications and are called isoperimetric inequalities; for instance see [[Chavel 1984](#); [2001](#)].

Nardulli [[2007](#), Theorem 7] gave an expansion of the isoperimetric profile of a compact Riemannian manifold \mathcal{M}^{n+1} as

$$(1) \quad I_{\mathcal{M}}(v) = \left(1 - \gamma_n \max_{p \in \mathcal{M}} S_{\mathcal{M}}(p) v^{2/(n+1)} + O(v^{4/(n+1)})\right) I_{\mathbb{R}^{n+1}}(v),$$

where $S_{\mathcal{M}}$ is the scalar curvature of \mathcal{M} and γ_n is a positive number. This was obtained after showing that small isoperimetric regions can be written as normal graphs over the boundary of a geodesic ball $B_{\mathcal{M}}(p, r)$ centered at some point p . So he reduced the isoperimetric problem to a finite-dimensional one.

In our situation, as we will see later, the presence of the obstacle $\partial\Omega$ is not negligible and in fact the second fundamental form of $\partial\Omega$ determines the first-order expansion of the profile I_Ω . Before going on, we recall that Bayle and Rosales [[2005](#)] have shown that

$$I_\Omega(v) = (1 + O(v)) I_{\mathbb{R}_+^{n+1}}(v),$$

where $I_{\mathbb{R}_+^{n+1}}(v) = (n+1) |B_+^{n+1}|^{1/(n+1)} v^{n/(n+1)}$. From the reduction of the isoperimetric problem to a finite-dimensional one ([Lemma 4.7](#)), we can determine the first coefficient of the asymptotic expansions of the profile of Ω near zero. Letting $v = |r B_+^{n+1}|$ in [Lemmas 4.7](#) and [4.8](#), we show that

$$I_\Omega(v) = \min_{p \in \partial\Omega} \left\{ I_{\mathbb{R}_+^{n+1}}(v) - \frac{n}{n+2} \frac{|B^n|}{|B_+^{n+1}|} H_{\partial\Omega}(p) v + O_p(v^{(n+2)/(n+1)}) \right\},$$

where $H_{\partial\Omega}(p)$ is the mean curvature of $\partial\Omega$ at p and $O_p(\rho)$ is a smooth function in p and ρ tending to zero uniformly with respect to p as ρ tends to zero. Hence:

Corollary 1.3. *With $\beta_n = \frac{n}{(n+1)(n+2)} \frac{|B^n|}{|B_+^{n+1}|^{(n+2)/(n+1)}}$, we have*

$$I_\Omega(v) \sim \left(1 - \beta_n \max_{p \in \partial\Omega} H_{\partial\Omega}(p) v^{1/(n+1)} + O(v^{2/(n+1)})\right) I_{\mathbb{R}_+^{n+1}}(v).$$

Choe, Ghomi and Ritoré [2007] showed that an isoperimetric region outside a convex domain in Euclidean space has no less perimeter than the area of a hemisphere, provided it encloses the volume of a half-ball. In this situation, from Corollary 1.3, we can weaken convexity by strict H-convexity (that is, the domain has nonnegative mean curvature).

Corollary 1.4. *If Ω is a strictly H-convex smooth bounded domain of \mathbb{R}^{n+1} and v is small enough,*

$$I_{\mathbb{R}^{n+1} \setminus \Omega}(v) > I_{\mathbb{R}_+^{n+1}}(v).$$

This, therefore, moves toward the question of isoperimetric inequalities inside H-convex domains.

Druet [2002] proved a local sharp isoperimetric inequality, with the aim of answering a question from [Morgan and Johnson 2000]: In a compact Riemannian manifold (\mathcal{M}^{n+1}, g) whose scalar curvature satisfies $\max_{p \in \partial \mathcal{M}} S_g(p) < n(n+1)K_0$ for some $K_0 \in \mathbb{R}$, the perimeter of the solution to the isoperimetric problem with small volume is strictly greater than that of an isoperimetric region in the space form of constant sectional curvature K_0 . In this direction we have obtained:

Corollary 1.5. *Suppose Ω is a bounded smooth domain in a Riemannian manifold (\mathcal{M}^{n+1}, g) . Let Ω_0 be a bounded smooth domain in any other Riemannian manifold $(\mathcal{M}_0^{n+1}, g_0)$ with mean curvatures satisfying $\max_{p \in \partial \Omega} H_{\partial \Omega}(p) < \max_{p \in \partial \Omega_0} H_{\partial \Omega_0}$. Then, if v is small enough,*

$$I_{\Omega}(v) > I_{\Omega_0}(v).$$

2. Preliminaries

We denote by $N_{\partial \Omega}$ the unit interior normal vector field along $\partial \Omega$. We consider an oriented orthonormal frame field $(E_1, \dots, E_n, N_{\partial \Omega})$ of \mathcal{M} along $\partial \Omega$, and use it to introduce geodesic normal coordinates in a neighborhood (in $\partial \Omega$) of a point $p \in \partial \Omega$ with coordinates $x' = (x^1, \dots, x^n) \in \mathbb{R}^n$. We set

$$f^p(x') := \exp_p^{\partial \Omega}(x^i E_i).$$

This choice of coordinates induces coordinate vector fields on $\partial \Omega$:

$$Y_i(x') = f_*(\partial_{x^i}), \quad \text{for } i = 1, \dots, n.$$

For any vector field Y on $T\partial \Omega$, we define $S(Y) = \nabla_Y N_{\partial \Omega}$, where ∇ is the connection on \mathcal{M} .

Now consider a local parametrization of a neighborhood of p in \mathcal{M} by

$$F^p(x) := \exp_{f^p(x')}^{\mathcal{M}}(x^{n+1} N_{\partial \Omega}), \quad x = (x', x^{n+1}) \in \mathbb{R}^{n+1}.$$

This yields the coordinate vector fields in \mathcal{M}

$$\begin{aligned} X_i(x) &:= F_*^P(\partial_{x^i}), \quad i = 1, \dots, n, \\ X_{n+1}(x) &:= F_*^P(\partial_{x^{n+1}}). \end{aligned}$$

In the whole paper, for any vector fields $X, Y \in T\mathcal{M}$, we define $\langle X, Y \rangle := g(X, Y)$. We denote by R_p and $R_p^{\mathcal{M}}$ the Riemannian tensors of $\partial\Omega$ and \mathcal{M} , respectively.

Lemma 2.1. *Near the point $F^P(x', 0) = f^P(x')$,*

$$X_i = Y_i + x^{n+1}S(Y_i) + \frac{1}{2}(x^{n+1})^2 R_p^{\mathcal{M}}(N_{\partial\Omega}, Y_i)N_{\partial\Omega} + O(|x^{n+1}|^3).$$

Near $p = F^P(0)$ we have

$$\langle Y_i, Y_j \rangle = \delta_{ij} + \frac{1}{3}\langle R_p(E_k, E_i)E_l, E_j \rangle x^k x^l + O(|x|^3).$$

Proof. By construction,

$$\nabla_{X_{n+1}}^k X_{n+1}|_{f(x')} = 0 \quad \text{for any integer } k \geq 1.$$

Now since $\nabla_{X_{n+1}} X_i|_{f(x')} = \nabla_{X_i} X_{n+1}|_{f(x')} = S(Y_i)$ and $X_{n+1}|_{f(x')} = N_{\partial\Omega}$, we get

$$\begin{aligned} \nabla_{X_{n+1}}^2 X_i|_{f(x')} &= \nabla_{X_{n+1}}(\nabla_{X_i} X_{n+1})|_{f(x')} \\ &= R_p^{\mathcal{M}}(N_{\partial\Omega}, Y_i)N_{\partial\Omega} + \nabla_{X_i} \nabla_{X_{n+1}} X_{n+1}|_{f(x')} = R_p^{\mathcal{M}}(N_{\partial\Omega}, Y_i)N_{\partial\Omega}. \end{aligned}$$

For the proof of the last expansions, see for example [Pacard and Xu 2009, Proposition 2.1]. \square

This lemma affords the next proposition, which gives expansions of the metric $g_{\alpha\beta} := \langle X_\alpha, X_\beta \rangle$ in a neighborhood of $p \in \partial\Omega$ in \mathcal{M} with $\alpha, \beta \in \{1, \dots, n, n+1\}$.

Proposition 2.2. *In a neighborhood of p ,*

$$\begin{aligned} g_{ij} &= \delta_{ij} + 2\langle S(Y_i), Y_j \rangle x^{n+1} + \frac{1}{3}\langle R_p(E_k, E_i)E_l, E_j \rangle x^k x^l \\ &\quad + (\langle R_p^{\mathcal{M}}(N_{\partial\Omega}, E_i)N_{\partial\Omega}, E_j \rangle + \langle S(Y_i), S(Y_j) \rangle)(x^{n+1})^2 + O(|x|^3), \\ g_{in+1} &= O(|x|^3), \quad g_{n+1n+1} = 1. \end{aligned}$$

We fix the following notation:

$$\mathbb{R}_+^{n+1} := \{x = (x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : x^{n+1} > 0\}.$$

Let B^{n+1} be the unit ball of \mathbb{R}^{n+1} centered at the origin. We define

$$B_+^{n+1} = B^{n+1} \cap \mathbb{R}_+^{n+1} \quad \text{and} \quad S_+^{n+1} := \partial B_+^{n+1}.$$

We will denote by $\Theta : B^n \rightarrow S_+^n$ the inverse of the stereographic projection from the south pole. $\Theta = (\Theta^1, \dots, \Theta^n, \Theta^{n+1})$ is a conformal parametrization of S_+^n

and, for any $z = (z^1, \dots, z^n) \in B^n$,

$$\Theta(z) = (z\mu(z), \mu(z) - 1) = \left(\frac{2z^1}{1+|z|^2}, \dots, \frac{2z^n}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2} \right)$$

with conformal factor given by

$$(2) \quad \mu(z) := \frac{2}{1+|z|^2}.$$

We often use the projection of Θ on $\mathbb{R}^n \times \{0\}$ and denote it by

$$(3) \quad \tilde{\Theta}(z) := (z, 0) \mu(z).$$

The next lemma collect some useful properties of the function Θ . We omit the proof, which can be obtained with elementary computations.

Lemma 2.3. *For every $i, j, l = 1, \dots, n$,*

$$\begin{aligned} \Theta_i^{n+1} &= -\mu \Theta^i, & \tilde{\Theta}_i &= -\Theta^i \tilde{\Theta} + \mu E_i, & \langle \Theta_i, \Theta_j \rangle &= \mu^2 \delta_{ij}, \\ \Delta_{S^n} &= \Delta_{\mathbb{R}^n} - \langle \Theta_{ii}, \Theta_k \rangle \Theta^k \partial_k. \end{aligned}$$

Here Θ_l is the partial derivative of Θ in the variable z^l .

