SEPARATION OF RELATIVELY QUASICONVEX SUBGROUPS

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We show that if all hyperbolic groups are residually finite, these statements follow: In relatively hyperbolic groups with peripheral structures consisting of finitely generated nilpotent subgroups, relatively quasiconvex subgroups are separable; geometrically finite subgroups of nonuniform lattices in rank one symmetric spaces are separable; Kleinian groups are subgroup separable. We also show that LERF for finite volume hyperbolic 3-manifolds would follow from LERF for closed hyperbolic 3-manifolds.

We prove these facts by reducing, via combination and filling theorems, the separability of a relatively quasiconvex subgroup of a relatively hyperbolic group G to that of a quasiconvex subgroup of a hyperbolic quotient $\bar{G}$. A result of Agol, Groves, and Manning is then applied.

1. Main results

A subgroup H of a group G is called *separable* if for any $g \in G \setminus H$ there is a homomorphism π onto a finite group such that $\pi(g) \notin \pi(H)$. A group is said to be *residually finite* if the trivial subgroup is separable, and *subgroup separable* or *LERF* if every finitely generated subgroup is separable. For example, Hall [1949] showed that free groups are LERF. It follows from a theorem of Mal'cev [1958] that polycyclic (and in particular finitely generated nilpotent) groups are LERF. A group is called *slender* if every subgroup is finitely generated. Polycyclic groups are also slender, by a result of Hirsch [1938].

Given a relatively hyperbolic group with peripheral structure consisting of LERF and slender subgroups, we study separability of relatively quasiconvex subgroups. This is connected, via filling constructions, to residual finiteness of hyperbolic groups. It is not known whether all word-hyperbolic groups are residually finite. Consequences of a positive or negative answer to this question have been explored by several authors; see for example [Agol et al. 2009; Kapovich and Wise 2000; Long and Niblo 1991; Minasyan 2006; Osin 2007; Wise 2002].

Theorem 1.1 [Agol et al. 2009]. *If all hyperbolic groups are residually finite, then every quasiconvex subgroup of a hyperbolic group is separable.*

We extend this result, answering a question in [Agol et al. 2009], as follows:

Theorem 1.2. *Suppose that all hyperbolic groups are residually finite. If G is a torsion free relatively hyperbolic group with peripheral structure consisting of subgroups that are LERF and slender, then any relatively quasiconvex subgroup of G is separable.*

This extension has some interesting corollaries. A *pinched Hadamard manifold* is a simply connected Riemannian manifold with pinched negative curvature. Bowditch [1995] gave several equivalent definitions of geometrical finiteness for discrete subgroups of the isometry group of a pinched Hadamard manifold, generalizing the notion of geometrical finiteness in Kleinian groups. The next theorem summarizes some useful facts about these groups. (Statement (1) can be found in [Farb 1998] or [Bowditch 1999]; statement (2) follows from the Margulis lemma; and statement (3) is [Hruska 2008, Corollary 1.3].)

Theorem 1.3. *Let X be a pinched Hadamard manifold and G a geometrically finite subgroup of* Isom(X).

(1) *G is hyperbolic relative to a collection of representatives of conjugacy classes of maximal parabolic subgroups.*

(2) *Maximal parabolic subgroups of G are virtually nilpotent.*

(3) *A subgroup H of G is relatively quasiconvex if and only if H is geometrically finite.*

Rank one symmetric spaces are pinched Hadamard manifolds.

Corollary 1.4. *Suppose that all hyperbolic groups are residually finite. Let G be a discrete, geometrically finite subgroup of the isometry group of a rank one symmetric space. (For example, G could be a lattice.) All the geometrically finite subgroups of G are separable.*

If the symmetric space is $\mathbb{H}^3$, more can be said — see Section 5 for the proof.

Corollary 1.5. *If all hyperbolic groups are residually finite, then all finitely generated Kleinian groups are LERF.*

We will prove Theorem 1.2 by combining one of Martínez-Pedroza's combination theorems [2009] with Theorem 1.1 and the Dehn filling technique of [Groves and Manning 2008; Osin 2007]. We next give a more detailed discussion.

Definition 1.6. A relatively quasiconvex subgroup H of G is called *fully quasiconvex* if for any subgroup $P \in \mathcal{P}$ and any $f \in G$, either $H \cap P^f$ is finite or $H \cap P^f$ is a finite index subgroup of P^f. (Here $P^f = f P f^{-1}$.)

Using the work in [Martínez-Pedroza 2009], we show the following.

Theorem 1.7. *Let G be a group hyperbolic relative to a collection of slender and LERF subgroups. Suppose that Q is a relatively quasiconvex subgroup of G and g is an element of G not in Q. Then there exists a fully quasiconvex subgroup H that contains Q and does not contain g.*

Remark 1.8. When G is a finite-volume hyperbolic 3-manifold group and Q is the fundamental group of a quasifuchsian surface, Theorem 1.7 can be proved using geometric arguments like those in [Cooper and Long 1999; Cooper and Long 2001]. Such geometric arguments were applied (differently) to separability questions in [Agol et al. 2001]; see also [Wise 2006].

Using results of [Agol et al. 2009] and Theorem 1.7, we prove Theorem 1.9, a separation result of relatively quasiconvex subgroups by maps onto word hyperbolic groups. (Part of this theorem can be interpreted as saying that G is "quasiconvex extended residually hyperbolic".)

Theorem 1.9. *Let G be a torsion free group hyperbolic relative to a collection of slender and LERF subgroups. For any relatively quasiconvex subgroup Q of G and any element $g \in G$ such that $g \notin Q$, there is a fully quasiconvex subgroup H of G, and a surjective homomorphism $\pi : G \to \overline{G}$ such that*

(1) $Q < H$,

(2) $\overline{G}$ *is a word-hyperbolic group,*

(3) $\pi(H)$ *is a quasiconvex subgroup of $\overline{G}$, and*

(4) $\pi(g) \notin \pi(H)$.

We can prove Theorem 1.2 from Theorem 1.9 as follows:

Proof of Theorem 1.2. Let $Q < G$ be relatively quasiconvex, and let $g \in G \setminus Q$. By Theorem 1.9, there is a fully quasiconvex $H < G$ containing Q but not g, and a quotient $\pi : G \to K$ such that $\pi(g) \notin \pi(H)$, K is hyperbolic, and $\pi(K)$ is quasiconvex.

Assuming all hyperbolic groups are residually finite, Theorem 1.1 implies that there is a finite group F and a quotient $\phi : K \to F$ such that $\phi(\pi(g)) \notin \phi(\pi(H))$. Since $\phi(\pi(H))$ contains $\phi(\pi(Q))$, the map $\phi \circ \pi$ serves to separate g from Q. $\square$

Remark 1.10. The torsion-free hypotheses in Theorems 1.9 and 1.2 are not really necessary. We sketch the necessary changes to our argument in Appendix B. If one is primarily interested in Theorem 1.2 in the special case of virtually polycyclic peripheral subgroups, we have a simple argument, pointed out to us by the referee:

Let G be a group hyperbolic relative to a collection $\mathcal{P} = \{P_1, \dots, P_m\}$ of virtually polycyclic subgroups. An easy argument shows that each P_i contains a finite index normal subgroup that is torsion free. Moreover, G contains only finitely many finite

order nonparabolic elements, up to conjugacy [Osin 2006, Theorem 4.2]. It then
follows from Osin's version of the relatively hyperbolic Dehn filling theorem that
there is a filling $G \xrightarrow{\pi} G(N_1', \dots, N_m')$ such that $G(N_1', \dots, N_m')$ is hyperbolic and
no nontrivial torsion element of G is in the kernel of π. Assuming hyperbolic
groups are residually finite, the group $G(N_1', \dots, N_m')$ has a torsion-free subgroup
S of finite index. The preimage $G_0 = \pi^{-1}(S)$ is a torsion-free finite index subgroup
of G. Again under the assumption that hyperbolic groups are residually finite,
Theorem 1.2 implies that G_0 is QCERF, and so, applying Corollary 2.4 below, G
must also be QCERF.

The paper is organized as follows. In Section 2.5, we define relatively hyperbolic
groups in a way that suits this paper and is equivalent to the more standard defi-
nitions. In Section 3, we recall a combination theorem for relatively quasiconvex
subgroups from [Martínez-Pedroza 2009] and prove Theorem 1.7. In Section 4,
we recall some definitions and results on fillings of relatively hyperbolic groups
and prove Theorem 1.9. In Section 5, we give two applications to separability
questions on hyperbolic 3-manifolds: Corollary 1.5 and Proposition 5.3. In Ap-
pendix A, we prove a result we need on the equivalence of various definitions of
relative quasiconvexity, and in Appendix B, we sketch how to prove our results in
the presence of torsion.

2. Preliminaries

2.1. *Separability.* Let G be a group. Recall that the *profinite topology* on G is
the smallest topology on G in which all finite index subgroups and their cosets are
closed. The group G is residually finite if and only if this topology is Hausdorff.
A subgroup H is separable if and only if it is a closed subset of G in this topology.

Given a subgroup $G_0 < G$, one can ask whether the profinite topology on G_0
coincides with the topology induced by the profinite topology on G. In general,
the topologies are quite different, but in case G_0 is finite index, the topologies
coincide.

