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A graded $\mathbb{F}_p$-algebra A with action of the Steenrod algebra $\mathscr{A}^*$ is said to be Steenrod presentable if there is a polynomial ring $P = \mathbb{F}_p[u_1, \dots, u_n]$ with an action of $\mathscr{A}^*$ and an $\mathscr{A}^*$-invariant ideal $I \subset P$ such that $A = P/I$ and the induced action of $\mathscr{A}^*$ on P/I is the given one. It is shown that an action φ of a simple compact Lie group G on a homogeneous Kähler manifold $X = G/H$ has a Steenrod presentable equivariant cohomology for almost all primes p if and only if φ is conjugate to the standard action by left translation. Application to the case $H = T$ a maximal torus reproduces a former result of the author: namely, that every topological G-action on G/T is conjugate to the standard action by left translation with isotropy group a maximal torus.

1. Introduction

Suppose X to be a space, and let $A = H^*(X; \mathbb{F}_p)$ be its cohomology with coefficients in the prime field $\mathbb{F}_p$. Then on A there is an unstable action of the p-Steenrod algebra $\mathscr{A}^*$. On the other hand, given a presentation $A = P/I$, for an ideal $I \subset P$ where P is the polynomial algebra $P = \mathbb{F}_p[h_1, \dots, h_n]$, with deg $h_i = d_i$, one might ask whether the given action of $\mathscr{A}^*$ is induced by an action of $\mathscr{A}^*$ on the polynomial algebra that leaves the defining ideal stable. In the case $p \neq 2$ and d_i prime to p for all i, a necessary condition condition is given by a theorem of Adams and Wilkerson [1980]; see also [Smith 1995, Theorem 10.5.1]. In particular it follows from this theorem that the polynomial ring P must be the invariant ring

$$P = \mathbb{F}_p[x_1, \dots, x_n]^W, \quad \deg x_i = 2,$$

where $W \subset \mathrm{GL}(n, \mathbb{F}_p)$ is a finite group of order $d_1 \dots d_n$ generated by pseudoreflections acting on $\mathbb{F}_p[x_1, \dots, x_n] = \mathbb{F}_p[V]$ in the standard way [Smith 1995; 1997].

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This implies that the degrees d_i can only assume certain precise values, which are exactly the Weyl–Coxeter degrees of the group W; see for example [Smith 1995, p. 199].

In the following, we say that the $\mathbb{F}_p$-algebra A with an unstable action of $\mathscr{A}^*$ is *Steenrod presentable* if there is a polynomial ring $P = \mathbb{F}_p[x_1, \dots, x_n]^W$ with the standard action of the Steenrod algebra $\mathscr{A}^*$ and an $\mathscr{A}^*$-stable ideal $I \subset P$ such that $A \cong P/I$ with the induced $\mathscr{A}^*$-module structure.

As the main example of Steenrod presentable $\mathbb{F}_p$-algebras, we consider the cohomology of homogeneous spaces $X = G/H$, where $H \subset G$ is a closed connected maximal rank subgroup of a compact connected Lie group G. Then there is the standard fibration

$$G/H \longrightarrow B_H \longrightarrow B_G,$$

where B_K is the classifying space for the topological group K. If $H \subset G$ is a subgroup of maximal rank and if nor G neither H have p-torsion, the ring $H^*(G/H; \mathbb{F}_p)$ has a presentation

$$H^*(G/H; \mathbb{F}_p) \cong \frac{H^*(B_H; \mathbb{F}_p)}{H^*_+(B_G; \mathbb{F}_p) \cdot H^*(B_H; \mathbb{F}_p)}$$

such that the action of the Steenrod algebra $\mathscr{A}^*$ on $H^*(G/H; \mathbb{F}_p)$ is induced by the standard action of $\mathscr{A}^*$ on the ring $H^*(B_H; \mathbb{F}_p)$.

So, throughout this note we shall assume that $p \neq 2$ and that B_G and B_H do not have p-torsion for all primes to be considered.