Observe that all hypersurfaces nearby a geodesic sphere centered at $p \in \partial\Omega$ with radius r can be parametrized by a mapping $G : B^n \rightarrow \mathcal{M}$ defined by

$$(4) \quad G(z) := F^p(r(1+\omega)\tilde{\Theta}(z), r(1+\omega)\Theta^{n+1}(z)),$$

for some $p \in \partial\Omega$ and $\omega : S_+^n \rightarrow \mathbb{R}$. By construction, since $\Theta^{n+1} = 0$ on ∂S_+^n ,

$$\partial \Sigma_{p,r,\omega} \subset \partial\Omega.$$

Given $p \in \partial\Omega$ and $\omega : S_+^n \rightarrow \mathbb{R}$, throughout this paper, the expression $\Sigma_{p,r,\omega}$ will denote the hypersurface $F^p(r(1+\omega)S_+^n)$, while $E_{p,r,\omega}$ will denote the set bounded by $\Sigma_{p,r,\omega}$ and $\partial\Omega$.

Notation. Any expression of the form $L_p(\omega)$ denotes a linear combination of the function ω together with its derivatives of order up to 2 with respect to the vector fields Θ_i . Similarly, $\bar{L}_p(\omega)$ will denote such a linear combination with first derivatives. The coefficients of L_p or \bar{L}_p might depend on r and p but, for all $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $r \in (0, 1)$ and $p \in \partial\Omega$ such that

$$\|L_p(\omega)\|_{\mathcal{C}^{k,a}(\bar{S}_+^n)} \leq c \|\omega\|_{\mathcal{C}^{k+2,a}(\bar{S}_+^n)}, \quad \|\bar{L}_p(\omega)\|_{\mathcal{C}^{k,a}(\bar{S}_+^n)} \leq c \|\omega\|_{\mathcal{C}^{k+1,a}(\bar{S}_+^n)}.$$

Given $a \in \mathbb{N}$, an expression of the form $Q_p^a(\omega)$ (respectively, $\bar{Q}_p^a(\omega)$) will denote a nonlinear operator in the function ω together with its Θ_i -derivatives of order up to 2 (respectively, order 1). The coefficients of the Taylor expansion of $Q_p^a(\omega)$ in powers of ω and its partial derivatives might depend on r and p and, given

$k \in \mathbb{N}$, there exists a constant $c > 0$ independent of $r \in (0, 1)$ and $p \in \mathcal{M}$ such that $Q_p^a(0) = 0$ and

$$\begin{aligned} \|\mathcal{Q}_p^a(\omega_1) - \mathcal{Q}_p^a(\omega_2)\|_{\mathcal{C}^{k,\alpha}(\overline{S_+^n})} \\ \leq c \left(\|\omega_1\|_{\mathcal{C}^{k+2,\alpha}(\overline{S_+^n})} + \|\omega_2\|_{\mathcal{C}^{k+2,\alpha}(\overline{S_+^n})} \right)^{a-1} \|\omega_1 - \omega_2\|_{\mathcal{C}^{k+2,\alpha}(\overline{S_+^n})}, \end{aligned}$$

provided $\|\omega_i\|_{\mathcal{C}^{k+2,\alpha}(\overline{S_+^n})} \leq 1$. Also

$$\begin{aligned} \|\bar{\mathcal{Q}}_p^a(\omega_1) - \bar{\mathcal{Q}}_p^a(\omega_2)\|_{\mathcal{C}^{k,\alpha}(\overline{S_+^n})} \\ \leq c \left(\|\omega_1\|_{\mathcal{C}^{k+1,\alpha}(\overline{S_+^n})} + \|\omega_2\|_{\mathcal{C}^{k+1,\alpha}(\overline{S_+^n})} \right)^{a-1} \|\omega_1 - \omega_2\|_{\mathcal{C}^{k+1,\alpha}(\overline{S_+^n})}, \end{aligned}$$

provided $\|\omega_i\|_{\mathcal{C}^{k+2,\alpha}(\overline{S_+^n})} \leq 1$. We also agree that any term denoted by $\mathbb{C}_p(r^d)$ is a smooth function on S_+^n that might depend on p but satisfies

$$\|\mathbb{C}_p(r^d)\|_{\mathcal{C}^{k,\alpha}(\overline{S_+^n})} \leq c r^d,$$

for a constant c independent of p .

3. Mean curvature expansion of $\Sigma_{p,r,\omega}$

We now develop the expansion of the mean curvature $H(p, r, \omega)$ of a hypersurface $\Sigma_{p,r,\omega}$ in terms of r and ω . The proof is similar to the one in [Fall and Mahmoudi 2008], and we just sketch it for the reader's convenience.

Let $z \mapsto G(z)$ parametrize $\Sigma_{p,r,\omega}$ as defined in (4).

Notation. With an abuse of notation, at the point p , we let

$$\Theta := \Theta^j E_j + \Theta^{n+1} N_{\partial\Omega} = \tilde{\Theta} + \Theta^{n+1} N_{\partial\Omega}, \quad \Theta_i := \partial_{z^i} \Theta^j E_j + \partial_{z^i} \Theta^{n+1} N_{\partial\Omega},$$

while at the point $G(z)$, we define the vector fields

$$\Upsilon := \Theta^j X_j + \Theta^{n+1} X_{n+1} = \tilde{\Upsilon} + \Theta^{n+1} X_{n+1}, \quad \Upsilon_i := \partial_{z^i} \Theta^j X_j + \partial_{z^i} \Theta^{n+1} X_{n+1}.$$

We also set

$$\omega_j := \partial_{z^j} \omega, \quad \omega_{ij} := \partial_{z^i} \partial_{z^j} \omega.$$

From the notation, it is clear that the tangent space of $\Sigma_{p,r,\omega}$ is spanned by the vector fields

$$(5) \quad Z_j = G_*(\partial_{z^j}) = r(1 + \omega) \Upsilon_j + r \omega_j \Upsilon, \quad j = 1, \dots, n.$$

Letting $g_{ij}^\Sigma := \langle Z_i, Z_j \rangle$ be the first fundamental form of $\Sigma_{p,r,\omega} (= \Sigma)$, we conclude using Proposition A.1 that

$$(6) \quad (1 + \omega)^{-2} r^{-2} g_{ij}^\Sigma = \mu^2 + 2r \langle S(\tilde{\Theta}_j), \tilde{\Theta}_i \rangle \Theta^{n+1} + \mathbb{C}(r^2) + rL(\omega) + Q(\omega).$$

3.1. The normal vector field. We expand the unit normal to $\Sigma_{p,r,\omega}$. Define the vector field

$$\tilde{N}_\Sigma := -r \Upsilon + \alpha^j Z_j.$$

It is the outer normal field (not necessarily unitary) along $\Sigma_{p,r,\omega}$ if we can determine α^j so that its tangential components $\langle \tilde{N}_\Sigma, Z_j \rangle$ vanish. This leads to a linear system for α^j .

From (15)–(16) there follows the expansion

$$\langle \Upsilon, Z_j \rangle = r\omega_j + 2r^2\Theta^{n+1}\langle S(\tilde{\Theta}), \tilde{\Theta}_j \rangle + \mathcal{O}(r^3) + r^2 L(\omega) + r^2 Q(\omega),$$

Using (6) and some algebraic calculations, one obtains

$$\begin{aligned} (7) \quad \alpha^j \langle Z_j, Z_i \rangle &= r \langle \Upsilon, Z_j \rangle \\ &= r^2 (\omega_i + 2r\Theta^{n+1}\langle S(\tilde{\Theta}), \tilde{\Theta}_i \rangle + \mathcal{O}(r^2) + r L(\omega) + Q(\omega)); \end{aligned}$$

hence straightforward computations imply that

$$\alpha^k \langle \Theta_i, \Theta_k \rangle = \omega_i + 2r\Theta^{n+1}\langle S(\tilde{\Theta}), \tilde{\Theta}_i \rangle + \mathcal{O}(r^2) + r L(\omega) + Q(\omega).$$

Also using (7), we have

$$\begin{aligned} \langle \tilde{N}_\Sigma, \tilde{N}_\Sigma \rangle &= r^2 \langle \Upsilon, \Upsilon \rangle - 2r\alpha^k \langle Z_k, \Upsilon \rangle + \alpha_l \alpha_k \langle Z_k, Z_l \rangle \\ &= r^2 (1 + 2r\Theta^{n+1}\langle S(\tilde{\Theta}), \tilde{\Theta} \rangle + \mathcal{O}(r^2) + r L(\omega) + Q(\omega)) - \alpha_l \alpha_k \langle Z_k, Z_l \rangle \\ &= r^2 (1 + 2r\Theta^{n+1}\langle S(\tilde{\Theta}), \tilde{\Theta} \rangle + \mathcal{O}(r^2) + r L(\omega) + Q(\omega)). \end{aligned}$$

From this we deduce that

$$|\tilde{N}_\Sigma|^{-1} = r^{-1} (1 - r\Theta^{n+1}\langle S(\tilde{\Theta}), \tilde{\Theta} \rangle + \mathcal{O}(r^2) + r L(\omega) + Q(\omega)).$$

Therefore the unit normal can be expanded as

$$N_\Sigma = \frac{\tilde{N}_\Sigma}{|\tilde{N}_\Sigma|} = - (1 - r\Theta^{n+1}\langle S(\tilde{\Theta}), \tilde{\Theta} \rangle) \Upsilon + \alpha_k Z_k + (\mathcal{O}(r^2) + r L(\omega) + Q(\omega))_\alpha X_\alpha.$$

3.2. The second fundamental form. We turn to the expansion of the coefficients of the second fundamental form.

By definition $\nabla_{Z_i} Z_j \simeq DZ_j/dz^i$, whence we readily get the expansions

$$\begin{aligned} r^{-1} \nabla_{Z_i} Z_j &= \omega_{ij} \Upsilon + \omega_j \Upsilon_i + \omega_i \Upsilon_j + (1 + \omega) \Upsilon_{ij} + r \Theta_i^\alpha \Theta_j^\beta \nabla_{X_\alpha} X_\beta \\ &\quad + (\mathcal{O}(r^2) + r L(\omega) + Q(\omega))_\alpha X_\alpha, \end{aligned}$$

so using (15)–(16), we get

$$\begin{aligned} r^{-1} \langle N_\Sigma, \nabla_{Z_i} Z_j \rangle = & - (1 + \omega) (1 - r \Theta^{n+1} \langle S(\tilde{\Theta}), \tilde{\Theta} \rangle) \langle \Upsilon_{ij}, \Upsilon \rangle - \omega_{ij} + \alpha^k \langle \Upsilon_{ij}, \Upsilon_k \rangle \\ & + r (\Theta^{n+1} \langle S(\tilde{\Theta}_i), \tilde{\Theta}_j \rangle - \Theta_i^{n+1} \langle S(\tilde{\Theta}), \tilde{\Theta}_j \rangle - \Theta_j^{n+1} \langle S(\tilde{\Theta}), \tilde{\Theta}_i \rangle) \\ & + \mathbb{O}(r^2) + r L(\omega) + Q(\omega). \end{aligned}$$

