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THE ENTROPY FORMULAS FOR THE CR HEAT EQUATION AND THEIR APPLICATIONS ON PSEUDOHERMITIAN $(2n+1)$-MANIFOLDS

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We derive Perelman's and Nash-type entropy formulas for the CR heat equation on closed pseudohermitian $(2n+1)$-manifolds and show that it is monotone nonincreasing if its pseudohermitian Ricci curvature minus $(n+1)/2$ times the pseudohermitian torsion is nonnegative. As results, we are able to obtain an integral version of the subgradient estimate for the CR heat equation and an upper bound estimate for the first positive eigenvalue of the sublaplacian by using the CR Bochner formulas and CR Harnack-type inequality. As a byproduct, we obtain a sharp lower bound estimate for the first positive eigenvalue of the sublaplacian on a closed pseudohermitian $(2n+1)$-manifold.

1. Introduction

In 2002, Hamilton and Perelman used Ricci flow to solve the Poincaré conjecture and the Thurston geometrization conjecture on 3-manifolds. A key observation of Perelman was that the Ricci flow is the gradient flow of the Perelman functional when we enlarge the system to the coupled Ricci flow. From this point of view, it is interesting to find the CR analogue of the Ricci flow in a pseudohermitian 3-manifold.

Let (M^{2n+1}, J, θ) be a pseudohermitian manifold, as discussed in Section 2. In this paper, we study monotonicity formulas of Hamilton entropy [1988] and Perelman entropy [2002] for the CR heat equation on (M^{2n+1}, J, θ). The derivation of both of entropy formulas resembles the Li and Yau's [1986] gradient estimate for the heat equation. From this point of view, there is a corresponding problem in pseudohermitian manifolds (see (4-1)). Then it is natural to find analogous entropy formulas for the CR heat equation and apply them to pseudohermitian manifolds

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by using the CR Bochner formula (3-4) and the CR Harnack-type inequality (5-1) on a closed pseudohermitian $(2n+1)$-manifold.

The key lemma in this paper, Lemma 3.2, derives the CR version of the Bochner formula. This formula (3-2) involves an extra third-order operator P that characterizes CR-pluriharmonic functions [Lee 1988; Graham and Lee 1988] and has no analogue in the Riemannian case. After integrating by parts, we are able to relate this extra operator to the CR Paneitz operator P_0 as in (1-2) and (3-4).

Definition 1.1. Let (M^{2n+1}, J, θ) be a complete pseudohermitian manifold. Let

$$P\varphi = \sum_{\alpha=1}^{n} (\varphi_{\bar{\alpha}}{}^{\bar{\alpha}}{}_{\beta} + inA_{\beta\alpha}\varphi^{\alpha})\theta^{\beta} = (P_{\beta}\varphi)\theta^{\beta} \quad \text{for } \beta = 1, 2, \dots, n,$$

which is an operator that characterizes CR-pluriharmonic functions; see [Lee 1988] for $n = 1$ and [Graham and Lee 1988] for $n \geq 2$. Here

$$P_{\beta}\varphi = \sum_{\alpha=1}^{n} (\varphi_{\bar{\alpha}}{}^{\bar{\alpha}}{}_{\beta} + inA_{\beta\alpha}\varphi^{\alpha}) \tag{1-1}$$

and $\bar{P}\varphi = (\bar{P}_{\beta}\varphi)\theta^{\bar{\beta}}$, the conjugate of P.

We also define

$$P_0\varphi = 4(\delta_b(P\varphi) + \bar{\delta}_b(\bar{P}\varphi)),$$

which is the so-called CR Paneitz operator P_0. Here δ_b is the divergence operator that takes $(1,0)$-forms to functions by $\delta_b(\sigma_{\alpha}\theta^{\alpha}) = \sigma_{\alpha,}{}^{\alpha}$ and $\bar{\delta}_b(\sigma_{\bar{\alpha}}\theta^{\bar{\alpha}}) = \sigma_{\bar{\alpha},}{}^{\bar{\alpha}}$. If we define $\partial_b\varphi = \varphi_{\alpha}\theta^{\alpha}$ and $\bar{\partial}_b\varphi = \varphi_{\bar{\alpha}}\theta^{\bar{\alpha}}$, then the formal adjoint of ∂_b on functions (with respect to the Levi form and the volume form $d\mu$) is $\partial_b^* = -\delta_b$.

Note that for a closed pseudohermitian $(2n+1)$-manifold (M, J, θ), one has

$$-\int_M \langle P\varphi + \bar{P}\varphi, d_b\varphi\rangle \, d\mu = \frac{1}{4}\int_M P_0\varphi \cdot \varphi \, d\mu. \tag{1-2}$$

Here $d_b\varphi = \varphi_{\alpha}\theta^{\alpha} + \varphi_{\bar{\alpha}}\theta^{\bar{\alpha}}$. Moreover, for a closed pseudohermitian $(2n+1)$-manifold (M, J, θ) of zero torsion [Graham and Lee 1988],

$$P_0\varphi = \mathscr{L}_n\mathscr{L}_{\bar{n}}\varphi = \Delta_b^2\varphi + n^2T^2\varphi.$$

Here $\mathscr{L}_n\varphi = -\Delta_b\varphi + inT\varphi = -2\varphi_{\bar{\alpha}}{}^{\bar{\alpha}}$.

See [Graham and Lee 1988; Hirachi 1993; Lee 1988] for details about these operators.

Remark 1.2 [Hirachi 1993; Graham and Lee 1988]. Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with $n \geq 2$. Then a smooth real-valued function f satisfies $P_0 f = 0$ on M if and only if $P_{\beta} f = 0$ on M. This holds also for a closed pseudohermitian 3-manifold of zero torsion.

Let M be the boundary of a connected strictly pseudoconvex domain $\Omega \subset C^{n+1}$ and $P_\beta f = 0$ for a smooth real-valued function f on M. Then f is the boundary value of a pluriharmonic function $u (\partial\bar{\partial} u = 0)$ in Ω. Also, if Ω is simply connected, there exists a holomorphic function w in Ω such that $\mathrm{Re}(w) = u$ and $u|_M = f$.

Definition 1.3. On a closed pseudohermitian $(2n+1)$-manifold (M, J, θ), we call the Paneitz operator P_0 with respect to (J, θ) *essentially positive* if there exists a constant $\Lambda > 0$ such that

$$\int_M P_0\varphi \cdot \varphi \, d\mu \geq \Lambda \int_M \varphi^2 \, d\mu. \tag{1-3}$$

for all real C^∞ smooth functions $\varphi \in (\ker P_0)^\perp$, that is, those perpendicular to the kernel of P_0 in the L^2 norm with respect to the volume form $d\mu = \theta \wedge (d\theta)^n$.

We say that P_0 is *nonnegative* if

$$\int_M P_0\varphi \cdot \varphi \, d\mu \geq 0$$

for all real C^∞ smooth functions φ.

Remark 1.4. Suppose (M, J, θ) is a closed pseudohermitian 3-manifold. The positivity of P_0 is a CR invariant in that it is independent of the choice of the contact form θ.

Let (M, J, θ) be a closed pseudohermitian 3-manifold with zero torsion. Then the corresponding CR Paneitz operator is essentially positive [Chang et al. 2007; Cao and Chang 2007].

Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with $n \geq 2$. Then the corresponding CR Paneitz operator is always nonnegative [Graham and Lee 1988; Chang and Chiu 2009a].

Now we consider the CR heat equation

$$\Big(\Delta_b - \frac{\partial}{\partial t}\Big) u(x, t) = 0. \tag{1-4}$$

Let u be a positive solution of (1-4), and let $\varphi = -\log u$. We first define the energy functional for the CR heat equation (1-4) by

$$\mathcal{F}(\varphi) = \int_M |\nabla_b \varphi|^2 e^{-\varphi} \, d\mu. \tag{1-5}$$

By applying the integral version of CR Bochner formula (3-1), we obtain an entropy formula:

Theorem 1.5. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$(\mathrm{Ric} - \tfrac{1}{2}(n+1)\,\mathrm{Tor})(Z, Z) \geq 0 \quad \textit{for all } Z \in T_{1,0}.$$

Also assume that the CR Paneitz operator P_0 is nonnegative if $n = 1$. Let u be a smooth positive solution of (1-4) and let $\varphi = -\ln u$. Then $\mathcal{F}(\varphi)$ is monotone nonincreasing along the CR heat equation and the monotonicity is strict unless the solution u is constant.

Next we consider the so-called Nash-type entropy [Nash 1958]. Let $u(x,t)$ be a positive solution to the CR heat equation $(\Delta_b - \partial/\partial t)u = 0$ with $\int_M u\,d\mu = 1$. We define

$$N(u,t) = -\int_M (\ln u)u\,d\mu \tag{1-6}$$

and

$$\widetilde{N}(u,t) = \frac{n}{n+2}N(u,t) - \frac{(n+1)(n+2)}{2}(\ln t + 1).$$

By applying the CR Harnack-type estimate of Proposition 5.1, we obtain another entropy formula:

Theorem 1.6. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with zero torsion and nonnegative pseudohermitian Ricci tensors. Suppose $u(x,t)$ is the positive solution of (1-4) on $M \times [0,\infty)$ with $\int_M u\,d\mu = 1$ and a positive smooth CR-pluriharmonic function as an initial. Then*

$$\frac{d}{dt}\widetilde{N}(u,t) = \int_M \Big(\frac{n}{n+2}|\nabla_b\varphi|^2 - \varphi_t - \frac{(n+1)(n+2)}{2t}\Big)u\,d\mu \le 0$$

for all $t \in (0,\infty)$ with $\varphi = -\ln u$.

By Theorem 1.6, we have the integral version of the subgradient estimate for the positive solution of (1-4):

Corollary 1.7. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with zero torsion and nonnegative pseudohermitian Ricci tensors. Suppose $u(x,t)$ is the positive solution of (1-4) on $M \times [0,\infty)$ with a positive smooth CR-pluriharmonic function as an initial. Then there exists a constant C_1 such that*

$$\int_M |\nabla_b u^{1/2}|^2\,d\mu \le \frac{C_1}{t} \quad \text{on } M \times [0,\infty).$$

Remark 1.8. Theorem 1.6 and Corollary 1.7 can be extended to a complete (with respect to Carnot–Carathéodory distance) noncompact pseudohermitian $(2n+1)$-manifold whenever one has the CR version of the sublaplacian comparison theorem. Indeed, this is the case for a $(2n+1)$-dimensional Heisenberg group H^n [Chang et al. ≥ 2010, Theorem 1.3]. See [Ni 2004; Cao and Yau 1992; Jerison and Sánchez-Calle 1986; Sánchez-Calle 1984] for related work.

As a byproduct we also obtain a sharp bound estimate for the first positive eigenvalue of sublaplacian on a pseudohermitian $(2n+1)$-manifold. For a complete Riemannian manifold (N^m, g), by using the standard Laplacian comparison theorem, S.-Y. Cheng [1975] obtained an upper bound for the first positive eigenvalue μ_1 of the Laplacian:

$$\mu_1 \le \tfrac{1}{4}(m-1)^2 \quad \text{if } \mathrm{Ric}(g) \ge -(m-1)g.$$

On the other hand, for a complete pseudohermitian manifold, the sublaplacian comparison theorem is not available at this stage. However, by using the monotonicity formula (3-5) of the entropy $\mathcal{F}(\varphi)$ for the CR heat equation, we can obtain an upper bound for the first positive eigenvalue λ_1 of the sublaplacian Δ_b in a complete noncompact pseudohermitian $(2n+1)$-manifold M.