Suppose a compact connected Lie group K is acting in a reasonable way on $X = G/H$. Then X is totally nonhomologous to zero in the fibration

$$X \longrightarrow X_K \longrightarrow B_K,$$

where $X_K = E_K \times_K X$ is the Borel construction. Write $H^*(X; \mathbb{F}_p) = P/I_0$, $P = \mathbb{F}_p[h_1, \dots, h_n]$, where the ideal $I_0 \subset P$ is generated by a set $g_1, \dots, g_n$ of multiplicative generators of the invariant ring $R_G = H^*(B_G; \mathbb{F}_p) \subset H^*(B_H; \mathbb{F}_p)$. As can be shown in the same way as in the proof of [Hauschild 1986, Theorem 1.1], the equivariant cohomology $H^*_K(X; \mathbb{F}_p) = H^*(X_K; \mathbb{F}_p)$ is a graded algebra over $R = H^*(B_K; \mathbb{F}_p)$, which can be written as $H^*_K(X; \mathbb{F}_p) = P_R/I$, where $P_R = R \otimes P$ and I is an ideal generated by homogeneous elements of the form $1 \otimes g_j - r_j$, where the r_j are elements of the ideal $R_+ P_R$ generated by the augmentation ideal of R. On the ring $P_R = H^*(B_K; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(B_H; \mathbb{F}_p)$ there is the natural unstable $\mathscr{A}^*$-module structure and the equivariant cohomology is *Steenrod presentable* if I is stable under this $\mathscr{A}^*$-action inducing the given $\mathscr{A}^*$-action on the quotient. Moreover, since the isomorphism $H^*(X; \mathbb{F}_p) \cong H^*_K(X; \mathbb{F}_p)/H^*_+(B_K; \mathbb{F}_p) H^*_K(X; \mathbb{F}_p)$ is induced by the inclusion $i \colon X \to X_K$ of the fiber, the Steenrod presentation of

$H_K^*(X;\mathbb{F}_p)$ induces the Steenrod presentation of $H^*(X;\mathbb{F}_p)$. For more information on Steenrod powers acting on equivariant cohomology, see [Allday and Puppe 1993; Quillen 1971].

2. Steenrod powers and rational cohomology

Observe that $X = G/H$ is now the fiber of two fibrations, and that in both fibrations it is totally nonhomologous to zero. Consequently there is the canonical epimorphism $j^*\colon H^*(B_H;\mathbb{F}_p) \to H^*(X;\mathbb{F}_p)$ induced by the inclusion $j\colon X \to B_H$ of the fiber. Moreover, let i^* be induced by the inclusion $i\colon X \to X_K$ of the fiber in the Borel fibration. Both maps commute of course with the respective $\mathscr{A}^*$-module structures.

Observation 1. *The equivariant cohomology $H_K^*(X;\mathbb{F}_p)$ is Steenrod presentable if and only if there is a homomorphism $J\colon H^*(B_H;\mathbb{F}_p) \to H_K^*(X;\mathbb{F}_p)$ making the following diagram commute*:

$$\begin{array}{ccc} & & H_K^*(X;\mathbb{F}_p) \\ & \overset{J}{\nearrow} & \big\downarrow{\scriptstyle i^*} \\ H^*(B_H;\mathbb{F}_p) & \xrightarrow[j^*]{} & H^*(X;\mathbb{F}_p) \end{array}$$

Proof. Let $\pi : X_K \to B_K$ be the projection in the Borel fibration, and then consider the homomorphism $\pi^* \otimes J\colon H^*(B_K;\mathbb{F}_p) \otimes H^*(B_H;\mathbb{F}_p) \to H_K^*(X;\mathbb{F}_p)$. This map is surjective and commutes with the respective $\mathscr{A}^*$-actions. Let $I = \operatorname{Ker}(\pi^* \otimes J)$; then $H_K^*(X;\mathbb{F}_p) = (H^*(B_K;\mathbb{F}_p) \otimes H^*(B_H;\mathbb{F}_p))/I$ is a Steenrod presentation.

