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SUSANNE PUMPLÜN



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# WITT GROUPS OF HERMITIAN FORMS OVER A BRAUER–SEVERI VARIETY

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**Let $X$ be a Brauer–Severi variety over a field $k$ of characteristic not 2, and let $D$ be a division algebra over $k$ with a $k$-linear involution. We investigate Witt groups of certain hermitian forms over $D \otimes_k \mathbb{O}_X$.**

## Introduction

Let $k$ be a field. For any Brauer–Severi variety over $k$ with structure morphism $\tau : X \to \operatorname{Spec}(k)$, the base change morphism $\tau^* : W(k) \to W(X)$ between the Witt rings of $k$ and of $X$ was shown to be surjective in [Pumplün 1998b; 2000], provided that char $k \neq 2$. The Witt groups of symmetric bilinear forms over $X$ with values in a line bundle that generates Pic $X$ were calculated in [Pumplün 1999]. In this paper, we see that the method involved in both proofs, that is, the killing of certain cohomology groups, carries over to the setting of hermitian forms over finite separable field extensions of $k$ with a $k$-linear involution. Moreover, the method employed in [Pumplün 1998a] to prove that $\tau^* : W(k) \to W(X)$ is an isomorphism, if $X$ is the Brauer–Severi variety associated to a central simple algebra of odd index, generalizes to Witt groups of $\varepsilon$-hermitian forms.

The content of the paper is as follows. Let $A$ be an algebra over $k$ together with a $k$-linear involution $\sigma$. After the preliminaries in Section 1, Section 2 deals with in certain special cases the injectivity and surjectivity of the group homomorphism $U_\tau : W^\varepsilon(A) \to W^\varepsilon(A \otimes_k \mathbb{O}_X)$. The extension theorem in Section 3 generalizes [Arason 1980, Erster Schritt] to hermitian space. Section 4 proves Theorem 8, which generalizes Horrocks's theorem [Barth and Hulek 1978]. Together with the results on extension groups in Section 5, the extension theorem is used to prove that for a separable field extension $l/k$ with a $k$-linear involution $\sigma$ with char $k \neq 2$,

$$U_\tau : W^1(l) \to W^1(l \otimes_k \mathbb{O}_X)$$

is surjective. This result can be found in Section 6. We finish in Section 7 with a brief look at the case that $X = \mathbb{P}^1_k$, where $k$ is a field of characteristic not 2 and

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$D$ is a division algebra over $k$ with a $k$-linear involution $\sigma$. Then $U_\tau : W^\varepsilon(D) \to W^\varepsilon(D \otimes \mathcal{O}_X)$ is bijective for $\varepsilon = \pm 1$. A strategy for a possible proof of the same result for $X = \mathbb{P}^n_k$ is discussed in Section 7.2.

For the basic terminology and results on extension groups, the reader is referred to [Hartshorne 1977] and [Hilton and Stammbach 1971].

## 1. Basic terminology

**1.1.** Let $X$ be a scheme. By an $\mathcal{O}_X$-algebra, we will always mean an associative $\mathcal{O}_X$-algebra that is unital and locally free of finite constant rank as $\mathcal{O}_X$-module. Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra with an $\mathcal{O}_X$-linear involution $\sigma$. Let $\varepsilon \in H^0(X, \mathcal{A})$ be an element of the center of $\mathcal{A}$ such that $\varepsilon\sigma(\varepsilon) = 1$. Let $\mathcal{M}$ be a vector bundle over $X$ that is locally free of finite rank as a right $\mathcal{A}$-module. Put $\mathcal{M}^* = \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ for the dual sheaf considered as a right $\mathcal{A}$-module $ma = \sigma(a)m$ through the involution $\sigma$ for all $a$ in $\mathcal{A}$, $m$ in $\mathcal{M}$. Then $*$ is an exact contravariant duality functor; see [Knus 1991, page 75]. We canonically identify $\mathcal{M}$ and $\mathcal{M}^{**}$.

A isomorphism $h : \mathcal{M} \to \mathcal{M}^*$ is called a (nondegenerate) *$\varepsilon$-hermitian form* if $h = \varepsilon h^*$, and $(\mathcal{M}, h)$ is called an *$\varepsilon$-hermitian space over $\mathcal{A}$*. Two $\varepsilon$-hermitian spaces $(\mathcal{M}, h)$ and $(\mathcal{M}', h')$ over $X$ are *isometric*, written as $(\mathcal{M}, h) \cong (\mathcal{M}', h')$, if there is an $\mathcal{O}_X$-linear isomorphism $f : \mathcal{M} \to \mathcal{M}'$ such that $f^*hf = h'$. The *orthogonal sum* $(\mathcal{M}_1, h_1) \perp (\mathcal{M}_2, h_2)$ of two $\varepsilon$-hermitian spaces $(\mathcal{M}_1, h_1)$ and $(\mathcal{M}_2, h_2)$ is defined as the $\varepsilon$-hermitian space