Observing that

$$\begin{aligned} \langle \Upsilon_{ij}, \Upsilon \rangle &= \langle \Theta_{ij}, \Theta \rangle + 2r \Theta^{n+1} \langle S(\tilde{\Theta}), \tilde{\Theta}_{ij} \rangle + \mathbb{O}(r^2) + r L(\omega) + Q(\omega), \\ \langle \Upsilon_{ij}, \Upsilon_k \rangle &= \langle \Theta_{ij}, \Theta_k \rangle + \mathbb{O}(r) + r L(\omega) + Q(\omega), \end{aligned}$$

we obtain with a little work:

Proposition 3.1. *The second fundamental form of the $\Sigma_{p,r,\omega}$ has the expansion*

$$\begin{aligned} r^{-1} \langle N_\Sigma, \nabla_{Z_i} Z_j \rangle = & - (1 + \omega - r \Theta^{n+1} \langle S(\tilde{\Theta}), \tilde{\Theta} \rangle) \langle \Theta_{ij}, \Theta \rangle \\ & - (\omega_{ij} + 2r \Theta^{n+1} \langle S(\tilde{\Theta}), \tilde{\Theta}_{ij} \rangle) + \alpha^k \langle \Theta_{ij}, \Theta_k \rangle \\ & + r (\Theta^{n+1} \langle S(\tilde{\Theta}_i), \tilde{\Theta}_j \rangle - \Theta_i^{n+1} \langle S(\tilde{\Theta}), \tilde{\Theta}_j \rangle - \Theta_j^{n+1} \langle S(\tilde{\Theta}), \tilde{\Theta}_i \rangle) \\ & + \mathbb{O}(r^2) + r L(\omega) + Q(\omega). \end{aligned}$$

Let $H(p, r, \omega)$ be the mean curvature of the hypersurface $\Sigma_{p,r,\omega}$. Contracting with the metric, (6), and using also Lemma 2.3, we have:

Proposition 3.2. *In the notation above, we have*

$$\begin{aligned} r H(p, r, \omega) = & n - (\Delta_{S_+^n} \omega + n\omega) + r \Theta^{n+1} ((n+3) \langle S(\tilde{\Theta}), \tilde{\Theta} \rangle - \langle S(E_i), E_i \rangle) \\ & + \mathbb{O}(r^2) + r L(\omega) + Q(\omega) \end{aligned}$$

in $\Sigma_{p,r,\omega}$ and

$$\langle N_{\partial \Sigma_{p,r,\omega}}^{\Sigma_{p,r,\omega}}, N_{\partial \Sigma_{p,r,\omega}}^{\partial \Omega} \rangle = - \frac{\partial \omega}{\partial \eta} + r^2 \bar{L}(\omega) + \bar{Q}(\omega) \quad \text{on } \partial \Sigma_{p,r,\omega},$$

where $\eta := -N_{\partial \Omega}$ is the outer unit normal to ∂S_+^n and N_B^A is the normal of B in A .

Proof. We first determine $N_{\partial \Sigma_{p,r,\omega}}^{\partial \Omega}$. Fix a parametrization $s \mapsto \bar{\Theta}(s) \in S^{n-1} = \partial B^n$ of the unit sphere. Noting that $\Theta(\bar{\Theta}(s)) = \bar{\Theta}(s)$, we see that the mapping

$$s \mapsto \bar{G}(s) := f^p(r(1+\omega)\bar{\Theta}(s)) = F^p(r(1+\omega)\bar{\Theta}(s), 0)$$

parametrizes $\partial \Sigma_{p,r,\omega} \subset \partial \Omega$ and hence its tangent space is spanned by

$$\bar{Z}_i = r(1+\omega) \bar{\Upsilon}_i + r \partial_{s^i} \omega \bar{\Upsilon}, \quad i = 1, \dots, n-1,$$

where

$$\bar{\Upsilon} := \bar{\Theta}^j Y_j, \quad \bar{\Upsilon}_i := \partial_{s^i} \bar{\Theta}^j Y_j.$$

Therefore, setting

$$(8) \quad \tilde{N}_{\partial \Sigma_{p,r,\omega}}^{\partial \Omega} = -r \bar{\Upsilon} + \beta^k \bar{Z}_k,$$

we need only to find β^k , $k = 1, \dots, n$ so that it is orthogonal to \bar{Z}_i . This can be found in [Pacard and Xu 2009, Lemma 2.1] and one has

$$(9) \quad \beta^k \langle \bar{Z}_k, \bar{Z}_i \rangle = r \partial_{s^i} \omega$$

while $r^{-2} \langle \bar{Z}_k, \bar{Z}_i \rangle = \langle \bar{\Theta}_k, \bar{\Theta}_i \rangle (1 + \mathbb{O}(r^2) + r^2 L(\omega) + Q(\omega))$ and $|\tilde{N}_{\partial \Sigma_{p,r,\omega}}^{\partial \Omega}|^{-1} = r^{-1}(1 + Q(\omega))$.

We now determine $N_{\partial \Sigma_{p,r,\omega}}^{\Sigma_{p,r,\omega}}$. To this aim, we denote by ν the unit outer normal to the unit disc B^n and similarly as we have expanded $\tilde{N}_{\partial \Sigma_{p,r,\omega}}^{\partial \Omega}$, we let

$$\begin{aligned} \tilde{N}_{\partial \Sigma}^{\Sigma} &= G_*(\partial_\nu)|_{\partial B_1^n} + \gamma^k \bar{Z}_k = -r(1 + \omega)N_{\partial \Omega} + r \partial_\nu \omega \tilde{\Upsilon}|_{\partial B_1^n} + \gamma^k \bar{Z}_k \\ &= -r(1 + \omega)N_{\partial \Omega} + r \partial_\nu \omega \bar{\Upsilon} + \gamma^k \bar{Z}_k. \end{aligned}$$

Here, we have used that $\partial_\nu \tilde{\Theta}|_{\partial B_1^n} = 0$ and $\partial_\nu \Theta^{n+1}|_{\partial B_1^n} = -1$. Noting that $\langle N_{\partial \Omega}, \bar{Z}_j \rangle$ vanishes and $\langle \bar{\Upsilon}, \bar{Z}_j \rangle = r \omega_j$, we see that $\tilde{N}_{\partial \Sigma}^{\Sigma} \in T \Sigma_{p,r,\omega}$ is normal to $\partial \Sigma_{p,r,\omega}$ if

$$\gamma^k \langle \bar{Z}_k, \bar{Z}_j \rangle = r^2 \omega_j \partial_\nu \omega.$$

We also deduce that $|\tilde{N}_{\partial \Sigma}^{\Sigma}|^{-1} = r^{-1}(1 + Q(\omega))$.

Collecting these with the fact that $\langle N_{\partial \Sigma}^{\Sigma}, N_{\partial \Sigma}^{\partial \Omega} \rangle = 0$ when $\omega = 0$, we obtain

$$\langle N_{\partial \Sigma}^{\Sigma}, N_{\partial \Sigma}^{\partial \Omega} \rangle = -\frac{\partial \omega}{\partial \eta} + r^2 \bar{L}(\omega) + \bar{Q}^2(\omega) \quad \text{on } \partial \Sigma_{p,r,\omega},$$

because $\partial_\nu \omega = \frac{\partial \omega}{\partial \eta}$ (since $\mu = 1$). □

4. Proof of Theorem 1.2 and expansions of the isoperimetric profile I_Ω

P. Bérard and D. Meyer [1982, Appendix C] have shown by a localization argument that the isoperimetric profile of a compact Riemannian manifold asymptotically approaches that of \mathbb{R}^{n+1} . V. Bayle and C. Rosales [2005, Proposition 2.1] proved that the relative isoperimetric profile of a domain Ω of a Riemannian manifold behaves like the profile of the half space \mathbb{R}_+^{n+1} . Precisely, with

$$\begin{aligned} I(r) &\asymp I_\Omega(|r B_+^{n+1}|) = \min_{\substack{E \subset \Omega \\ |E|_g = |r B_+^{n+1}|}} \mathcal{P}_g(E, \Omega), \\ I_+(r) &:= I_{\mathbb{R}_+^{n+1}}(|r B_+^{n+1}|) = \mathcal{P}(r B_+^{n+1}, \mathbb{R}_+^{n+1}), \end{aligned}$$

they proved:

Proposition 4.1. *For any $\varepsilon > 0$, there exists $r_0(\varepsilon) > 0$ such that*

$$(1 - \varepsilon)I_+(r) \leq I(r) \leq (1 + \varepsilon)I_+(r), \quad \text{whenever } r \leq r_0.$$

Notice that from this upper bound, an isoperimetric region with small volume must touch the boundary (perpendicularly) because otherwise it would contradict

Bérard and Meyer's lower bound. Moreover this upper bound will help after suitable scaling together with the Heintze–Karcher inequality to obtain a uniform bound for the mean curvature of the minimizing hypersurface trapping a small volume; see [Morgan and Johnson 2000, Section 2].

We start by proving the following regularity result which were obtained in [Morgan and Johnson 2000] and under weaker assumptions in [Nardulli 2006] in compact Riemannian manifolds.

Lemma 4.2. *There exists $r_0 > 0$ such that if $r \in (0, r_0)$ and $E \subset \Omega$ is any set such that $\mathcal{P}_g(E, \Omega) = I(r)$, there exist $p \in \partial\Omega$ and $\omega^{p,r} : S_+^n \rightarrow \mathbb{R}$ such that*

$$\overline{\partial E \cap \Omega} = F^p(r(1 + \omega^{p,r} \bar{S}_+^n))$$

with $\|\omega^{p,r}\|_{\mathcal{C}^{2,\alpha}(S_+^n)} + \|\omega^{p,r}\|_{\mathcal{C}^{1,\alpha}(\bar{S}_+^n)} \rightarrow 0$ as $r \rightarrow 0$.

Proof. Let $E_j \subset \Omega$ such that $\mathcal{P}_g(E_j, \Omega) = I(r_j)$, $r_j \rightarrow 0$ as $j \rightarrow +\infty$. Define $\Omega_j := (1/r_j)\Omega$ and $E'_j := (1/r_j)E_j$, so $|E'_j|_{g_j} = |B_+^{n+1}|$ and $\mathcal{P}_{g_j}(E'_j, \Omega_j) = (1/r_j^n)\mathcal{P}_g(E_j, \Omega) \leq c' I_+(1)$.

Following [Morgan and Johnson 2000, Section 2] with the help of Proposition 4.1, we may assume that there exists a constant $R > 0$ such that

$$\text{diam}_{g_j}(E'_j) \leq R$$

and since $\partial E'_j$ intersects $\partial\Omega_j$, then

$$\sup_{e \in \partial E_j} \text{dist}_{g_j}(e, \partial\Omega_j) \leq \text{diam}_{g_j}(E'_j) \leq R.$$

We can let $p_j \in \partial\Omega_j$ and $U_j \subset \mathbb{R}_+^{n+1}$ be such that $E'_j = F_j(U_j)$, where the map $F_j : \gamma_j B_+^{n+1} \rightarrow \Omega_j$ is defined by $F_j(\cdot) := (1/r_j)F^{p_j}(r_j(\cdot))$ and $\gamma_j \rightarrow \infty$ as $j \rightarrow \infty$.