On the other hand, given a Steenrod presentation

$$H_K^*(X;\mathbb{F}_p) = (H^*(B_K;\mathbb{F}_p) \otimes H^*(B_H;\mathbb{F}_p))/I,$$

and $J\colon H^*(B_H;\mathbb{F}_p) \to H_K^*(X;\mathbb{F}_p)$ given by

$$H^*(B_H;\mathbb{F}_p) \ni \xi_H \mapsto 1 \otimes \xi_H \mod I,$$

then J commutes with the $\mathscr{A}^*$-actions and $i^* \circ J = j^*$. □

Let X, X' be spaces such that the rational cohomology rings $H^*(X;\mathbb{Q})$ and $H^*(X';\mathbb{Q})$ are finitely generated as graded $\mathbb{Q}$-algebras. Then we have to define what it means for a homomorphism $\theta\colon H^*(X;\mathbb{Q}) \leftarrow H^*(X';\mathbb{Q})$ to commute with Steenrod powers for almost all primes p. Let $y_1, \ldots, y_m \in H^*(X';\mathbb{Q})$ be a set of multiplicative generators; similarly, let $x_1, \ldots, x_n \in H^*(X;\mathbb{Q})$ be a set of multiplicative generators. Then $\theta(y_i) = p_i(x_1, \ldots, x_n) \in H^*(X;\mathbb{Q})$ are polynomials. Let Prime_θ be the (finite) subset of primes which appear as divisors of the denominators of the coefficients of the p_i. Then for all $p \notin \mathrm{Prime}_\theta$ there are unique homomorphisms $\theta_p, \bar{\theta}_p$ which make the following diagram commute [Adams and Mahmud 1976]:

$$\begin{array}{ccc} H^*(X;\mathbb{Q}) & \xleftarrow{\ \theta\ } & H^*(X';\mathbb{Q}) \\ \uparrow & & \uparrow \\ H^*(X;\mathbb{Z}_{(p)}) & \xleftarrow{\ \theta_p\ } & H^*(X';\mathbb{Z}_{(p)}) \\ \downarrow & & \downarrow \\ H^*(X;\mathbb{F}_p) & \xleftarrow{\ \bar{\theta}_p\ } & H^*(X';\mathbb{F}_p) \end{array}$$

Here the vertical maps are induced by the canonical maps $\mathbb{Z}_{(p)} \to \mathbb{Q}$ and $\mathbb{Z}_{(p)} \to \mathbb{F}_p$ respectively. We say that θ commutes with the Steenrod powers for almost all primes p if the $\bar{\theta}_p$ commute with Steenrod powers for $p \notin \mathrm{Prime}_\theta$.

Definition 2. Let K be a compact Lie group acting on $X = G/H$. Then we say that the rational equivariant cohomology $H^*_K(X;\mathbb{Q})$ is Steenrod presentable if there is a lifting J of the edge homomorphism j^*

$$\begin{array}{ccc} & & H^*_K(X;\mathbb{Q}) \\ & \overset{J}{\nearrow} & \downarrow i^* \\ H^*(B_H;\mathbb{Q}) & \xrightarrow[j^*]{} & H^*(X;\mathbb{Q}) \end{array}$$

such that $\bar{J}_p\colon H^*(B_H;\mathbb{F}_p) \to H^*_K(X;\mathbb{F}_p)$ commutes with Steenrod powers for almost all p.

A homogeneous space G/H such that $\operatorname{rank} G = \operatorname{rank} H$ is Kähler if and only if $H = Z(K)$ is the centralizer of a (not necessarily maximal) torus K, or, equivalently, if H is conjugate to an isotropy group of the adjoint representation [Besse 1987, Chapter 8].

Here is the main theorem of this article.