Corollary 1.9. *Let (M, J, θ) be a complete noncompact pseudohermitian $(2n+1)$-manifold. Suppose that*

$$(\mathrm{Ric} - \tfrac{1}{2}(n+1)\,\mathrm{Tor})(Z, Z) \ge -k_1|Z|^2, \quad \text{for all } Z \in T_{1,0}$$

where k_1 is a positive constant. Also assume that the CR Paneitz operator P_0 is nonnegative if $n = 1$. Then

$$\lambda_1 \le \frac{(2n-1)n}{n+1} k_1.$$

A. Greenleaf [1985] proved the pseudohermitian analogue of Lichnerowicz's theorem [1958] for the lower bound of the first positive eigenvalue λ_1 of the sublaplacian for a pseudohermitian manifold M^{2n+1} with $n \ge 3$. More precisely, under the condition on the Webster–Ricci curvature and the pseudohermitian torsion that

$$(\mathrm{Ric} - \tfrac{1}{2}(n+1)\,\mathrm{Tor})(Z, Z) \ge k|Z|^2 \quad \text{for } Z \in T_{1,0}$$

and for some positive constant k, one has $\lambda_1 \ge nk/(n+1)$.

S.-Y. Li and H.-S. Luk [2004] proved the same result for $n = 1$ and $n = 2$. However, in the case $n = 1$, they needed a condition depending not only on the Webster–Ricci curvature and the pseudohermitian torsion, but also on a covariant derivative of the pseudohermitian torsion. Now by using the entropy formula (3-5) for the CR heat equation, we can obtain a sharp lower bound on the first positive eigenvalue λ_1 of the sublaplacian Δ_b on a closed pseudohermitian $(2n+1)$-manifold M. Our argument works for all $n \ge 1$.

Corollary 1.10. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$(\mathrm{Ric} - \tfrac{1}{2}(n+1)\,\mathrm{Tor})(Z, Z) \ge k_2|Z|^2 \quad \text{for all } Z \in T_{1,0},$$

and for k_2 a positive constant. Also assume that the CR Paneitz operator P_0 is nonnegative if $n = 1$. Then

$$\lambda_1 \geq \frac{n}{n+1} k_2.$$

This was proved by H.-L. Chiu [2006] for $n = 1$.

Moreover, we can obtain an effective lower bound on the first positive eigenvalue λ_1 under a general curvature condition on a closed pseudohermitian $(2n+1)$-manifold. See [Chang and Chiu 2007; Chang and Chiu 2009b] for $n = 1$.

Corollary 1.11. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with essentially positive CR Paneitz operator P_0. Suppose that*

$$(\mathrm{Ric} - \tfrac{1}{2}(n+1)\,\mathrm{Tor})(Z, Z) \geq -k_3 |Z|^2 \quad \textit{for all } Z \in T_{1,0}$$

and for k_3 a nonnegative constant. Then there exist constants C_2 and C_3 such that

$$\lambda_1 \geq \max\left\{ \frac{-nk_3 + \sqrt{n^2 k_3^2 + 3(n+1) C_2 \Lambda}}{2(n+1)},\ C_3(n, k_3, \tau_0, d_M) \right\}.$$

Here $\tau_0 = \max|A_{\alpha\gamma}|$ and d_M is the diameter of M with respect to the Carnot–Carathéodory distance.

2. Preliminaries

We first give a brief introduction to pseudohermitian geometry; see [Lee 1988] for more details. Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ with $\dim_R \xi = 2n$. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the integrability condition that if X and Y are in ξ, then $[JX, Y] + [X, JY]$ and $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$ are in ξ too. A CR structure J can extend to $\mathbb{C} \otimes \xi$, which then decomposes into the direct sum of $T_{1,0}$ and $T_{0,1}$, the eigenspaces of J with respect to i and $-i$, respectively. A manifold M with a CR structure is called a CR manifold. A pseudohermitian structure compatible with ξ is a CR structure J compatible with ξ together with a choice of contact form θ. Such a choice determines a unique real vector field T transverse to ξ, which is called the characteristic vector field of θ, such that $\theta(T) = 1$ and $\mathcal{L}_T \theta = 0$ or $d\theta(T, \cdot) = 0$. Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM \otimes \mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, where $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}$, and where T is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$, satisfies

$$d\theta = i h_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}} \tag{2-1}$$

for some positive definite hermitian matrix of functions $(h_{\alpha\bar{\beta}})$. Because we can actually choose Z_α so that $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$, we henceforth assume this.

The Levi form $\langle\,\cdot\,,\,\cdot\,\rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by

$$\langle Z, W\rangle_{L_\theta} = -i\langle d\theta, Z\wedge \overline{W}\rangle.$$

We can extend $\langle\,\cdot\,,\,\cdot\,\rangle_{L_\theta}$ to $T_{0,1}$ by defining $\langle \bar{Z}, \overline{W}\rangle_{L_\theta} = \overline{\langle Z, W\rangle}_{L_\theta}$ for all $Z, W\in T_{1,0}$. The Levi form induces naturally a Hermitian form $\langle\,\cdot\,,\,\cdot\,\rangle_{L_\theta^*}$ on the dual bundle of $T_{1,0}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $d\mu = \theta\wedge(d\theta)^n$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. For example $\langle\varphi, \psi\rangle = \int_M \varphi\overline{\psi}\, d\mu$ for functions φ and ψ.

The pseudohermitian connection of (J, θ) is the connection ∇ on $TM\otimes\mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = \theta_\alpha{}^\beta\otimes Z_\beta, \quad \nabla Z_{\bar\alpha} = \theta_{\bar\alpha}{}^{\bar\beta}\otimes Z_{\bar\beta}, \quad \nabla T = 0,$$

where $\theta_\alpha{}^\beta$ are the 1-forms uniquely determined by the equations

$$d\theta^\beta = \theta^\alpha\wedge\theta_\alpha{}^\beta + \theta\wedge\tau^\beta, \quad 0 = \tau_\alpha\wedge\theta^\alpha, \quad 0 = \theta_\alpha{}^\beta + \theta_{\bar\beta}{}^{\bar\alpha}, \tag{2-2}$$

By the Cartan lemma, we can write $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma} = A_{\gamma\alpha}$. The curvature of the Webster–Stanton connection in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar\alpha}\}$ is

$$\Pi_\beta{}^\alpha = \overline{\Pi_{\bar\beta}{}^{\bar\alpha}} = d\theta_\beta{}^\alpha - \theta_\beta{}^\gamma\wedge\theta_\gamma{}^\alpha,$$
$$\Pi_0{}^\alpha = \Pi_\alpha{}^0 = \Pi_0{}^{\bar\beta} = \Pi_{\bar\beta}{}^0 = \Pi_0{}^0 = 0.$$

Webster showed that $\Pi_\beta{}^\alpha$ can be written

$$\Pi_\beta{}^\alpha = R_\beta{}^\alpha{}_{\rho\bar\sigma}\theta^\rho\wedge\theta^{\bar\sigma} + W_\beta{}^\alpha{}_\rho\theta^\rho\wedge\theta - W^\alpha{}_{\beta\bar\rho}\theta^{\bar\rho}\wedge\theta + i\theta_\beta\wedge\tau^\alpha - i\tau_\beta\wedge\theta^\alpha, \tag{2-3}$$

where the coefficients satisfy

$$R_{\beta\bar\alpha\rho\bar\sigma} = \overline{R_{\alpha\bar\beta\sigma\bar\rho}} = R_{\bar\alpha\beta\bar\sigma\rho} = R_{\rho\bar\alpha\beta\bar\sigma} \quad\text{and}\quad W_{\beta\bar\alpha\gamma} = W_{\gamma\bar\alpha\beta}.$$

We will denote components of covariant derivatives with indices preceded by a comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0, \alpha, \bar\alpha\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar\alpha}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $f_\alpha = Z_\alpha f$, $f_{\alpha\bar\beta} = Z_{\bar\beta}Z_\alpha f - \theta_\alpha{}^\gamma(Z_{\bar\beta})Z_\gamma f$ and $f_0 = Tf$ for f a (smooth) function.

For a real function f, the subgradient $\nabla_b f\in\xi$ is defined by $\langle Z, \nabla_b f\rangle_{L_\theta} = df(Z)$ for all vector fields Z tangent to contact plane. Locally $\nabla_b f = \sum_\alpha f_{\bar\alpha}Z_\alpha + f_\alpha Z_{\bar\alpha}$. We can use the connection to define the subhessian as the complex linear map

$$(\nabla^H)^2 f : T_{1,0}\oplus T_{0,1}\to T_{1,0}\oplus T_{0,1}, \quad Z\mapsto \nabla_Z\nabla_b f.$$

Also

$$\Delta_b f = \mathrm{Tr}((\nabla^H)^2 f) = \textstyle\sum_\alpha (f_{\alpha\bar\alpha} + f_{\bar\alpha\alpha}).$$

The Webster–Ricci tensor and the torsion tensor on $T_{1,0}$ are defined by

$$\mathrm{Ric}(X,Y)=R_{\alpha\bar\beta}X^{\alpha}Y^{\bar\beta}\quad\text{and}\quad \mathrm{Tor}(X,Y)=i\textstyle\sum_{\alpha,\beta}(A_{\bar\alpha\bar\beta}X^{\bar\alpha}Y^{\bar\beta}-A_{\alpha\beta}X^{\alpha}Y^{\beta}),$$

where $X=X^{\alpha}Z_{\alpha},\ Y=Y^{\beta}Z_{\beta}$ and $R_{\alpha\bar\beta}=R_{\gamma}{}^{\gamma}{}_{\alpha\bar\beta}$. The Webster scalar curvature is $R=R_{\alpha}{}^{\alpha}=h^{\alpha\bar\beta}R_{\alpha\bar\beta}$.

3. Perelman's entropy formula

Let (M,J,θ) be a closed pseudohermitian $(2n+1)$-manifold. In this section, we first want to show that the entropy formula for the CR heat equation is monotone nonincreasing.

First we recall the following CR Bochner formulas.

Lemma 3.1. *Let (M,J,θ) be a closed pseudohermitian $(2n+1)$-manifold. For a (smooth) real function f on M, we have*

$$\tag{3-1} \tfrac12\Delta_b|\nabla_b f|^2=|(\nabla^H)^2f|^2+\langle\nabla_b f,\nabla_b\Delta_b f\rangle + (2\,\mathrm{Ric}-(n-2)\,\mathrm{Tor})((\nabla_b f)_{\mathbb C},(\nabla_b f)_{\mathbb C})+2\langle J\nabla_b f,\nabla_b f_0\rangle$$

and also [Chang and Chiu 2009a]

$$\tag{3-2} \tfrac12\Delta_b|\nabla_b f|^2=|(\nabla^H)^2f|^2+(1+\tfrac{2}{n})\langle\nabla_b f,\nabla_b\Delta_b f\rangle + (2\,\mathrm{Ric}-(n+2)\,\mathrm{Tor})((\nabla_b f)_{\mathbb C},(\nabla_b f)_{\mathbb C}) -\tfrac{4}{n}\langle Pf+\overline{P}f,d_bf\rangle,$$

where $(\nabla_b f)_{\mathbb C}=\sum_\alpha f_{\bar\alpha}Z_\alpha$ is the corresponding complex $(1,0)$-vector filed $\nabla_b f$.

Lemma 3.2 [Chang and Chiu 2009a]. *Let (M,J,θ) be a closed pseudohermitian $(2n+1)$-manifold. For a (smooth) real function f on M, we have*

$$\tag{3-3} \int_M\langle J\nabla_b f,\nabla_b f_0\rangle\,d\mu=-n\int_M(f_0)^2\,d\mu$$

and

$$\tag{3-4} n^2\int_M(f_0)^2\,d\mu=\int_M(\Delta_b f)^2d\mu + 2n\int_M\mathrm{Tor}((\nabla_b f)_{\mathbb C},(\nabla_b f)_{\mathbb C})\,d\mu-\frac12\int_M P_0f\cdot f\,d\mu,$$

where P_0 is the CR Paneitz operator.

Now we can derive the following entropy formula for the CR heat equation (1-4) on a closed pseudohermitian $(2n+1)$-manifold.

