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By using the reflecting diffusion process and a conformal change of metric, a generalized maximum principle is established for (unbounded) time-space functions on a class of noncompact Riemannian manifolds with (nonconvex) boundary. As applications, Li–Yau-type gradient and Harnack inequalities are derived for the Neumann semigroup on a class of noncompact manifolds with (nonconvex) boundary. These generalize some previous ones obtained for the Neumann semigroup on compact manifolds with boundary. As a byproduct, the gradient inequality for the Neumann semigroup derived by Hsu on a compact manifold with boundary is confirmed on these noncompact manifolds.

1. Introduction

Suppose M is a d-dimensional connected complete Riemannian manifold, and let $L = \Delta + Z$, where Z is a C^1 vector field satisfying the curvature-dimension condition of Bakry and Émery [1984] given by

$$\Gamma_2(f, f) := \tfrac{1}{2}L|\nabla f|^2 - \langle \nabla Lf, \nabla f\rangle \geq \frac{(Lf)^2}{m} - K|\nabla f|^2 \quad \text{for } f \in C^\infty(M) \tag{1-1}$$

for some constants $K \geq 0$ and $m > d$. By [Qian 1998, page 138], this condition is equivalent to

$$\mathrm{Ric} - \nabla Z - \frac{Z \otimes Z}{m-d} \geq -K. \tag{1-2}$$

When $Z = 0$ and M is either without boundary or compact and with a convex boundary ∂M, Li and Yau [1986] found a now-famous gradient estimate for the (Neumann) semigroup P_t generated by L:

$$|\nabla \log P_t f|^2 - \alpha\partial_t \log P_t f \leq \frac{d\alpha^2}{2t} + \frac{d\alpha^2 K}{4(\alpha-1)} \quad \text{for } t > 0 \text{ and } \alpha > 1 \tag{1-3}$$

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for all positive $f \in C_b(M)$. We note that in [Li and Yau 1986] the second term on the right side of (1-3) is $d\alpha^2 K/(\sqrt{2}(\alpha-1))$, but $\sqrt{2}$ here can be replaced by 4 according to a refined calculation; see for example [Davies 1989].

As an application, (1-3) implies a parabolic Harnack inequality for P_t:

$$P_t f(x) \le \left(\frac{t+s}{t}\right)^{d\alpha/2} (P_{t+s} f(y)) \exp\left(\frac{\alpha\rho(x,y)^2}{4s} + \frac{\alpha K d s}{4(\alpha-1)}\right) \quad \text{for } t>0 \text{ and } x,y\in M, \tag{1-4}$$

where $\alpha > 1$ and $f \in C_b(M)$ is positive. From this Harnack inequality, one obtains Gaussian-type heat kernel bounds for P_t; see [Li and Yau 1986; Davies 1989].

The gradient estimate (1-3) has been extended and improved in several papers. See for example [Bakry and Qian 1999] for an improved version for $\alpha = 1$ with $Z \neq 0$ and $\partial M = \varnothing$, and see [Wang 1997] for an extension to a compact manifold with nonconvex boundary. The aim of this paper is to investigate the gradient and Harnack inequalities for P_t on noncompact manifolds with (nonconvex) boundary.

Recall that the key step of Li and Yau's argument for the gradient estimate (1-3) is to apply the maximum principle to the reference function

$$G(t,x) := t(|\nabla \log P_t f|^2 - \alpha\partial_t \log P_t f)(x) \quad \text{for } t \in [0,T] \text{ and } x \in M.$$

When M is compact without boundary, the maximum principle says that for any smooth function G on $[0,T]\times M$ with $G(0,\cdot)\le 0$ and $\sup G > 0$, there exists a maximal point of G at which $\nabla G = 0$, $\partial_t G \ge 0$, and $\Delta G \le 0$. When M is compact with a convex boundary, the same assertion holds for the above specified function G, as observed in [Li and Yau 1986, proof of Theorem 1.1]. In [1997], J. Wang extended this maximum principle on a compact manifold with nonconvex boundary by taking

$$G(t,x) = t(\phi|\nabla \log P_t f|^2 - \alpha\partial_t \log P_t f)(x) \quad \text{for } t \in [0,T] \text{ and } x \in M$$

for a nice function ϕ compensating the concavity of the boundary.

As for a noncompact manifold without boundary, Li and Yau [1986] established the gradient estimate by applying the maximal principle to a sequence of functions with compact support that approximate the original function G. An alternative is to apply directly the following generalized maximum principle:

Lemma 1.1 [Yau 1975]. *For any bounded smooth function G on $[0,T]\times M$ with $G(0,\cdot)\le 0$ and $\sup G > 0$, there exists a sequence $\{(t_n,x_n)\}_{n\ge1}\subset[0,T]\times M$ such that*

(i) $0 < G(t_n,x_n) \uparrow \sup G$ *as* $n\uparrow\infty$, *and*

(ii) *for any* $n \ge 1$,

$$LG(t_n,x_n)\le 1/n, \quad |\nabla G(t_n,\cdot)(x_n)| \le 1/n, \quad \partial_t G(t_n,x_n)\ge 0.$$

To apply this generalized maximal principle for the gradient estimate, one has to first confirm the boundedness of $G(t,\cdot) := t(|\nabla \log P_t f|^2 - \alpha\partial_t \log P_t f)$ on $[0,T]\times M$ for $T>0$.

Since the boundedness of this type of reference function is unknown when M is noncompact with a nonconvex boundary, we shall establish a generalized maximum principle on a class of noncompact manifolds with boundary for not necessarily bounded functions. Applying this principle to a suitable reference function G, we derive the Li–Yau-type gradient and Harnack inequalities for Neumann semigroups. To establish such a maximum principle, we adopt a localization argument so that the classical maximum principle can be applied.

For M noncompact without boundary, Li and Yau [1986] used such a localization argument to apply the maximal principle to functions with compact support; they then passed to the desired global estimate by taking a limit. To do this, they constructed cut-off functions using ρ_o, the Riemannian distance function to a fixed point $o\in M$. It turns out that this argument works also when ∂M is convex; see Section 2.1. For the nonconvex case, we will use the conformal change of metric introduced in [Wang 2007] to make a nonconvex boundary convex; see Section 2.2.

Assumption A. The manifold M is connected and complete with boundary ∂M and such that either

(1) ∂M is convex, or

(2) the second fundamental form of ∂M is bounded, the sectional curvature of M is bounded from above, and the injectivity radius $\mathrm{i}_{\partial M}$ of ∂M is positive.