Lemma 2.2. *Let $G_0 < G$ be a finite index subgroup, and let $C_0 \subseteq G_0$. The following
conditions are equivalent*:

(1) C_0 *is closed in the profinite topology on G_0.*

(2) C_0 *is closed in the profinite topology on G.*

(3) $C_0 = C \cap G_0$ *for some set C that is closed in the profinite topology on G.*

Proof. Suppose C_0 is closed in G_0. Since the finite index subgroups of G_0 and their
cosets generate the topology on G_0, we can write C_0 as a finite union of arbitrary

intersections of cosets

$$(2\text{-}1) \qquad C_0 = \bigcup_{i=1}^{n} \bigcap_{i \in I_j} g_i K_i,$$

where every K_i is a finite index subgroup of G_0. But since G_0 is finite index in G, each of the K_i appearing in Equation (2-1) is also finite index in G. Thus C_0 is a finite union of intersections of closed sets in the profinite topology on G, so C_0 is closed in G. Thus condition (1) implies condition (2).

Trivially, condition (2) implies condition (3). To see that condition (3) implies condition (1), we first establish the following.

Claim 2.3. *If $g \in G$, and $K < G$ is finite index, then $gK \cap G_0$ is closed in the profinite topology on G_0.*

Proof. Let $K_0 = K \cap G_0$. Since K_0 is finite index in K, the subgroup K is a finite union of cosets $K = \bigcup_{i=1}^{p} g_i K_0$, and so $gK = \bigcup_{i=1}^{p} gg_i K_0$ is as well. Since a coset of K_0 in G either lies inside G_0 or in its complement, $gK \cap G_0$ is a finite union of cosets of K_0 in G_0. Since K_0 is finite index in G_0, these cosets are closed in the profinite topology on G_0 and therefore $gK \cap G_0$ is closed as well. □

Suppose then that $C_0 = C \cap G_0$ for C closed in G. Then $C = \bigcup_{i=1}^{n} \bigcap_{i \in I_j} g_i K_i$, where now the K_i are arbitrary finite index subgroups of G, and the g_i are arbitrary elements of G. We thus have

$$C_0 = C \cap G_0 = \bigcup_{i=1}^{n} \bigcap_{i \in I_j} (g_i K_i) \cap G_0.$$

By the claim, every $(g_i K_i) \cap G_0$ appearing here is closed in G_0, and so C_0 is also closed in the profinite topology on G_0. □

Corollary 2.4. *Let $H < G$ be a pair of groups, and let G_0 be a finite index subgroup of G. Let $H_0 = H \cap G_0$. The subgroup H is separable in G if and only if H_0 is separable in G_0.*

Another easy corollary is that LERFness and QCERFness are commensurability invariants.

2.5. *Relative hyperbolicity.* Several authors have studied relative hyperbolicity, using different but equivalent definitions. The definition we use here is based on [Osin 2006]. Let G be a group, let $\mathcal{P}$ denote a collection of subgroups $\{P_1, \ldots, P_m\}$, and let S be a finite generating set assumed to be symmetric, that is, $S = S^{-1}$. Denote by $\Gamma(G, \mathcal{P}, S)$ the Cayley graph of G with respect to the generating set $S \cup \bigcup \mathcal{P}$. Given a path p between vertices in $\Gamma(G, \mathcal{P}, S)$, we will call the initial vertex p_- and the terminal vertex p_+. The path p determines a word Label(p) in the alphabet $S \cup \bigcup \mathcal{P}$; this word represents an element g such that $p_+ = p_- g$.

Definition 2.6 (weak relative hyperbolicity). The pair $(G, \mathcal{P})$ is *weakly relatively hyperbolic* if there is an integer $\delta \geq 0$ such that $\Gamma(G, \mathcal{P}, S)$ is δ-hyperbolic. We may also say that G is *weakly hyperbolic relative to* $\mathcal{P}$.

Definition 2.7 [Osin 2006]. Let q be a combinatorial path in the Cayley graph $\Gamma(G, \mathcal{P}, S)$. Subpaths of q with at least one edge are called *nontrivial*. For $P_i \in \mathcal{P}$, a P_i-*component* of q is a maximal nontrivial subpath s of q with Label(s) a word in the alphabet P_i. When we don't need to specify the index i, we will refer to P_i-components as $\mathcal{P}$-*components*.

Two $\mathcal{P}$-components s_1 and s_2 are *connected* if their vertices belong to the same left coset of P_i for some i. A $\mathcal{P}$-component s of q is *isolated* if it is not connected to a different $\mathcal{P}$-component of q. The path q is *without backtracking* if every $\mathcal{P}$-component of q is isolated.

A vertex v of q is called *phase* if it is not an interior vertex of a $\mathcal{P}$-component s of q. Let p and q be paths between vertices in $\Gamma(G, \mathcal{P}, S)$. The paths p and q are *k-similar* if

$$\max\{\text{dist}_S(p_-, q_-), \text{dist}_S(p_+, q_+)\} \leq k,$$

where dist$_S$ is the metric induced by the finite generating set S (as opposed to the metric in $\Gamma(G, \mathcal{P}, S)$).

Remark 2.8. A geodesic path q in $\Gamma(G, \mathcal{P}, S)$ is without backtracking, all $\mathcal{P}$-components of q consist of a single edge, and all vertices of q are phase.

Definition 2.9 (bounded coset penetration, or BCP). The pair $(G, \mathcal{P})$ satisfies the *BCP property* if for any $\lambda \geq 1$, $c \geq 0$, and $k \geq 0$, there is an integer $\epsilon(\lambda, c, k) > 0$ such that if p and q are any two k-similar (λ, c)-quasigeodesics in $\Gamma(G, \mathcal{P}, S)$ without backtracking, these conditions hold:

(i) The sets of phase vertices of p and q are contained in the closed $\epsilon(\lambda, c, k)$-neighborhoods of each other, with respect to the metric dist$_S$.

(ii) If s is any $\mathcal{P}$-component of p such that $\text{dist}_S(s_-, s_+) > \epsilon(\lambda, c, k)$, then there exists a $\mathcal{P}$-component t of q that is connected to s.

(iii) If s and t are connected $\mathcal{P}$-components of p and q respectively, then

$$\max\{\text{dist}_S(s_-, t_-), \text{dist}_S(s_+, t_+)\} \leq \epsilon(\lambda, c, k).$$

Remark 2.10. Our definition of the BCP property corresponds to the conclusion of [Osin 2006, Theorem 3.23].

Definition 2.11 (relative hyperbolicity). The pair $(G, \mathcal{P})$ is *relatively hyperbolic* if the group G is weakly hyperbolic relative to $\mathcal{P}$ and the pair $(G, \mathcal{P})$ satisfies the BCP property. If $(G, \mathcal{P})$ is relatively hyperbolic, then we say G is *hyperbolic relative to* $\mathcal{P}$; if there is no ambiguity, we just say that G is relatively hyperbolic.

Remark 2.12. Definition 2.11 is equivalent to [Osin 2006, Definition 2.35] for finitely generated groups: To see that Osin's definition implies Definition 2.11, apply Osin's Theorem 3.23; to see the converse, apply Osin's Lemma 7.9 and Theorem 7.10. For the equivalence of Osin's definition and the various other definitions of relative hyperbolicity, see [Hruska 2008] and the references therein.

The definition of relative hyperbolicity is independent of finite generating set S.

3. Combination of parabolic and quasiconvex subgroups

In this section, G will be hyperbolic relative to a finite collection of subgroups $\mathcal{P} = \{P_1, \ldots, P_m\}$, and S will be a finite generating set for G. Denote by $\Gamma(G, \mathcal{P}, S)$ the Cayley graph of G with respect to the generating set $S \cup \bigcup \mathcal{P}$.

3.1. *Parabolic and quasiconvex subgroups.*

Definition 3.2. The *peripheral* subgroups of G are the elements of $\mathcal{P}$. A subgroup of G is called *parabolic* if it can be conjugated into a peripheral subgroup.

Proposition 3.3 [Osin 2006, Proposition 2.36]. *Let $P_i, P_j \in \mathcal{P}$, and let $g \in G$. Unless $i = j$ and $g \in P_i$, the intersection $P_i^g \cap P_j$ is finite.*

In particular, if Q is a subgroup of G, then any infinite maximal parabolic subgroup of Q is of the form $Q \cap P_i^f$ for some $f \in Q$ and some $P_i \in \mathcal{P}$.

Definition 3.4 [Osin 2006, Definition 4.9]. A subgroup Q of G is called *quasiconvex relative to $\mathcal{P}$* (or simply *relatively quasiconvex* when the collection $\mathcal{P}$ is fixed) if there exists a constant $\sigma \geq 0$ such that the following holds: Let $f, g \in Q$, and let p be an arbitrary geodesic path from f to g in the Cayley graph $\Gamma(G, \mathcal{P}, S)$. For any vertex $v \in p$, there exists a vertex $w \in Q$ such that $\mathrm{dist}_S(v, w) \leq \sigma$, where dist_S is the word metric induced by S.

Remark 3.5. For more on different definitions of relative quasiconvexity in the literature, see Appendix A.

Theorem 3.6 [Hruska 2008, Theorem 9.1]. *Let Q be a finitely generated relatively quasiconvex subgroup of G. The number of infinite maximal parabolic subgroups of Q up to conjugacy in Q is finite. Also, if $\mathbb{O}$ is a set of representatives of these conjugacy classes, then Q is hyperbolic relative to $\mathbb{O}$.*

Remark 3.7. Hruska [2008] uses an extended definition of relative hyperbolicity that includes some countable but nonfinitely generated groups. Using this extended definition, the assumption of finite generation in Theorem 3.6 is superfluous.