Lemma 3.3. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. If $u(x,t)$ is the positive smooth solution of (1-4) on $M \times [0,\infty)$. Let $\varphi = -\ln u$ and let $F = u^{\alpha}$ for any constant $\alpha \neq 0, 1/2$. Then we have*

$$
\begin{aligned}
\text{(3-5)}\quad & \alpha^2(n+1)\frac{d}{dt}\mathscr{F}(\varphi) \\
&= \frac{1}{3(1-2\alpha)}\int_M F^{\frac{1}{\alpha}-2}\Big((4n\alpha+3)(\Delta_b F)^2+(4\alpha-3)n\sum_{\alpha=1}^{n}|F_{\alpha\bar{\alpha}}+F_{\bar{\alpha}\alpha}|^2\Big)\,d\mu \\
&\quad + \frac{4n(4\alpha-3)}{3(1-2\alpha)}\int_M F^{\frac{1}{\alpha}-2}\Big(\sum_{\alpha,\beta=1}^{n}|F_{\alpha\beta}|^2+\sum_{\alpha\neq\beta=1}^{n}|F_{\alpha\bar{\beta}}|^2\Big)\,d\mu \\
&\quad + \frac{4n(4\alpha-3)}{3(1-2\alpha)}\int_M F^{\frac{1}{\alpha}-2}(\mathrm{Ric}-\tfrac{1}{2}(n+1)\,\mathrm{Tor})((\nabla_b F)_{\mathbb{C}},(\nabla_b F)_{\mathbb{C}})\,d\mu \\
&+\frac{24n\alpha^2-2(8n-3)\alpha-3}{12\alpha^2}\int_M F^{\frac{1}{\alpha}-4}|\nabla_b F|^4\,d\mu+\frac{2\alpha^2(4\alpha-3)}{1-2\alpha}\int_M P_0 F^{\frac{1}{2\alpha}}\cdot F^{\frac{1}{2\alpha}}\,d\mu.
\end{aligned}
$$

Proof. By using $\partial\varphi/\partial t = \Delta_b\varphi - |\nabla_b\varphi|^2$, we first compute

$$
\begin{aligned}
\frac{d}{dt}\int_M|\nabla_b\varphi|^2 u\,d\mu = \frac{d}{dt}\int_M(\Delta_b\varphi)u\,d\mu &= \int_M(2\Delta_b\varphi-|\nabla_b\varphi|^2)\Delta_b u\,d\mu \\
&= \int_M e^{-\varphi}(2\Delta_b\varphi-|\nabla_b\varphi|^2)(|\nabla_b\varphi|^2-\Delta_b\varphi)\,d\mu \\
&= \int_M e^{-\varphi}(-2(\Delta_b\varphi)^2+3\Delta_b\varphi|\nabla_b\varphi|^2-|\nabla_b\varphi|^4)\,d\mu.
\end{aligned}
$$

Now let $F = e^{-\alpha\varphi}(=u^{\alpha})$ for some constant $\alpha \neq 0$. Then

$$
|\nabla_b\varphi|^2 = \alpha^{-2}F^{-2}|\nabla_b F|^2 \quad\text{and}\quad \Delta_b\varphi = \alpha^{-1}(F^{-2}|\nabla_b F|^2 - F^{-1}\Delta_b F).
$$

Thus

$$
\begin{aligned}
\text{(3-6)}\quad & \alpha^2\frac{d}{dt}\int_M|\nabla_b\varphi|^2 u\,d\mu \\
&= -2\int_M F^{\alpha^{-1}-2}(\Delta_b F)^2\,d\mu + (4\alpha-3)\alpha^{-1}\int_M F^{\alpha^{-1}-3}\Delta_b F|\nabla_b F|^2\,d\mu \\
&\qquad - (2\alpha^2-3\alpha+1)\alpha^{-2}\int_M F^{\alpha^{-1}-4}|\nabla_b F|^4\,d\mu.
\end{aligned}
$$

Secondly, we will deal with the term $\int_M F^{\alpha^{-1}-3}\Delta_b F|\nabla_b F|^2\,d\mu$ in (3-6). We need the identities

$$
\text{(3-7)}\quad 0 = \int_M \delta_b(F^{\alpha^{-1}-2}\Delta_b F\nabla_b F)\,d\mu
$$

$$
\begin{aligned}
&= \int_M F^{\alpha^{-1}-2}(\Delta_b F)^2\, d\mu + \int_M \langle \nabla_b (F^{\alpha^{-1}-2}\Delta_b F), \nabla_b F\rangle\, d\mu \\
&= \int_M F^{\alpha^{-1}-2}(\Delta_b F)^2\, d\mu + (\alpha^{-1}-2)\int_M F^{\alpha^{-1}-3}\Delta_b F|\nabla_b F|^2\, d\mu \\
&\qquad\qquad + \int_M F^{\alpha^{-1}-2}\langle \nabla_b \Delta_b F, \nabla_b F\rangle\, d\mu
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \int_M \delta_b (F^{\alpha^{-1}-2}\nabla_b |\nabla_b F|^2)\, d\mu \\
&= \int_M F^{\alpha^{-1}-2}\Delta_b |\nabla_b F|^2\, d\mu + (\alpha^{-1}-2)\int_M F^{\alpha^{-1}-3}\langle \nabla_b F, \nabla_b |\nabla_b F|^2\rangle\, d\mu \\
&= \int_M F^{\alpha^{-1}-2}\Delta_b |\nabla_b F|^2\, d\mu - (\alpha^{-1}-2)\int_M F^{\alpha^{-1}-3}\Delta_b F|\nabla_b F|^2\, d\mu \\
&\qquad - (\alpha^{-1}-2)(\alpha^{-1}-3)\int_M F^{\alpha^{-1}-4}|\nabla_b F|^4\, d\mu.
\end{aligned} \tag{3-8}
$$

Now, by using the Bochner formula (3-1) to replace the term $\Delta_b |\nabla_b F|^2$ in the last equality of (3-8) and by using (3-3), we get

$$
\begin{aligned}
&\int_M F^{\alpha^{-1}-2}\langle \nabla_b F, \nabla_b \Delta_b F\rangle\, d\mu = \\
&-\int_M F^{\alpha^{-1}-2}|(\nabla^H)^2 F|^2\, d\mu - \int_M F^{\alpha^{-1}-2}(2\,\mathrm{Ric} - (n-2)\,\mathrm{Tor})((\nabla_b F)_{\mathbb{C}}, (\nabla_b F)_{\mathbb{C}}) d\mu \\
&\qquad + 2n\int_M F^{\alpha^{-1}-2}(F_0)^2\, d\mu + \tfrac{1}{2}(\alpha^{-1}-2)\int_M F^{\alpha^{-1}-3}\Delta_b F|\nabla_b F|^2\, d\mu \\
&\qquad + \tfrac{1}{2}(\alpha^{-1}-2)(\alpha^{-1}-3)\int_M F^{\alpha^{-1}-4}|\nabla_b F|^4\, d\mu.
\end{aligned} \tag{3-9}
$$

Combining (3-7) and (3-9), we get

$$
\begin{aligned}
&-(\alpha^{-1}-2)\int_M F^{\alpha^{-1}-3}\Delta_b F|\nabla_b F|^2\, d\mu \\
&\quad = \int_M F^{\alpha^{-1}-2}(\Delta_b F)^2\, d\mu + \tfrac{1}{2}(\alpha^{-1}-2)(\alpha^{-1}-3)\int_M F^{\alpha^{-1}-4}|\nabla_b F|^4\, d\mu \\
&\qquad\qquad + \tfrac{1}{2}(\alpha^{-1}-2)\int_M F^{\alpha^{-1}-3}\Delta_b F|\nabla_b F|^2\, d\mu \\
&\qquad - \int_M F^{\alpha^{-1}-2}|(\nabla^H)^2 F|^2\, d\mu + 2n\int_M F^{\alpha^{-1}-2}(F_0)^2\, d\mu \\
&\qquad - \int_M F^{\alpha^{-1}-2}(2\,\mathrm{Ric} - (n-2)\,\mathrm{Tor})((\nabla_b F)_{\mathbb{C}}, (\nabla_b F)_{\mathbb{C}})\, d\mu.
\end{aligned}
$$

We may rewrite this as

$$
\begin{aligned}
\text{(3-10)}\quad & \int_M F^{\alpha^{-1}-3}\Delta_b F|\nabla_b F|^2\,d\mu \\
&= -\frac{2}{3}\frac{\alpha}{1-2\alpha}\int_M F^{\alpha^{-1}-2}(\Delta_b F)^2\,d\mu + \frac{1}{3}\frac{3\alpha-1}{\alpha}\int_M F^{\alpha^{-1}-4}|\nabla_b F|^4\,d\mu \\
&\quad + \frac{2}{3}\frac{\alpha}{1-2\alpha}\int_M F^{\alpha^{-1}-2}|(\nabla^H)^2F|^2\,d\mu - \frac{4}{3}\frac{n\alpha}{1-2\alpha}\int_M F^{\alpha^{-1}-2}(F_0)^2\,d\mu \\
&\quad + \frac{2}{3}\frac{\alpha}{1-2\alpha}\int_M F^{\alpha^{-1}-2}(2\,\mathrm{Ric}-(n-2)\,\mathrm{Tor})((\nabla_b F)_{\mathbb{C}}, (\nabla_b F)_{\mathbb{C}})\,d\mu \\
&= -\frac{2}{3}\frac{\alpha}{1-2\alpha}\int_M F^{\alpha^{-1}-2}\big((\Delta_b F)^2 - \tfrac{1}{2}\textstyle\sum_{\alpha=1}^n|F_{\alpha\bar\alpha}+F_{\bar\alpha\alpha}|^2\big)\,d\mu \\
&\quad + \frac{4}{3}\frac{\alpha}{1-2\alpha}\int_M F^{\alpha^{-1}-2}\Big(\sum_{\alpha,\beta=1}^n|F_{\alpha\beta}|^2 + \sum_{\alpha,\beta=1,\alpha\neq\beta}^n|F_{\alpha\bar\beta}|^2\Big)\,d\mu \\
&\quad - \frac{n\alpha}{1-2\alpha}\int_M F^{\alpha^{-1}-2}(F_0)^2\,d\mu + \frac{3\alpha-1}{3\alpha}\int_M F^{\alpha^{-1}-4}|\nabla_b F|^4\,d\mu \\
&\quad + \frac{2}{3}\frac{\alpha}{1-2\alpha}\int_M F^{\alpha^{-1}-2}(2\,\mathrm{Ric}-(n-2)\,\mathrm{Tor})((\nabla_b F)_{\mathbb{C}}, (\nabla_b F)_{\mathbb{C}})\,d\mu,
\end{aligned}
$$

where $\alpha \neq 0, 1/2$ and the last equation follows from the identity

$$
|(\nabla^H)^2F|^2 = 2\Big(\sum_{\alpha,\beta=1}^n|F_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^n|F_{\alpha\bar\beta}|^2\Big) + \frac{1}{2}\sum_{\alpha=1}^n|F_{\alpha\bar\alpha}+F_{\bar\alpha\alpha}|^2 + \frac{n}{2}(F_0)^2.
$$

Finally, we deal with the term $\int_M F^{\alpha^{-1}-2}(F_0)^2\,d\mu$. By putting $f = F^{1/(2\alpha)}$ in (3-4), we obtain

$$
\begin{aligned}
& n^2\int_M F^{\alpha^{-1}-2}(F_0)^2\,d\mu \\
&= \int_M F^{\alpha^{-1}-2}(\Delta_b F)^2\,d\mu + \frac{1-2\alpha}{\alpha}\int_M F^{\alpha^{-1}-3}\Delta_b F|\nabla_b F|^2\,d\mu \\
&\qquad + \frac{(1-2\alpha)^2}{4\alpha^2}\int_M F^{\alpha^{-1}-4}|\nabla_b F|^4\,d\mu - 2\alpha^2\int_M P_0F^{(2\alpha)^{-1}}\cdot F^{(2\alpha)^{-1}}\,d\mu \\
&\qquad\qquad + 2n\int_M F^{\alpha^{-1}-2}\,\mathrm{Tor}((\nabla_b F)_{\mathbb{C}}, (\nabla_b F)_{\mathbb{C}})\,d\mu.
\end{aligned}
$$