Recall that the Riemannian distance function $\rho_{\partial M}$ to the boundary is smooth on the set $\{\rho_{\partial M} < \mathrm{i}_{\partial M}\}$.

Let N be the inward unit normal vector field on ∂M. The second fundamental form of ∂M is

$$\mathrm{II}(X,Y) = -\langle \nabla_X N, Y\rangle \quad \text{for } X, Y \in T\partial M.$$

The boundary ∂M is called convex if $\mathrm{II}\ge 0$. We are now ready to state our generalized maximal principle for possibly unbounded functions.

Theorem 1.2. *Let M satisfy A, and let L satisfy (1-2). Let $T>0$, and let G be a smooth function on $[0,T]\times M$ such that $NG|_{\partial M}\ge 0$, $G(0,\cdot)\le 0$ and $\sup G>0$. Then for any $\varepsilon>0$, there exists a sequence $\{(t_n,x_n)\}_{n\ge 1}\subset (0,T]\times M$ such that Lemma 1.1(i) holds and for any $n\ge 1$*

$$LG(t_n,x_n) \le \frac{G(t_n,x_n)^{1+\varepsilon}}{n}, \qquad |\nabla G(t_n,x_n)| \le \frac{G(t_n,x_n)^{1+\varepsilon}}{n},$$
$$\partial_t G(t_n,x_n) \ge 0.$$

Applying Theorem 1.2 to a proper choice of function G, we will derive the Li–Yau-type gradient estimate (1-5). We shall prove that the reflecting diffusion process X_t generated by L on M is non explosive, so that the corresponding Neumann semigroup P_t can be formulated as

$$P_t f(x) = \mathsf{E}^x f(X_t) \quad \text{for } t \geq 0,\, x \in M, \text{ and } f \in C_b(M),$$

where E^x is the expectation taken for $X_0 = x$.

Theorem 1.3. *Let M satisfy A, and suppose L satisfies (1-2) with $\|Z\|_\infty < \infty$. Then the reflecting L-diffusion process on M is nonexplosive and the corresponding Neumann semigroup P_t satisfies these assertions*:

(i) *If ∂M is convex, then (1-3) holds with m in place of d.*

(ii) *If ∂M is nonconvex with $\mathrm{II} \geq -\sigma$ for some $\sigma > 0$, then for any bounded $\phi \in C^\infty(M)$ with $\phi \geq 1$ and $N \log \phi|_{\partial M} \geq 2\sigma$, the gradient inequality*

$$|\nabla \log P_t f|^2 - \alpha \partial_t \log P_t f \leq \frac{m(1+\varepsilon)\alpha^2}{2(1-\varepsilon)t} + \frac{m\alpha^2 K(\phi,\varepsilon,\alpha)}{4(\alpha - \|\phi\|_\infty)} \tag{1-5}$$

holds for all positive $f \in C_b(M)$, $\alpha > \|\phi\|_\infty$, $t > 0$, $\varepsilon \in (0,1)$ and

$$K(\phi,\varepsilon,\alpha) := \frac{1+\varepsilon}{1-\varepsilon}\left(K + \tfrac{1}{\varepsilon}\|\nabla \log\phi\|_\infty^2 + \tfrac{1}{2}\sup(-\phi^{-1}L\phi) + \frac{m\alpha^2\|\nabla\log\phi\|_\infty^2(1+\varepsilon)}{8(\alpha-\|\phi\|_\infty)^2\varepsilon(1-\varepsilon)}\right).$$

We emphasize that the results in Theorem 1.3 are new for noncompact manifolds with boundary. When M is compact with a convex boundary, the first assertion was proved in [Li and Yau 1986] by using the classical maximum principle on compact manifolds, while when M is compact with a nonconvex boundary, an inequality similar to (1-5) was proved in [Wang 1997] by using the "interior rolling R-ball" condition.

These two theorems will be proved in Sections 2 and 3. By a standard argument due to Li and Yau [1986], the gradient estimate (1-5) implies a Harnack inequality. Let $\rho(x, y)$ be the Riemannian distance between $x, y \in M$, that is, the infimum of the length of all smooth curves in M that link x and y.

Corollary 1.4. *In the situation of Theorem 1.3 the Neumann semigroup P_t satisfies*

$$P_t f(x) \leq \left(\frac{t+s}{t}\right)^{m(1+\varepsilon)\alpha/2(1-\varepsilon)} (P_{t+s} f(y)) \exp\left(\frac{\alpha\rho(x,y)^2}{4s} + \frac{\alpha m K(\phi,\varepsilon,\alpha)s}{4(\alpha-\|\phi\|_\infty)}\right) \tag{1-6}$$

for all positive $f \in C_b(M)$, $t, \varepsilon \in (0,1)$, $\alpha > \|\phi\|_\infty$ and $x, y \in M$. In particular, if ∂M is convex, then (1-4) holds with m in place of d and for all $\alpha > 1$.

To derive explicit inequalities for the nonconvex case, we shall take a specific choice of ϕ as in [Wang 2007]. Let $\mathrm{i}_{\partial M}$ be the injectivity radius of ∂M, and let $\rho_{\partial M}$ be the Riemannian distance to the boundary. We shall take $\phi = \varphi \circ \rho_{\partial M}$ for a nice reference function φ on $[0, \infty)$. More precisely, let the sectional curvature satisfy $\mathrm{Sect}_M \le k$ and $-\sigma \le \mathrm{II} \le \gamma$ for some $k, \sigma, \gamma > 0$. Let

$$h(s) = \cos(\sqrt{k}\, s) - (\gamma/\sqrt{k}) \sin(\sqrt{k}\, s) \quad \text{for } s \ge 0.$$