In case all the peripheral subgroups are slender, relatively quasiconvex subgroups are necessarily finitely generated; see [Hruska 2008, Corollary 9.2].

3.8. *Combination of quasiconvex subgroups.* For $g \in G$, we denote by $|g|_S$ the distance from g to the identity element in the word metric induced by S.

Theorem 3.9 [Martínez-Pedroza 2009, Theorem 1.1]. *Let Q be a relatively quasiconvex subgroup of G, and let P be a maximal parabolic subgroup of G. Suppose that $P^f = P_i$ for some $P_i \in \mathcal{P}$ and $f \in G$.*

There are constants $C = C(Q, P) \geq 0$ and $c = c(Q, P) \geq 0$ with the following property. Suppose $D \geq C$, and R is a subgroup of P such that $P \cap Q < R$ and $|g|_S > D$ for every element $g \in R \setminus Q$. It follows that

(1) *the subgroup $H = \langle Q \cup R \rangle$ is relatively quasiconvex and the natural map $Q *_{Q \cap R} R \to H$ is an isomorphism;*

(2) *every parabolic subgroup of H is conjugate in H to a parabolic subgroup of Q or R;*

(3) *for any $g \in H$, either g belongs to Q or any geodesic from 1 to g in the relative Cayley graph $\Gamma(G, \mathcal{P}, S)$ has at least one P_i-component t such that $|t|_S > D - c$.*

Proof. Conclusions (1) and (2) rephrase [Martínez-Pedroza 2009, Theorem 1.1]. The proof of conclusion (3) is divided into two cases: $P \in \mathcal{P}$ and $P \notin \mathcal{P}$.

Case 1: $P \in \mathcal{P}$. We summarize part of the argument for [Martínez-Pedroza 2009, Theorem 1.1] for conclusions (1) and (2); we then explain how conclusion (3) follows in this case.

Let $g \in Q *_{Q \cap R} R \setminus Q$. The element g has a *normal form*

$$(3\text{-}1) \qquad\qquad g = g_1 h_1 \cdots g_k h_k,$$

where $g_j \in Q \setminus Q \cap R$ for $1 < j \leq k$, $h_j \in R \setminus Q \cap R$ for $1 \leq j < k$, either $g_1 = 1$ or $g_1 \in Q \setminus Q \cap R$, and either $h_k = 1$ or $h_k \in R \setminus Q \cap R$. We use the normal form to produce a path o in $\Gamma(G, \mathcal{P}, S)$ from 1 to the image of g by the natural map $Q *_{Q \cap R} R \to H$ as follows. For each j between 1 and k, let u_j be a geodesic path in $\Gamma(G, \mathcal{P}, S)$ from $g_1 h_1 \cdots h_j$ to $g_1 h_1 \cdots h_j g_j$ (so that $\mathrm{Label}(u_j)$ represents g_j). Similarly, let v_j be a geodesic path from $g_1 h_1 \cdots g_{j-1}$ to $g_1 h_1 \cdots g_{j-1} h_j$ (so that $\mathrm{Label}(v_j)$ represents h_j). A path o from 1 to g in $\Gamma(G, \mathcal{P}, S)$ is given by

$$o = u_1 v_1 \ldots u_k v_k.$$

See Figure 1.

Each subsegment v_j is part of a $\mathcal{P}$-component t_j of the path o. Let D be as in the hypothesis of the theorem. The penultimate inequality in the proof of Claim 2 in the proof of [Martínez-Pedroza 2009, Lemma 5.1] is

$$(3\text{-}2) \qquad |t_j|_S = \mathrm{dist}_S((t_j)_-, (t_j)_+) > D - 2M(P, Q, \sigma),$$

$\Gamma(G, \mathcal{P}, S)$
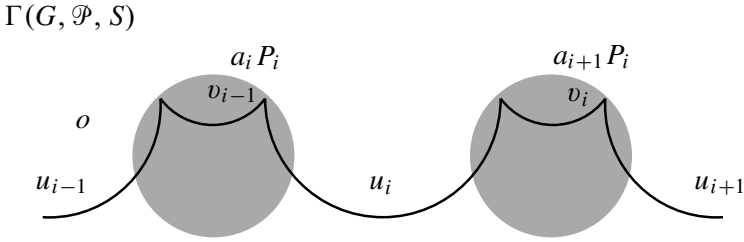
Figure 1. Part of the polygonal path o in $\Gamma(G, \mathcal{P}, S)$. For each i, the P_i-component of o containing the subsegment v_j is long with respect to the S-metric. This implies that the path o is a quasi-geodesic with different endpoints.

where σ is the quasiconvexity constant for Q and $M(P, Q, \sigma)$ is the constant provided by [Martínez-Pedroza 2009, Lemma 4.2] for the subgroups Q, P, and the constant σ. Let η be the constant from [Martínez-Pedroza 2009, Proposition 3.1]. If $D - 2M(P, Q, \sigma) > \eta$, then o is a $(\lambda, 0)$-quasigeodesic with distinct endpoints. Let $C = \eta + 2M(P, Q, \sigma)$.

It follows from the argument just sketched that if $D > C$, then the natural map $Q *_{Q \cap R} R \to H$ is an isomorphism. It can further be shown that H is a relatively quasiconvex subgroup and that the parabolic subgroups of H are conjugate into Q or R by elements of H. See [Martínez-Pedroza 2009, Lemmas 5.2 and 5.3] for details. In other words, conclusions (1) and (2) hold for $P \in \mathcal{P}$ and C as above.

If $g \in H \setminus Q$ and p is a geodesic from 1 to g in $\Gamma(G, \mathcal{P}, S)$, then the $(\lambda, 0)$-quasigeodesic o and the geodesic p are 0-similar. Since o has a $\mathcal{P}$-component of S-length at least $D - 2M(P, Q, \sigma)$, the BCP property of Definition 2.9 implies that p has a $\mathcal{P}$-component of S-length at least $D - 2M(P, Q, \sigma) - 2\epsilon(0, \lambda, 0)$. In the case that $f = 1$ and $P \in \mathcal{P}$, we have verified conclusion (3) of the theorem for c equal to

$$c(Q, P) = 2M(P, Q, \sigma) + 2\epsilon(0, \lambda, 0).$$

Case 2: $P \notin \mathcal{P}$, but $P^f \in \mathcal{P}$ for some $f \in G \setminus P$. Since $P^f \in \mathcal{P}$ and Q^f is relatively quasiconvex, by [Martínez-Pedroza 2009, Theorem 1.1] and Case 1, all three conclusions of Theorem 3.9 hold for Q^f and P^f and some constants $C' = C(Q^f, P^f) > 0$ and $c' = c(Q^f, P^f) > 0$. Define

$$C = C' + 2|f|_S + 3\epsilon(1, 0, |f|_S) \quad \text{and} \quad c = c' + 2|f|_S + 2\epsilon(1, 0, |f|_S),$$

where $\epsilon(1, 0, |f|_S)$ is the BCP constant from Definition 2.9. Now we show that the theorem holds for the subgroups P and Q, and the constants C and c. Let R be a subgroup of Q satisfying the hypothesis of the theorem for a constant $D > C$.

If $r \in R \setminus Q$, then $|r|_S \geq D$, by hypothesis. It follows that $|r^f|_S \geq D - 2|f|_S \geq C'$. Therefore:

- The subgroup $H^f = \langle Q^f \cup R^f \rangle$ is relatively quasiconvex and the natural map $Q^f *_{(Q \cap R)^f} R^f \to H^f$ is an isomorphism. Since relative quasiconvexity is preserved by conjugation, $H = \langle Q \cup R \rangle$ is relatively quasiconvex. Obviously the map $Q *_{Q \cap R} R \to H$ is also an isomorphism. In other words, conclusion (1) holds for Q and P and the constant C.

- Every parabolic subgroup of H^f is conjugate in H^f to a parabolic subgroup of Q^f or R^f. Parabolicity is preserved under conjugation, so conclusion (2) holds for the subgroups Q, R, and H, and the constant C.

- For any $h \in H^f$, either $h \in Q^f$ or any geodesic from 1 to h in the relative Cayley graph $\Gamma(G, \mathcal{P}, S)$ has at least one P_i-component t that satisfies $|t|_S > D - 2|f|_S - c'$.

It remains to see why conclusion (3) holds with the chosen constant c. Let $g \in H \setminus Q$ and let p be a geodesic from 1 to g in $\Gamma(G, \mathcal{P}, S)$. We must show that p has a P_i-component of S-length at least $D - c$. Let q be a geodesic from 1 to fgf^{-1}. Since fgf^{-1} belongs to $H^f \setminus Q^f$, the geodesic q has a P_i-component u of S-length at least $D - 2|f|_S - c'$. Let r be the geodesic starting at f, with the same label as p. Thus r joins f to fg, and the geodesics q and r are $|f|_S$-similar; see Figure 2.