$$\left(\mathcal{M}_1 \oplus \mathcal{M}_2, \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}\right),$$

with the element

$$\left[\begin{smallmatrix} h_1 & 0 \\ 0 & h_2 \end{smallmatrix}\right] \in \mathcal{H}om(M_1 \oplus M_2, M_1^* \oplus M_2^*)$$

denoted by $h_1 \perp h_2$. Given an $\varepsilon$-hermitian space $(\mathcal{M}, h)$ and a right $\mathcal{A}$-submodule $\mathcal{N} \subset \mathcal{M}$, always assumed to be locally a direct summand of $\mathcal{M}$ that is locally free of finite rank as a right $\mathcal{A}$-module, with inclusion $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$,

$$\mathcal{A}^\perp = \ker(\mathcal{M} \xrightarrow{h} \mathcal{M}^* \xrightarrow{\iota^*} \mathcal{N}^*)$$

is a right $\mathcal{A}$-submodule of $\mathcal{M}$, the *orthogonal complement* of $\mathcal{N}$ in $(\mathcal{M}, h)$. An $\varepsilon$-hermitian space $(\mathcal{M}, h)$ is called *metabolic* if $\mathcal{M}$ contains a subbundle $\mathcal{N}$ that is locally free of finite rank as a right $\mathcal{A}$-module such that $\mathcal{N} = \mathcal{N}^\perp$, making the short exact sequence

$$0 \to \mathcal{N} \xrightarrow{\iota} \mathcal{M} \xrightarrow{\iota^* h} \mathcal{N}^* \to 0$$

exact. Given a locally free right $\mathcal{A}$-module of finite rank $\mathcal{P}$,

$$H^\varepsilon(\mathcal{P}) = \left(\mathcal{P} \oplus \mathcal{P}^*, \begin{bmatrix} 0 & 1 \\ \varepsilon & 0 \end{bmatrix}\right)$$

is a metabolic space, the *hyperbolic space* of $\mathcal{P}$. An $\varepsilon$-hermitian space $(\mathcal{M}, h)$ is *hyperbolic* if $(\mathcal{M}, h) \cong H^\varepsilon(\mathcal{P})$ for a suitable $\mathcal{P}$. Two $\varepsilon$-hermitian spaces $(\mathcal{M}, h)$ and $(\mathcal{M}', h')$ over $X$ are *Witt equivalent*, written as $(\mathcal{M}, h) \sim (\mathcal{M}', h')$, if there exist metabolic $\varepsilon$-hermitian spaces $(\mathcal{M}_1, h_1)$ and $(\mathcal{M}_2, h_2)$ such that

$$(\mathcal{M}, h) \perp (\mathcal{M}_1, h_1) \cong (\mathcal{M}', h') \perp (\mathcal{M}_2, h_2).$$

Witt equivalence is an equivalence relation and the set of equivalence classes

$$W^\varepsilon(\mathcal{A}) = \{\, [(\mathcal{M}, h)] \mid (\mathcal{M}, h) \text{ is an } \varepsilon\text{-hermitian space} \,\},$$

together with the addition canonically induced by the orthogonal sum, is a group, the *Witt group of $\varepsilon$-hermitian spaces*.

**1.2.** Let $Y$ be a scheme and $\tau : Y \to X$ a morphism of schemes. For a vector bundle $\mathcal{F}$ over $X$, $\tau^*\mathcal{F} \cong \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is a vector bundle over $Y$, and $\tau^*\mathcal{A} \cong \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is an algebra over $Y$ with involution $\sigma \otimes 1$; for every locally free right $\mathcal{A}$-module $\mathcal{M}$ of finite rank, $\tau^*\mathcal{M} \cong \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is a locally free right $\tau^*\mathcal{A}$-module of finite rank. Given an $\varepsilon$-hermitian space $(\mathcal{M}, h)$ over $\mathcal{A}$, $\tau^*(\mathcal{M}, h) \cong (\tau^*\mathcal{M}, \tau^*h)$ is a $\tau^*\varepsilon$-hermitian space over $\tau^*\mathcal{A}$. The morphism $\tau$ induces a group homomorphism

$$U_\tau : W^\varepsilon(\mathcal{A}) \to W^{\varepsilon'}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_Y), \quad (\mathcal{M}, h) \mapsto (\mathcal{M}, h) \otimes_{\mathcal{A}} (\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$$

where $\varepsilon' = \tau^*\varepsilon$. If $\pi : Z \to Y$ is another morphism of schemes, then $U_{\tau\circ\pi} = U_\pi \circ U_\tau$.