Substituting this into (3-10), we find that its left side is equal to $(n+1)^{-1}$ times

$$
-\frac{\alpha}{3(1-2\alpha)}\int_M F^{\alpha^{-1}-2}\Big((2n+3)(\Delta_b F)^2 - n\sum_{\alpha=1}^n|F_{\alpha\bar\alpha}+F_{\bar\alpha\alpha}|^2\Big)\,d\mu
$$

$$+\frac{4n\alpha}{3(1-2\alpha)}\int_M F^{\alpha^{-1}-2}\Big(\sum_{\alpha,\beta=1}^{n}|F_{\alpha\beta}|^2+\sum_{\alpha,\beta=1,\ \alpha\neq\beta}^{n}|F_{\alpha\bar\beta}|^2\Big)\,d\mu$$
$$+\frac{4n\alpha}{3(1-2\alpha)}\int_M F^{\alpha^{-1}-2}(\mathrm{Ric}-\tfrac{1}{2}(n+1)\,\mathrm{Tor})((\nabla_b F)_{\mathbb{C}},(\nabla_b F)_{\mathbb{C}})\,d\mu$$
$$+\frac{6(2n+1)\alpha-(4n+3)}{12\alpha}\int_M F^{\alpha^{-1}-4}|\nabla_b F|^4\,d\mu+\frac{2\alpha^3}{1-2\alpha}\int_M P_0F^{(2\alpha)^{-1}}\cdot F^{(2\alpha)^{-1}}\,d\mu.$$

Then Lemma 3.3 follows by substituting this identity into (3-6). □

Proof of Theorem 1.5. By using the Cauchy–Schwarz inequality

$$\sum_{\alpha=1}^{n}|F_{\alpha\bar\alpha}+F_{\bar\alpha\alpha}|^2\geq\tfrac{1}{n}(\Delta_bF)^2,$$

we obtain

(3-11)
$$\frac{1}{3(1-2\alpha)}\Big((4n\alpha+3)(\Delta_bF)^2+(4\alpha-3)n\sum_{\alpha=1}^{n}|f_{\alpha\bar\alpha}+f_{\bar\alpha\alpha}|^2\Big)\leq\frac{4(n+1)\alpha}{3(1-2\alpha)}(\Delta_bF)^2$$

for $\alpha<1/2$ with $\alpha\neq0$, and $24n\alpha^2-2(8n-3)\alpha-3\leq0$ if and only if

$$\frac{(8n-3)-\sqrt{(8n-3)^2+72n}}{24n}\leq\alpha\leq\frac{(8n-3)+\sqrt{(8n-3)^2+72n}}{24n}.$$

The CR Paneitz operator is always nonnegative for a closed pseudohermitian $(2n+1)$-manifold with $n\geq2$. Then Theorem 1.5 follows from the Lemma 3.3 easily by choosing the constant α so that it lies within the interval

$$[((8n-3)-\sqrt{(8n-3)^2+72n})/24n,0].$$ □

However, if the pseudohermitian torsion is zero on a closed pseudohermitian 3-manifold, then the corresponding CR Paneitz operator is nonnegative. Therefore:

Corollary 3.4. *Let (M,J,θ) be a closed pseudohermitian 3-manifold with zero pseudohermitian torsion and nonnegative Tanaka–Webster curvature. Then the energy functional $\mathcal{F}(\varphi)$ is monotone nonincreasing along the CR heat equation and the monotonicity is strict unless the solution u is constant.*

4. The Nash-type entropy formula and subgradient estimate

We now prove the monotonicity formula for $\widetilde{N}(u,t)$ and derive the integral version of the subgradient estimate for the CR heat equation on $M^n\times[0,\infty)$.

Let $u(x,t)$ be a positive solution to the CR heat equation

$$\Big(\Delta_b-\frac{\partial}{\partial t}\Big)u=0\quad\text{with}\quad\int_M u\,d\mu=1.$$

Proof of Theorem 1.6 and Corollary 1.7. By a simple calculation using

$$\frac{d}{dt}\int_M \varphi e^{-\varphi}\,d\mu=\int_M |\nabla_b\varphi|^2 e^{-\varphi}\,d\mu=\mathscr{F}(\varphi)$$

and

$$\frac{d}{dt}\int_M e^{-\varphi}\,d\mu=0,$$

we find

$$\begin{aligned}\frac{d}{dt}\widetilde{N}&=\frac{n}{n+2}\mathscr{F}(\varphi)-\frac{(n+1)(n+2)}{2t}\\&=\int_M\left(\frac{n}{n+2}|\nabla_b\varphi|^2-\varphi_t-\frac{(n+1)(n+2)}{2t}\right)u\,d\mu,\end{aligned}$$

where φ satisfies $e^{-\varphi}=u$. All these together with Proposition 5.1 imply

$$\frac{d}{dt}\widetilde{N}=\frac{n}{n+2}\mathscr{F}(\varphi)-\frac{(n+1)(n+2)}{2t}\leq 0 \tag{4-1}$$

and then

$$\int_M |\nabla_b\varphi|^2 e^{-\varphi}\,d\mu\leq\frac{(n+1)(n+2)^2}{2nt}\quad\text{for all } t\in(0,\infty).$$

But

$$\int_M |\nabla_b\varphi|^2 e^{-\varphi}\,d\mu=4\int_M |\nabla_b e^{-\frac{\varphi}{2}}|^2\,d\mu=4\int_M |\nabla_b u^{\frac{1}{2}}|^2 d\mu.$$

Hence there exists a constant $C_1=C(n,\int_M d\mu)$ such that

$$\int_M |\nabla_b u^{\frac{1}{2}}|^2\,d\mu\leq\frac{C_1}{t}.$$

This completes the proof of Theorem 1.6, Corollary 1.7. □

5. The CR Harnack-type inequality

In this section, we establish the CR Harnack-type inequality for the positive solution of (1-4) on $M\times[0,\infty)$.

Proposition 5.1. *Let (M, J,θ) be a closed pseudohermitian $(2n+1)$-manifold of zero torsion and nonnegative pseudohermitian Ricci tensors. If $u(x,t)$ is the positive solution of (1-4) on $M\times[0,\infty)$ such that $P_\beta u=0$ at $t=0$, then u satisfies the estimate*

$$\frac{n}{n+2}|\nabla_b\psi|^2+\psi_t-\frac{(n+1)(n+2)}{2t}\leq 0\quad\text{on } M\times[0,\infty). \tag{5-1}$$

Here $\psi(x,t)=\ln u(x,t)$.

Before we give a proof of Proposition 5.1, we need a series of lemmas.

Lemma 5.2. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold of zero torsion. If $u(x,t)$ is a solution of*

$$\Big(\Delta_b - \frac{\partial}{\partial t}\Big)u(x,t) = 0 \quad \text{on } M \times [0,\infty) \text{ with } P_\beta u(x,0) = 0,$$

then $P_\beta u(x,t) = 0$ for all $t \in [0,\infty)$.

Proof. From Remark 1.2, we have $P_0 u = 0$ if and only if $P_\beta u = 0$ and

$$P_0 u = ({\Delta_b}^2 u + nT^2 u).$$

It follows that $\Delta_b P_0 u = P_0 \Delta_b u$. Applying P_0 to the heat equation, we obtain

$$\Big(\Delta_b - \frac{\partial}{\partial t}\Big)P_0 u(x,t) = 0 \quad \text{on } M \times [0,\infty) \text{ with } P_0 u(x,0) = 0.$$

Hence the lemma follows from the maximum principle and Remark 1.2. □

Lemma 5.3. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Let $\psi = \ln f$ for $f > 0$. Then*

$$\begin{aligned}(5\text{-}2)\quad & 4\langle P\psi + \overline{P}\psi, d_b\psi\rangle_{L_\theta^*} \\ &= 4f^{-2}\langle Pf + \overline{P}f, d_b f\rangle_{L_\theta^*} - 2\langle\nabla_b\psi, \nabla_b|\nabla_b\psi|^2\rangle + 2f^{-1}\Delta_b f|\nabla_b\psi|^2.\end{aligned}$$

Proof. Let $Q(x) = |\nabla_b\psi|^2(x)$. We compute

$$\begin{aligned}\nabla_b Q &= Q_{\bar\alpha}Z_\alpha + Q_\alpha Z_{\bar\alpha} \\ &= 2\nabla_b(\psi_\alpha\psi_{\bar\alpha}) \\ &= 2(f^{-4}(f^2 f_\alpha f_{\bar\alpha\bar\beta} + f^2 f_{\bar\alpha} f_{\alpha\bar\beta} - 2ff_{\bar\alpha}f_{\bar\beta}f_\alpha))Z_\beta + \text{complex conjugate}.\end{aligned}$$

It follows that

$$\begin{aligned}P_\beta\psi &= \psi_{\bar\alpha\alpha\beta} + inA_{\beta\alpha}\psi_{\bar\alpha} \\ &= f^{-4}(-f^2 f_\alpha f_{\bar\alpha\beta} + f^3 f_{\bar\alpha\alpha\beta} - f^2 f_{\bar\alpha} f_{\alpha\beta} - f^2 f_\beta f_{\bar\alpha\alpha} + 2ff_\alpha f_\beta f_{\bar\alpha}) \\ &\qquad\qquad + inA_{\beta\alpha}f^{-1}f_{\bar\alpha} \\ &= f^{-1}P_\beta f - \tfrac{1}{2}Q_\beta - f^{-2}f_\beta f_{\bar\alpha\alpha} \\ &= f^{-1}P_\beta f - \tfrac{1}{2}Q_\beta - \psi_\beta f^{-1}f_{\bar\alpha\alpha}.\end{aligned}$$

Thus

$$\begin{aligned}4\langle P\psi + \overline{P}\psi, d_b\psi\rangle_{L_\theta^*} &= 4\langle(P_\beta\psi)\theta^\beta + (\overline{P}_\beta\psi)\theta^{\bar\beta}, \psi_\beta\theta^\beta + \psi_{\bar\beta}\theta^{\bar\beta}\rangle_{L_\theta^*} \\ &= 4((P_\beta\psi)\psi_{\bar\beta} + (\overline{P}_\beta\psi)\psi\psi_\beta) \\ &= 4(f^{-1}P_\beta f - \tfrac{1}{2}Q_\beta - \psi_\beta f^{-1}f_{\bar\alpha\alpha})\psi_{\bar\beta} + \text{complex conjugate} \\ &= 4f^{-2}\langle Pf + \overline{P}f, d_b f\rangle_{L_\theta^*} - 2\langle\nabla_b\psi, \nabla_b|\nabla_b\psi|^2\rangle + 2(f^{-1}\Delta_b f|\nabla_b\psi|^2).\end{aligned}$$

□

Lemma 5.4. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Let $u(x,t)$ be the positive smooth solution of (1-4) on $M \times [0, \infty)$. Suppose that*

$$(2\,\mathrm{Ric} - (n+2)\,\mathrm{Tor})(Z, Z) \geq -k_0|Z|^2 \quad \textit{for all } Z \in T_{1,0}$$

and k_0 a nonnegative constant. Then the function

$$G = t(|\nabla_b \psi|^2 + (1 + 2/n)\psi_t) \tag{5-3}$$

satisfies the inequality

$$\begin{aligned}(\Delta_b - \tfrac{\partial}{\partial t})G \geq &-\frac{2n}{n+2}\langle \nabla_b \psi, \nabla_b G\rangle \\ &+ \frac{2n}{(n+1)(n+2)^2 t} G\Big(G - \frac{(n+1)(n+2)^2}{2n}\Big) \\ &- k_0 t |\nabla_b \psi|^2 - 8n^{-1} t u^{-2} \langle P u + \overline{P} u, d_b u\rangle_{L^*_\theta}.\end{aligned}$$

Here $\psi(x,t) = \ln u(x,t)$.