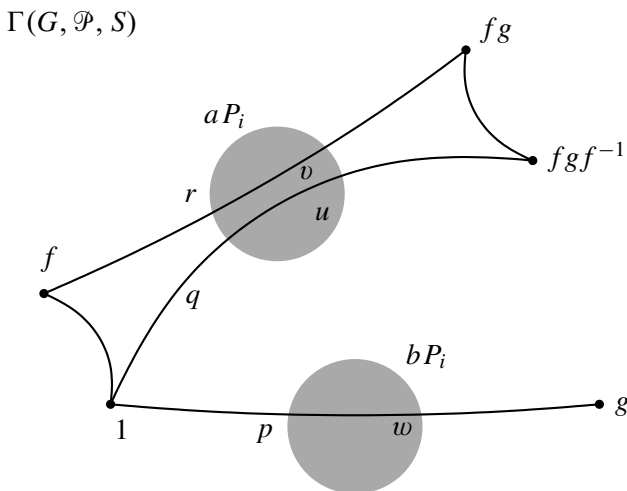By the BCP property, r has a P_i-component v of S-length at least

$$D - 2|f|_S - c' - 2\epsilon(1, 0, |f|_S) = D - c.$$

Since $p = f^{-1}r$ and r have the same labels, it follows that p has a P_i-component w of S-length at least $D - c$. □

Corollary 3.10 [Martínez-Pedroza 2009, Lemma 5.4]. *Suppose that G, Q, P, and R are as assumed in Theorem 3.9 and that $Q \cap P$ is a proper subgroup of R.*

If $\{K_1, \ldots, K_n\}$ is a set of representatives of the maximal infinite parabolic subgroups of Q up to conjugacy in Q, such that $K_1 = P \cap Q$, then $\{R, K_2, \ldots, K_n\}$ is a set of representatives of the maximal parabolic subgroups of H up to conjugacy in H.

Proof. By Theorem 3.9(2), a maximal parabolic subgroup of H is conjugate to R or K_i for some $i = 2, \ldots n$. Hence $\{R, K_2, \ldots, K_n\}$ is a set of representatives of maximal parabolic subgroups. That all these subgroups are different up to conjugacy follows from the algebraic structure of H as an amalgamated product. In particular, since K_i and K_j are not conjugate in Q, they are not conjugate in $Q *_{Q \cap R} R$. The subgroup $R < Q *_{Q \cap R} R$ is not conjugate to a subgroup of Q since $Q \cap R$ is a proper subgroup of R. □

$\Gamma(G, \mathcal{P}, S)$

Figure 2. The geodesics q and r are $|f|_S$-similar and q contains a large $\mathcal{P}$-component u. By the BCP-property, r has a large $\mathcal{P}$-component v. Since p and r have the same word-label, p has a large $\mathcal{P}$-component w.

Proof of Theorem 1.7. Suppose that every subgroup in $\mathcal{P}$ is LERF and slender. Let Q be a relatively quasiconvex subgroup of G, and let g be an element of G not in Q. Let $\{K_1, \ldots, K_n\}$ be a set of representatives of maximal infinite parabolic subgroups of Q up to conjugacy in Q; such a set exists by Theorem 3.6. By Proposition 3.3, for each K_i there is a peripheral subgroup $P_i \in \mathcal{P}$ and $f_i \in G$ such that $K_i < P_i^{f_i}$.

We will construct an ascending sequence of relatively quasiconvex subgroups

$$Q = Q_0 < Q_1 < \cdots < Q_n = H$$

such that for each $k \in \{1, \ldots, n\}$ the following properties hold.

(i) For $0 \leq j \leq k$, the subgroup $Q_k \cap P_j^{f_j}$ is finite index in $P_j^{f_j}$.

(ii) $\{Q_k \cap P_1^{f_1}, \ldots, Q_k \cap P_k^{f_k}, K_{k+1}, \ldots, K_n\}$ is a set of representatives of the maximal parabolic subgroups of Q_k up to conjugation in Q_k.

(iii) $g \notin Q_k$.

It will follow that $H = Q_n$ is a fully quasiconvex subgroup that contains Q and does not contain the element g.

Choose a geodesic p from 1 to g in the relative Cayley graph $\Gamma(G, \mathcal{P}, S)$. Let L be an upper bound for the S-length of the P_k-components of the path p.

We now show how to construct Q_k, assuming that Q_{k-1} is already constructed. Let C and c be the constants provided by Theorem 3.9 for the subgroups Q_{k-1} and $P_k^{f_k}$. Since $P_k^{f_k}$ is slender, $K_k = Q_{k-1} \cap P_k^{f_k}$ is finitely generated. Define a finite set $F \subset P_k^{f_k} \setminus K_k$ by

$$F = \begin{cases} \{p \in P_k^{f_k} \setminus K_k \mid |p|_S \le L + C + c\} \cup \{g\} & \text{if } g \in P_k^{f_k}, \\ \{p \in P_k^{f_k} \setminus K_k \mid |p|_S \le L + C + c\} & \text{otherwise.} \end{cases}$$

Because $P_k^{f_k} \cong P_k$ is LERF, we may find a finite index subgroup R_k of $P_k^{f_k}$ such that $K_k < R_k$ and $f \notin R_k$ for all $f \in F$. In particular, $|h|_S > L + C + c$ for any $h \in R_k \setminus Q_{k-1}$. Let $Q_k = \langle Q_{k-1} \cup R_k \rangle$. Note that the hypotheses of the combination Theorem 3.9 (and hence those of Corollary 3.10) are satisfied for the relatively quasiconvex subgroup Q_{k-1} and the parabolic subgroup R_k.

We now verify properties (i)–(iii) for the subgroup Q_k just constructed. The first follows because Q_k contains $Q_j \cap P_j^{f_j}$ for each j between 1 and $k-1$, and also contains R_k. Corollary 3.10 implies that property (ii) is satisfied. By Theorem 3.9(3), any geodesic in the Cayley graph $\Gamma(G, S \cup \bigcup \mathcal{P})$ from 1 to an element of Q_k that is not in $Q_{k-1} \cup R_k$ has a P_k-component of S-length greater than $L + C$; it follows that the element g does not belong to Q_k, and so property (iii) is also satisfied. This concludes the construction of the group Q_k, and the theorem follows by taking $H = Q_n$. $\square$

4. Fillings of relatively hyperbolic groups

Suppose G is torsion free and hyperbolic relative to a collection of subgroups $\mathcal{P} = \{P_1, \ldots, P_m\}$, and let $S \subset G$ be a finite generating set of G. Suppose that $S \cap P_i$ is a generating set of P_i for each i.

Definition 4.1. A *filling* of G is determined by a collection of subgroups $\{N_i\}_{i=1}^m$ such that N_i is a normal subgroup of P_i for each i; these subgroups are called *filling kernels*. The quotient of G by the normal subgroup generated by $\bigcup_{i=1}^m N_i$ is denoted by $G(N_1, \ldots, N_m)$.

The following result is due (in the present setting) independently to Groves and Manning [2008, Theorem 7.2 and Corollary 9.7] and to Osin [2007, Theorem 1.1]. (Osin actually proves a more general result, in which G may have torsion.)

Theorem 4.2 [Groves and Manning 2008; Osin 2007]. *Let F be a finite subset of G. There exists a constant B depending on G, $\mathcal{P}$, S, and F with the following property. If a collection of filling kernels $\{N_i\}_{i=1}^m$ satisfies $|f|_S > B$ for every nontrivial $f \in \bigcup_i N_i$, then*

  (1) *the natural map $\iota_i : P_i / N_i \to G(N_1, \ldots, N_m)$ is injective,*

  (2) *$G(N_1, \ldots, N_m)$ is hyperbolic relative to $\{\iota_i(P_i/N_i)\}_{i=1}^m$, and*

(3) *the projection* $G \to G(N_1, \ldots, N_m)$ *is injective on* F.

4.3. *Fillings and quasiconvex subgroups.* Let H be a relatively quasiconvex subgroup of G. A filling $G \to G(N_1, \ldots, N_m)$ is an H-*filling* if $N_i^g \subset P_i^g \cap H$ whenever $H \cap P_i^g$ is nontrivial.

Theorem 4.4 [Agol et al. 2009, Propositions 4.3 and 4.5]. *Let $H < G$ be a finitely generated relatively quasiconvex subgroup and let $g \in G \setminus H$. There is a finite subset $F \subset G$ depending on H and g with the following property.*

If $\pi : G \to G(N_1, \ldots, N_m)$ is an H-filling that is injective on F, then $\pi(H)$ is a relatively quasiconvex subgroup of $G(N_1, \ldots, N_m)$, and $\pi(g) \notin \pi(H)$.

Remark 4.5. Agol, Groves and Manning use in [2009, Propositions 4.3 and 4.5] a different definition of relative quasiconvexity than the one we use here. In particular, their definition requires the subgroup to be finitely generated. We show in Appendix A (specifically Corollary A.11) that their definition is equivalent to the one we use here under the assumption that the subgroup is finitely generated.

Proof of Theorem 1.9. Suppose every subgroup in $\mathcal{P}$ is LERF and slender. Let Q be a relatively quasiconvex subgroup of G, and let g be an element of G not in Q. By Theorem 1.7, there is a fully quasiconvex subgroup H that contains Q and not g. We must choose filling kernels $\{N_i\}_{i=1}^m$ such that $\pi : G \to G(N_1, \ldots, N_m)$ is an H-filling, $G(N_1, \ldots, N_m)$ is a word hyperbolic group, $\pi(H)$ is a quasiconvex subgroup, and $\pi(g) \notin \pi(H)$.