Then h is the unique solution to the differential equation $h'' + kh = 0$ with boundary conditions $h(0) = 1$ and $h'(0) = -\gamma$. By the Laplacian comparison theorem for $\rho_{\partial M}$ (see [Kasue 1984, Theorem 0.3] or [Wang 2007]),

$$\Delta \rho_{\partial M} \ge \frac{(d-1)h'}{h}(\rho_{\partial M}) \quad \text{and} \quad \rho_{\partial M} < \mathrm{i}_{\partial M} \wedge h^{(-1)}(0), \tag{1-7}$$

where $h^{(-1)}(0) = (1/\sqrt{k}) \arcsin(\sqrt{k}/\sqrt{k+\gamma^2})$ is the first zero point of h. Fix a positive number $r_0 \le \mathrm{i}_{\partial M} \wedge h^{(-1)}(0)$, and let

$$\delta = \frac{2\sigma(1-h(r_0))^{d-1}}{\int_0^{r_0} (h(s) - h(r_0))^{d-1}\, \mathrm{d}s},$$
$$\varphi(r) = 1 + \delta \int_0^r (h(s) - h(r_0))^{1-d}\, \mathrm{d}s \int_{s \wedge r_0}^{r_0} (h(u) - h(r_0))^{d-1}\, \mathrm{d}u.$$

It is easy to see that $\varphi \circ \rho_{\partial M}$ is differentiable with a Lipschitzian gradient. By a simple approximation argument, we may apply Theorem 1.3 and Corollary 1.4 to $\phi = \varphi \circ \rho_{\partial M}$; see [Wang 2007, page 1436].

Obviously, (1-7) and $N = \nabla \rho_{\partial M}$ imply

$$\Delta \varphi \circ \rho_{\partial M} \ge -\delta \quad \text{and} \quad N \log \varphi \circ \rho_{\partial M}|_{\partial M} = \varphi'(0)/\varphi(0) = 2\sigma.$$

Moreover, by [Wang 2007, (20)] we have

$$\delta \le 2\sigma d r_0^{-1} \quad \text{and} \quad \varphi(r_0) \le 1 + \sigma d r_0.$$

Thus, for $\phi := \varphi \circ \rho_{\partial M}$ we have

$$-\phi^{-1} L\phi \le 2\sigma d r_0^{-1} + 2\sigma \|Z\|_\infty, \quad \|\nabla \log \phi\|_\infty^2 \le 4\sigma^2,$$
$$\|\phi\|_\infty \le \varphi(r_0) \le 1 + \sigma d r_0.$$

Combining these with Theorem 1.3 and Corollary 1.4, we obtain these explicit inequalities on a class of nonconvex and noncompact manifolds:

Corollary 1.5. *Let* $\mathrm{i}_{\partial M} > 0$, *and suppose* $\gamma \ge \mathrm{II} \ge -\sigma$ *and* $\mathrm{Sect}_M \le k$ *for some* $\gamma, \sigma, k > 0$. *If* (1-2) *holds and* $\|Z\|_\infty < \infty$, *then for any positive number*

$$r_0 \le \min\{\mathrm{i}_{\partial M}, (1/\sqrt{k}) \arcsin(\sqrt{k}/\sqrt{k+\gamma^2})\},$$

the inequalities

$$|\nabla \log P_t f|^2 - \alpha\partial_t \log P_t f \le \frac{m(1+\varepsilon)\alpha^2}{2(1-\varepsilon)t} + \frac{m\alpha^2 K_\varepsilon}{4(\alpha-1-\sigma d r_0)}$$

and

$$P_t f(x) \le \Big(\frac{t+s}{t}\Big)^{m(1+\varepsilon)\alpha/2(1-\varepsilon)} (P_{t+s} f(y)) \exp\bigg(\frac{\alpha\rho(x,y)^2}{4s} + \frac{m\alpha K_\varepsilon s}{4(\alpha-1-\sigma d r_0)}\bigg) \quad \text{for } x, y \in M$$

hold for all positive $f \in C_b(M)$, $t > 0$, $\varepsilon \in (0,1)$, $\alpha > 1+\sigma d r_0$, *and*

$$K_\varepsilon = \frac{1+\varepsilon}{1-\varepsilon}\bigg(K + \frac{4\sigma^2}{\varepsilon} + \frac{\sigma d}{r_0} + \sigma\|Z\|_\infty + \frac{m\alpha^2\sigma^2(1+\varepsilon)}{2(\alpha-1-\sigma d r_0)^2\varepsilon(1-\varepsilon)}\bigg).$$

Combining our gradient estimate with an approximation and a probabilistic argument, we can derive the gradient estimate (1-9) for a class of noncompact manifolds:

Theorem 1.6. *Let M satisfy A, and let L satisfy (1-2) with $\|Z\|_\infty < \infty$. Let κ_1 and κ_2 be positive elements of $C_b(M)$ such that*

$$\text{(1-8)} \qquad \mathrm{Ric} - \nabla Z \ge -\kappa_1 \quad \textit{and} \quad \mathrm{II} \ge -\kappa_2$$

hold on M and ∂M, respectively. Then

$$\text{(1-9)} \qquad |\nabla P_t f|(x) \le \mathsf{E}^x\bigg(|\nabla f|(X_t)\exp\Big(\int_0^t \kappa_1(X_s)\,\mathrm{d}s + \int_0^t \kappa_2(X_s)\,\mathrm{d}l_s\Big)\bigg)$$

holds for all $f \in C_b^1(M)$, $t > 0$, and $x \in M$.

Inequality (1-9) was first derived by Hsu [2002] on a compact manifold with boundary. In [2002, Theorem 3.7], Hsu applied the Itô formula to $F(U_t, T-t) := U_t^{-1}\nabla P_{T-t} f(X_t)$, where U_t is the horizontal lift of X_t on the frame bundle $O(M)$. Since M is compact, the (local) martingale part of this process is a real martingale (it may not be for noncompact M). Then the desired gradient estimate followed immediately from [2002, Corollary 3.6]. In Section 4, we will prove the boundedness of $\nabla P_{(\cdot)} f$ on $[0,T]\times M$ for any $T > 0$ and $f \in C_b^1(M)$, which leads to a simple proof of (1-9) for a class of noncompact manifolds.

2. Proof of Theorem 1.2

We consider the convex case and pass to the nonconvex case using the conformal change of metric constructed in [Wang 2007]. Without loss of generality, we may assume that $\sup G := \sup_{[0,T]\times M} > 1$. (Otherwise, we simply replace G by mG for a sufficiently large $m > 0$.)