For j fixed and sufficiently large, let $h_j := (F_j)_*(g_j)$ be the metric induced by F_j on \mathbb{R}_+^{n+1} . Then U_j minimizes the perimeter $\mathcal{P}_{h_j}(\cdot, \gamma_j B_+^{n+1})$ in $(\gamma_j B_+^{n+1}, h_j)$ among sets enclosing its volume $|U_j|_{h_j} = |B_+^{n+1}|$ and also intersects $\partial\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \{0\}$ perpendicularly.

Now since h_j is converging to the Euclidean metric, we get $\text{diam}(U_j) \leq c$ for every large j . So $\mathcal{P}_{h_j}(U_j, \mathbb{R}_+^{n+1}) \leq c$, which implies that $\mathcal{P}(U_j, \mathbb{R}_+^{n+1}) \leq c$. Hence by compactness there exists $U \subset \mathbb{R}_+^{n+1}$ such that $D1_{U_j} \xrightarrow{*} D1_U$. Furthermore by the trace theorem,

$$1_{U_j}|_{\mathbb{R}^n \times \{0\}} \xrightarrow{L^1} 1_U|_{\mathbb{R}^n \times \{0\}}.$$

Now to see that U is a minimizer, we take $V \subset \mathbb{R}_+^{n+1}$ such that $|V| = |B_+^{n+1}|$ and define $c_j \rightarrow 1$ such that $c_j|V|_{h_j} = |B_+^{n+1}|$ (this is possible since h_j too converges

to the Euclidean metric). But then we have

$$\mathcal{P}_{h_j}(U_j, \mathbb{R}_+^{n+1}) \leq c_j^{n/(n+1)} \mathcal{P}_{h_j}(V, \mathbb{R}_+^{n+1})$$

and this, together with the semicontinuity of the perimeter, implies that

$$\mathcal{P}(U, \mathbb{R}_+^{n+1}) \leq \mathcal{P}(V, \mathbb{R}_+^{n+1}).$$

We conclude that U is a minimizer in \mathbb{R}_+^{n+1} among sets that enclose the volume $|B_+^{n+1}|$; therefore U is a half-ball. Finally, again by results from [Morgan and Johnson 2000, Section 2], we have smooth convergence because mean curvatures are bounded. Hence we may assume that there exists $\omega^{p_j, r_j} \in \mathcal{C}^{2, \alpha}(S_+^n)$ such that

$$(10) \quad \Sigma_j := \partial U_j \cap \mathbb{R}_+^{n+1} = (1 + \omega^{p_j, r_j}) S_+^n$$

with $\|\omega^{p_j, r_j}\|_{\mathcal{C}^{2, \alpha}(S_+^n)}$ tending to zero as $j \rightarrow \infty$.

We now estimate the free boundary, $\mathcal{H}^n(\partial \Sigma_j)$, by slicing with hyperplanes $\mathbb{R}^n \times \{0\} + hN_{\partial \Omega}$ with $h \in \mathbb{R}$. For terminology, we refer the reader to [Morgan 1988]. In the following, with an abuse of notation, we will call Σ_j the integer rectifiable current associated to the set Σ_j . We define $m_j(h)$ by

$$m_j(h) := \mathcal{H}^n(\Sigma_j \cap \{d < h\}) = \mathcal{P}(U_j, \{d_j < h\}) \quad \text{for } 0 < h < \frac{1}{2},$$

where $\mathbb{R}^{n+1} \ni x \mapsto d(x) = x^{n+1}$ is the distance function from $\partial \mathbb{R}_+^{n+1} = \mathbb{R}^n \times \{0\}$. For $h \geq 0$ we consider the slice

$$\langle \Sigma_j, d, h_+ \rangle = (\partial \Sigma_j) \llcorner \{d > h\} - \partial(\Sigma_j \llcorner \{d > h\}).$$

We deduce that $\langle \Sigma_j, d, 0_+ \rangle = \partial \Sigma_j$. From [Morgan 1988, Section 4.11, (3)], we then get

$$\mathcal{H}^{n-1}(\langle \Sigma_j, d, 0_+ \rangle) \leq \text{Lip}(d) \liminf_{h \searrow 0} \frac{m_j(h)}{h} = \liminf_{h \searrow 0} \frac{m_j(h)}{h},$$

the equality being a consequence of $\text{Lip}(d) = 1$. Since $m_j(h)$ is increasing the same argument yields, for \mathcal{L}^1 a.e. $h > 0$,

$$\mathcal{H}^{n-1}(\langle \Sigma_j, d, h_+ \rangle) \leq m'_j(h).$$

By (10) and Lemma A.3,

$$\begin{aligned} m_j(h) &= \mathcal{P}(U_j, \{d < h\}) = \mathcal{H}^n((1 + \omega^{p_j, r_j}) S_+^n \cap \{d < h\}) \\ &\leq h(1 + O(r_j)) \mathcal{H}^{n-1}(\langle \Sigma_j, d, h_+ \rangle) \leq h(1 + O(r_j)) m'_j(h). \end{aligned}$$

Hence we get $m_j(h) \leq 2h m'_j(h)$, which is equivalent to

$$\left(\frac{m_j(h)}{h} + 2m_j(h) \right)' \geq 0$$

for every \mathcal{L}^1 a.e. $\frac{1}{2} > h > 0$. From this and the fact that m_j is increasing we conclude that

$$\mathcal{H}^{n-1}(\partial \Sigma_j) \leq \mathcal{H}^n(\Sigma_j \setminus \{d < h\}) \left(\frac{1}{h} + 2 \right) \quad \text{for every } h \in (0, \frac{1}{2}).$$

From this together with [Lemma A.3](#) we have

$$\mathcal{H}^{n-1}(\partial \Sigma_j) \leq c \mathcal{H}^n(\Sigma_j) \leq \tilde{c} \quad \text{for any large } j.$$

Consequently, Σ_j is an integral current, and by compactness [[Morgan 1988](#), 5.5], $\partial \Sigma_j$ converges weakly to ∂S_+^n . Since mean curvatures of Σ_j are bounded (see also [[Morgan and Johnson 2000](#), (2.4)]), $C^{1,\alpha}$ convergence up to the free boundary follows by [[Grüter and Jost 1986](#)]. Hence finally we can assume that $\omega^{p_j, r_j} \in \mathcal{C}^{1,\alpha}(\overline{S_+^n})$ if j is sufficiently large with

$$\partial E'_j \cap \overline{\Omega}_j = \frac{1}{r_j} F^{p_j}(r_j(1 + \omega^{p_j, r_j}) \overline{S_+^n}),$$

and $\|\omega^{p_j, r_j}\|_{\mathcal{C}^{2,\alpha}(S_+^n)} + \|\omega^{p_j, r_j}\|_{\mathcal{C}^{1,\alpha}(\overline{S_+^n})} \rightarrow 0$ as $r \rightarrow 0$. \square

Remark 4.3. When applying the first compactness result, namely $D1_{U_j} \xrightarrow{*} D1_U$, we also have (by Rellich's theorem) that

$$1_{U_j} \xrightarrow{L^1} 1_U.$$

Since $U = B_+^{n+1}$, by [[Nardulli 2006](#)], $\partial U_j \cap \mathbb{R}_+^{n+1}$ can be written as a normal graph over S_+^n by a smooth function ω^{p_j, r_j} for which $\|\omega^{p_j, r_j}\|_{\mathcal{C}^{2,\alpha}(S_+^n)} \rightarrow 0$ as $r \rightarrow 0$.

Note also that $C^{k,\alpha}$ regularity and estimates of ω^{p_j, r_j} can be obtained by a bootstrap argument using [Proposition 3.2](#), as in [[Morgan and Johnson 2000](#); [Nardulli 2006](#)].

The following lemma shows the smoothness of the *center of mass* $c(r, p, \omega) \in \partial \Omega$ of the hypersurface $\Sigma_{p,r,\omega} := F^p(r(1 + \omega) \overline{S_+^n})$ as a function in r , p and ω . The proof can be obtained from [[Nardulli 2007](#), Lemmas 1.3–1.4], with slight modifications.

Lemma 4.4. *There exists a smooth map $c : \mathbb{R} \times \partial \Omega \times \mathcal{C}^{2,\alpha}(\overline{S_+^n}) \rightarrow \partial \Omega$ such that*

$$\int_{\Sigma_{p,r,\omega}} (F^c)^{-1}(z) d\text{vol}_{\Sigma_{p,r,\omega}} = 0.$$

Moreover there exists a smooth vector field $X_{p,r,\omega}$ on $T_p \partial \Omega$ such that

$$X_{p,0,\omega} = \frac{\int_{S_+^n} (1 + \omega)^n \tilde{\Theta} \sqrt{\|d\omega\|^2 + (1 + \omega)^2} d\sigma}{\int_{S_+^n} (1 + \omega)^{n-1} \sqrt{\|d\omega\|^2 + (1 + \omega)^2} d\sigma}.$$

and $c(r, p, \omega) = \exp_p^{\partial \Omega}(r X_{p,r,\omega})$. (Here $d\omega$ is the differential of ω .)

According to [Proposition 3.2](#), with $H(p, r, \omega)$ being the mean curvature of $\Sigma_{p,r,\omega}$, we define $T(p, r, \cdot) : \mathcal{C}^{2,\alpha}(\overline{S_+^n}) \rightarrow \mathcal{C}^{0,\alpha}(\overline{S_+^n})$

$$(11) \quad \int_{S_+^n} T(p, r, \omega) \omega' d\sigma := \int_{S_+^n} r H(p, r, \omega) \omega' d\sigma - \oint_{\partial S_+^n} \langle N_{\partial \Sigma_{p,r,\omega}}^{\partial \Omega}, N_{\partial \Sigma_{p,r,\omega}}^{\Sigma_{p,r,\omega}} \rangle \omega' ds,$$

for every $\omega' \in L^2$.

Consider the eigenvalue problem

$$\Delta_{S_+^n} u = \lambda u \quad \text{in } S_+^n, \quad \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial S_+^n.$$

Letting

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$$

be the eigenvalues, up to a reflection, we know that $\lambda_k = n + k - 1$ and the eigenspaces corresponding to $\lambda_0 = 0$ and $\lambda_1 = n$ are

$$(12) \quad \Lambda_0 := \text{span}\{1\}, \quad \Lambda_1 := \text{span}\{\Theta^1, \dots, \Theta^n\}.$$

We denote by Π_0 and Π_1 the L^2 projections onto these spaces and we define

$$\Pi := \text{Id} - \Pi_1 - \Pi_0, \quad \Pi_1^\perp := \Pi_0 + \Pi.$$

We recall that $E_{p,r,\omega}$ is the set bounded by the hypersurface $\Sigma_{p,r,\omega}$ and $\partial \Omega$.

Definition 4.5. A set $E_{p,r,\omega}$ is called a *pseudo-half-ball* if $\Pi \circ T(p, r, \omega) \equiv 0$, or equivalently

$$\Pi_1^\perp \circ T(p, r, \omega) \equiv \text{Const} \in \mathbb{R},$$

where $T(p, r, \omega)$ is defined in [\(11\)](#).