By Theorem 3.6, there is a set $\{K_1, \ldots, K_n\}$ of representatives of the infinite maximal parabolic subgroups of H up to conjugacy in H. For each $r \in \{1, \ldots, n\}$, there is an integer $i_r \in \{1, \ldots, m\}$ and an element $f_r \in G$ such that $K_r^{f_r}$ is a finite index subgroup of P_{i_r}. The index i_r is determined by r, but there may be many distinct f_r with this property. On the other hand, there will be only finitely many conjugates K_i^g in P_{i_r}. Let I_r be the intersection of these conjugates; the group I_r is a finite index normal subgroup of P_{i_r}. For $i \in \{1, \ldots, m\}$ define the subgroup M_i of P_i as

$$M_i = \begin{cases} P_i & \text{if } \{r \mid i_r = i\} = \varnothing, \\ \bigcap \{I_r \mid i_r = i\} & \text{if } \{r \mid i_r = i\} \neq \varnothing. \end{cases}$$

Put another way, if some conjugate of H intersects P_i nontrivially,

$$M_i = \bigcap \{K_r^g \mid K_r^g \cap P_i \neq \{1\}, r \in \{1, \ldots, n\}, \text{ and } g \in G\}.$$

In other words, M_i is the intersection of P_i with all the K_r^g that lie in P_i. Because M_i is a finite intersection of finite index normal subgroups of P_i, it is a finite index normal subgroup of P_i. From the definition, $M_i < K_r^g$ whenever $r \in \{r \mid i_r = i\}$ and $K_r^g < P_i$.

By Theorem 4.4, there is a finite subset $F \subset G$ such that if $\pi : G \to \overline{G}$ is any H-filling that is injective on F, then $\pi(g) \notin \pi(H)$ and $\pi(H)$ is relatively quasiconvex. Let $B > 0$ be the constant from Theorem 4.2, applied to this finite subset F.

Since each P_i is residually finite, there is some finite index $\hat{N}_i \lhd P_i$ such that $|p|_S > B$ for all $p \in \hat{N}_i \setminus \{1\}$. Let $N_i = \hat{N}_i \cap M_i$. Since N_i is an intersection of two finite index normal subgroups of P_i, it is a finite index normal subgroup of P_i. Moreover, $|p|_S > B$ for every $p \in N_i \setminus \{1\}$.

Lemma 4.6. $G \to G(N_1, \ldots, N_m)$ *is an H-filling.*

Proof. Let $P_i \in \mathscr{P}$, $f \in G$, and suppose that $H \cap P_i^f \neq \{1\}$. We must show that $N_i^f < H$. Obviously it suffices to show that $M_i^f < H$. Since G is torsion free, $H \cap P_i^f$ is infinite and $H \cap P_i^f = K_r^h$ for some $h \in H$. The group

$$M_i^f = \bigcap \{K_r^g \mid K_r^g \cap P_i^f \neq \{1\}, r \in \{1, \ldots, n\}, \text{ and } g \in G\}$$

must therefore be contained in $H \cap P_i^f$, and thus in H. $\square$

We claim the subgroup $H < G$ and the filling $G \to \overline{G} := G(N_1, \ldots, N_m)$ satisfy the conclusions of Theorem 1.9. Indeed, (1) is satisfied by construction. By Theorem 4.2(2), the quotient $\overline{G}$ is hyperbolic relative to a set of finite groups; hence $\overline{G}$ is hyperbolic, which establishes (2). According to Lemma 4.6, $G \to \overline{G}$ is an H-filling. By Theorem 4.2(3), $G \to \overline{G}$ is injective on F. Since the peripheral subgroups are slender, H is finitely generated [Hruska 2008, Corollary 9.2]. We therefore may apply Theorem 4.4 to obtain finally parts (3) and (4). $\square$

5. Applications to 3-manifolds

5.1. *Kleinian groups.* A *Kleinian group* is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$, the group of isometries of hyperbolic 3-space. We now will show that if all hyperbolic groups are residually finite, then all finitely generated Kleinian groups are LERF.

Proof of Corollary 1.5. By Selberg's lemma [Alperin 1987], every finitely generated Kleinian group contains a torsion-free subgroup of finite index. Applying Corollary 2.4, it suffices to consider torsion-free Kleinian groups. Let G be a torsion-free Kleinian group and let $H < G$ be a finitely generated subgroup.

Case 1: $\mathbb{H}^3/G$ has finite volume. Suppose first that H is not geometrically finite. By a result of [Canary 1996, Corollary 8.3] together with the positive solution to the tameness conjecture [Agol 2004; Calegari and Gabai 2006], H must be a *virtual fiber subgroup* of G. This implies that there is a finite index subgroup of $G_0 < G$ whose intersection H_0 with H is normal in G_0, and such that $G_0/H_0 \cong \mathbb{Z}$. The group H_0 is clearly separable in G_0, so H is separable in G by Corollary 2.4.

If H is geometrically finite, then it is relatively quasiconvex by Theorem 1.3, and we may apply Theorem 1.2.

Case 2: $\mathbb{H}^3/G$ has infinite volume. In this case it follows from the Scott core theorem and Thurston's geometrization theorem for Haken manifolds that G is isomorphic to a geometrically finite Kleinian group G'; see for example [Matsuzaki and Taniguchi 1998, Theorem 4.10]. If H' is the image of H in G', then obviously H is separable in G if and only if H' is separable in G'. We therefore may as well assume that G is geometrically finite to begin with.

Since G is geometrically finite and infinite covolume, every finitely generated subgroup of G is geometrically finite by an argument of Thurston; for a proof, see [Morgan 1984, Proposition 7.1] or [Matsuzaki and Taniguchi 1998, Theorem 3.11]. In particular, H is geometrically finite, so Theorem 1.3 implies that H is a relatively quasiconvex subgroup of G. To finish, we apply Theorem 1.2 again. □

5.2. *Subgroup separability in finite volume 3-manifolds.*

Proposition 5.3. *If all fundamental groups of compact hyperbolic 3-manifolds are QCERF, then all fundamental groups of finite volume hyperbolic 3-orbifolds are LERF.*

Proof. Let $G = \pi_1(M)$, where M is some finite volume hyperbolic 3-orbifold. Applying Corollary 2.4, we may pass to a finite cover and assume that M is an orientable manifold. It follows that G is hyperbolic relative to some finite collection $\mathcal{P} = \{P_i, \ldots, P_m\}$ of subgroups, each of which is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Now suppose $Q < G$ is some finitely generated subgroup. We must show that Q is separable. If Q is geometrically infinite, then we argue as we did in the proof of Corollary 1.5 that Q is the fundamental group of a virtual fiber, and thus separable. We may thus suppose that Q is geometrically finite, and therefore a relatively quasiconvex subgroup of G.

Let $g \in G \setminus Q$. We then apply Theorem 1.7 to enlarge Q to a fully quasiconvex subgroup H not containing g. Let $F \subset G$ be the finite set obtained by applying Theorem 4.4 to G, H, and g. Let B_1 be the constant from Theorem 4.2 applied to G and F with respect to some generating set S for G. We will choose a cyclic filling kernel $N_i < P_i$ for each $P_i \in \mathcal{P}$. The hyperbolic Dehn surgery theorem of Thurston [Thurston 1980; Hodgson and Kerckhoff 2005] implies there is some constant B_2 such that if the generators of the N_i are chosen to have length greater that B_2, then the filling $G(N_1, \ldots, N_m)$ will be the (orbifold) fundamental group of a hyperbolic orbifold obtained by attaching orbifold solid tori to the boundary components of a compact core of M. Let $B = \max\{B_1, B_2\}$.

For each $P_i \in \mathcal{P}$, we choose some cyclic $N_i < P_i$. For each i, let $n_i \in P_i$ satisfy $|n_i|_S > B$. There are at most finitely many conjugates $P_i^{t_1}, \ldots P_i^{t_k}$ such that $P_i^{t_j} \cap H$ is nonempty; for each such j, the group $P_i^{t_j} \cap H$ is finite index in $P_i^{t_j}$. If there are no such conjugates, we choose $N_i = \langle n_i \rangle$. Otherwise, we let $N_i = \langle n_i^\alpha \rangle$, where the power $\alpha \in \mathbb{N}$ is chosen so that $n_i^\alpha \in H^{t_j^{-1}}$ for each j, and so $|n_i^\alpha|_S > B$.

With the $N_i < P_i$ chosen as above, the H-filling $G(N_1, \ldots, N_m)$ is a compact hyperbolic 3-orbifold group. Let $\pi : G \to G(N_1, \ldots, N_m)$ be the quotient map. By Theorem 4.4, $\pi(H)$ is a quasiconvex subgroup of $G(N_1, \ldots, N_m)$ that does not contain $\pi(g)$. By assumption, compact hyperbolic 3-manifold groups (and thus compact hyperbolic 3-orbifold groups) are QCERF. There is therefore some finite group F and some $\phi: G(N_1, \ldots, N_m) \to F$ with $\phi(\pi(g)) \notin \phi(\pi(H))$. Since $\phi(\pi(H))$ contains $\phi(\pi(Q))$, we have separated g from Q in a finite quotient.   □

Remark 5.4. No part of the proof of Proposition 5.3 relies essentially on the results in this paper or those in [Agol et al. 2009]; the proof depends only on facts about hyperbolic 3-manifolds and 3-orbifolds that could be deduced by geometric arguments, based on the Gromov–Thurston 2π theorem. On the other hand, the proposition illustrates nicely the general principle that a relatively hyperbolic group that can be approximated by QCERF Dehn fillings must itself be QCERF.