Theorem 3. *Let G be a simple compact connected Lie group and $H \subset G$ be a closed connected subgroup of maximal rank such that $X = G/H$ is Kähler and let G act topologically on $X = G/H$. Then the following statements are equivalent.*

(i) *The equivariant cohomology $H^*_G(X;\mathbb{Q})$ is Steenrod presentable.*

(ii) *The group G acts transitively on X with an isotropy group conjugate to K, where K is a maximal rank subgroup of G isomorphic to H by an automorphism of G which is inner with the possible exception of the even Spin groups.*

(iii) *There is an isomorphism $H^*_G(X;\mathbb{Q}) \cong R_H$ as R_G-algebras.*

As a corollary, we recover an earlier result from [Hauschild 1985]. (See also [Hauschild 1986] and the introduction of [Hauschild 2006], where the uniqueness problem for locally smooth $\mathrm{SU}(n+1)$-actions on $\mathrm{SU}(n+1)/S(U(n-1)\times U(2))$ is considered.)

Theorem 4. *Let G be a simple compact connected Lie group, and let $T \subset G$ be a maximal torus. Let G act nontrivially on $X = G/T$ via φ. Then up to conjugacy, φ is the standard transitive G-action on X with isotropy group conjugate to T.*

Proof (for a proof using obstruction theory, see the Appendix). Write $H^*(B_T;\mathbb{Q}) = \mathbb{Q}[x_1,\dots,x_n]$, $\deg x_i = 2$. Let $R_G = H^*(B_G;\mathbb{Q})$ and write

$$H^*_G(X;\mathbb{Q}) = \frac{R_G[X_1,\dots,X_n]}{I}, \quad \deg X_i = 2.$$

Define $J(x_i) = \overline{X}_i$, where the $\overline{X}_i$ is the class of X_i. Let p be a prime such that J_p and $\bar{J}_p$ are defined.

The values of the Steenrod powers $\mathcal{P}^k(x_i)$ and $\mathcal{P}^k(\overline{X}_i)$ are completely determined by the instability conditions, that is, we have $\mathcal{P}^k(x_i) = x_i^p$ for $k=1$ and $\mathcal{P}^k(x_i) = 0$ for $k > 1$. The same holds in $H^*_G(X;\mathbb{F}_p)$; that is, $\mathcal{P}^k(\overline{X}_i) = \overline{X}_i^p$ for $k=1$ and $\mathcal{P}^k(\overline{X}_i) = 0$ for $k > 1$. It follows that $\mathcal{P}^k \bar{J}_p(x_i) = \bar{J}_p \mathcal{P}^k(x_i)$ for all i. By simple induction using the Cartan rule, one gets the relation $\mathcal{P}^k \circ \bar{J}_p = \bar{J}_p \circ \mathcal{P}^k$ for all $k \geq 0$ and almost all primes p. So, the equivariant cohomology is Steenrod presentable and the result follows from Theorem 3. □

3. A proof of the main theorem

The following definitions synthesize certain cohomological properties of symplectic manifolds and are taken from the paper [Allday 1998]. We consider cohomology with coefficients in a field $\mathbb{Q}$, with char $\mathbb{Q} = 0$. As a coefficient field of cohomology, the symbol $\mathbb{Q}$ will be omitted in this paragraph.

Definition 5. Let X be a Poincaré duality space over $\mathbb{Q}$ with formal dimension $2n$.

(i) The space X is said to be c-symplectic (that is, cohomologically symplectic) if there is $w \in H^2(X)$ such that $w^n \neq 0$.

(ii) If X is c-symplectic, for $0 \leq j \leq n$, consider the map $w^j \colon H^{n-j}(X) \to H^{n+j}(X)$, defined as $a \mapsto w^j a$, for all $a \in H^{n-j}(X)$. Then X is said to satisfy the hard Lefschetz condition if w^j is an isomorphism for all j. In this case X is also said to be c-Kähler.

Let X be a c-symplectic space with $w \in H^2(X)$ as in the definition above. Let G be a compact connected Lie group acting on X. Then $g^*(w) = w$ for all $g \in G$. In this way any action of a compact connected Lie group on a c-symplectic space is considered to be a cohomologically symplectic action.