2.1. ***Convex ∂M.*** Fix $o \in M$, and let ρ_o be the Riemannian distance to the point o. Since ∂M is convex, there exists a minimal geodesic in M of length $\rho(x, y)$ that links any x and y in M; see for example [Wang 2005a, Proposition 2.1.5]. So, by (1-2) and a comparison theorem (see [Qian 1998])

$$L\rho_o \le \sqrt{K(m-1)}\,\coth\bigl(\sqrt{K/(m-1)}\,\rho_o\bigr)$$

holds outside $\{o\}\cup\text{cut}(o)$, where $\text{cut}(o)$ is the cut locus of o. In the sequel, we will set $L\rho_o = 0$ on $\text{cut}(o)$ so that this implies

$$L\sqrt{1+\rho_o^2} \le c_1 \quad \text{on } M \tag{2-1}$$

for some constant $c_1 > 0$.

Let $h \in C_0^\infty([0,\infty))$ be decreasing such that

$$h(r) = \begin{cases} 1 & \text{if } r \le 1, \\ \exp(-(3-r)^{-1}) & \text{if } r \in [2,3), \\ 0 & \text{if } r \ge 3. \end{cases}$$

Obviously, for any $\varepsilon > 0$ we have

$$\sup_{[0,\infty)} \bigl\{|h^{\varepsilon-1}h''| + |h^{\varepsilon-1}h'|\bigr\} < \infty. \tag{2-2}$$

Let $W = \sqrt{1+\rho_o^2}$, and take $\varphi_n = h(W/n)$ for $n \ge 1$. Then

$$\{\varphi_n = 1\} \uparrow M \quad \text{as } n \uparrow \infty. \tag{2-3}$$

So, according to (2-1) and (2-2),

$$\begin{aligned} |\nabla \log \varphi_n| &\le \frac{c}{n\varphi_n^\varepsilon}, \\ \varphi_n^{-1} L\varphi_n &= \frac{h'(W/n)}{nh(W/n)} LW + \frac{h''(W/n)}{n^2 h(W/n)} |\nabla W|^2 \ge -\frac{c}{n\varphi_n^\varepsilon} \end{aligned} \tag{2-4}$$

holds for some constant $c > 0$ and all $n \ge 1$.

Let $G_n(t,x) = \varphi_n(x)G(t,x)$ for $t \in [0,T]$ and $x \in M$. Since G_n is continuous with compact support, there exists $(t_n, x_n) \in [0,T]\times M$ such that

$$G_n(t_n, x_n) = \max_{[0,T]\times M} G_n.$$

By (2-3) and that $\sup G > 1$, we have $\lim_{n\to\infty} G(t_n, x_n) = \sup G > 1$. By renumbering from a sufficient large n_0, we may assume that $G_n(t_n, x_n)$ is greater than 1 and is increasing in n. In particular, Lemma 1.1(i) holds and

$$\varphi_n(x_n) \ge 1/G(t_n, x_n) \quad \text{for } n \ge 1. \tag{2-5}$$

Moreover, since $G_n(0,\cdot)\le 0$, we have $t_n>0$ and $\partial_t G(t_n,x_n)\ge 0$ for $n\ge 1$. Thus, it remains to confirm that

$$\begin{aligned}|\nabla G(t_n,x_n)| &\le cG(t_n,x_n)^{1+\varepsilon}/n \quad \text{and}\\ LG(t_n,x_n) &\le cG(t_n,x_n)^{1+\varepsilon}/n \quad \text{for } n\ge 1\end{aligned} \tag{2-6}$$

for some constant $c>0$. Indeed, by using a subsequence $\{(t_{mn},x_{mn})\}_{n\ge 1}$ for $m\ge c$ to replace $\{(t_n,x_n)\}_{n\ge 1}$, one may reduce (2-6) with some $c>0$ to that with $c=1$.

Since x_n is the maximal point of G_n, we have $\nabla G_n(t_n,x_n)=0$ if $x_n\in M\setminus\partial M$. If $x_n\in\partial M$, we have $NG_n(t_n,x_n)\le 0$. Recall that $NG(t_n,\cdot)\ge 0$ and $G(t_n,x_n)>0$. Then, noting that $N\rho_0\le 0$ together with $h'\le 0$ implies $N\varphi_n\ge 0$, we conclude that $NG_n(t_n,x_n)\ge 0$. Hence, $NG_n(t_n,x_n)=0$. Moreover, since x_n is the maximal point of $G_n(t_n,\cdot)$ on the closed manifold ∂M, we have $UG_n(t_n,x_n)=0$ for all $U\in T\partial M$. Therefore, $\nabla G_n(t_n,x_n)=0$ also holds for $x_n\in\partial M$. Combining this with (2-4) and (2-5), we obtain

$$|\nabla G(t_n,x_n)|\le \frac{G(t_n,x_n)}{\varphi_n(x_n)}|\nabla\varphi_n|\le \frac{cG(t_n,x_n)^{1+\varepsilon}}{n},$$

which proves the first inequality in (2-6).

Finally, by (2-4), the inequality

$$\varphi_n L_n G+GL_n\varphi_n+2\langle\nabla G,\nabla\varphi_n\rangle\ge \varphi_n L_n G-\frac{c\varphi_n^{1-\varepsilon}}{n}G-\frac{2c\varphi_n^{1-\varepsilon}}{n}|\nabla G| =: \Phi$$

holds on $\{G_n>0\}\setminus \mathrm{cut}(o)$. By Lemma 2.1 below we obtain at the point (t_n,x_n) that

$$LG\le \frac{c}{n\varphi_n^{\varepsilon}}G+\frac{2c}{n\varphi_n}|\nabla G|.$$

Combining this with (2-5) and the first inequality in (2-6), we get

$$LG(t_n,x_n)\le \frac{c}{n}G^{1+2\varepsilon}(t_n,x_n)$$

for some constant $c>0$ and all $n\ge 1$. Since $\varepsilon>0$ is arbitrary, we may replace ε by $\varepsilon/2$ (recall that $G(t_n,x_n)\ge 1$). This proves the second inequality in (2-6).

Lemma 2.1. *The reflecting L-diffusion process is nonexplosive, and for any Φ in $C_b(M)$ such that*

$$\Phi\le LG_n=GL\varphi_n+\varphi_n LG+2\langle\nabla\varphi_n,\nabla G\rangle \quad \textit{on } \{G_n>0\}\setminus\mathrm{cut}(o),$$

we have $\Phi(t_n,x_n)\le 0$ for all $n\ge 1$.