6. Acknowledgments

Appendix A. The equivalence of definitions of quasiconvexity

This paper relies heavily on [Martínez-Pedroza 2009] and [Agol et al. 2009] for properties of relatively quasiconvex subgroups of relatively hyperbolic groups. These papers use different definitions of relative quasiconvexity, but we show here that the two definitions agree, at least for finitely generated subgroups.

A few words about the literature. Dahmani [2003] and Osin [2006] studied classes of subgroups of relatively hyperbolic groups that they called *relatively quasiconvex*, a notion intended to generalize that of quasiconvexity in hyperbolic groups. Dahmani's definition was dynamical, whereas Osin's was Definition 3.4, also used in [Martínez-Pedroza 2009]. Hruska [2008] gave several definitions of relative quasiconvexity in the setting of countable (but not necessarily finitely generated) relatively hyperbolic groups, including definitions based on Osin's and Dahmani's, and showed that they are equivalent. The authors of [Agol et al. 2009] were mainly interested in relatively hyperbolic structures on groups that were already hyperbolic, and used a definition of relative quasiconvexity (based more closely on the usual metric notion of quasiconvexity) different from any of those in [Hruska 2008]. The definition in [Agol et al. 2009] applies only to a finitely generated subgroup of a finitely generated relatively hyperbolic group.

Throughout this section G will be hyperbolic, relative to a finite collection of subgroups $\mathcal{P} = \{P_1, \ldots, P_n\}$, and S will be a finite generating set for G. The

cusped space (recalled below) for G with respect to $\mathcal{P}$ and S will be denoted by $X(G, \mathcal{P}, S)$, and $d(\cdot, \cdot)$ will denote the path metric on the cusped space. In particular, we will not need the word metric on G, but only the metric induced by this path metric. For detailed definitions and background on cusped spaces for relatively hyperbolic groups, see [Groves and Manning 2008] and [Hruska 2008]; we do sketch the construction and recall some terminology here.

Let A be a discrete metric space with metric ρ. The *combinatorial horoball based on A* is a graph $\mathcal{H}(A)$ with vertex set $A \times \mathbb{Z}_{\geq 0}$, such that

- (a, n) is connected by an edge to $(a, n + 1)$ for any $a \in A$, and
- for $n \geq 1$, (a, n) is connected to (a', n) whenever $\rho(a, a') \leq 2^n$.

Edges of the first type are called *vertical*; edges of the second type are called *horizontal*. We say that a vertex (a, n) of $\mathcal{H}(A)$ has *depth n*. If A is a subset of a path metric space Y, we may *attach a horoball to Y along A* by gluing $A \subseteq Y$ to $A \times \{0\} \subset \mathcal{H}(A)$, and taking the obvious path metric on the union. If G is finitely generated by S, we take Y to be the Cayley graph of G with respect to S; any subset of G inherits a discrete metric from the path metric on the Cayley graph. The *cusped space $X(G, \mathcal{P}, S)$* is the space obtained by simultaneously attaching horoballs to Y along all left cosets tP for $P \in \mathcal{P}$. A vertex v of $X(G, \mathcal{P}, S)$ corresponds either to a group element, if it lies in the Cayley graph of G, or otherwise to a triple (tP, g, n), where tP is a left coset of an element of $\mathcal{P}$, the element g lies in tP, and $n > 0$ is the depth of v in the attached horoball $\mathcal{H}(tP)$. In what follows we do not distinguish between v and the corresponding group element or triple.

The group G is hyperbolic relative to $\mathcal{P}$ if and only if the space $X(G, \mathcal{P}, S)$ is Gromov hyperbolic; see for example [Groves and Manning 2008, Theorem 3.25]. If so, then G acts on $X(G, \mathcal{P}, S)$ *geometrically finitely*,[1] meaning in particular this:

(1) Given any $n \geq 0$, let X_n be the subset of $X(G, \mathcal{P}, S)$ obtained by deleting all vertices of height greater than n. Then G acts cocompactly on X_n, which is an example of what Hruska calls a *truncated space* for the action of G on $X(G, \mathcal{P}, S)$.

(2) For fixed n, there are only finitely many components of $X(G, \mathcal{P}, S) \setminus X_n$, up to the action of G. (The components of $X(G, \mathcal{P}, S) \setminus X_{n-1}$ are called *n-horoballs* for $n \geq 1$. If n is understood, we call them *horoballs*. A 0-horoball is a 1-neighborhood of a 1-horoball, and is equal to $\mathcal{H}(tP)$ for some coset tP of some $P \in \mathcal{P}$.)

Relatively quasiconvex subgroups according to Agol, Groves, and Manning. Let H be a relatively hyperbolic group, let $\mathcal{D} = \{D_1, \ldots, D_m\}$ be the peripheral subgroups of H, and let T be a finite generating set for H. Let $\phi \colon H \to G$ be a

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[1]Hruska uses the term "cofinitely".

homomorphism. If every $\phi(D_i) \in \mathcal{D}$ is conjugate in G into some $P_j \in \mathcal{P}$, we say that the map ϕ *respects the peripheral structure on H*.

One can extend such a map ϕ to a map $\check{\phi}$ between zero-skeletons of cusped spaces as follows: For each $D_i \in \mathcal{D}$, choose an element $c_i \in G$ (of minimal length) and some P_{j_i} such that $\phi(D_i) \subseteq c P_{j_i} c^{-1}$. For $h \in H$, we have $\check{\phi}(h) = \phi(h)$. For a vertex $(s D_i, h, n)$ in a horoball of $X(H, \mathcal{D}, T)$, define

$$\check{\phi}(s D_i, h, n) = (\phi(s) c_i P_{j_i}, \phi(h) c_i, n).$$

Lemma A.1 [Agol et al. 2009, Lemma 3.1]. *Let $\phi \colon H \to G$ be a homomorphism that respects the peripheral structure on H. The extension $\check{\phi}$ defined above is H-equivariant and Lipschitz. If ϕ is injective, then $\check{\phi}$ is proper.*

Recall that for $C \geq 0$ a subset A of a geodesic metric space X is C-*quasiconvex* if every geodesic with endpoints in A lies in a C-neighborhood of A, and is *quasiconvex* if it is C-quasiconvex for some C. The following is slightly paraphrased:

Definition A.2 [Agol et al. 2009, Definition 3.11]. Let G be as above, and let $H < G$ be finitely generated by a set T. We say that H is (*QC-AGM*) *relatively quasiconvex* in $(G, \mathcal{P})$ if, for some finite collection of subgroups $\mathcal{D}$ of H,

(1) H is hyperbolic relative to $\mathcal{D}$, and

(2) if $\iota \colon H \to G$ is the inclusion, then the map $\check{\iota} \colon X(H, \mathcal{D}, T)^0 \to X(G, \mathcal{P}, S)^0$ described in Lemma A.1 has quasiconvex image.

Relatively quasiconvex subgroups according to Hruska. The following definition is [Hruska 2008, Definition 6.6], where it is called *QC-3*. Hruska shows there that this definition is equivalent to several others, including our Definition 3.4.

Definition A.3. We say a subgroup $H \leq G$ is (*QC-H*) *relatively quasiconvex* if the following holds. Let (X, ρ) be some (any) proper Gromov hyperbolic space on which $(G, \mathcal{P})$ acts geometrically finitely. Let $X - U$ be some (any) truncated space for G acting on X. For some (any) basepoint $x \in X - U$, there is a constant $\mu \geq 0$ such that whenever c is a geodesic in X with endpoints in the orbit Hx, we have $c \cap (X - U) \subseteq \mathcal{N}_\mu(Hx)$, where the neighborhood is taken with respect to the metric ρ on X.

Remark A.4. The meaning of "some (any)" in Definition A.3 just means that the word "some" can be replaced by "any" without affecting which subgroups of G are (QC-H) relatively quasiconvex. Thus this definition has some nondefinitional content, established in [Hruska 2008, Proposition 7.5 and 7.6].

Definition A.5. Let $A \subset X = X(G, \mathcal{P}, S)$ be a horoball, and let $R > 0$. A geodesic γ *penetrates the horoball A to depth R* if there is a point $p \in \gamma \cap A$ at distance at least R from $X \setminus A$. We say that *A is R-penetrated by the subgroup H* if there is a geodesic γ with endpoints in H that penetrates the horoball A to depth R.

Proposition A.6. *Let $H < G$ be (QC-H) relatively quasiconvex. Then there is a constant $R = R(G, \mathcal{P}, S, H)$ such that whenever a 0-horoball is R-penetrated by H, the intersection of H with the stabilizer of that horoball is infinite.*

Before the proof, we quote a proposition from [Hruska 2008] and prove two lemmas.

Proposition A.7 [Hruska 2008, Proposition 9.4]. *Suppose G has a proper, left invariant metric d, and suppose xH and yK are arbitrary left cosets of subgroups of G. For each constant L, there is a constant $L' = L'(G, d, xH, yK)$ such that in the metric space (G, d), we have*

$$\mathcal{N}_L(xH) \cap \mathcal{N}_L(yK) \subseteq \mathcal{N}_{L'}(xHx^{-1} \cap yKy^{-1}).$$

Lemma A.8. *Let H be a (QC-H) relatively quasiconvex subgroup of G. Let A be a 0-horoball of $X(G, \mathcal{P}, S)$ whose stabilizer is P^t for $P \in \mathcal{P}$. If A is R-penetrated by H for all $R > 0$, then $H \cap P^t$ is infinite.*

Proof. It suffices to show that, for every $M > 0$, there is some h in $H \cap P^t$ with $d(1, h) > M$.