Letting $\Xi \in T_p \partial \Omega$ be such that $\Pi_1 \omega = \langle \Xi, \tilde{\Theta} \rangle$, we get by [Lemma 4.4](#)

$$(13) \quad c(r, p, \omega) = p + \frac{|S_+^n|}{n+1} r \Xi + r^2 \{L_p(\omega) + \mathbb{C}(r) + Q_p(\omega)\}^a E_a.$$

From the expansion of the volume of the sets $E_{p,r,\omega}$ ([Lemma A.4](#)), we define

$$\Phi(p, r, \omega) := r^{-n-1} |E_{p,r,\omega}|_g - |B_+^{n+1}| = \int_{S_+^n} \omega d\sigma + \mathbb{C}(r) + \int_{S_+^n} (\mathbb{C}(r) \omega + \hat{Q}_p(\omega)) d\sigma.$$

It turns out that

$$\Phi(p, 0, 0) = 0, \quad \frac{\partial \Phi(p, 0, 0)}{\partial \omega} [u] = \Pi_0 u.$$

Now, we can associate to any hypersurface $\Sigma_{p,r,\omega}$ the smooth mapping

$$\begin{aligned} \Psi : \partial \Omega \times (0, 1) \times \mathcal{C}^{2,\alpha}(\overline{S_+^n}) &\rightarrow T_p \partial \Omega \times \Pi \mathcal{C}^{0,\alpha}(\overline{S_+^n}) \times \mathbb{R}, \\ \Psi(p, r, \omega) &:= \left(\frac{n+1}{|S_+^n|} X_{p,r,\omega}, \Pi \circ T(p, r, \omega), -n \Phi(p, r, \omega) \right). \end{aligned}$$

Lemma 4.6. *There exist $r_0 > 0$ and $c_0 > 0$ such that for any $p \in \partial\Omega$ and $r \in (0, r_0)$, there exists a unique smooth $\omega^{p,r} \in \mathcal{C}^{2,\alpha}(\overline{S_+^n})$ with*

$$\|\omega^{p,r}\|_{\mathcal{C}^{2,\alpha}(\overline{S_+^n})} \leq c_0 r_0$$

such that $\Psi(p, r, \omega^{p,r}) = (0, 0, 0)$, namely

$$c(r, p, \omega^{p,r}) = p, \quad \Pi_1^\perp \circ T(p, r, \omega^{p,r}) \in \mathbb{R} \quad \text{and} \quad |E_{p,r,\omega^{p,r}}|_g = |r B_+^{n+1}|$$

for every $r \in (0, r_0)$.

Proof. We make the identification $\mathcal{C}^{k,\alpha}(\overline{S_+^n}) \equiv T_p \partial\Omega \times \Pi \mathcal{C}^{k,\alpha}(\overline{S_+^n}) \times \mathbb{R}$. Any $u \in \mathcal{C}^{k,\alpha}(\overline{S_+^n})$ has a decomposition $u = \langle \Xi, \tilde{\Theta} \rangle + \Pi u + u_0 = \langle \Xi, \tilde{\Theta} \rangle + w$.

It is easy to see that $\Psi(p, 0, 0) = (0, 0, 0)$ while

$$\left\langle \frac{\partial \Psi}{\partial \omega}(p, 0, 0)[u], u' \right\rangle = \int_{S_+^n} \nabla_{S_+^n} w \nabla_{S_+^n} w' - n w w' d\sigma + \int_{S_+^n} \langle \Xi, \tilde{\Theta} \rangle \langle \Xi', \tilde{\Theta}' \rangle d\sigma.$$

Since $\frac{\partial \Psi}{\partial \omega}(p, 0, 0)$ is an isomorphism from $\mathcal{C}^{2,\alpha}(\overline{S_+^n})$ in to $\mathcal{C}^{0,\alpha}(\overline{S_+^n})$, the lemma follows by the implicit function theorem. \square

By choosing r_0 small enough in Lemma 4.6, we may assume that the hypersurfaces $\Sigma_{p,r,\omega^{p,r}}$ are embedded in Ω for any $r \in (0, r_0)$, since $\|\omega^{p,r}\|_{\mathcal{C}^{1,\alpha}(\overline{S_+^n})} \rightarrow 0$ as $r_0 \rightarrow 0$. For simplicity, we will call $E_{p,r} := E_{p,r,\omega^{p,r}}$ the sets bounded by $\Sigma_{p,r,\omega^{p,r}}$ and $\partial\Omega$.

The preceding lemma yields, for any fixed $r \in (0, r_0)$, a manifold of pseudo-half-balls diffeomorphic to $\partial\Omega$ and having volume $|r B_+^{n+1}|$ equal to

$$\mathfrak{C}_r := \{E_{p,r,\omega^{p,r}} \subset \Omega : \Psi(p, r, \omega^{p,r}) = (0, 0, 0), \|\omega^{p,r}\|_{\mathcal{C}^{2,\alpha}(\overline{S_+^n})} \leq c_0 r_0, p \in \partial\Omega\}.$$

Lemma 4.7. *If $r \ll 1$, then*

$$I(r) = \inf_{E \in \mathfrak{C}_r} \mathcal{P}_g(E, \Omega) = \inf_{p \in \partial\Omega} \mathcal{P}_g(E_{p,r}, \Omega),$$

where $E_{p,r}$, $p \in \partial\Omega$, ranges over the elements of \mathfrak{C}_r .

Proof. Let E be a solution to the isoperimetric problem with $|E|_g = |r B_+^{n+1}|$. We have to check that E belongs to \mathfrak{C}_r if r is small enough. Assume that $r \ll 1$. Lemma 4.2 implies that $\partial E \cap \overline{\Omega} = F^q(r(1 + u^{q,r})\overline{S_+^n})$ for some $q \in \partial\Omega$ and $\|u^{q,r}\|_{\mathcal{C}^{2,\alpha}(\overline{S_+^n})}$ approaching 0 as $r \rightarrow 0$.

Let $p \in \partial\Omega$ be the center of mass of ∂E . By (13),

$$\text{dist}_g(p, q) \leq c(r^2 + r \|u^{q,r}\|_{\mathcal{C}^{2,\alpha}(\overline{S_+^n})})$$

so if $r \ll 1$, we can find $v(p, r)$ with $\|v(p, r)\|_{\mathcal{C}^{2,\alpha}(\overline{S_+^n})} \rightarrow 0$ as $r \rightarrow 0$ such that $\overline{\partial E \cap \Omega} = \overline{\Sigma}_{p,r,v(p,r)}$. Since p is the center of mass, $X_{p,r,v(p,r)} = 0$. From the mean curvature expansions, we get $\Pi \circ T(p, r, v(p, r)) = 0$ because the mean

curvature of ∂E is constant and ∂E intersects $\partial\Omega$ perpendicularly. Consequently $\Psi(p, r, v(p, r)) = (0, 0, 0)$. We conclude that $E \in \mathfrak{C}_r$ if r is small enough. \square

Lemma 4.8. *For any $E_{p,r} \in \mathfrak{C}_r$, we have*

$$\mathcal{P}_g(E_{p,r}, \Omega) = \mathcal{P}(rB^{n+1}, \mathbb{R}_+^{n+1}) - \frac{n}{n+2} \frac{|B^n|}{|B_+^{n+1}|} H_{\partial\Omega}(p) |rB_+^{n+1}| + O_p(|rB_+^{n+1}|^{(n+2)/(n+1)}),$$

where $O_p(\rho)$ is smooth and tends to zero as $\rho \rightarrow 0$ uniformly in p .

Proof. Let $E_{p,r} \in \mathfrak{C}_r$. Differentiating $\Phi(p, r, \omega^{p,r}) = 0$ with respect to r , we deduce

$$\Pi_0 \omega^{p,r} = \int_{S_+^n} \omega^{p,r} d\sigma = -\frac{r}{n+2} \langle S_p(E_i), E_i \rangle \int_{S_+^n} \Theta^{n+1} d\sigma + O_p(r^2).$$

Together with Lemma A.3, this gives

$$\begin{aligned} & r^{-n} \mathcal{P}_g(E_{p,r}, \Omega) \\ &= \mathcal{P}(B^{n+1}, \mathbb{R}_+^{n+1}) + r \int_{S_+^n} (\langle S_p(E_i), E_i \rangle - \langle S_p(\tilde{\Theta}), \tilde{\Theta} \rangle) \Theta^{n+1} d\sigma + nr \int_{S_+^n} \omega_0^p d\sigma + O_p(r^2) \\ &= \mathcal{P}(B^{n+1}, \mathbb{R}_+^{n+1}) + \frac{2r}{n+2} \langle S_p(E_i), E_i \rangle \int_{S_+^n} \Theta^{n+1} d\sigma - \int_{S_+^n} \langle S_p(\tilde{\Theta}), \tilde{\Theta} \rangle \Theta^{n+1} d\sigma + O_p(r^2). \end{aligned}$$

Recall that $H_{\partial\Omega}(p) = -\frac{1}{n} \langle S_p(E_i), E_i \rangle$. Moreover since

$$\int_{S_+^n} \langle S_p(\tilde{\Theta}), \tilde{\Theta} \rangle \Theta^{n+1} d\sigma = \langle S_p(E_i), E_j \rangle \int_{S_+^n} \Theta^i \Theta^j \Theta^{n+1} d\sigma$$

and observing that $\int_{S_+^n} \Theta^i \Theta^j \Theta^{n+1} d\sigma = 0$ if $i \neq j$, we deduce that

$$\mathcal{P}_g(E_{p,r}, \Omega) = \mathcal{P}(rB^{n+1}, \mathbb{R}_+^{n+1}) - c_n H_{\partial\Omega}(p) |rB_+^{n+1}| + O_p(|rB_+^{n+1}|^{(n+2)/(n+1)}),$$

with

$$c_n = \frac{n}{|B_+^{n+1}|} \int_{S_+^n} \left(\frac{2}{n+2} - (\Theta^1)^2 \right) \Theta^{n+1} d\sigma = \frac{n}{n+2} \frac{|B^n|}{|B_+^{n+1}|}.$$

We have used equalities

$$(14) \quad \int_{S_+^n} \Theta^{n+1} d\sigma = \frac{\text{Area } S^{n-1}}{n} = |B^n|, \quad \int_{S_+^n} (\Theta^i)^2 \Theta^{n+1} d\sigma = \frac{\text{Area } S^{n-1}}{n(n+2)}. \quad \square$$

The proof of Theorem 1.2 is finalized by the following:

Lemma 4.9. *Let r_k be a sequence tending to 0 and let $E_k \subset \Omega$ satisfy $|E_k|_g = |r_k B_+^{n+1}|$ and $\mathcal{P}_g(E_k, \Omega) = I(r_k)$. Let $p_k \in \partial\Omega$ be the center of mass of ∂E_k , and suppose p_k converges to a point $p \in \partial\Omega$. Then*

$$H_{\partial\Omega}(p) = \max_{q \in \partial\Omega} H_{\partial\Omega}(q).$$

Proof. If k is large enough, $E_k = E_{p_k, r_k} \in \mathfrak{E}_{r_k}$ and also by [Lemma 4.7](#) we have that

$$\mathcal{P}_g(E_{p_k, r_k}, \Omega) = I(r_k) = \min_{q \in \partial\Omega} \mathcal{P}_g(E_{q, r_k}, \Omega)$$

where E_{p, r_k} , $p \in \partial\Omega$, denote the elements of \mathfrak{E}_{r_k} . Now by [Lemma 4.8](#), we have

$$-H_{\partial\Omega}(p_k) + O(p_k, r_k) = \min_{q \in \partial\Omega} (-H_{\partial\Omega}(q) + O(q, r_k))$$

with $|O(p_k, r_k)| \rightarrow 0$ and $\sup_{q \in \partial\Omega} |O(q, r_k)| \rightarrow 0$ when k tends to infinity. The lemma then follows by taking k to infinity. \square

Appendix A: Expansions for the area and enclosed volume

Consider a hypersurface $\partial E_{p, r, \omega} = \Sigma_{p, r, \omega}$. Recall that the mapping

$$z \mapsto G(z) = F^p(r(1+\omega)\tilde{\Theta}(z), r(1+\omega)\Theta^{n+1}(z))$$

(see [\(4\)](#) and [\(5\)](#)) parametrizes $\Sigma_{p, r, \omega}$ and its tangent space is spanned by the vector fields

$$Z_j = G_*(\partial_{z^j}) = r(1+\omega)\Upsilon_j + r\omega_j\Upsilon, \quad j = 1, \dots, n.$$

At the point $G(z)$, [Proposition 2.2](#) yields

$$\begin{aligned} (15) \quad \langle X_i, X_j \rangle &= \delta_{ij} + 2r(1+\omega)\langle S(\tilde{\Upsilon}_i), \tilde{\Upsilon}_j \rangle \Theta^{n+1} + \frac{1}{3}r^2(1+\omega)^2 \langle R_p(\tilde{\Theta}, \tilde{\Theta}_i)\tilde{\Theta}, \tilde{\Theta}_j \rangle \\ &\quad + (\langle R_p^{\mathcal{M}}(N_{\partial\Omega}, \tilde{\Theta}_i)N_{\partial\Omega}, \tilde{\Theta}_j \rangle + \langle S(\tilde{\Upsilon}_i), S(\tilde{\Upsilon}_j) \rangle)(\Theta^{n+1})^2 \\ &\quad + \mathbb{O}(r^3) + r^3L(\omega) + r^3Q(\omega), \end{aligned}$$

$$(16) \quad \langle X_i, X_{n+1} \rangle = \mathbb{O}(r^3) + r^3L(\omega) + r^3Q(\omega), \quad \langle X_{n+1}, X_{n+1} \rangle = 1.$$

Letting $g_{ij}^{\Sigma_{p, r, \omega}} := \langle Z_i, Z_j \rangle$, we then get the first fundamental form of $\Sigma_{p, r, \omega}$.

Proposition A.1.

$$\begin{aligned} &(1+\omega)^{-2}r^{-2}g_{ij}^{\Sigma_{p, r, \omega}} \\ &= \mu^2\delta_{ij} + (2\omega_j\langle S(\tilde{\Upsilon}), \tilde{\Upsilon}_i \rangle + 2\omega_i\langle S(\tilde{\Upsilon}), \tilde{\Upsilon}_j \rangle + 2(1+\omega)\langle S(\tilde{\Upsilon}_j), \tilde{\Upsilon}_i \rangle)r\Theta^{n+1} \\ &\quad + \omega_i\omega_j + \frac{1}{3}\langle R_p(\tilde{\Theta}, \tilde{\Theta}_i)\tilde{\Theta}, \tilde{\Theta}_j \rangle r^2 + \langle S(\tilde{\Upsilon}_j), S(\tilde{\Upsilon}_i) \rangle r^2(\Theta^{n+1})^2 \\ &\quad + (\Theta^{n+1})^2 \langle R_p^{\mathcal{M}}(N_{\partial\Omega}, \tilde{\Theta}_i)N_{\partial\Omega}, \tilde{\Theta}_j \rangle r^2 + \mathbb{O}(r^3) + r^2L(\omega) + rQ^2(\omega) + Q^3(\omega). \end{aligned}$$

Using this formula we can deduce the expansion of the volume form.

Lemma A.2. *Under the notation above, the volume form expands as*

$$\begin{aligned}
 r^n \sqrt{\det g^{\Sigma_{p,r,\omega}}} = & \mu^n + r \Theta^{n+1} \mu^{n-2} \langle S(\tilde{\Upsilon}_i), \tilde{\Upsilon}_i \rangle + n \omega \mu^n \\
 & + r \Theta^{n+1} \mu^{n-2} (3(n+1) \omega \langle S(\tilde{\Upsilon}_i), \tilde{\Upsilon}_i \rangle + 2 \omega_i \langle \tilde{\Upsilon}, \tilde{\Upsilon} \rangle) \\
 & + \frac{1}{2} r^2 (\Theta^{n+1})^2 \mu^{n-2} (\langle S(\tilde{\Upsilon}_i), S(\tilde{\Upsilon}_i) \rangle + \mu^{-2} |\langle S(\tilde{\Upsilon}_i), \tilde{\Upsilon}_i \rangle|^2 - 2 \mu^{-2} |\langle S(\tilde{\Upsilon}_i), \tilde{\Upsilon}_j \rangle|^2) \\
 & + \frac{1}{6} r^2 \mu^{n-2} \langle R_p(\tilde{\Theta}, \tilde{\Theta}_i) \tilde{\Theta}, \tilde{\Theta}_i \rangle + \frac{1}{2} r^2 (\Theta^{n+1})^2 \mu^{n-2} \langle R_p^{\mathcal{M}}(N_{\partial\Omega}, \tilde{\Theta}_i) N_{\partial\Omega}, \tilde{\Theta}_i \rangle \\
 & + \frac{1}{2} \mu^{n-2} (\omega_i^2 + n(n-1) \mu^2 \omega^2) + \mathcal{O}(r^3) + r^2 L(\omega) + r Q^2(\omega) + Q^3(\omega).
 \end{aligned}$$

Observe that

$$\begin{aligned}
 (17) \quad \langle S(Y_k), Y_l \rangle \Big|_{f^p(r(1+\omega)\tilde{\Theta})} & = \langle S(E_k), E_l \rangle + \langle T(\tilde{\Theta}, E_k), E_l \rangle + \mathcal{O}(r^2) + r L(\omega) + Q(\omega),
 \end{aligned}$$

where $T(Y_i, Y_k) = \nabla_{Y_i} \nabla_{Y_k} N_{\partial\Omega}$. In fact we have

$$Y_i \langle S(Y_k), Y_l \rangle = \langle T(Y_i, Y_k), Y_l \rangle + \langle \nabla_{Y_k} N_{\partial\Omega}, \nabla_{Y_i} Y_l \rangle.$$

By the parallel transport of the vector fields Y_j with respect to the connection $\nabla^{\partial\Omega}$ of $\partial\Omega$, we have $\nabla_{Y_i}^{\partial\Omega} Y_l \Big|_p = 0$. Since $\nabla_{Y_i} Y_l = \nabla_{Y_i}^{\partial\Omega} Y_l - \langle S(Y_i), Y_l \rangle N_{\partial\Omega}$, it follows that $\langle \nabla_{Y_k} N_{\partial\Omega}, \nabla_{Y_i} Y_l \rangle \Big|_p = 0$.

Lemma A.3. *The area of the hypersurface $\Sigma_{p,r,\omega}$ has the following expansion:*

$$\begin{aligned}
 & r^{-n} A(\Sigma_{p,r,\omega}) \\
 = & \mathcal{P}(B^{n+1}, \mathbb{R}_+^{n+1}) + r \int_{S_+^n} (\langle S(E_i), E_i \rangle - \langle S(\tilde{\Theta}), \tilde{\Theta} \rangle) \Theta^{n+1} d\sigma + n \int_{S_+^n} \omega d\sigma \\
 & + 3r(n+1) \int_{S_+^n} (\langle S(E_i), E_i \rangle - \langle S(\tilde{\Theta}), \tilde{\Theta} \rangle) \Theta^{n+1} \omega d\sigma \\
 & + 2r \int_{S_+^n} \Theta^{n+1} \langle S(\tilde{\Theta}), \nabla_{S^n} \omega \rangle d\sigma \\
 & + \frac{1}{2} r^2 \int_{S_+^n} \left((\langle S(E_i), S(E_i) \rangle - \langle S(\tilde{\Theta}), S(\tilde{\Theta}) \rangle) \right. \\
 & \quad \left. + (\langle S(E_i), E_i \rangle - \langle S(\tilde{\Theta}), \tilde{\Theta} \rangle)^2 \right) (\Theta^{n+1})^2 d\sigma \\
 & - \frac{1}{2} r^2 \int_{S_+^n} (|\langle S(\tilde{\Theta}), \tilde{\Theta} \rangle|^2 - 2|\langle S(E_i), E_i \rangle|^2 + |\langle S(E_j), S(E_i) \rangle|^2) d\sigma \\
 & - \frac{1}{6} r^2 \int_{S_+^n} \text{Ric}_p(\tilde{\Theta}, \tilde{\Theta}) d\sigma \\
 & - \frac{1}{2} r^2 \int_{S_+^n} (\text{Ric}_p^{\mathcal{M}}(\tilde{\Theta}, \tilde{\Theta}) + \langle R_p^{\mathcal{M}}(N_{\partial\Omega}, \tilde{\Theta}) N_{\partial\Omega}, \tilde{\Theta} \rangle) (\Theta^{n+1})^2 d\sigma \\
 & + \frac{1}{2} \int_{S_+^n} (|\nabla_{S^n} \omega|^2 + n(n-1) \omega^2) d\sigma + \mathcal{O}_p(r^3) + \int_{S_+^n} (r^2 L(\omega) + r Q^2(\omega) + Q^3(\omega)) d\sigma.
 \end{aligned}$$

Use [Proposition 2.2](#) and (17) to have the volume form of $B^{\mathcal{M}}(p, \rho)$ in \mathcal{M} for ρ small:

$$\begin{aligned} \rho^{-n} \sqrt{\det g_{ij}} &= 1 + \rho \Theta^{n+1} (\langle S(E_i), E_i \rangle + \rho \langle T(\tilde{\Theta}), E_i \rangle) \\ &\quad + \frac{1}{2} \rho^2 \langle R_p^{\mathcal{M}}(N_{\partial\Omega}, E_i) N_{\partial\Omega}, E_i \rangle + \frac{1}{6} \rho^2 \langle R_p(\tilde{\Theta}), E_i \rangle \tilde{\Theta}, E_i \rangle \\ &\quad + \rho^2 (\Theta^{n+1})^2 \langle R_p^{\mathcal{M}}(N_{\partial\Omega}, E_i) N_{\partial\Omega}, E_i \rangle + \frac{1}{2} \rho^2 (\Theta^{n+1})^2 \langle S(E_i), S(E_i) \rangle \\ &\quad + \frac{1}{8} \rho^2 (\Theta^{n+1})^2 |\langle S(E_i), E_j \rangle|^2 - \frac{1}{4} \rho^2 (\Theta^{n+1})^2 |\langle S(E_i), E_k \rangle|^2 + \mathcal{O}(\rho^3). \end{aligned}$$

Integration over the set $\rho \leq r(1 + \omega)$ gives the expansion of the volume bounded by $\Sigma_{p, \omega, r}$ and $\partial\Omega$.