Proof. we first prove the lemma for $n = 1$. Let $F = t(|\nabla_b \psi|^2 + 3\psi_t)$. First differentiating (5-3) with respect to t, we have

$$\begin{aligned} F_t &= t^{-1}F + t(|\nabla_b \psi|^2 + 3\psi_t)_t \\ &= t^{-1}F + t(4|\nabla_b \psi|^2 + 3\Delta_b \psi)_t = t^{-1}F + t(8\langle \nabla_b \psi, \nabla_b \psi_t\rangle + 3\Delta_b \psi_t). \end{aligned} \tag{5-4}$$

By using the CR version of the Bochner formula (3-2) and Lemma 5.3, one obtains

$$\begin{aligned} \Delta_b F &= t(\Delta_b |\nabla_b \psi|^2 + 3\Delta_b \psi_t) \\ &= t\big(2|(\nabla^H)^2 \psi|^2 + 6\langle \nabla_b \psi, \nabla_b \Delta_b \psi\rangle \\ &\qquad + 2(2\,\mathrm{Ric} - 3\,\mathrm{Tor})((\nabla_b \psi)_{\mathbb{C}}, (\nabla_b \psi)_{\mathbb{C}}) \\ &\qquad\qquad - 8\langle P\psi + \overline{P}\psi, d_b \psi\rangle_{L^*_\theta} + 3\Delta_b \psi_t\big) \\ &\geq t\big(4|\psi_{11}|^2 + (\Delta_b \psi)^2 + 6\langle \nabla_b \psi, \nabla_b \Delta_b \psi\rangle - k_0|\nabla_b \psi|^2 \\ &\qquad\qquad - 8\langle P\psi + \overline{P}\psi, d_b \psi\rangle_{L^*_\theta} + 3\Delta_b \psi_t\big) \\ &= t\big(4|\psi_{11}|^2 + (\Delta_b \psi)^2 + 6\langle \nabla_b \psi, \nabla_b \Delta_b \psi\rangle - k_0|\nabla_b \psi|^2 \\ &\qquad - 8u^{-2}\langle P u + \overline{P} u, d_b u\rangle_{L^*_\theta} + 4\psi_t |\nabla_b \psi|^2 \\ &\qquad\qquad + 4\langle \nabla_b \psi, \nabla_b |\nabla_b \psi|^2\rangle + 3\Delta_b \psi_t\big). \end{aligned} \tag{5-5}$$

Here we have used the inequalities

$$|(\nabla^H)^2 \psi|^2 = 2|\psi_{11}|^2 + \tfrac{1}{2}(\Delta_b \psi)^2 + \tfrac{1}{2}\psi_0^2 \geq 2|\psi_{11}|^2 + \tfrac{1}{2}(\Delta_b \psi)^2 \tag{5-6}$$

and

$$(2\,\mathrm{Ric} - 3\,\mathrm{Tor})((\nabla_b \psi)_{\mathbb{C}}, (\nabla_b \psi)_{\mathbb{C}}) \geq -k_0|(\nabla_b \psi)_{\mathbb{C}}|^2 = -\tfrac{1}{2}k_0|\nabla_b \psi|^2$$

and $\psi_t = u_t/u = \Delta_b u/u$. Applying the formula

$$\Delta_b \psi = \psi_t - |\nabla_b \psi|^2 = \tfrac{1}{3} t^{-1} F - \tfrac{4}{3} |\nabla_b \psi|^2 \tag{5-7}$$

in combination with (5-4) and (5-5), we conclude

$$\begin{aligned}\Big(\Delta_b - \frac{\partial}{\partial t}\Big) F \geq -t^{-1} F + t\big(&4|\psi_{11}|^2 + (\Delta_b \psi)^2 + 6\langle \nabla_b \psi, \nabla_b \Delta_b \psi \rangle \\ &+ 4\langle \nabla_b \psi, \nabla_b |\nabla_b \psi|^2 \rangle - 8\langle \nabla_b \psi, \nabla_b \psi_t \rangle \\ &- k_0 |\nabla_b \psi|^2 + 4\psi_t |\nabla_b \psi|^2 - 8u^{-2} \langle Pu + \bar{P}u, d_b u \rangle_{L_\theta^*}\big) \\ = -t^{-1} F + t\big(&-\tfrac{2}{3} t^{-1} \langle \nabla_b \psi, \nabla_b F \rangle - \tfrac{4}{3} \langle \nabla_b \psi, \nabla_b |\nabla_b \psi|^2 \rangle + 4|\psi_{11}|^2 \\ &+ (\Delta_b \psi)^2 - k_0 |\nabla_b \psi|^2 + 4\psi_t |\nabla_b \psi|^2 - 8u^{-2} \langle Pu + \bar{P}u, d_b u \rangle_{L_\theta^*}\big).\end{aligned}$$

Now it is easy to see that $\langle \nabla_b \psi, \nabla_b |\nabla_b \psi|^2 \rangle = 4\,\mathrm{Re}(\psi_{11} \psi_{\bar{1}} \psi_{\bar{1}}) + \Delta_b \psi |\nabla_b \psi|^2$. Thus

$$\begin{aligned}-\tfrac{4}{3} \langle \nabla_b \psi, \nabla_b |\nabla_b \psi|^2 \rangle &= -\tfrac{16}{3}\,\mathrm{Re}(\psi_{11} \psi_{\bar{1}} \psi_{\bar{1}}) - \tfrac{4}{3} \Delta_b \psi |\nabla_b \psi|^2 \\ &\geq -4|\psi_{11}|^2 - \tfrac{16}{9} |\psi_{\bar{1}}|^4 - \tfrac{4}{3} \Delta_b \psi |\nabla_b \psi|^2 \\ &= -4|\psi_{11}|^2 - \tfrac{4}{9} |\nabla_b \psi|^4 - \tfrac{4}{3} \Delta_b \psi |\nabla_b \psi|^2.\end{aligned}$$

Here we have used the basic inequality $2\,\mathrm{Re}(zw) \leq \epsilon |z|^2 + \epsilon^{-1} |w|^2$ for all $\epsilon > 0$. All these imply

$$\begin{aligned}\big(\Delta_b - \tfrac{\partial}{\partial t}\big) F &\geq -t^{-1} F - \tfrac{2}{3} \langle \nabla_b \psi, \nabla_b F \rangle + t\big((\Delta_b \psi)^2 + \tfrac{8}{3} \Delta_b \psi |\nabla_b \psi|^2 + \tfrac{32}{9} |\nabla_b \psi|^4 \\ &\qquad - k_0 |\nabla_b \psi|^2 - 8u^{-2} \langle Pu + \bar{P}u, d_b u \rangle_{L_\theta^*}\big) \\ &\geq -\tfrac{2}{3} \langle \nabla_b \psi, \nabla_b F \rangle + \tfrac{1}{9} t^{-1} F(F - 9) - k_0 t |\nabla_b \psi|^2 \\ &\qquad - 8u^{-2} \langle Pu + \bar{P}u, d_b u \rangle_{L_\theta^*}.\end{aligned}$$

This completes the proof for $n = 1$. For $n \geq 2$, we need more inequalities: For any smooth function f, we need

$$|(\nabla^H)^2 f|^2 \geq 2 \sum_{\alpha,\beta=1}^{n} |f_{\alpha\beta}|^2 + 2 \sum_{\alpha\neq\beta=1}^{n} |f_{\alpha\bar{\beta}}|^2 + \frac{1}{2} \sum_{\alpha=1}^{n} |f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha}|^2. \tag{5-8}$$

and

$$\begin{aligned}\langle \nabla_b f, \nabla_b |\nabla_b f|^2 \rangle \leq (n+2) \sum_{\alpha,\beta=1}^{n} |f_{\alpha\beta}|^2 &+ (n+2) \sum_{\alpha\neq\beta=1}^{n} |f_{\alpha\bar{\beta}}|^2 \\ &+ (\Delta_b f + |\nabla_b f|^2) |\nabla_b f|^2 \\ &+ \frac{(n+2)(n-1)}{4(n+1)} \sum_{\alpha=1}^{n} |f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha}|^2.\end{aligned} \tag{5-9}$$

We first claim (5-8) holds. Since

$$|(\nabla^H)^2 f|^2 = 2\sum_{\alpha,\beta=1}^n (f_{\alpha\beta} f_{\bar\alpha\bar\beta} + f_{\alpha\bar\beta} f_{\bar\alpha\beta}) = 2\sum_{\alpha,\beta=1}^n (|f_{\alpha\beta}|^2 + |f_{\alpha\bar\beta}|^2)$$
$$= 2\Big(\sum_{\alpha,\beta=1}^n |f_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^n |f_{\alpha\bar\beta}|^2 + \sum_{\alpha=1}^n |f_{\alpha\bar\alpha}|^2\Big),$$
$$\sum_{\alpha=1}^n |f_{\alpha\bar\alpha}|^2 = \frac{1}{4}\sum_{\alpha=1}^n (|f_{\alpha\bar\alpha} + f_{\bar\alpha\alpha}|^2 + f_0^2) = \frac{1}{4}\sum_{\alpha=1}^n |f_{\alpha\bar\alpha} + f_{\bar\alpha\alpha}|^2 + \tfrac{1}{4} n f_0^2.$$

It follows that

$$|(\nabla^H)^2 f|^2 = 2\Big(\sum_{\alpha,\beta=1}^n |f_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^n |f_{\alpha\bar\beta}|^2\Big) + \frac{1}{2}\sum_{\alpha=1}^n |f_{\alpha\bar\alpha} + f_{\bar\alpha\alpha}|^2 + \tfrac{1}{2} n f_0^2$$
$$\geq 2\Big(\sum_{\alpha,\beta=1}^n |f_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^n |f_{\alpha\bar\beta}|^2\Big) + \frac{1}{2}\sum_{\alpha=1}^n |f_{\alpha\bar\alpha} + f_{\bar\alpha\alpha}|^2.$$

Secondly, we claim (5-9) holds. We first derive

$$\begin{aligned}
&\big\langle \nabla_b f, \nabla_b |\nabla_b f|^2 \big\rangle \\
&= 4\sum_{\alpha,\beta=1}^n \mathrm{Re}(f_{\alpha\beta} f_{\bar\alpha} f_{\bar\beta} + f_{\alpha\bar\beta} f_{\bar\alpha} f_\beta) \\
&= 4\,\mathrm{Re}\Big(\sum_{\alpha,\beta=1}^n f_{\alpha\beta} f_{\bar\alpha} f_{\bar\beta} + \sum_{\alpha\neq\beta=1}^n f_{\alpha\bar\beta} f_{\bar\alpha} f_\beta\Big) + 2\sum_{\alpha=1}^n (f_{\alpha\bar\alpha} + f_{\bar\alpha\alpha})|f_\alpha|^2 \\
&\leq (n+2)\Big(\sum_{\alpha,\beta=1}^n |f_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^n |f_{\alpha\bar\beta}|^2\Big) + \frac{4}{n+2}\sum_{\alpha,\beta=1}^n |f_\alpha|^2 |f_\beta|^2 \\
&\qquad + \frac{4}{n+2}\sum_{\alpha\neq\beta=1}^n |f_\alpha|^2 |f_\beta|^2 + 2\sum_{\alpha=1}^n (f_{\alpha\bar\alpha} + f_{\bar\alpha\alpha})|f_\alpha|^2 \\
&= (n+2)\Big(\sum_{\alpha,\beta=1}^n |f_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^n |f_{\alpha\bar\beta}|^2\Big) + \frac{1}{n+2}|\nabla_b f|^4 \\
&\qquad\qquad + \frac{4}{n+2}\sum_{\alpha\neq\beta=1}^n |f_\alpha|^2 |f_\beta|^2 + 2\sum_{\alpha=1}^n (f_{\alpha\bar\alpha} + f_{\bar\alpha\alpha})|f_\alpha|^2.
\end{aligned}$$

Here we used the identity $\sum_{\alpha,\beta=1}^n |f_\alpha|^2 |f_\beta|^2 = (\sum_{\alpha=1}^n |f_\alpha|^2)^2 = \frac{1}{4}|\nabla_b f|^4$.