**1.3.** ***Affine schemes.*** Let $X = \operatorname{Spec} R$ be an affine scheme. Under the usual categorical equivalence, vector bundles over $X$ can be identified with finitely generated projective $R$-modules. For an algebra $A$ over $R$ with an $R$-linear involution $\sigma$, with $A$ always assumed to be finitely generated projective of constant rank as an $R$-module, $W^\varepsilon(A)$ canonically identifies with $W^\varepsilon(\tilde{A})$, the Witt group of $\varepsilon$-hermitian forms over the $\mathcal{O}_X$-algebra $\tilde{A}$, the sheaf of $\mathcal{O}_X$-algebras associated to $A$. Under this identification, the base change homomorphism $W^\varepsilon(\tilde{A}) \to W^\varepsilon(\tilde{A} \otimes_{\mathcal{O}_X} \mathcal{O}_Y)$, for a morphism $Y = \operatorname{Spec} R' \to X = \operatorname{Spec} R$, corresponds to the base change $W^\varepsilon(A) \to W^\varepsilon(A \otimes_R R')$ from $R$ to the $R$-algebra $R'$.

**1.4.** ***Brauer–Severi varieties.*** Let $k$ be a field. If $B$ is a central simple algebra over $k$ of $\dim_k B = n^2$, then $B \cong \operatorname{Mat}_s(D)$ for a central division algebra $D$ over $k$. Let $r = \exp A$ be the order of $B$ in the Brauer group $\operatorname{Br} k$. Let $k'/k$ be a finite separable field extension that is a maximal subfield of $D$, so $[k' : k] = d$. Let $X$ be the Brauer–Severi variety associated with $B$ and let $X' = X \times_k k'$. Then $X' \cong \mathbb{P}^{n-1}_{k'}$. We know that $\operatorname{Pic} X \cong \mathbb{Z}$ and that there is an element $\mathcal{L}$ generating $\operatorname{Pic} X$ with $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \cong \mathcal{O}_{X'}(r)$. $X \cong \mathbb{P}^{n-1}_k$ if and only if $r = 1$, if and only if $X$ has a rational point [Artin 1982]. In that case $\mathcal{L} = \mathcal{O}_X(1)$. We define $\mathcal{L}(0) = \mathcal{O}_X$,

$\mathcal{L}(m) = \mathcal{L} \otimes \cdots \otimes \mathcal{L}$ ($m$-times) for $m > 0$ and $\mathcal{L}(m) = \mathcal{L}^\vee \otimes \cdots \otimes \mathcal{L}^\vee$ ($(-m)$-times) for $m < 0$, where $m \in \mathbb{Z}$.

**1.5.** ***Facts on vector bundles over proper schemes.*** Let $X$ be a proper scheme over $k$, and let $l/k$ be an algebraic field extension. The theorem of Krull and Schmidt holds for vector bundles over $X$, that is, every vector bundle on $X$ can be decomposed as a direct sum of indecomposable vector bundles, which is unique up to isomorphism and order of summands [Arason et al. 1992, p. 1324]. Moreover, nonisomorphic vector bundles on $X$ extend to nonisomorphic vector bundles on $X_l = X \times_k l$ for every separable algebraic field extension $l/k$ [ibid., p. 1325].

Let $l/k$ be a separable finite field extension of degree $s = [l : k]$. If $\mathcal{N}$ is a vector bundle on $X_l$, the direct image $\pi_* \mathcal{N}$ of $\mathcal{N}$ under the projection morphism $\pi : X_l \to X$ is a vector bundle on $X$ denoted by $\mathrm{tr}_{l/k}(\mathcal{N})$ [ibid., pp. 1362 and 1329].

The canonical projection $\pi : X_l \to X$ is an affine flat morphism [ibid., p. 1329], and the direct image $\mathcal{B} = \pi_* \mathcal{O}_{X_l}$ is an $\mathcal{O}_X$-algebra that is locally free of rank $s$ as an $\mathcal{O}_X$-module, that is,

$$\mathrm{tr}_{l/k}(\mathcal{O}_{X_l}) = \pi_* \mathcal{O}_{X_l} \cong \mathcal{O}_X^s.$$

The assignment $\mathcal{F} \to \pi_* \mathcal{F}$ gives an equivalence of categories from quasicoherent $\mathcal{O}_{X_l}$-modules to quasicoherent $\mathcal{O}_X$-modules that are $\mathcal{B}$-modules at the same time [Hartshorne 1977, p. 145, Example 5.17]. This equivalence matches locally free $\mathcal{O}_{X_l}$-modules of finite rank with locally free $\mathcal{B}$-modules of finite rank and, in particular, $\mathrm{Pic}(X_l)$ with $\mathrm{Pic}(\mathcal{B})$.

## 2. Certain special cases

**2.1.** ***On the injectivity of $U_\tau$.*** Let $A$ be an algebra over $k$ together with a $k$-linear involution $\sigma$. Let $n \geq 2$.