Proof. Let X_t be the reflecting L-diffusion process generated by L, and let U_t be its horizontal lift on the frame bundle $O(M)$. By the Itô formula for $\rho_o(X_t)$

found by Kendall [1987] for $\partial M = \varnothing$ and by the fact that $N\rho_o|_{\partial M} \le 0$ when ∂M is nonempty but convex, we have

$$\mathrm{d}\rho_o(X_t) = \sqrt{2}\langle \nabla \rho_o(X_t), U_t \,\mathrm{d}B_t\rangle + L\rho_o(X_t)\,\mathrm{d}t - \mathrm{d}l_t + \mathrm{d}l_t', \tag{2-7}$$

where B_t is the d-dimensional Brownian motion, where $L\rho_o$ is taken to be zero on $\{o\}\cup \mathrm{cut}(o)$, and where l_t and l_t' are two increasing processes such that l_t' increases only when $X_t = o$, while l_t increases only when $X_t \in \mathrm{cut}(o)\cup\partial M$ (note that $l_t' = 0$ for $d \ge 2$). Combining this with (2-1) we obtain

$$\mathrm{d}\sqrt{1+\rho_o^2(X_t)} \le \mathrm{d}M_t + L\sqrt{1+\rho_o^2(X_t)}\,\mathrm{d}t \le \mathrm{d}M_t + c_1\,\mathrm{d}t$$

for some martingale M_t. This implies immediately that X_t does not explode.

Now, let us take $X_0 = x_n$. Since $h' \le 0$, it follows from (2-7) that

$$\mathrm{d}\varphi_n(X_t) \ge \sqrt{2}\langle \nabla\varphi_n(X_t), U_t\,\mathrm{d}B_t\rangle + L\varphi_n(X_t)\,\mathrm{d}t, \tag{2-8}$$

where we set $L\varphi_n = 0$ on $\mathrm{cut}(o)$ as above.

On the other hand, since $NG(t_n,\cdot\,) \ge 0$, we may apply the Itô to $G(t_n, X_t)$ to obtain

$$\mathrm{d}G(t_n, X_t) \ge \sqrt{2}\langle \nabla G(t_n, X_t), U_t\,\mathrm{d}B_t\rangle + LG(t_n, X_t)\,\mathrm{d}t. \tag{2-9}$$

Because $G_n(t_n, x_n) > 0$, there exists an $r > 0$ such that $G_n > 0$ on $B(x_n, r)$, the geodesic ball in M centered at x_n with radius r. Let

$$\tau = \inf\{t \ge 0 : \; X_t \notin B(x_n, r)\}.$$

Then (2-8) and (2-9) imply

$$\mathrm{d}G_n(t_n, X_t) \ge \mathrm{d}M_t + LG_n(t_n,\cdot\,)(X_t)\,\mathrm{d}t \ge \mathrm{d}M_t + \Phi(t_n, X_t)\,\mathrm{d}t \quad \text{for } t \le \tau$$

for some martingale M_t. Since $G_n(t_n, X_t) \le G_n(t_n, x_n)$ and $X_0 = x_n$, this implies that

$$0 \ge \mathsf{E}G_n(t_n, X_{t\wedge\tau}) - G_n(t_n, x_n) \ge \mathsf{E}\int_0^{t\wedge\tau} \Phi(t_n, X_s)\,\mathrm{d}s.$$

Therefore, the continuity of Φ implies that

$$\Phi(t_n, x_n) = \lim_{t\to 0}\frac{1}{\mathsf{E}(t\wedge\tau)}\mathsf{E}\int_0^{t\wedge\tau}\Phi(t_n, X_s)\,\mathrm{d}s \le 0. \qquad \square$$

2.2. *Nonconvex* ∂M. Under our assumptions on M, there exists a constant $R > 1$ and a function $\phi \in C^\infty(M)$ such that

$$1 \le \phi \le R, \quad |\nabla\phi| \le R, \quad N\log\phi|_{\partial M} \ge \sigma.$$

By [Wang 2007, Lemma 2.1], the boundary ∂M is convex under the new metric $\langle\cdot,\cdot\rangle' := \phi^{-2}\langle\cdot,\cdot\rangle$. Let $L' = \phi^2 L$. By [Wang 2007, Equation (9)], the vector

$U' := \phi U$ is unit under the new metric for any unit vector $U \in TM$, and the corresponding Ricci curvature satisfies

$$\text{(2-10)} \quad \operatorname{Ric}'(U', U') \geq \phi^2 \operatorname{Ric}(U, U) + \phi\Delta\phi - (d-3)|\nabla\phi|^2 \\ - 2(U\phi)^2 + (d-2)\phi \operatorname{Hess}_\phi(U, U).$$

Let Δ' be the Laplacian induced by the new metric. By [Wang 2007, Lemma 2.2], we have

$$L' := \phi^2 L = \Delta' + (d-2)\phi\nabla\phi + \phi^2 Z =: \Delta' + Z'.$$

Noting that

$$\nabla'_X Y = \nabla_X Y - \langle X, \nabla \log\phi\rangle Y - \langle Y, \nabla \log\phi\rangle X + \langle X, Y\rangle \nabla \log\phi \quad \text{for } X, Y \in TM,$$

we have

$$\begin{aligned} \langle \nabla_{U'} Z', U'\rangle' &= \langle \nabla_U Z', U\rangle - \langle Z', \nabla \log\phi\rangle \\ &= \phi^2 \langle \nabla_U Z, U\rangle + (U\phi^2)\langle Z, U\rangle + (d-2)(U\phi)^2 \\ &\qquad + (d-2)\phi \operatorname{Hess}_\phi(U, U) - \langle Z', \nabla \log\phi\rangle. \end{aligned}$$

Combining this with (2-10) and the properties of ϕ mentioned above, we find a constant $c_1 > 0$ such that

$$\text{(2-11)} \quad \operatorname{Ric}'(U,' U') - \langle \nabla'_{U'} Z', U'\rangle' \geq \phi^2(\operatorname{Ric} - \nabla Z)(U, U) - c_1 \quad \text{for } |U| = 1.$$

Moreover, since

$$\begin{aligned} (Z' \otimes' Z')(U', U') := (\langle Z', U'\rangle')^2 = \phi^{-2}\langle Z', U\rangle^2 \\ &\leq 2(d-2)^2\langle \nabla\phi, U\rangle^2 + 2\phi^2\langle Z, U\rangle^2 \\ &\leq 2(d-2)^2 R^2 + 2\phi^2 (Z \otimes Z)(U, U), \end{aligned}$$

it follows from (1-2) and (2-11) that

$$\operatorname{Ric}' - \nabla' Z' - \frac{Z' \otimes' Z'}{2(m-d)} \geq -\phi^2 K - c_2 \geq -K'$$

holds for the metric $\langle \cdot, \cdot \rangle'$ and some constants $c_2, K' > 0$. Therefore, we may apply Lemma 2.1 to L' on the convex Riemannian manifold $(M, \langle \cdot, \cdot \rangle')$ to conclude that the desired sequence $\{(t_n, x_n)\}$ exists.