Lemma A.4. *The following expansion holds:*

$$\begin{aligned} &r^{-n-1} |E(p, r, \omega)|_g \\ &= \frac{1}{n+1} \mathcal{P}(B^{n+1}, \mathbb{R}_+^{n+1}) + \frac{r}{n+2} \langle S(E_i), E_i \rangle \int_{S_+^n} \Theta^{n+1} d\sigma + \int_{S_+^n} \omega d\sigma \\ &\quad + \frac{r^2}{n+3} \left(\frac{1}{8} |\langle S(E_i), E_i \rangle|^2 + \frac{1}{2} \langle S(E_i), S(E_i) \rangle - \frac{1}{4} |\langle S(E_i), E_j \rangle|^2 \right) \int_{S_+^n} (\Theta^{n+1})^2 d\sigma \\ &\quad + -\frac{r^2}{6(n+3)} \int_{S_+^n} \text{Ric}_p(\tilde{\Theta}, \tilde{\Theta}) d\sigma - \frac{r^2}{2(n+3)} \text{Ric}_p^{\mathcal{M}}(N_{\partial\Omega}, N_{\partial\Omega}) \int_{S_+^n} (\Theta^{n+1})^2 d\sigma \\ &\quad + r \langle S(E_i), E_i \rangle \int_{S_+^n} \Theta^{n+1} \omega d\sigma + \frac{n}{2} \int_{S_+^n} \omega^2 d\sigma + O_p(r^3) \\ &\quad + \int_{S_+^n} (\mathcal{O}(r^2) \omega + \mathcal{O}(r) \hat{Q}^2(\omega) + \hat{Q}^3(\omega)) d\sigma, \end{aligned}$$

where $\hat{Q}^a(\omega)$ is a polynomial in ω of degree at least a , with smooth coefficients depending on p , Θ and maybe on r but uniformly bounded by a constant depending only on Ω .

Appendix B: Existence of orthogonally intersecting hemispheres of constant mean curvature centered on ∂B

Let E be an open smooth subset of Ω and $\Sigma := \partial E \cap \Omega$. Assume that the boundary of $\partial\Sigma$ is nonempty and is contained in $\partial\Omega$. From the first variation of area (see for instance [\[Ros and Vergasta 1995\]](#)), E is a critical point for the perimeter functional under variations that keep the volume invariant if and only if

$$nH_\Sigma \equiv \text{const in } \Sigma \quad \text{and} \quad \langle N_{\partial\Sigma}^\Sigma, N_{\partial\Sigma}^{\partial\Omega} \rangle_g = 0 \quad \text{in } \partial\Sigma,$$

where for $B \subset A$, the expression N_B^A denotes the unit outer normal of B in A while H_Σ is the mean curvature of Σ .

We have seen in [Section 4](#) that solutions E_r to the isoperimetric problem trapping a volume $|r B_+^{n+1}|$ have mean curvatures $H_{\partial E_r}$ blowing up and in fact $H_{\partial E_r} \sim n/r$. Moreover their boundaries are normal graphs over a hemisphere centered at some point in $\partial\Omega$. It is therefore natural to study the existence of disk-type solutions to the problem

$$(18) \quad H_\Sigma \equiv \frac{n}{r} \text{ in } \Sigma, \quad \partial\Sigma \subset \partial\Omega \quad \text{and} \quad \langle N_{\partial\Sigma}^\Sigma, N_{\partial\Sigma}^{\partial\Omega} \rangle_g = 0 \text{ on } \partial\Sigma.$$

Proposition B.1. *There exist $r_0 > 0$ and a smooth function $f : (0, r_0) \times \partial\Omega \rightarrow \mathbb{R}$ such that for every $r \in (0, r_0)$, if p is a critical point of $f(r, \cdot)$ then (18) admits a solution $\Sigma_{p,r}$ which is a normal graph over $F^p(r S_+^n)$. Furthermore*

$$\|f(r, \cdot) - H_{\partial\Omega}\|_{C^1(\partial\Omega)} \leq c r,$$

for some positive constant c .

Let us describe the proof of this. We have to recall that we look for stationary sets with a given profile for the total energy functional

$$\mathcal{E}_r(E) = \mathcal{P}_g(E, \Omega) + \frac{n}{r}|E|_g.$$

The set $Z_r := \{F^p(r B_+^{n+1}), p \in \partial\Omega\}$ is a manifold of approximate solutions for \mathcal{E}_r . Indeed, by [Lemma A.3](#) and [Lemma A.4](#),

$$\mathcal{E}_r(F^p(r B_+^{n+1})) = \mathcal{O}(r).$$

Due to the invariance by translations when $\partial\Omega = \mathbb{R}^n$ is “flat”, the linearized mean curvature operator together with the orthogonality conditions (see [Proposition 3.2](#)) may have small (possibly zero) eigenvalues, so we cannot invert it to apply a fixed point argument to solve the problem. However we will perturb Z_r to a manifold \tilde{Z}_r of critical points for \mathcal{E} modulo n Lagrange-multipliers. The second step is to show that in fact \tilde{Z}_r is a *natural constraint* for \mathcal{E} , namely a critical point of $\mathcal{E}|_{\tilde{Z}_r}$ is also stationary for \mathcal{E} . For that we use an argument of Kapouleas [[1991](#)] which was successfully employed in [[Pacard and Xu 2009](#)] to obtain constant mean curvature spheres in Riemannian manifolds.

This method is also closely related to variational-perturbative methods introduced by Ambrosetti and Badiale [[1998](#)] and subsequently used with success to get existence and multiplicity results for a wide class of variational problems in some perturbative settings. We refer to [[Ambrosetti and Malchiodi 2006](#)] for more details and related applications.

At this point, the reader may wish to review [Proposition 3.2](#).

Define

$$\langle \mathcal{L}_0(u), u' \rangle := \int_{S_+^n} (\nabla_{S_+^n} u \nabla_{S_+^n} u' - n u u') d\sigma.$$

Since the kernel of this operator is Λ_1 (see (12)), by the Fredholm theorem there exists a unique $\bar{\omega}^p \in \mathcal{C}^{2,\alpha}(S_+^n)$ such that

$$\begin{cases} \mathcal{L}_0 \bar{\omega} = \Theta^{n+1}((n+3)\langle S(\tilde{\Theta}), \tilde{\Theta} \rangle - \langle S(E_i), E_i \rangle) & \text{in } S_+^n, \\ \frac{\partial \bar{\omega}}{\partial \eta} = 0 & \text{on } \partial S_+^n, \end{cases}$$

because of the evenness of the right hand side. Moreover $\bar{\omega}^p$ satisfies

$$(19) \quad n \int_{S_+^n} \bar{\omega}^p d\sigma = \int_{S_+^n} \Theta^{n+1}((n+3)\langle S(\tilde{\Theta}), \tilde{\Theta} \rangle - \langle S(E_i), E_i \rangle) d\sigma.$$

Fixed point argument.

Lemma B.2. *for every $p \in \partial\Omega$ and r small, there exist a unique $\hat{\omega}^{p,r}$ and a vector field $\Xi_{p,r}$ on $T_p\partial\Omega$ such that*

$$(20) \quad \begin{cases} rH(p, r, r\bar{\omega}^p + \hat{\omega}) = n & \text{in } S_n^+, \\ \langle N_{\partial\Sigma}^\Sigma, N_{\partial\Sigma}^{\partial\Omega} \rangle = \langle \Xi_{p,r}, \tilde{\Theta} \rangle & \text{on } S^{n-1}. \end{cases}$$

Proof. We recall that Π_1 is the L^2 projection on Λ_1 , the space spanned by Θ^i , $i = 1, \dots, n$. For any $v \in L^2(S_+^n)$, we decompose it as

$$v = \hat{\omega} + \langle \Xi, \tilde{\Theta} \rangle = \omega_1 + \omega_0 + \langle \Xi, \tilde{\Theta} \rangle,$$

where $\Pi_0\omega = \omega_0 = \int_{S_+^n} \omega d\sigma$ and $\hat{\omega} = \Pi_1^\perp v$. Recalling the definition of T in (11), we define

$$\mathcal{L}_{p,r} : \mathcal{C}^{2,\alpha}(\bar{S}_+^n) \rightarrow \mathcal{C}^{0,\alpha}(\bar{S}_+^n)$$

by setting

$$\langle \mathcal{L}_{p,r}v, v' \rangle := \left\langle \frac{\partial T(r, p, r\bar{\omega}^p + \omega)}{\partial \omega}(p, r, 0)[v], v' \right\rangle - \oint_{S^{n-1}} \langle \Xi, \tilde{\Theta} \rangle \langle \Xi', \tilde{\Theta} \rangle ds,$$

for all $v' \in L^2$. By Proposition 3.2,

$$\begin{aligned} (\mathcal{L}_{p,r}v, v') &= \int_{S_+^n} (\nabla_{S^n} \hat{\omega} \nabla_{S^n} \hat{\omega}' - n\hat{\omega}\hat{\omega}') d\sigma - \oint_{S^{n-1}} \langle \Xi, \tilde{\Theta} \rangle \langle \Xi', \tilde{\Theta} \rangle ds \\ &\quad + r \int_{S_+^n} v' L(\hat{\omega}) d\sigma + r \oint_{S^{n-1}} v' \bar{L}(\hat{\omega}) ds. \end{aligned}$$

Since $\int_{S_+^n} |\nabla \omega_1|^2 d\sigma \geq 2(n+1) \int_{S_+^n} |\omega_1|^2 d\sigma$, it is easy to see that

$$\Pi \circ \mathcal{L}_{p,r} \geq \frac{1}{2} + o_r(1), \quad \Pi_0 \circ \mathcal{L}_{p,r} \leq -n + o_r(1), \quad \Pi_1 \circ \mathcal{L}_{p,r} \leq -\frac{|S_+^n|}{n+1} + o_r(1),$$

where $o_r(1)$ is a function in r (maybe depending on p) which tends to zero (uniformly in p) as $r \rightarrow 0$. From this, we deduce that $\mathcal{L}_{p,r}$ is uniformly invertible and there exists a constant independent of p and r such that

$$\|\mathcal{L}_{p,r}^{-1}\|_{L^2} \leq C \quad \text{for any } p \in \partial\Omega, \quad r \ll 1.$$

Now the system (20) is equivalent to the fixed point equation

$$v = (\mathcal{L}_{p,r})^{-1} \{ \mathbb{O}_p(r^2) + \mathcal{Q}_p(\hat{w}) \},$$

where $\mathcal{Q}^2(\hat{w})$ is the quadratic part of the mapping T defined in (11). By elliptic regularity theory, in a small ball of radius cr^2 in $\Pi_1^\perp \mathcal{C}^{2,\alpha}(\bar{S}_+^n) \times T_p \partial\Omega$, the above equation has a unique solution $(\hat{w}^{p,r}, \Xi_{p,r})$ such that (20) is satisfied. \square

Since the implicit function theorem also applies, one has the smoothness of $p \mapsto \omega^{p,r}$ and $p \mapsto \Xi_{p,r} \in T_p \partial\Omega$. Moreover differentiating the mean curvature equation in p , using standard elliptic regularity theory, we can deduce that

$$\|\omega^{(\cdot),r}\|_{C^{2,\alpha} \times \mathcal{C}^1(\partial\Omega)} + \|\Xi(\cdot),r\|_{\mathcal{C}^1(\partial\Omega)} \leq cr^2$$

for some constant $c > 0$ independent of r .