Now we compute the last term in the inequality above.

$$
\begin{aligned}
&\sum_{\alpha=1}^{n}(f_{\alpha\bar{\alpha}}+f_{\bar{\alpha}\alpha})|f_\alpha|^2\\
&\quad=\Big(\sum_{\alpha=1}^{n}(f_{\alpha\bar{\alpha}}+f_{\bar{\alpha}\alpha})\Big)\Big(\sum_{\beta=1}^{n}|f_\beta|^2\Big)-\sum_{\alpha=1}^{n}(f_{\alpha\bar{\alpha}}+f_{\bar{\alpha}\alpha})\Big(\sum_{\beta=1\neq\alpha}^{n}|f_\beta|^2\Big)\\
&\quad\leq\frac{1}{2}\Delta_b f\,|\nabla_b f|^2+\sum_{\alpha=1}^{n}|f_{\alpha\bar{\alpha}}+f_{\bar{\alpha}\alpha}|\Big(\sum_{\beta=1\neq\alpha}^{n}|f_\beta|^2\Big)\\
&\quad\leq\frac{1}{2}\Delta_b f|\nabla_b f|^2+\frac{(n-1)(n+2)}{8(n+1)}\sum_{\alpha=1}^{n}|f_{\alpha\bar{\alpha}}+f_{\bar{\alpha}\alpha}|^2\\
&\qquad\qquad+\frac{2(n+1)}{(n-1)(n+2)}\sum_{\alpha=1}^{n}\Big(\sum_{\beta=1\neq\alpha}^{n}|f_\beta|^2\Big)^2.
\end{aligned}
$$

Substituting this inequality our previous computation of $\langle\nabla_b f,\,\nabla_b|\nabla_b f|^2\rangle$, we get

$$
\begin{aligned}
&\langle\nabla_b f,\,\nabla_b|\nabla_b f|^2\rangle\\
&\quad\leq(n+2)\Big(\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2+\sum_{\alpha\neq\beta=1}^{n}|f_{\alpha\bar{\beta}}|^2\Big)+\Delta_b f|\nabla_b f|^2\\
&\qquad+\frac{(n-1)(n+2)}{4(n+1)}\sum_{\alpha=1}^{n}|f_{\alpha\bar{\alpha}}+f_{\bar{\alpha}\alpha}|^2+\frac{1}{n+2}|\nabla_b f|^4\\
&\qquad+\frac{4(n+1)}{(n+2)(n-1)}\sum_{\alpha=1}^{n}\Big(\sum_{\beta=1\neq\alpha}^{n}|f_\beta|^2\Big)^2+\frac{4}{n+2}\sum_{\alpha\neq\beta=1}^{n}|f_\alpha|^2|f_\beta|^2\\
&\quad\leq(n+2)\Big(\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2+\sum_{\alpha\neq\beta=1}^{n}|f_{\alpha\bar{\beta}}|^2\Big)+\Delta_b f|\nabla_b f|^2\\
&\qquad+\frac{(n+2)(n-1)}{4(n+1)}\sum_{\alpha=1}^{n}|f_{\alpha\bar{\alpha}}+f_{\bar{\alpha}\alpha}|^2+\frac{1}{n+2}|\nabla_b f|^4\\
&\qquad+\frac{4(n+1)}{(n+2)(n-1)}\Big(\sum_{\alpha=1}^{n}(\sum_{\beta=1\neq\alpha}^{n}|f_\beta|^2)^2+\sum_{\alpha\neq\beta=1}^{n}|f_\alpha|^2|f_\beta|^2\Big)\\
&\quad=(n+2)\Big(\sum_{\alpha,\beta=1}^{n}|f_{\alpha\beta}|^2+\sum_{\alpha\neq\beta=1}^{n}|f_{\alpha\bar{\beta}}|^2\Big)+\Delta_b f|\nabla_b f|^2\\
&\qquad+\frac{(n+2)(n-1)}{4(n+1)}\sum_{\alpha=1}^{n}|f_{\alpha\bar{\alpha}}+f_{\bar{\alpha}\alpha}|^2+|\nabla_b f|^4.
\end{aligned}
$$

Here we have used the identity

$$\sum_{\alpha=1}^{n}\Big(\sum_{\beta=1\neq\alpha}^{n}|f_\beta|^2\Big)^2+\sum_{\alpha\neq\beta=1}^{n}|f_\alpha|^2|f_\beta|^2=(n-1)\Big(\sum_{\alpha=1}^{n}|f_\alpha|^2\Big)^2=\frac{n-1}{4}|\nabla_b f|^4.$$

Now as before we differentiate (5-3) with respect to t, getting

$$\begin{aligned} G_t-t^{-1}G&=t(|\nabla_b\psi|^2+(1+2/n)\psi_t)_t\\ &=t(2(1+1/n)|\nabla_b\psi|^2+(1+2/n)\Delta_b\psi)_t\\ &=t(4(1+1/n)\langle\nabla_b\psi,\nabla_b\psi_t\rangle+(1+2/n)\Delta_b\psi_t). \end{aligned}\tag{5-10}$$

By using the CR version of the Bochner formula (3-2) and Lemma 5.3, one obtains

$$\begin{aligned} \Delta_b G&=t(\Delta_b|\nabla_b\psi|^2+(1+2/n)\Delta_b\psi_t)\\ &=t\big(2|(\nabla^H)^2\psi|^2+2(1+2/n)\,\langle\nabla_b\psi,\nabla_b\Delta_b\psi\rangle\\ &\qquad+2(2\,\mathrm{Ric}-(n+2)\,\mathrm{Tor})((\nabla_b\psi)_{\mathbb{C}},(\nabla_b\psi)_{\mathbb{C}})\\ &\qquad-(8/n)\langle P\psi+\bar P\psi,d_b\psi\rangle_{L^*_\theta}+(1+2/n)\Delta_b\psi_t\big)\\ &\geq t\big(2|(\nabla^H)^2\psi|^2+2(1+2/n)\langle\nabla_b\psi,\nabla_b\Delta_b\psi\rangle-k_0|\nabla_b\psi|^2\\ &\qquad-(8/n)\langle P\psi+\bar P\psi,d_b\psi\rangle_{L^*_\theta}+(1+2/n)\Delta_b\psi_t\big)\\ &=t\big(2|(\nabla^H)^2\psi|^2+2(1+2/n)\langle\nabla_b\psi,\nabla_b\Delta_b\psi\rangle-k_0|\nabla_b\psi|^2\\ &\qquad-(8/n)u^{-2}\langle Pu+\bar Pu,d_bu\rangle_{L^*_\theta}+4/n\,\psi_t|\nabla_b\psi|^2\\ &\qquad+(4/n)\langle\nabla_b\psi,\nabla_b|\nabla_b\psi|^2\rangle+(1+2/n)\Delta_b\psi_t\big). \end{aligned}\tag{5-11}$$

Here we have used the inequality

$$(2\,\mathrm{Ric}-(n+2)\,\mathrm{Tor})((\nabla_b\psi)_C,(\nabla_b\psi)_C)\geq -k_0|(\nabla_b\psi)_C|^2=-\tfrac{1}{2}k_0|\nabla_b\psi|^2$$

and

$$\psi_t=u_t/u=\Delta_b u/u.$$

Applying the formula

$$\Delta_b\psi=\psi_t-|\nabla_b\psi|^2=\frac{n}{(n+2)t}G-\frac{2(n+1)}{n+2}|\nabla_b\psi|^2\tag{5-12}$$

together with (5-10) and (5-11), we conclude

$$\begin{aligned}
&\Big(\Delta_b-\frac{\partial}{\partial t}\Big)G+t^{-1}G\\
&\quad\ge t\big(2|(\nabla^H)^2\psi|^2+2(1+2/n)\langle\nabla_b\psi,\nabla_b\Delta_b\psi\rangle\\
&\qquad\qquad+(4/n)\langle\nabla_b\psi,\nabla_b|\nabla_b\psi|^2\rangle-4(1+1/n)\langle\nabla_b\psi,\nabla_b\psi_t\rangle\\
&\qquad\qquad-k_0|\nabla_b\psi|^2+(4/n)\psi_t|\nabla_b\psi|^2-(8/n)u^{-2}\langle Pu+\overline{P}u,d_bu\rangle_{L^*_\theta}\big)\\
&\quad=t\Big(2|(\nabla^H)^2\psi|^2-\frac{4}{n+2}\langle\nabla_b\psi,\nabla_b|\nabla_b\psi|^2\rangle-k_0|\nabla_b\psi|^2\\
&\qquad\qquad+(4/n)\psi_t|\nabla_b\psi|^2-(8/n)u^{-2}\langle Pu+\overline{P}u,d_bu\rangle_{L^*_\theta}\Big)\\
&\qquad\qquad-\frac{2n}{(n+2)}\langle\nabla_b\psi,\nabla_bG\rangle.
\end{aligned}$$

Now, by (5-8) and (5-9), the Cauchy–Schwarz inequality and by applying (5-12), we finally have

$$\begin{aligned}
&\Big(\Delta_b-\frac{\partial}{\partial t}\Big)G+t^{-1}G\\
&\quad\ge-\frac{2n}{(n+2)}\langle\nabla_b\psi,\nabla_bG\rangle+t\Big(\frac{2}{n+1}\sum_{\alpha=1}^{n}|\psi_{\alpha\bar\alpha}+\psi_{\bar\alpha\alpha}|^2+\frac{8}{n(n+2)}\psi_t|\nabla_b\psi|^2\\
&\qquad\qquad-k_0|\nabla_b\psi|^2-\frac{8}{n}u^{-2}\langle Pu+\overline{P}u,d_bu\rangle_{L^*_\theta}\Big)\\
&\quad\ge-\frac{2n}{(n+2)}\langle\nabla_b\psi,\nabla_bG\rangle+t\Big(\frac{2}{n(n+1)}(\Delta_b\psi)^2+\frac{8}{n(n+2)}\psi_t|\nabla_b\psi|^2\\
&\qquad\qquad-k_0|\nabla_b\psi|^2-\frac{8}{n}u^{-2}\langle Pu+\overline{P}u,d_bu\rangle_{L^*_\theta}\Big)\\
&\quad=-\frac{2n}{(n+2)}\langle\nabla_b\psi,\nabla_bG\rangle+t\Big(\frac{2n}{(n+1)(n+2)^2t^2}G^2+\frac{8}{n(n+2)^2}|\nabla_b\psi|^4\\
&\qquad\qquad-k_0|\nabla_b\psi|^2-\frac{8}{n}u^{-2}\langle Pu+\overline{P}u,d_bu\rangle_{L^*_\theta}\Big).\qquad\square
\end{aligned}$$

Proof of Proposition 5.1. Applying Lemma 5.4 to ψ by setting $A_{\alpha\gamma}=0$ and $k_0=0$, and by using Lemma 5.2 for all t, we get

$$\langle Pu+\overline{P}u,d_bu\rangle=0.$$

Then we have

$$\begin{aligned}
&\Big(\Delta_b-\frac{\partial}{\partial t}\Big)G\\
&\quad\ge-\frac{2n}{(n+2)}\langle\nabla_b\varphi,\nabla_bG\rangle+\frac{1}{t}\frac{2n}{(n+1)(n+2)^2}G\Big(G-\frac{(n+1)(n+2)^2}{2n}\Big).
\end{aligned}\tag{5-13}$$

Proposition 5.1 follows if G is at most $(n+1)(n+2)^2/(2n)$. If not, at the maximum point (x_0, t_0) of G on $M \times [0, T]$ for some $T > 0$, we have

$$G(x_0, t_0) > (n+1)(n+2)^2/2n.$$

Clearly, $t_0 > 0$, because $G(x, 0) = 0$. By the fact that (x_0, t_0) is a maximum point of G on $M \times [0, T]$, we have

$$\Delta_b G(x_0, t_0) \le 0, \qquad \nabla_b G(x_0, t_0) = 0, \qquad G_t(x_0, t_0) \ge 0.$$

Combining this with (5-13), we have

$$0 \ge \frac{1}{t_0} \frac{2n}{(n+1)(n+2)^2} G(x_0, t_0)\Big(G(x_0, t_0) - \frac{(n+1)(n+2)^2}{2n}\Big),$$

which is a contradiction. Hence $G \le (n+1)(n+2)^2/(2n)$. □

6. An upper bound estimate for the first positive eigenvalue

By using the entropy formula (3-5) for the CR heat equation, we can obtain an upper bound estimate for the first positive eigenvalue λ_1 of the sublaplacian Δ_b on a complete noncompact pseudohermitian $(2n+1)$-manifold M when its pseudohermitian Ricci curvature minus $(n+1)/2$ times the pseudohermitian torsion has a negative lower bound.