3. Proofs of Theorem 1.3 and Corollary 1.4

Proof of Theorem 1.3. When ∂M is convex, Lemma 2.1 ensures that X_t does not explode. If ∂M is nonconvex, this can be confirmed by reparametrizing the time of the process. More precisely, let X'_t be the reflecting diffusion process on M generated by $L' := \phi^2 L$ constructed in Section 2.2. Since $L' = \Delta' + Z'$

satisfies (1-2) for some $K > 0$ on the convex manifold $(M, \langle\cdot,\cdot\rangle')$, the process X'_t generated by L' is nonexplosive by Lemma 2.1. Since $X_t = X'_{\xi^{-1}(t)}$, where ξ^{-1} is the inverse of

$$t \mapsto \xi(t) = \int_0^t \phi^2(X'_s)\,\mathrm{d}s,$$

we have $t\|\phi\|_\infty^{-2} \le \xi^{-1}(t) \le t$, and the process X_t is nonexplosive as well.

Let $f \in C_b^1(M)$ be strictly positive, and let $u(t,x) = \log P_t f(x)$. For a fixed number $T > 0$, we will apply Theorem 1.2 to the reference function

$$G(t,x) = t\big\{\phi(x)|\nabla u|^2(t,x) - \alpha u_t(t,x)\big\} \quad \text{for } t \in [0,T] \text{ and } x \in M.$$

Note that $\mathrm{II} \ge -\sigma$ and $N \log\phi \ge 2\sigma$ imply

$$\begin{aligned} N\phi &\ge 2\sigma\phi, \\ N|\nabla P_t f|^2 &= 2\,\mathrm{Hess}_{P_t f}(N, \nabla P_t f) = 2\mathrm{II}(\nabla P_t f, \nabla P_t f) \ge -2\sigma|\nabla P_t f|^2. \end{aligned}$$

Since $P_t f$ and hence u_t satisfy the Neumann boundary condition, this implies that

$$NG = t\Big\{(N\phi)|\nabla u|^2 + \frac{\phi}{(P_t f)^2} N|\nabla P_t f|^2\Big\} \ge t\{2\sigma\phi|\nabla u|^2 - 2\sigma\phi|\nabla u|^2\} = 0 \quad \text{on } \partial M.$$

According to [Ledoux 2000, (1.14)], inequality (1-2) implies

$$L|\nabla u|^2 - 2\langle\nabla Lu, \nabla u\rangle \ge -2K|\nabla u|^2 + \frac{|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}. \tag{3-1}$$

By multiplying this inequality by ε and (1-1) by $2(1-\varepsilon)$ and by combining the results, we obtain

$$L|\nabla u|^2 \ge 2\langle\nabla Lu, \nabla u\rangle - 2K|\nabla u|^2 + \frac{2(1-\varepsilon)(Lu)^2}{m} + \frac{\varepsilon|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}.$$

It is also easy to check that $Lu = u_t - |\nabla u|^2$ and $\partial_t|\nabla u|^2 = 2\langle\nabla u, \nabla u_t\rangle$. Then we arrive at

$$\begin{aligned} &(L-\partial_t)|\nabla u|^2 \\ &\quad\ge \frac{2(1-\varepsilon)}{m}(|\nabla u|^2 - u_t)^2 + \frac{\varepsilon|\nabla|\nabla u|^2|^2}{2|\nabla u|^2} - 2\langle\nabla u, \nabla|\nabla u|^2\rangle - 2K|\nabla u|^2. \end{aligned} \tag{3-2}$$

On the other hand,

$$\begin{aligned} -\alpha(L-\partial_t)u_t &= 2\alpha\langle\nabla u, \nabla u_t\rangle = 2\langle\nabla u, \nabla(\phi|\nabla u|^2 - t^{-1}G)\rangle \\ &= 2\phi\langle\nabla u, \nabla|\nabla u|^2\rangle + 2|\nabla u|^2\langle\nabla u, \nabla\phi\rangle - 2t^{-1}\langle\nabla u, \nabla G\rangle. \end{aligned}$$

Combining this with (3-2), we obtain

$$\begin{aligned}(L-\partial_t)G = &-\frac{G}{t}+t\big(\phi(L-\partial_t)|\nabla u|^2+|\nabla u|^2L\phi+2\langle\nabla\phi,\nabla|\nabla u|^2\rangle\big)\\ &+t\big(2\phi\langle\nabla u,\nabla|\nabla u|^2\rangle+2|\nabla u|^2\langle\nabla u,\nabla\phi\rangle-2t^{-1}\langle\nabla u,\nabla G\rangle\big)\\ \geq &-\frac{G}{t}+\frac{2(1-\varepsilon)\phi t}{m}(|\nabla u|^2-u_t)^2+\frac{\varepsilon\phi t|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}-2K\phi t|\nabla u|^2\\ &-2|\nabla u|\cdot|\nabla G|-2t|\nabla u|^3|\nabla\phi|-2t|\nabla\phi|\cdot|\nabla|\nabla u|^2|+t|\nabla u|^2L\phi.\end{aligned}$$

Noting that

$$\frac{\varepsilon\phi t|\nabla|\nabla u|^2|^2}{2|\nabla u|^2}-2t|\nabla\phi|\cdot|\nabla|\nabla u|^2|\geq-\frac{2t|\nabla\phi|^2|\nabla u|^2}{\varepsilon\phi},$$

we get

$$\begin{aligned}(L-\partial_t)G\geq&-\frac{G}{t}+\frac{2(1-\varepsilon)\phi t}{m}(|\nabla u|^2-u_t)^2-2K\phi t|\nabla u|^2-2|\nabla u|\cdot|\nabla G|\\ &-2t|\nabla u|^3|\nabla\phi|+t|\nabla u|^2L\phi-\frac{2t|\nabla\phi|^2|\nabla u|^2}{\varepsilon\phi}.\end{aligned}\tag{3-3}$$

We assume that $\sup G>0$, otherwise the proof is done. Since $G(0,\cdot)=0$ and $NG|_{\partial M}\geq 0$, we can apply Theorem 1.2. Let $\{(t_n,x_n)\}$ be fixed in Theorem 1.2 with, for example, $\varepsilon=1/2$. Then,

$$(L-\partial_t)G(t_n,x_n)\leq\frac{G^{3/2}(t_n,x_n)}{n}\quad\text{and}\quad|\nabla G|(t_n,x_n)\leq\frac{G^{3/2}(t_n,x_n)}{n}.\tag{3-4}$$

From now on, we evaluate functions at the point (t_n,x_n), so that $t=t_n$.