Definition 6. Let X be a c-symplectic space with c-symplectic class $\omega \in H^2(X)$. Let a torus G act on X. Then the action is said to be cohomologically Hamiltonian (c-Hamiltonian) if $\omega \in \mathrm{Im}\{i^*\colon H^2_G(X) \to H^2(X)\}$, where $i\colon X \to X_G$ is the inclusion of the fiber in the bundle $X_G \to B_G$.

The main reason we have restricted ourselves to homogeneous spaces G/H with the Kähler property is the following result, which can be considered a generalization of a theorem of Atiyah [1983] (see also [Guillemin and Sternberg 1982; Audin 1991, Corollary 4.2.3]). For the definition of uniformity see [Allday and Puppe 1993, Definition 3.6.17]. For other consequences of the Kähler property, see [Allday et al. 2002].

Theorem 7 [Allday 1998]. *Let the r-torus $G = T^r$ act on a closed c-symplectic manifold X in an effective, uniform, c-Hamiltonian way. Then X^G has at least $r+1$ connected components.*

The conditions of the theorem are always satisfied if X is totally nonhomologous to zero in the Borel fibration [Allday and Puppe 1993]. Let G be a torus and suppose G is acting on a c-symplectic manifold X with vanishing odd cohomology. As we have seen before, the equivariant cohomology can be written as $H^*_G(X) = R_G[h_1, \dots, h_n]/I$ where $R_G = H^*(B_G)$ and the $h_1, \dots, h_n$ is a system of homogeneous multiplicative generators, I the defining ideal. Let $X^G = F_1 + F_2 + \cdots + F_s$ be the decomposition of the fixed space X^G into its connected components. Then for every α, $1 \le \alpha \le s$, we choose a point $p_\alpha \in F_\alpha$ and define a prime ideal P_α as the kernel of the composed homomorphism $R_G[h_1, \dots, h_n] \to H^*_G(X) \to H^*_G(p_\alpha) \cong R_G$. Here the first homomorphism is the natural projection and the second is given by restricting equivariant cohomology classes to $E_G \times_G \{p_\alpha\}$. Then the radical of I is given by $\sqrt{I} = \bigcap_\alpha P_\alpha$. Moreover there is a natural bijection between the primary components of the ideal I and the connected components of X^G. For more details on these standard facts on equivariant cohomology see [Allday and Puppe 1993; Hsiang 1975]. The following lemma is an immediate consequence of the result of Allday.

Lemma 8. *Let the r-torus $G = T^r$ act on a closed c-symplectic manifold X with vanishing odd cohomology. Suppose G is acting on X in an effective, uniform, c-Hamiltonian way. Then there exists a connected component F of X^G such that the prime ideal $P \subset R[h_1, \dots, h_n]$ belonging to F is of the kind $P = (h_1 - \beta_1, \dots, h_n - \beta_n)$ with $\beta_i \in R^{\deg h_i}$ and some $\beta_i \neq 0$.*

Proof of the main theorem. (i) ⇒ (ii): Let $R_G = H^*(B_G)$ and let $R_H = H^*(B_H) \cong \mathbb{Q}[h_1, \dots, h_n]$. Suppose $H^*_G(X) = (R_G \otimes_{\mathbb{Q}} R_H)/I_G$ to be a Steenrod presentation. Let $T \subset G$ be a maximal torus; then the equivariant cohomology of the induced T-action is given by $H^*_T(X) \cong H^*_G(X) \otimes_{R_G} R_T$. Let $I_T \subset R_G \otimes_{\mathbb{Q}} R_T$ be the ideal generated by I_G, that is, $I_T = I_G \cdot (R_T \otimes_{\mathbb{Q}} R_H)$; then $H_T^*(X) \cong R_T[h_1, \ldots, h_n]/I_T$. By the previous lemma there is a connected component $F \subset X^T$ of the fixed set X^T such that the corresponding prime ideal has the form $P = (h_1 - \beta_1, \ldots, h_n - \beta_n)$ with $(\beta_1, \ldots, \beta_n) \neq 0$. In particular, the restriction homomorphism $H_T^*(X) \to H_T^*(\{p\}) \cong R_T$, $p \in F$ is nontrivial. Let $G_p \subset G$ be the isotropy group of p. It follows from the commutativity of the diagram