First we consider the case of $n = 1$. In this case, we need the nonnegativity of the CR Paneitz operator P_0 to get an upper bound estimate.

Proposition 6.1. *Suppose (M, J, θ) is a complete noncompact pseudohermitian 3-manifold with nonnegative CR Paneitz operator P_0. Suppose that*

$$(\mathrm{Ric} - \mathrm{Tor})(Z, Z) \ge -k_1 |Z|^2 \quad \textit{for all } Z \in T_{1,0},$$

with k_1 a positive constant. Then $\lambda_1 \le k_1/2$.

Proof. Assume that f is a (unit) first eigenfunction of $-\Delta_b$; namely, $-\Delta_b f = \lambda_1 f$ and $\int_M f^2\, d\mu = 1$. It is known that $f > 0$. Now we let $u : M \times [0, \infty) \to \mathbb{R}$ be the solution of the CR heat equation

$$\big(\Delta_b - \tfrac{\partial}{\partial t}\big) u(x, t) = 0, \quad \text{with } u(x, 0) = f(x)^2,$$

and let φ be defined as before by $e^{-\varphi} = u$. Since $e^{-\varphi(0)/2} = f$ and f is the first eigenfunction, we have

$$\lambda_1 = -\frac{\Delta_b f}{f} = \tfrac{1}{4}(2\Delta_b \varphi - |\nabla_b \varphi|^2) \quad \text{at } t = 0$$

and equivalently

$$\Delta_b \varphi = 2\lambda_1 + \tfrac{1}{2} |\nabla_b \varphi|^2. \tag{6-1}$$

Applying the Lemma 3.3 with any constant $\alpha \neq 0,\ 1/2$, for $n=1$, we have at $t=0$

$$\begin{aligned}
0 &= \int_M 4\lambda_1 \Delta_b u\, d\mu = \int_M (2\Delta_b\varphi - |\nabla_b\varphi|^2)\Delta_b u\, d\mu = \frac{d}{dt}\int_M |\nabla_b\varphi|^2 u\, d\mu \\
&= \frac{4\alpha}{3(1-2\alpha)}\int_M (\Delta_b\varphi - \alpha|\nabla_b\varphi|^2)^2 u\, d\mu + \frac{24\alpha^2-10\alpha-3}{24}\int_M |\nabla_b\varphi|^4 u\, d\mu \\
&\quad + \frac{2(4\alpha-3)}{3(1-2\alpha)}\int_M \big(|\varphi_{11} - \alpha\varphi_1\varphi_1|^2 + (\mathrm{Ric}-\mathrm{Tor})((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}})\big) u\, d\mu \\
&\qquad + \frac{4\alpha-3}{1-2\alpha}\int_M P_0 u^{1/2}\cdot u^{1/2}\, d\mu.
\end{aligned}$$

Because $(\mathrm{Ric}-\mathrm{Tor})((\nabla_b\varphi)_{\mathbb{C}}, (\nabla_b\varphi)_{\mathbb{C}}) \geq -k_0|(\nabla_b\varphi)_{\mathbb{C}}|^2 = -(k_0/2)|\nabla_b\varphi|^2$, the Paneitz operator P_0 is nonnegative by assumption, and from (6-1) for $\alpha < 1/2$ with $\alpha \neq 0$ we then have

$$\begin{aligned}
0 &\geq -\frac{4\alpha}{3(1-2\alpha)}\int_M (\Delta_b\varphi - \alpha|\nabla_b\varphi|^2)^2 u\, d\mu \\
&\qquad - \frac{24\alpha^2-10\alpha-3}{24}\int_M |\nabla_b\varphi|^4 u\, d\mu + \frac{(4\alpha-3)}{3(1-2\alpha)}k_1\int_M |\nabla_b\varphi|^2 u\, d\mu \\
&= -\frac{4\alpha}{3(1-2\alpha)}\int_M \big(4\lambda_1^2 + 2(1-2\alpha)\lambda_1|\nabla_b\varphi|^2 + ((1-2\alpha)^2/4)|\nabla_b\varphi|^4\big) u\, d\mu \\
&\qquad - \frac{24\alpha^2-10\alpha-3}{24}\int_M |\nabla_b\varphi|^4 u\, d\mu + \frac{(4\alpha-3)}{3(1-2\alpha)}k_1\int_M |\nabla_b\varphi|^2 u\, d\mu \\
&= \frac{1}{3(1-2\alpha)}\Big(-16\alpha\lambda_1^2\int_M u\, d\mu + [(4\alpha-3)k_1 - 8\alpha(1-2\alpha)\lambda_1]\int_M |\nabla_b\varphi|^2 u\, d\mu \\
&\qquad - \frac{(4\alpha-3)\,(1-4\alpha^2)}{8}\int_M |\nabla_b\varphi|^4 u\, d\mu\Big).
\end{aligned}$$

Noting that at $t=0$

$$4\lambda_1 = 4\int_M |\nabla_b f|^2\, d\mu = \int_M |\nabla_b\varphi|^2 u\, d\mu \quad \text{and} \quad \int_M u\, d\mu = 1, \tag{6-2}$$

we obtain

$$\begin{aligned}
0 &\geq \frac{1}{3(1-2\alpha)}\Big(-16\alpha\lambda_1^2 + 4\lambda_1((4\alpha-3)k_1 - 8\alpha(1-2\alpha)\lambda_1) \\
&\qquad - \frac{(4\alpha-3)(1-4\alpha^2)}{8}\int_M |\nabla_b\varphi|^4 u\, d\mu\Big) \\
&= \frac{4\alpha-3}{3(1-2\alpha)}\Big(4\lambda_1(4\alpha\lambda_1 + k_1) - ((1-4\alpha^2)/8)\int_M |\nabla_b\varphi|^4 u\, d\mu\Big).
\end{aligned}$$

We substitute the Hölder inequality

$$\int_M |\nabla_b \varphi|^4 u\, d\mu \geq \Big(\int_M |\nabla_b \varphi|^2 u\, d\mu\Big)^2 = 16\lambda_1^2 \tag{6-3}$$

into the inequality above to conclude

$$0 \geq \frac{2(4\alpha-3)\lambda_1}{3(1-2\alpha)}((4\alpha^2+8\alpha-1)\lambda_1+2k_0) \quad \text{for } -1/2 \leq \alpha < 1/2 \text{ with } \alpha \neq 0.$$

Therefore

$$(4\alpha^2+8\alpha-1)\lambda_1+2k_1 \geq 0 \quad \text{for } -1/2 \leq \alpha < 1/2 \text{ with } \alpha \neq 0.$$

The result follows by letting $\alpha = -1/2$ to get the optimal upper bound of λ_1. □

Proposition 6.2. *Suppose (M, J, θ) is a complete noncompact pseudohermitian $(2n+1)$-manifold for $n \geq 2$. Suppose that*

$$(\mathrm{Ric} - ((n+1)/2)\,\mathrm{Tor})(Z, Z) \geq -k_1|Z|^2 \quad \textit{for all } Z \in T_{1,0},$$

with k_1 a positive constant. Then

$$\lambda_1 \leq \frac{(2n-1)n}{n+1} k_1.$$

Proof. Let f be a normalized eigenfunction of $-\Delta_b$ corresponding to the first positive eigenvalue λ_1. Let u and φ be the same in the proof of Proposition 6.1. As before, by applying the Lemma 3.3 with $F = u^\alpha$ for any constant $\alpha \neq 0, 1/2$, with $n \geq 2$, we see that the following vanishes at $t = 0$:

$$\frac{1}{3(1-2\alpha)}\int_M F^{\alpha^{-1}-2}\Big((4n\alpha+3)(\Delta_b F)^2 + (4\alpha-3)n\sum_{\alpha=1}^n |F_{\alpha\bar\alpha}+F_{\bar\alpha\alpha}|^2\Big)\,d\mu$$
$$+\frac{4n(4\alpha-3)}{3(1-2\alpha)}\int_M F^{\alpha^{-1}-2}\Big(\sum_{\alpha,\beta=1}^n |F_{\alpha\beta}|^2 + \sum_{\alpha\neq\beta=1}^n |F_{\alpha\bar\beta}|^2\Big)\,d\mu$$
$$+\frac{4n(4\alpha-3)}{3(1-2\alpha)}\int_M F^{\alpha^{-1}-2}(\mathrm{Ric} - \tfrac{1}{2}(n+1)\,\mathrm{Tor})((\nabla_b F)_{\mathbb{C}}, (\nabla_b F)_{\mathbb{C}})\,d\mu$$
$$+\frac{24n\alpha^2-2(8n-3)\alpha-3}{12\alpha^2}\int_M F^{\alpha^{-1}-4}|\nabla_b F|^4 d\mu + \frac{2\alpha^2(4\alpha-3)}{1-2\alpha}\int_M P_0 u^{1/2}\cdot u^{1/2} d\mu$$

Since $(\mathrm{Ric} - \frac{1}{2}(n+1)\,\mathrm{Tor})((\nabla_b F)_{\mathbb{C}}, (\nabla_b F)_{\mathbb{C}}) \geq -k_1|(\nabla_b F)_{\mathbb{C}}|^2 = -\frac{1}{2}k_1|\nabla_b F|^2$, by the Cauchy–Schwarz inequality (3-11), the Paneitz operator P_0 is always non-negative for $n \geq 2$, and from (6-1), (6-2), (6-3), for $-1/2(2n-1) \leq \alpha < 1/2$ with

$\alpha \neq 0$, we then have

$$\begin{aligned}
0 \geq &-\frac{4(n+1)\alpha}{3(1-2\alpha)}\int_M (\Delta_b\varphi - \alpha|\nabla_b\varphi|^2)^2 u\, d\mu + \frac{2n(4\alpha-3)}{3(1-2\alpha)}k_1\int_M |\nabla_b\varphi|^2 u\, d\mu \\
&\qquad - \frac{24n\alpha^2 - 2(8n-3)\alpha - 3}{12}\int_M |\nabla_b\varphi|^4 u\, d\mu \\
= &-\frac{4(n+1)\alpha}{3(1-2\alpha)}\int_M (4\lambda_1^2 + 2(1-2\alpha)\lambda_1|\nabla_b\varphi|^2 + ((1-2\alpha)^2/4)|\nabla_b\varphi|^4)u\, d\mu \\
&+ \frac{2n(4\alpha-3)}{3(1-2\alpha)}k_1\int_M |\nabla_b\varphi|^2 u\, d\mu - \frac{24n\alpha^2 - 2(8n-3)\alpha - 3}{12}\int_M |\nabla_b\varphi|^4 u\, d\mu \\
= &\frac{4\alpha-3}{3(1-2\alpha)}\Big(8(2(n+1)\alpha\lambda_1 + nk_1)\lambda_1 \\
&\qquad - \frac{(1-2\alpha)(2(2n-1)\alpha+1)}{4}\int_M |\nabla_b\varphi|^4 u\, d\mu\Big). \\
\geq &\frac{4(4\alpha-3)}{3(1-2\alpha)}\lambda_1\big((4(2n-1)\alpha^2 + 8\alpha - 1)\lambda_1 + 2nk_1\big).
\end{aligned}$$

Therefore

$$(4(2n-1)\alpha^2 + 8\alpha - 1)\lambda_1 + 2nk_1 \geq 0$$

for $-1/(2(2n-1)) \leq \alpha < 1/2$ with $\alpha \neq 0$. Then the result follows by choosing $\alpha = -1/(2(2n-1))$ to get an upper bound of λ_1. □

Corollary 1.9 follows easily from Propositions 6.1 and 6.2. □

7. A lower bound estimate for the first positive eigenvalue

By using the entropy formula (3-5) for the CR heat equation, we can derive a sharp lower bound estimate for the first positive eigenvalue λ_1 of the sublaplacian Δ_b on a closed pseudohermitian $(2n+1)$-manifold M.