Let $\mu=|\nabla u|^2/G$. We have

$$|\nabla u|^2-u_t=\Big(\mu-\frac{(\mu t-1)\phi}{\alpha t}\Big)G=\frac{\mu t(\alpha-\phi)+\phi}{\alpha t}G.$$

Combining this with (3-3) and (3-4), we arrive at

$$\begin{aligned}&\frac{2(1-\varepsilon)\phi(\mu t(\alpha-\phi)+\phi)^2}{m\alpha^2t}G^2\\ &\leq\frac{G^{3/2}}{n}+\frac{G}{t}+\frac{2\sqrt{\mu}G^2}{n}+2t|\nabla\phi|(\mu G)^{3/2}+(2k\phi+2\varepsilon^{-1}\phi^{-1}|\nabla\phi|^2-L\phi)\mu tG.\end{aligned}\tag{3-5}$$

Since it is easy to see that

$$(\mu t(\alpha-\phi)+\phi)^2\geq\max\big\{\phi^2,4\mu t(\alpha-\phi)\phi,(2t(\alpha-\phi))^{3/2}\sqrt{\phi}\mu^{3/2}\big\},$$

we may multiply both sides of (3-5) by $t(\mu t(\alpha-\phi)+\phi)^{-2}G^{-2}$ to obtain

$$\begin{aligned}\frac{2(1-\varepsilon)\phi}{m\alpha^2} &\le \frac{c't}{n(1\wedge\sqrt{G})}+\frac{1}{\phi^2 G}+\frac{2K+2\varepsilon^{-1}|\nabla\log\phi|^2-\phi^{-1}L\phi}{4(\alpha-\phi)G}t\\ &\qquad\qquad+\frac{|\nabla\log\phi|\sqrt{t\phi}}{(\alpha-\phi)^{3/2}\sqrt{2G}}\\ &\le \frac{c't}{n(1\wedge\sqrt{G})}+\frac{1}{\phi^2 G}+\frac{2K+2\varepsilon^{-1}|\nabla\log\phi|^2-\phi^{-1}L\phi}{4(\alpha-\phi)G}\\ &\qquad\qquad+\frac{|\nabla\log\phi|^2 m\alpha^2(1+\varepsilon)t}{16(\alpha-\phi)^3\varepsilon(1-\varepsilon)G}+\frac{2(1-\varepsilon)\varepsilon\phi}{m\alpha^2(1+\varepsilon)}\end{aligned}$$

for some constant $c'>0$. Taking $n\to\infty$ and noting that $\phi\ge 1$, we conclude that $\theta:=\sup G$ satisfies

$$\begin{aligned}\frac{2(1-\varepsilon)}{m\alpha^2(1+\varepsilon)} \le \frac{1}{\theta}\Big(1+\frac{2K+2\varepsilon^{-1}\|\nabla\log\phi\|_\infty^2+\sup(-\phi^{-1}L\phi)}{4(\alpha-\|\phi\|_\infty)}T\\ +\frac{\|\nabla\log\phi\|_\infty^2 m\alpha^2(1+\varepsilon)T}{16(\alpha-\|\phi\|_\infty)^3\varepsilon(1-\varepsilon)}\Big).\end{aligned}$$

Combining this with $\theta\ge G(T,x)=T(\phi(x)|\nabla u|^2(T,x)-\alpha u_t(T,x))$ for $x\in M$, we obtain

$$\begin{aligned}&\phi(x)|\nabla u|^2(T,x)-\alpha u_t(T,x)\\ &\qquad\le \frac{m\alpha^2(1+\varepsilon)}{2(1-\varepsilon)}\Big(\frac{1}{T}+\frac{2K+2\varepsilon^{-1}\|\nabla\log\phi\|_\infty^2+\sup(-\phi^{-1}L\phi)}{4(\alpha-\|\phi\|_\infty)}\\ &\qquad\qquad+\frac{\|\nabla\log\phi\|_\infty^2 m\alpha^2(1+\varepsilon)}{16(\alpha-\|\phi\|_\infty)^3\varepsilon(1-\varepsilon)}\Big)\end{aligned}$$

for all $x\in M$. Then the proof is completed since $T>0$ is arbitrary. □

Proof of Corollary 1.4. By Theorem 1.3, the proof is standard according to [Li and Yau 1986]. For $x,y\in M$, let $\gamma:[0,1]\to M$ be the shortest curve in M linking x and y such that $|\dot\gamma|=\rho(x,y)$. Then, for any $s,t>0$ and $f\in C_b^\infty(M)$, it follows from (1-5) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\log P_{t+rs}f(\gamma_r) &= s\partial_u\log P_u f(\gamma_r)|_{u=t+rs}+\langle\dot\gamma_r,\nabla P_{t+rs}f(\gamma_r)\rangle\\ &\ge \frac{s}{\alpha}|\nabla\log P_{t+rs}f|^2(\gamma_r)-\rho(x,y)|\nabla\log f|(\gamma_r)\\ &\qquad -s\Big(\frac{m(1+\varepsilon)\alpha}{2(1-\varepsilon)(t+rs)}+\frac{m\alpha K(\phi,\varepsilon,\alpha)}{4(\alpha-1\|\phi\|_\infty)}\Big)\\ &\ge -\frac{\alpha}{4s}-s\Big(\frac{m(1+\varepsilon)\alpha}{2(1-\varepsilon)(t+rs)}+\frac{m\alpha K(\phi,\varepsilon,\alpha)}{4(\alpha-\|\phi\|_\infty)}\Big).\end{aligned}$$

We complete the proof by integrating with respect to $\mathrm{d}r$ over [0, 1]. □

4. Proof of Theorem 1.6

We first provide a simple proof of (1-9) under an extra assumption that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$ for any $T > 0$; we then drop this assumption by an approximation argument.