Variational argument. By Lemma B.2, fixing $r > 0$ small, for any $p \in \partial\Omega$, we have a unique hypersurface $\Sigma_{p,r} := \Sigma_{p,r,\omega^{p,r}}$ which is embedded because the $\mathcal{C}^{1,\alpha}$ bound (up to the boundary) of $\omega^{p,r} := r\bar{\omega}^p + \hat{w}^{p,r}$ tends to zero as $r \rightarrow 0$. This now yields for fixed $r > 0$ a manifold \tilde{Z}_r of sets $E^{p,r} \subset \Omega$, $p \in \partial\Omega$, bounded by $\Sigma_{p,r}$ and $\partial\Omega$ which is homeomorphic to $\partial\Omega$. We have to show that \tilde{Z}_r is a natural constraint for \mathcal{E} . For that we define the reduced functional $\varphi_r : \partial\Omega \rightarrow \mathbb{R}$ by

$$(21) \quad \varphi_r(p) = \mathcal{E}(E^{p,r}) = \mathcal{P}_g(E^{p,r}, \Omega) - \frac{n}{r} |E^{p,r}|_g,$$

for any $E^{p,r} \in \tilde{Z}_r$.

Lemma B.3. *Let φ_r be given by (21). There exists $r_0(\Omega) > 0$ such that for any $r \in (0, r_0)$, if p is a critical point of φ_r then $\Xi_{p,r} = 0$.*

Proof. Let p be a critical point of φ_r . Then for any vector field Ξ on $T_p \partial\Omega$,

$$d\varphi_r(p)[\Xi] = 0.$$

If $q := \exp_p^{\partial\Omega}(t\Xi)$, then for t sufficiently small the surface $\Sigma_{q,r}$ is a graph over $\Sigma_{p,r}$ for some smooth function $w_{p,r,\Xi,t}$ with variation vector field $\zeta_{p,r,\Xi}$ in $T_p \mathcal{M}$ satisfying

$$\zeta_{p,r,\Xi} := \frac{\partial}{\partial t} w_{p,r,\Xi,t} \Big|_{t=0} N_{\partial\Sigma}^{\partial\Omega} \quad \text{on } \partial\Sigma_{p,r} \subset \partial\Omega,$$

where $N_{\partial\Sigma}^{\partial\Omega}$ is the normal of $\partial\Sigma_{p,r}$ in $\partial\Omega$.

It is easy to see that for any parallel transport, in $\partial\Omega$, Z along geodesics issued from p of Ξ we have the estimate

$$\|\zeta_{p,r,\Xi} - Z\| \leq cr \|\Xi\| \quad \text{on } \partial\Sigma_{p,r}.$$

Now the first variation of area and volume yield

$$\begin{aligned} 0 &= d\varphi_r(p)[\Xi], \\ 0 &= \int_{\Sigma_{p,r}} \left(H_{\Sigma_{p,r}} - \frac{n}{r} \right) \langle \zeta_{p,r,\Xi}, N_{\partial\Sigma}^\Sigma \rangle d\sigma + \oint_{\partial\Sigma_{p,r}} \langle \zeta_{p,r,\Xi}, N_{\partial\Sigma}^\Sigma \rangle ds. \end{aligned}$$

By construction,

$$H_{\Sigma_{p,r}} = \frac{n}{r} \quad \text{in } \Sigma_{p,r} \quad \text{and} \quad \langle N_{\partial\Sigma}^\Sigma, N_{\partial\Sigma}^{\partial\Omega} \rangle = \langle \Xi_{p,r}, \tilde{\Theta} \rangle \quad \text{on } \partial\Sigma_{p,r}$$

thus

$$\oint_{\partial\Sigma_{p,r}} \langle \zeta_{p,r,\Xi}, N_{\partial\Sigma}^{\partial\Omega} \rangle \langle \Xi_{p,r}, \tilde{\Theta} \rangle ds = 0.$$

We have

$$\langle \zeta_{p,r,\Xi}, N_{\partial\Sigma}^{\partial\Omega} \rangle = -\langle Z, \tilde{\Upsilon} \rangle + \langle \zeta_{p,r,\Xi} - Z, N_{\partial\Sigma}^{\partial\Omega} \rangle + \langle Z, N_{\partial\Sigma}^{\partial\Omega} + \tilde{\Upsilon} \rangle \quad \text{on } \partial\Sigma_{p,r}.$$

The expansions of the metric together with the normal $N_{\partial\Sigma}^{\partial\Omega}$ (see (8) and (9)) show that

$$N_{\partial\Sigma}^{\partial\Omega} + \tilde{\Upsilon} = O(r) \quad \text{while} \quad \tilde{\Upsilon} = \tilde{\Theta}(1 + O(r)).$$

Therefore we have the estimates

$$|\langle \zeta_{p,r,\Xi}, N_{\partial\Sigma}^{\partial\Omega} \rangle + \langle \Xi, \tilde{\Theta} \rangle| \leq cr \|\Xi\|.$$

This implies, also by Hölder inequality, that

$$\begin{aligned} \oint_{\partial\Sigma_{p,r}} \langle \Xi_{p,r}, \tilde{\Theta} \rangle \langle \Xi, \tilde{\Theta} \rangle ds &\leq cr \|\Xi\| \oint_{\partial\Sigma_{p,r}} \langle \Xi_{p,r}, \tilde{\Theta} \rangle ds \\ &\leq cr \|\Xi\| \left(\oint_{\partial\Sigma_{p,r}} ds \right)^{1/2} \left(\oint_{\partial\Sigma_{p,r}} |\langle \Xi_{p,r}, \tilde{\Theta} \rangle|^2 ds \right)^{1/2}. \end{aligned}$$

Using the expansion of the metric of small perturbed geodesic sphere (see [Pacard and Xu 2009, Lemma 2.1]) we find that

$$\oint_{\partial\Sigma_{p,r}} \langle \Xi_{p,r}, \tilde{\Theta} \rangle \langle \Xi, \tilde{\Theta} \rangle ds \leq cr \|\Xi\| r^{(n-1)/2} \left(\oint_{\partial\Sigma_{p,r}} |\langle \Xi_{p,r}, \tilde{\Theta} \rangle|^2 ds \right)^{1/2},$$

while

$$\frac{1}{2} \text{Area}(S^{n-1}) r^{n-1} \|\Xi\|^2 \leq n \oint_{\partial\Sigma_{p,r}} |\langle \Xi, \tilde{\Theta} \rangle|^2 ds.$$

Hence we have

$$\oint_{\partial \Sigma_{p,r}} \langle \Xi_{p,r}, \tilde{\Theta} \rangle \langle \Xi, \tilde{\Theta} \rangle ds \leq cr \left(\oint_{\partial \Sigma_{p,r}} |\langle \Xi, \tilde{\Theta} \rangle|^2 ds \right)^{1/2} \left(\oint_{\partial \Sigma_{p,r}} |\langle \Xi_{p,r}, \tilde{\Theta} \rangle|^2 ds \right)^{1/2}.$$

And, finally setting $\Xi = \Xi_{p,r}$, we obtain

$$\oint_{\partial \Sigma_{p,r}} |\langle \Xi_{p,r}, \tilde{\Theta} \rangle|^2 ds \leq cr \oint_{\partial \Sigma_{p,r}} |\langle \Xi_{p,r}, \tilde{\Theta} \rangle|^2 ds.$$

Consequently $\Xi_{p,r} = 0$ for r small. □

Using Lemmas A.3 and A.4 in Appendix A, we get

$$\begin{aligned} r^{-n} \mathcal{P}_g(E_{p,r}, \Omega) &= \mathcal{P}(B^{n+1}, \mathbb{R}_+^{n+1}) + r \int_{S_+^n} (\langle S(E_i), E_i \rangle - \langle S(\tilde{\Theta}), \tilde{\Theta} \rangle) \Theta^{n+1} d\sigma \\ &\quad + n \int_{S_+^n} \omega d\sigma + O_p(r^2), \\ r^{-1-n} |E_{p,r}|_g &= \frac{1}{n+1} \mathcal{P}(B^{n+1}, \mathbb{R}_+^{n+1}) + \frac{r}{n+2} \langle S(E_i), E_i \rangle \int_{S_+^n} \Theta^{n+1} d\sigma \\ &\quad + \int_{S_+^n} \omega d\sigma + O_p(r^2). \end{aligned}$$

This now gives (recalling (14))

$$\begin{aligned} r^{-n} \varphi_r(p) &= r^{-n} \mathcal{E}_r(E^{p,r}) \\ &= \frac{1}{n+1} \mathcal{P}(B^{n+1}, \mathbb{R}_+^{n+1}) \\ &\quad + r \int_{S_+^n} \left(\frac{2}{n+2} \langle S(E_i), E_i \rangle - \langle S(\tilde{\Theta}), \tilde{\Theta} \rangle \right) \Theta^{n+1} d\sigma + O_p(r^2) \\ &= \frac{1}{n+1} \mathcal{P}(B^{n+1}, \mathbb{R}_+^{n+1}) - \frac{n|B^n|}{(n+2)} r H_{\partial\Omega}(p) + O_p(r^2). \end{aligned}$$

We end the proof of Proposition B.1 by setting

$$f(r, p) := \frac{-(n+2)}{rn|B^n|} \left(r^{-n} \varphi(p) - \frac{1}{n+1} \mathcal{P}(B^{n+1}, \mathbb{R}_+^{n+1}) \right) = H_{\partial\Omega}(p) + O_p(r).$$

Using (19), we also get precise expansions of the area of the of constant mean curvature hypersurfaces we have constructed as well as the volume of the domain they enclose.

Corollary B.4. *For any $E^{p,r} \in \tilde{Z}_r$,*

$$\begin{aligned} r^{-n} \mathcal{P}_g(E^{p,r}, \Omega) &= \mathcal{P}(B^{n+1}, \mathbb{R}_+^{n+1}) - n|B^n| r H_{\partial\Omega}(p) + O_p(r^2), \\ r^{-n-1} |E^{p,r}|_g &= |B_+^{n+1}| - \frac{n+1}{n+2} |B^n| r H_{\partial\Omega}(p) + O_p(r^2). \end{aligned}$$

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