$$\begin{array}{ccc} H_T^*(X) & \longrightarrow & H_T^*(\{p\}) \cong R_T \\ \cup \uparrow & & \cup \uparrow \\ H_G^*(X) & \xrightarrow[\mathrm{res}_p]{} & H_G^*(G(p)) \cong R_{G_p} \end{array}$$

that the restriction homomorphism

$$\mathrm{res}_p \colon H_G^*(X) \to H_G^*(G(p)) \cong R_{G_p}$$

must also be nontrivial. Let $U = G_p^o$ be the connected component of the unit element in G_p, and let $\eta \colon H^*(B_{G_p}) \to H^*(B_U)$ be the homomorphism induced by the inclusion $U \subset G_p$. Then consider the composition $\theta = \eta \circ \mathrm{res}_p \circ J$

$$\theta \colon H^*(B_H) \xrightarrow{J} H_G^*(X) \xrightarrow{\mathrm{res}_p} H^*(B_{G_p}) \xrightarrow{\eta} H^*(B_U).$$

It follows from the construction and the hypothesis that θ commutes with the Steenrod powers in $\mathscr{A}^*$ for almost all primes p. Let LT be the Lie algebra of the maximal torus T. Let $\Sigma \subset LT$ be the kernel of the projection $LT \to T$. After [Adams and Mahmud 1976, Theorem 1.5] there is an $\mathbb{R}$-linear map $\phi \colon LT \to LT$ carrying $\Sigma \otimes \mathbb{Q}$ into $\Sigma \otimes \mathbb{Q}$ such that the following diagram is commutative.

$$\begin{array}{ccc} H^*(B_H) & \xrightarrow{\theta} & H^*(B_U) \\ \downarrow & & \downarrow \\ H^*(B_T) & \xrightarrow{\phi^*} & H^*(B_T) \end{array}$$

Here ϕ^* is the graded ring homomorphism induced by the linear map ϕ. The existence of this map is a consequence of [Adams and Mahmud 1976, Lemma 1.2]. The vertical maps are the homomorphisms induced by the standard fibrations $B_T \to B_H$ and $B_T \to B_U$. It follows from our assumption that θ is nontrivial, which implies that ϕ^* is also nontrivial. Observe that the map θ induces exactly the homomorphism $\bar{\theta} \colon H^*(G/H) \to H^*(G/U)$ induced by the map $G/U \to G/G_p \cong$

$G(p) \subset X = G/H$. This means that we have a commutative diagram

$$\begin{array}{ccc} H^*(X) & \xrightarrow{\bar{\theta}} & H^*(G/U) \\ \uparrow & & \uparrow \\ H^*(B_H) & \xrightarrow{\theta} & H^*(B_U) \end{array}$$

where the vertical maps are the edge homomorphisms for the fibrations $B_H \to B_G$ and $B_U \to B_G$, respectively. It follows that θ sends the ideal

$$H^*_+(B_G) \cdot H^*(B_H) \subset H^*(B_H)$$

generated by the invariants of the Weyl group in $H^*(B_H)$ into the ideal

$$H^*_+(B_G) \cdot H^*(B_U) \subset H^*(B_U)$$

generated by the same invariants in $H^*(B_U)$. Then ϕ^* sends the ideal

$$H^*_+(B_G) \cdot H^*(B_T) \subset H^*(B_T)$$

into the ideal

$$H^*_+(B_G) \cdot H^*(B_T) \subset H^*(B_T),$$

therefore inducing a graded and nontrivial homomorphism

$$\overline{\phi^*} \colon H^*(G/T) \longrightarrow H^*(G/T).$$

Since G is a simple Lie group we can apply [Hauschild 1985, Lemma 4.1]. Therefore $\overline{\phi^*}$ must be a surjective map and consequently must be an isomorphism. Now the commutative diagrams above induce a commutative diagram