Lemma 4.1. *If that $f \in C_b^1(M)$ is such that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$ for any $T > 0$, then (1-9) holds.*

Proof. For any $\varepsilon > 0$, let $\eta_s = \sqrt{\varepsilon + |\nabla P_{t-s} f|^2}(X_s)$ for $s \le t$. By the Itô formula, we have

$$\mathrm{d}\eta_s = \mathrm{d}M_s + \frac{L|\nabla P_{t-s} f|^2 - 2\langle \nabla L P_{t-s} f, \nabla P_{t-s} f\rangle}{2\sqrt{\varepsilon + |\nabla P_{t-s} f|^2)^2}}(X_s)\,\mathrm{d}s \\ - \frac{|\nabla|\nabla P_{t-s} f|^2|^2}{4(\varepsilon + |\nabla P_{t-s} f|^2)^{3/2}}(X_s)\,\mathrm{d}s + \frac{N|\nabla P_{t-s} f|^2}{2\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s)\,\mathrm{d}l_s$$

for $s \le t$, where M_s is a local martingale. Combining this with (1-8) and (3-1), with κ_1 in place of K_0, we obtain

$$\begin{aligned} \mathrm{d}\eta_s &\ge \mathrm{d}M_s - \frac{\kappa_1 |\nabla P_{t-s} f|^2}{\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s)\,\mathrm{d}s - \frac{\kappa_2 |\nabla P_{t-s} f|^2}{\sqrt{\varepsilon + |\nabla P_{t-s} f|^2}}(X_s)\,\mathrm{d}l_s \\ &\ge \mathrm{d}M_s - \kappa_1(X_s)\eta_s\,\mathrm{d}s - \kappa_2(X_s)\eta_s\,\mathrm{d}l_s \quad \text{for } s \le t. \end{aligned}$$

Now η_s is bounded on $[0, t]$, and by the proof of [Wang 2005b, Lemma 2.1] we have $\mathrm{E}e^{\lambda l_t} < \infty$ for all $\lambda > 0$. This implies that

$$[0, t] \ni s \mapsto \sqrt{\varepsilon + |\nabla P_{t-s} f|^2(X_s)}\exp\Big(\int_0^s \kappa_1(X_s)\,\mathrm{d}s + \int_0^s \kappa_2(X_s)\,\mathrm{d}l_s\Big)$$

is a submartingale for any $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ we conclude that

$$[0, t] \ni s \mapsto |\nabla P_{t-s} f|(X_s)\exp\Big(\int_0^s \kappa_1(X_s)\,\mathrm{d}s + \int_0^s \kappa_2(X_s)\,\mathrm{d}l_s\Big)$$

is a submartingale as well. □

According to Lemma 4.1, it suffices to confirm the boundedness of $|\nabla P_{(\cdot)} f|$ on $[0, T] \times M$ for any $T > 0$ and $f \in C_b^1(M)$. We shall start from $f \in C_0^\infty(M)$ with $Nf|_{\partial M} = 0$, then pass to $f \in C_b^1(M)$ by combining an approximation argument and Lemma 4.1.

Case a. Let $f \in C_0^\infty(M)$ with $Nf|_{\partial M} = 0$. We have

$$P_t f = f + \int_0^t P_s L f\,\mathrm{d}s. \tag{4-1}$$

Since Lf is bounded, there is a $c > 0$ such that $Lf + c \geq 1$. Applying Corollary 1.5 with for example $\alpha = 2 + \sigma d r_0$ and $\varepsilon = 1/2$, but using $Lf + c$ in place of f, we obtain

$$\begin{aligned} |\nabla P_s Lf| &= |\nabla P_s(Lf+c)| \\ &\leq \|Lf+c\|_\infty \Big(\alpha \|P_s L^2 f\|_\infty + \frac{m(1+\varepsilon)\alpha^2}{2(1-\varepsilon)s} + \frac{m\alpha^2 K_\varepsilon}{4(\alpha - 1 - \sigma d r_0)}\Big)^{1/2} \\ &\leq c_1/\sqrt{s} \quad \text{for } s \leq T \end{aligned}$$

for some constant $c_1 > 0$. Combining this with (4-1) we conclude that, for some constant $c_2 > 0$,

$$|\nabla P_t f| \leq |\nabla f| + \int_0^t \frac{c_1}{\sqrt{s}} \, \mathrm{d}s \leq c_2 \quad \text{for } t \leq T.$$

Case b. Let $f \in C_0^\infty(M)$. There exists a sequence of functions $\{f_n\}_{n\geq 1} \subset C_0^\infty(M)$ such that $Nf_n|_{\partial M} = 0$, $f_n \to f$ uniformly as $n \to \infty$, and $\|\nabla f_n\|_\infty \leq 1 + \|\nabla f\|_\infty$ holds for any $n \geq 1$; see for example [Wang 1994]. By Case a and Lemma 4.1, (1-9) holds with f_n in place of f, so that

$$\frac{|P_t f_n(x) - P_t f_n(y)|}{\rho(x,y)} \leq C \quad \text{for } t \leq T,\ n \geq 1,\ x \neq y$$

for some constant $C > 0$. Letting first $n \to 0$ and then $y \to x$, we conclude that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$.

Case c. Let $f \in C_b^\infty(M)$. Let $\{g_n\}_{n\geq 1} \subset C_0^\infty)$ be such that $0 \leq g_n \leq 1$, $|\nabla g_n| \leq 2$ and $g_n \uparrow 1$ as $n \uparrow \infty$. By Case b and Lemma 4.1, we may apply (1-9) to $g_n f$ in place of f such that

$$\frac{|P_t(g_n f)(x) - P_t(g_n f)(y)|}{\rho(x,y)} \leq C \quad \text{for } t \leq T,\ n \geq 1,\ x \neq y$$

for some constant $C > 0$. By the same reason as in Case b, we conclude that $|\nabla P_{(\cdot)} f|$ is bounded on $[0, T] \times M$.

Case d. Finally, for $f \in C_b^1(M)$, there exist $\{f_n\}_{n\geq 1} \subset C_b^\infty(M)$ such that $f_n \to f$ uniformly as $n \to \infty$ and $\|\nabla f_n\|_\infty \leq \|\nabla f\|_\infty + 1$ for any $n \geq 1$. The proof is completed by the same reason as in Cases b and c. □

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