Let μ be the quasiconvexity constant of Definition A.3 for H and the space X' that consists of all vertices in $X(G, \mathcal{P}, S)$ at depth 0. Let C be the constant given by Proposition A.7 such that $\mathcal{N}_\mu(H) \cap tP \subseteq \mathcal{N}_C(H \cap tPt^{-1})$, where the neighborhoods are taken in the cusped space.

Suppose that γ is a geodesic with endpoints in H that penetrates the horoball A to depth $M + C$. The first and last points of $\gamma \cap A$ are group elements a and b, both in the coset tP. Since H is (QC-H) relatively quasiconvex, a and b are elements of $\mathcal{N}_\mu(H) \cap tP$ and therefore (using Proposition A.7) there are elements h_1 and h_2 in $H \cap P^t$ such that $d(h_1, a) \le C$ and $d(h_2, b) \le C$. Since $d(a, b) \ge 2(M + C)$,

$$d(1, h_1^{-1}h_2) = d(h_1, h_2) \ge 2(M + C) - 2C \ge 2M > M. \qquad \square$$

Lemma A.9. *Let H be a (QC-H) relatively quasiconvex subgroup of G. Let μ be the quasiconvexity constant of Definition A.3 for H and the space $X' = X_0$ that is obtained from $X(G, \mathcal{P}, S)$ by deleting all vertices of positive depth.*

Let $R > 0$, and let A be a 0-horoball stabilized by P^t for $P \in \mathcal{P}$. If A is R-penetrated by H, then there is a horoball A' such that

(1) *$A' = hA$ for some $h \in H$,*

(2) *$d(A', 1) \le \mu$, and*

(3) *A' is R-penetrated by H.*

Proof. Suppose that γ is a geodesic with endpoints h_1 and h_2 in H that penetrates the horoball A to depth R. Let a and b be the first and last vertices of $\gamma \cap A$. By (QC-H) relative quasiconvexity, there is some $h \in H$ such that $d(a, h) \le \mu$.

The geodesic $h^{-1}\gamma$ goes between $h^{-1}h_2$ and $h^{-1}h_2$, and penetrates the horoball $A' = h^{-1}A$ to depth R. Moreover, $d(1, A') \leq d(1, h^{-1}a) = d(a, h) \leq \mu$.   □

Proof of Proposition A.6. Suppose there is no such number R. There must be a sequence of integers $R_i \to \infty$ and a sequence of 0-horoballs $\{A_i\}$ such that, for each i, the horoball A_i is R_i-penetrated by H but the intersection of the stabilizer of A_i with H is finite.

For $h \in H$, the stabilizer of hA_i is conjugate (by h) to the stabilizer of A_i. Using Lemma A.9, we can therefore assume that $d(1, A_i) \leq \mu$ for each i. By passing to a subsequence, we can therefore assume that the sequence $\{A_i\}$ is constant. It follows that A_0 is R_i-penetrated by H for all i. Lemma A.8 then implies that the intersection of H with the stabilizer of A_0 is infinite, which is a contradiction.   □

Equivalence of the two definitions.

Suppose G is a group hyperbolic relative to a finite collection of subgroups $\mathscr{P}$, and let S be a finite generating set for G. Let $X(G, \mathscr{P}, S)$ be the cusped space for G with respect to $\mathscr{P}$ and S, and let δ be its hyperbolicity constant.

Theorem A.10. *Let H be a finitely generated subgroup of G. Then H is (QC-H) relatively quasiconvex if and only if H is (QC-AGM) relatively quasiconvex.*

Proof. One direction is easy. Suppose that $H < G$ is (QC-AGM) relatively quasiconvex, is generated by the finite set T, and has peripheral subgroups $\mathscr{D}$. Recall that to define $\check{\iota}\colon X(H, \mathscr{D}, T)^0 \to X(G, \mathscr{P}, S)^0$, an element $c_i \in G$ was chosen for each $D_i \in \mathscr{D}$ so that $D_i \subset c_i P_{j_i} c_i^{-1}$. Let $C = \max\{d(1, c_i) \mid D_i \in \mathscr{D}\}$, and let C_q be the constant of quasiconvexity in the definition of (QC-AGM) quasiconvexity. As remarked at the beginning of the appendix, the cusped space $X = X(G, \mathscr{P}, S)$ is acted on geometrically finitely by G, and the subspace $X - U = X_0 \subset X(G, \mathscr{P}, S)$ obtained by deleting 1-horoballs is a truncated space for the action. Moreover, as explained in Remark A.4, it suffices to find a μ that works for this choice of X and $X - U$ and for the H-orbit of 1 in X. Let $x, y \in H$, and let γ be any geodesic joining them in X. Let z be a vertex of γ contained in X_0. By (QC-AGM), there is some point $w \in \check{\iota}(X(H, \mathscr{D}, T)^0)$ such that $d(z, w) \leq C_q$. It follows that w belongs to X_{C_q}, but any point in $\check{\iota}(H, \mathscr{D}, T) \cap X_{C_q}$ is at most $C + C_q$ away from some point in H. It follows that z is no further than $\mu := C + 2C_q$ from H, and so H is (QC-H) relatively quasiconvex.

For the other direction, let H be a subgroup of G, and suppose that H is (QC-H) relatively quasiconvex. Let $\mathscr{D}$ be a set of representatives of the H-conjugacy classes of infinite maximal parabolic subgroups of H. By [Hruska 2008, Theorem 9.1], H is hyperbolic relative to $\mathscr{D}$. By Lemma A.1, the inclusion $\iota\colon H \to G$ extends to a Lipschitz map of (0-skeletons of) cusped spaces $\check{\iota}\colon X(H, \mathscr{D}, T)^0 \to X(G, \mathscr{P}, S)^0$. We need to prove that the image $Y = \check{\iota}(X(H, \mathscr{D}, T)^0)$ of $\check{\iota}$ is quasiconvex.

Let R be the constant provided by Proposition A.6 for the subgroup H. Let $X' = X_{100\delta+R}$ be the subspace of $X(G, \mathcal{P}, S)$ consisting of all vertices at depth at most $100\delta + R$. Since H is (QC-H) relatively quasiconvex, there is a constant μ such that $\zeta \cap X' \subset \mathcal{N}_\mu(H) \subset \mathcal{N}_\mu(Y)$ for any geodesic ζ in $X(G, \mathcal{P}, S)$ with endpoints in H, where the neighborhoods are taken with respect to the metric on $X(G, \mathcal{P}, S)$.

Let x and y be vertices of Y and let γ be a geodesic between them. We will show that the vertices of γ are contained in the M-neighborhood of Y, where M is a constant independent of x, y, and γ. We divide the proof into five (not necessarily disjoint) cases.

Case 1: The points x and y lie deeper than 10δ in the same horoball. By recalling some easy properties of the geometry of horoballs, we will show that γ is contained in the M_1-neighborhood of Y, where $M_1 = 6$. First, the 10δ-horoball containing x and y is convex; see [Groves and Manning 2008, Lemma 3.26]. Second, any geodesic with the same endpoints as γ is Hausdorff distance at most 4 from γ. Finally, there is a geodesic γ' of a particularly nice form with the same endpoints as γ. The geodesic γ' is a *regular geodesic*, which means that all its edges are vertical except for at most three consecutive horizontal edges at maximum depth; see [Groves and Manning 2008, Lemma 3.10]. Since the vertical subsegments of γ' start at points in Y and are vertical, they stay in Y, and so γ' stays in a 2-neighborhood of Y. As γ is contained in a 4-neighborhood of γ', we have γ contained in a 6-neighborhood of Y.

Case 2: The points x and y are elements of H, they are in the neighborhood of radius μ of a horoball $\mathcal{H}(tP)$, and the geodesic γ penetrates the horoball $\mathcal{H}(tP)$ to depth larger than $100\delta + R$. In this case, we will approximate γ by a regular geodesic inside $\mathcal{H}(tP)$ with (possibly different) endpoints in Y. Without loss of generality, assume that x is the identity, and so $d(1, t) \leq \mu$.

By Proposition A.6, the intersection $H \cap P^t$ is infinite, from which it follows that $H \cap P^t = D^s$ for some $D \in \mathfrak{D}$ and $s \in H$. We claim s can be chosen so that $d(1, s) < K$ for a constant K independent of x, y, and γ. Indeed, we observe that the set

$$W = \{(r, P) \in G \times \mathcal{P} \mid d(1, r) \leq \mu, \#(H \cap P^r) = \infty\}$$

is finite. For each $w = (r, P) \in W$, choose $u_w \in H$ so that $H \cap P^r = D^u$ for some $D \in \mathfrak{D}$; we let K be the maximum of $d(1, u_w)$ over all $w \in W$.

We further claim that there is an element $y' \in H \cap P^t$ such that $d(y, y') \leq L$ for a constant L independent of x, y, and γ. Indeed, for each $w = (r, P)$ in the set W defined above, Proposition A.7 implies we can find an $L_w > 0$ such that

$$H \cap \mathcal{N}_\mu(rP) \subseteq \mathcal{N}_{L_w}(H \cap P^r);$$

we let L be the maximum L_w over all $w \in W$.