Proof of Corollary 1.10. Let f be a unit eigenfunction of Δ_b corresponding to the first positive eigenvalue λ_1. Let a be a positive constant such that $f+a>0$ on M. As before, let u be the solution of the CR heat equation $(\Delta_b - \partial/\partial t)u(x,t) = 0$ with the initial data $u(x,0) = (f(x)+a)^2$. Let $\varphi = -\log u$. Then, since $e^{-\varphi(0)/2} = f+a$ and f is the first eigenfunction of Δ_b, we have at $t=0$

$$\Delta_b\varphi = 2\lambda_1\frac{f}{f+a} + \tfrac{1}{2}|\nabla_b\varphi|^2. \tag{7-1}$$

Again from Lemma 3.3 with $F = u^\alpha$ for any constant $\alpha \neq 0, 1/2$, from (7-1), we have the following at $t=0$:

$$\text{(7-2)}\quad \frac{1}{3(1-2\alpha)}\int_M F^{\alpha^{-1}-2}\Big((4n\alpha+3)(\Delta_b F)^2+(4\alpha-3)n\sum_{\alpha=1}^{n}|F_{\alpha\bar\alpha}+F_{\bar\alpha\alpha}|^2\Big)\,d\mu$$
$$+\frac{4n(4\alpha-3)}{3(1-2\alpha)}\int_M F^{\alpha^{-1}-2}\Big(\sum_{\alpha,\beta=1}^{n}|F_{\alpha\beta}|^2+\sum_{\alpha\neq\beta=1}^{n}|F_{\alpha\bar\beta}|^2\Big)\,d\mu$$
$$+\frac{4n(4\alpha-3)}{3(1-2\alpha)}\int_M F^{\alpha^{-1}-2}(\mathrm{Ric}-\tfrac12(n+1)\,\mathrm{Tor})((\nabla_b F)_{\mathbb{C}},(\nabla_b F)_{\mathbb{C}})\,d\mu$$
$$+\frac{24n\alpha^2-2(8n-3)\alpha-3}{12\alpha^2}\int_M F^{\alpha^{-1}-4}|\nabla_b F|^4\,d\mu+\frac{2\alpha^2(4\alpha-3)}{1-2\alpha}\int_M P_0u^{1/2}\cdot u^{1/2}\,d\mu$$
$$=\alpha^2(n+1)\frac{d}{dt}\int_M|\nabla_b\varphi|^2u\,d\mu$$
$$=2\alpha^2(n+1)\int_M(2\Delta_b\varphi-|\nabla_b\varphi|^2)(-\lambda_1(f+a)f+|\nabla_b f|^2)\,d\mu$$
$$=8\alpha^2(n+1)\lambda_1\int_M\frac{f}{f+a}(-\lambda_1 f(f+a)+|\nabla_b f|^2)\,d\mu$$
$$=-2\alpha^2(n+1)a\lambda_1\int_M|\nabla_b\varphi|^2u^{1/2}\,d\mu.$$

Since $(\mathrm{Ric}-\frac12(n+1)\,\mathrm{Tor})((\nabla_b F)_{\mathbb{C}},(\nabla_b F)_{\mathbb{C}})\geq k_2|(\nabla_b F)_{\mathbb{C}}|^2=\frac12 k_2|\nabla_b F|^2$, by the Cauchy–Schwarz inequality (3-11) the Paneitz operator P_0 is nonnegative, and from (6-2) and (7-1) for $\alpha<1/2$ with $\alpha\neq0$, we then have

$$0\geq-\frac{4(n+1)\alpha}{3(1-2\alpha)}\int_M(\Delta_b\varphi-\alpha|\nabla_b\varphi|^2)^2u\,d\mu$$
$$-\frac{2n(4\alpha-3)}{3(1-2\alpha)}k_2\int_M|\nabla_b\varphi|^2u\,d\mu-2(n+1)a\lambda_1\int_M|\nabla_b\varphi|^2u^{1/2}\,d\mu$$
$$-\frac{24n\alpha^2-2(8n-3)\alpha-3}{12}\int_M|\nabla_b\varphi|^4u\,d\mu$$
$$=-\frac{4(n+1)\alpha}{3(1-2\alpha)}\int_M\big(4\lambda_1^2f^2u^{-1}+2(1-2\alpha)\lambda_1|\nabla_b\varphi|^2+((1-2\alpha)^2/4)|\nabla_b\varphi|^4$$
$$-2(1-2\alpha)a\lambda_1u^{-1/2}|\nabla_b\varphi|^2\big)u\,d\mu$$
$$-\frac{8n(4\alpha-3)}{3(1-2\alpha)}\lambda_1k_2-2(n+1)a\lambda_1\int_M|\nabla_b\varphi|^2u^{1/2}\,d\mu$$
$$-\frac{24n\alpha^2-2(8n-3)\alpha-3}{12}\int_M|\nabla_b\varphi|^4u\,d\mu$$

$$= \frac{4\alpha - 3}{3(1-2\alpha)}\bigg(8(2(n+1)\alpha\lambda_1 - nk_2)\lambda_1 + 2(n+1)(1-2\alpha)a\lambda_1 \int_M |\nabla_b\varphi|^2 u^{1/2}\, d\mu - \frac{(1-2\alpha)[2(2n-1)\alpha+1]}{4}\int_M |\nabla_b\varphi|^4 u\, d\mu\bigg).$$

Therefore,

$$8(2(n+1)\alpha\lambda_1 - nk_2)\lambda_1 + 2(n+1)(1-2\alpha)a\lambda_1 \int_M |\nabla_b\varphi|^2 u^{1/2}\, d\mu - \tfrac{1}{4}(1-2\alpha)(2(2n-1)\alpha+1)\int_M |\nabla_b\varphi|^4 u\, d\mu \geq 0,$$

for $\alpha < 1/2$ with $\alpha \neq 0$. We then let $\alpha \to (1/2)^-$ to get a sharp lower bound estimate of λ_1. □

Corollary 1.11 follows from Theorem 7.1, Corollary 7.2 and Proposition 7.3.□

Theorem 7.1. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with an essentially positive CR Paneitz operator P_0. Suppose that*

$$(\mathrm{Ric} - ((n+1)/2)\,\mathrm{Tor})(Z, Z) \geq -k_3|Z|^2 \quad \textit{for all } Z \in T_{1,0},$$

with k_3 a nonnegative constant. Suppose also that $\ker(\Delta_b + \lambda_1 I) \cap (\ker P_0)^\perp \neq \phi$. Then

$$\lambda_1 \geq \frac{-nk_3 + \sqrt{n^2k_3^2 + 3(n+1)\Lambda}}{2(n+1)}.$$

Proof. Since $\ker(\Delta_b + \lambda_1 I) \cap (\ker P_0)^\perp \neq \phi$, we may let $f \in (\ker P_0)^\perp$ be a normalized eigenfunction of Δ_b corresponding to the first positive eigenvalue λ_1. Then

$$\int_M P_0 u^{1/2} \cdot u^{1/2}\, d\mu = \int_M P_0 f \cdot f\, d\mu \geq \Lambda \int_M f^2\, d\mu = \Lambda. \tag{7-3}$$

Following the same computations as the previous theorem and (7-3), we have

$$8(2(n+1)\alpha\lambda_1 + nk_3)\lambda_1 - 6\Lambda + 2(n+1)(1-2\alpha)a\lambda_1 \int_M |\nabla_b\varphi|^2 u^{1/2}\, d\mu - \tfrac{1}{4}(1-2\alpha)(2(2n-1)\alpha+1)\int_M |\nabla_b\varphi|^4 u\, d\mu \geq 0.$$

Therefore by letting $\alpha \to (1/2)^-$, we get $8((n+1)\lambda_1 + nk_3)\lambda_1 - 6\Lambda \geq 0$. This implies

$$\lambda_1 \geq \frac{-nk_3 + \sqrt{n^2k_3^2 + 3(n+1)\Lambda}}{2(n+1)}. \qquad \square$$

However we have the decomposition [Chang et al. 2007, Section 5]

$$\ker(\Delta_b + \lambda_1 I) = E \oplus_{P_0} E^{\perp}, \quad \text{where } E \subset \ker P_0 \text{ and } E^{\perp} \subset (\ker P_0)^{\perp}.$$

Now let f be an eigenfunction of Δ_b corresponding to the first positive eigenvalue λ_1. Then $f = f^{\perp} \oplus f_{\ker}$.

Corollary 7.2. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold with an essentially positive CR Paneitz operator P_0. Suppose that*

$$(\mathrm{Ric} - ((n+1)/2)\,\mathrm{Tor})(Z, Z) \geq -k_3|Z|^2 \quad \textit{for all } Z \in T_{1,0}$$

with k_3 a nonnegative constant. Then if $\ker(\Delta_b + \lambda_1 I) \cap (\ker P_0) = \phi$, there exists a constant $0 < C_2 \leq 1$ such that

$$\lambda_1 \geq \frac{-nk_3 + \sqrt{n^2 k_3^2 + 3(n+1) C_2 \Lambda}}{2(n+1)}.$$

Proof. By assumption we have

$$0 < C_2 \leq \int_M (f^{\perp})^2 \, d\mu = 1 - \int_M (f_{\ker})^2 d\mu \leq 1.$$

Since P_0 is self adjoint, we may replace (7-3) by

$$\int_M P_0 u^{1/2} \cdot u^{1/2} \, d\mu = \int_M P_0 f \cdot f \, d\mu = \int_M P_0 f^{\perp} \cdot f^{\perp} \, d\mu \geq \Lambda \int_M (f^{\perp})^2 \, d\mu \geq C_2 \Lambda.$$

Then we are done. □

On the other hand, if $\ker(\Delta_b + \lambda_1 I) \cap (\ker P_0) \neq \phi$, then by using the Li–Yau gradient estimates [1986], we find from [Chang and Chiu 2007] this:

Proposition 7.3. *Let (M, J, θ) be a closed pseudohermitian $(2n+1)$-manifold. Suppose that*

$$(\mathrm{Ric} - ((n+1)/2)\,\mathrm{Tor})(Z, Z) \geq -k_3|Z|^2$$

for some nonnegative constant k_3. Suppose also that $\ker\,(\Delta_b + \lambda_1 I) \cap (\ker P_0) \neq \phi$. Then

$$\lambda_1 \geq C_3(n, k_3, \tau_0, d_M).$$

Here $\tau_0 = \max |A_{\alpha\gamma}|$ and d_M is the diameter of M with respect to the Carnot–Carathéodory distance.

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