$$\begin{array}{ccc} H^*(G/H) & \xrightarrow{\bar{\theta}} & H^*(G/U) \\ \downarrow & & \downarrow \\ H^*(G/T) & \xrightarrow{\overline{\phi^*}} & H^*(G/T) \end{array}$$

where the vertical maps are the respective inclusions of invariants under the Weyl groups WH, WU respectively. It follows that the homomorphism $\bar{\theta}$ must be injective which implies dimensions $cd_{\mathbb{Q}}(X) \leq cd_{\mathbb{Q}}(G/U)$ for the respective rational cohomology. But G/H and G/U are closed oriented manifolds and therefore $\dim X \leq \dim G/U$, which implies $\dim X = \dim G(p)$. It follows that $X = G/G_p$, that is, the action is transitive. Now X is 1-connected and therefore G_p must be connected, that is, $G_p = G_p^o = U$. It follows that $G/H = G/U$ and $\bar{\theta}$ is an isomorphism. By a theorem of Papadima [1986], the isomorphism $\overline{\phi^*}$ is induced by an automorphism of the root system of G. This implies that the root systems of the maximal rank subgroups H and U are conjugate by such an automorphism, and consequently, the groups H and U are conjugate by an automorphism which is inner with the possible exception of the Spin groups.

(ii) $\Rightarrow$ (i): We have $X_G = E_G \times_G G/U \cong E_G/U = B_U$. But $H \cong U$ and so $X_G \cong B_H$ and therefore $H^*_G(X) \cong R_H$ as R_G-algebras.

(iii) $\Rightarrow$ (i): Take $J = \mathrm{Id}\colon R_H \to R_H$. □

Appendix

Proof of Theorem 4 using obstruction theory. Let $\pi : X_G \to B_G$ be the projection, let $b \in B_G$, and let $X_b = \pi^{-1}(b) \subset X_G$ be the fiber over b. Let $i_b\colon X_b \to X_G$ be the corresponding inclusion. Then consider the extension problem

$$\begin{array}{ccc} X_b & \xrightarrow{\ j\ } & B_T \\ {\scriptstyle i_b}\big\downarrow & \nearrow{\scriptstyle j'} & \\ X_G & & \end{array}$$

The obstruction to extend the inclusion $j\colon X_b \to B_T$ to a map $j'\colon X_G \to B_T$ is to be found in the group $H^3(X_G, X_b; \pi_2(B_T))$. Consider the following piece of the long exact cohomology sequence of the pair (X_G, X_b).

$$H^2(X_G; \mathbb{Z}) \to H^2(X_b; \mathbb{Z}) \to H^3(X_G, X_b; \mathbb{Z}) \to H^3(X_G, \mathbb{Z}) \to \dots$$

Now the first arrow, induced by the inclusion of the fiber, is surjective whereas $H^3(X_G; \mathbb{Z}) = 0$. It follows $H^3(X_G, X_b; \mathbb{Z}) = 0$ and so $H^3(X_G, X_b; \mathbb{Z}^n) = 0$. We thus have a lifting $J = j'^*$ which gives rise to the commutative diagram

$$\begin{array}{ccc} & & H^*_G(X; \mathbb{Z}) \\ & \overset{J}{\nearrow} & \big\downarrow {\scriptstyle i_b^*} \\ H^*(B_T; \mathbb{Z}) & \xrightarrow[j^*]{} & H^*(X; \mathbb{Z}). \end{array}$$

By the definition of J as a map induced geometrically, we conclude that $H^*_G(X; \mathbb{Q})$ is Steenrod presentable. Using the equivalence between (i) and (ii) in Theorem 3 and the standard fact that two maximal tori are conjugate, the result follows. □

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