Recall that to define $\check{\iota} \colon X(H, \mathscr{D}, T)^0 \to X(G, \mathscr{P}, S)^0$, an element $c_i \in G$ was chosen for each $D_i \in \mathscr{D}$ so that $D_i \subset c_i P_{j_i} c_i^{-1}$. Let

$$(A-1) \qquad\qquad C = \max\{d(1, c_i) \mid D_i \in \mathscr{D}\}.$$

The subgroup D is equal to D_i for some i, and we set $c = c_i$ for the same i.

Consider the elements $(sD, s, 10\delta)$ and $(sD, y's, 10\delta)$ of $X(H, \mathscr{D}, T)$ and their corresponding images in Y given by $(scP, sc, 10\delta)$ and $(scP, y'sc, 10\delta)$. The points $(scP, sc, 10\delta)$ and $(scP, y'sc, 10\delta)$ belong to the same 10δ-horoball, which is convex in $X(G, \mathscr{P}, S)$, as we noted in Case 1. Also as noted in Case 1, there is a regular geodesic γ' joining the points $(scP, sc, 10\delta)$ and $(scP, y'sc, 10\delta)$; since the endpoints lie in Y, the geodesic γ' is contained in the 2-neighborhood of Y.

On the other hand, the endpoints of the geodesics γ and γ' are close, namely,

$$d(1, (scP, sc, 10\delta)) \leq d(1, s) + d(1, c) + 10\delta \leq K + C + 10\delta \quad \text{and}$$

$$d(y, (scP, y'sc, 10\delta)) \leq d(y, y') + d(y', (scP, y'sc, 10\delta)) \leq L + K + C + 10\delta.$$

Since $X(G, \mathscr{P}, S)$ is δ-hyperbolic, the Hausdorff distance between γ' and γ is at most the distance between endpoints plus 2δ. Thus if $M_2 = 2\delta + K + L + C + 10\delta$, then γ is contained in the M_2-neighborhood of Y. This completes this case.

Case 3: x and y are elements of H. We split γ into subsegments $\gamma_1, \gamma_2, \ldots, \gamma_k$ such that no γ_i contains any group element (depth 0 vertex) in its interior, but the endpoints of each γ_i are group elements. Observe that each γ_i is either a single edge or a geodesic segment contained in a 0-horoball. Furthermore, since H is (QC-H) relatively quasiconvex, the endpoints of each γ_i are contained in the μ-neighborhood of H. We claim that each γ_i is contained in the M_3-neighborhood of Y, where $M_3 = 110\delta + R + 3\mu + 2 + M_2$. If γ_i is an edge, then the claim is immediate, so we suppose γ_i is contained in a 0-horoball $\mathscr{A}$.

First, suppose γ_i does not penetrate $\mathscr{A}$ to depth $110\delta + R + 2\mu$. An easy argument shows that the length of a geodesic in a combinatorial horoball is at most twice its maximum depth plus 4, so we have $|\gamma_i| < 220\delta + 2R + 2\mu + 4$, and γ_i is therefore contained in the $(110\delta + R + 3\mu + 2)$-neighborhood of H. In particular, γ_i is contained in the M_3-neighborhood of Y.

Now suppose that γ_i does penetrate the horoball $\mathscr{A}$ to depth $110\delta + R + 2\mu$. Let h_1 and h_2 be elements of H that are at distance at most μ from the endpoints of γ_i, and let α be a geodesic between them. Since $X(G, \mathscr{P}, S)$ is δ-hyperbolic, the Hausdorff distance between γ_i and α is at most $2\delta + \mu$. It follows that α penetrates the horoball $\mathscr{A}$ to depth $100\delta + R + \mu$, and hence it satisfies the condition of Case 2. Therefore, γ_i is in the $(2\delta + \mu + M_2)$-neighborhood (and hence in the M_3-neighborhood) of Y.

Case 4: x and y lie at depth no more than 50δ in $X(G, \mathcal{P}, S)$. If $x \in Y$ lies in a 1-horoball, then $x = (tP, hc_i, n)$ for some $P \in \mathcal{P}$, some $h \in H$, some $i \in \{1, \dots, m\}$, and some $n \leq 50\delta$; otherwise, $x \in H$. In any case, there is an element $h_1 \in H$ such that $d(x, h_1) \leq 50\delta + C$, where C is the constant defined in (A-1). By the same argument, there is an element $h_2 \in H$ such that $d(x, h_2) \leq 50\delta + C$. Since $X(G, \mathcal{P}, S)$ is δ-hyperbolic, the Hausdorff distance between γ and any geodesic γ' between h_1 and h_2 is at most $52\delta + C$. We may apply Case 3 to γ', and deduce that γ is contained in the M_4-neighborhood of Y, where $M_4 = 52\delta + C + M_3$.

Case 5: Either x or y lies inside a 50δ-horoball, but we are not in Case 1. Here we follow the proof of the last case of [Agol et al. 2009, Proposition 3.12]. If x or y lies in a horoball, that point is connected by a vertical path to a point in the right coset Hc_i at depth 0 in $X(G, \mathcal{P}, S)$. It's therefore possible to modify γ (by appending and deleting (mostly) vertical paths lying in a 3-neighborhood of Y) to a 10δ-local geodesic γ' with endpoints within C of H; the geodesic γ is contained in a 3-neighborhood of $\gamma' \cup Y$. By [Bridson and Haefliger 1999, III.H.1.13(3)], γ' is a $(7/3, 2)$-quasigeodesic. Since quasigeodesics track geodesics, there is a constant L_Q depending only on δ and C, and a geodesic γ'' with endpoints in H such that the Hausdorff distance between γ' and γ'' is at most L_Q. By Case 3, γ'' is contained in the M_2-neighborhood of Y. Let $M_5 = 3 + L_Q + M_2$, and observe that γ is contained in the M_5-neighborhood of Y.

Finally, we set $M = \max\{M_1, \dots, M_5\}$, and note that M does not depend on the vertices x and y of Y, or on the geodesic γ joining them. It follows that $Y = \breve{\imath}(X(H, \mathcal{D}, T)^0)$ is M-quasiconvex in $X(H, \mathcal{D}, T)$, and so H is (QC-AGM) relatively quasiconvex in $(G, \mathcal{P})$. $\qquad\square$

By applying the main result of Hruska [2008] on the equivalence of various definitions of relative quasiconvexity — our Definition 3.4 is Hruska's (QC-5), and our Definition A.3 (QC-H) is Hruska's (QC-3) — we obtain a useful fact:

Corollary A.11. *Suppose G is hyperbolic relative to $\mathcal{P}$, and let H be a finitely generated subgroup of G. The following are equivalent:*

(1) *H is a relatively quasiconvex subgroup of G, in the sense of Definition 3.4.*

(2) *H is a relatively quasiconvex subgroup of G, in the sense of Definition A.2.*

Appendix B. Extending the main result in the presence of torsion

Here we give some idea of the changes necessary to prove Theorem 1.9 (and hence Theorem 1.2) in the presence of torsion. Let G be a group hyperbolic relative to a finite collection $\mathcal{P}$ of LERF and slender subgroups, and let H be some relatively quasiconvex subgroup of G.

The main difference is that we must deal with the possibility that our relatively quasiconvex subgroup has finite but nontrivial maximal parabolic subgroups. Since a finite subgroup of a relatively hyperbolic group may intersect arbitrary sets of parabolic subgroups, we have to ignore these intersections. This is already handled in the arguments of Section 3 by only amalgamating with parabolic subgroups that have infinite intersection with H to obtain the fully quasiconvex subgroup Q.

In Section 4, it is necessary to modify the definition of H-filling as follows:

Definition B.1 (alternative definition in the presence of torsion). Suppose H is a relatively quasiconvex subgroup of G. A filling $G \to G(N_1, \ldots, N_m)$ is an H-*filling* if $N_i^g \subset P_i^g \cap H$ whenever $H \cap P_i^g$ is infinite.

We must check that the results from [Agol et al. 2009, Section 4.2] still hold with the new definition. (We do not know how to prove the result about height from Section 4.3 in this more general setting, but we do not need it for our argument.)

Examining the proofs in [Agol et al. 2009], the reader may check that it suffices to extend the technical Lemma 4.2 therein. We sketch how to do so, for the experts: In that lemma, the hypothesis of an H-filling is used to deduce the existence of a nontrivial element of H that is also in a conjugate of a filling kernel fixing a certain horoball; this follows from the fact that a geodesic between elements of H penetrates that horoball deeply. The heart of the argument is showing that if the geodesic penetrates the horoball deeply, the intersection of H with the horoball stabilizer is infinite. In the torsion-free setting, it suffices to show that the intersection is nontrivial. The proof in the presence of torsion is given in the previous appendix as Proposition A.6. With this proposition, one can prove the extended version of Lemma 4.2 in a straightforward manner, choosing slightly different constants to take the constant R from Proposition A.6 into account.

The proofs of [Agol et al. 2009, Propositions 4.3 and 4.5] go through in exactly the same way, and we obtain the same statement as Theorem 4.4 above, but with the new meaning of H-filling. Using Osin's Dehn filling result in place of Theorem 4.2, the rest of the proof of Theorem 1.9 goes through as written, with the exception that each mention of a condition of the form "$A \cap B \neq \{1\}$" for A and B subgroups of G should be replaced by "$A \cap B$ is infinite".

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