**Theorem 1.** *Let $X$ be a $k$-scheme with a rational point. Then*

$$U_\tau : W^\varepsilon(A) \to W^\varepsilon(A \otimes_k \mathcal{O}_X) \quad \textit{is injective.}$$

*Proof.* Pick a $k$-rational point in $X$, that is, a $k$-morphism $\delta : \operatorname{Spec} a, k \to X$. Then $\tau\delta = \mathrm{id}$ on $\operatorname{Spec} k$; hence $U_\delta U_\tau = \mathrm{id}$ on $W^\varepsilon(A \otimes_k \mathcal{O}_X)$, implying that $U_\tau$ is injective. □

A similar trick as used in [Pumplün 1998a] gives us the next result:

**Theorem 2.** *Let $X$ be a Brauer–Severi variety associated to a central simple algebra of odd index. Then*

$$U_\tau : W^\varepsilon(A) \to W^\varepsilon(A \otimes_k \mathcal{O}_X) \quad \textit{is injective.}$$

*Proof.* Let $B \cong \mathrm{Mat}_s(D)$ be the central simple algebra associated to $X$ and let $k'/k$ be a finite separable field extension that is a maximal subfield of the division algebra $D$, which is hence of odd degree. Define $X' = X \times_k k'$. Let $(M_1, h_1)$ and $(M_2, h_2)$ be two $\varepsilon$-hermitian spaces over $A$ such that

$$(M_1, h_1) \otimes_A (A \otimes_k \mathcal{O}_X) \sim (M_2, h_2) \otimes_A (A \otimes_k \mathcal{O}_X).$$

Then the same equivalence holds with $X$ replaced by $X'$, which implies

$$(M_1, h_1) \otimes_A (A \otimes_k k') \sim (M_2, h_2) \otimes_A (A \otimes_k k')$$

by Theorem 1. The assertion now follows from [Knus 1991, (10.3.1), p. 62]. □

**Theorem 3.** *Let $X$ be a Brauer–Severi variety associated to a central simple algebra of odd index. Let $A$ be a division algebra over $k$ and suppose* $\operatorname{char} k \neq 2$. *Let $(M_1, h_1)$ and $(M_2, h_2)$ be two $\varepsilon$-hermitian spaces over $A$ that become isometric over $A \otimes_k \mathcal{O}_X$. Then $(M_1, h_1) \cong (M_2, h_2)$.*

*Proof.* Since

$$(M_1, h_1) \otimes_A (A \otimes_k \mathcal{O}_X) \cong (M_2, h_2) \otimes_A (A \otimes_k \mathcal{O}_X)$$

we have $(M_1, h_1) \sim (M_2, h_2)$ by Theorem 1. By [Knus 1991, (10.3.3), p. 63], this yields $(M_1, h_1) \cong (M_2, h_2)$. □

**2.2.** ***On the surjectivity of $U_\tau$.*** Let $A$ be an algebra over $k$ (for example, quadratic étale or central simple) together with a $k$-linear involution $\sigma$. Let $X$ be a scheme over $k$ and let $k'/k$ be a separable odd degree field extension. Let $X' = X \times_k k'$ and $A' = A \otimes_k k'$. Observe that $A' \otimes_{k'} \mathcal{O}_{X'} \cong A \otimes_k \mathcal{O}_{X'}$.

**Theorem 4.** *If $U_\tau : W^\varepsilon(A') \to W^\varepsilon(A \otimes_k \mathcal{O}_{X'})$ is surjective, then so is*

$$U_\tau : W^\varepsilon(A) \to W^\varepsilon(A \otimes_k \mathcal{O}_X).$$

*Proof.* Let $\operatorname{tr}_{k'/k} : k' \to k$ be the trace of the extension $k'/k$. Its $A$-linear extension $\mathrm{id} \otimes \operatorname{tr}_{k'/k} : A \otimes_k k' \to A$ is an involution trace form in the sense of [Knus 1991, (7.3.2), p. 41]. Both maps induce group homomorphisms $\operatorname{tr}_{k'/k} : W(k') \to W(k)$ and $\operatorname{tr}_{k'/k} : W(X') \to W(X)$, and $T : W^\varepsilon(A \otimes_k k') \to W^\varepsilon(A)$ and $T : W^\varepsilon(A \otimes_k \mathcal{O}_{X'}) \to W^\varepsilon(A \otimes_k \mathcal{O}_X)$. As in [Knus 1991, p. 62], we can show that

$$T(U_\tau(\mathcal{M}, h) \otimes (\mathcal{F}, \gamma)) \sim (\mathcal{M}, h) \otimes \operatorname{tr}_{k'/k}(\mathcal{F}, \gamma)$$

or, equivalently,

$$T(((\mathcal{M}, h) \otimes_{\mathcal{A}} (\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})) \otimes (\mathcal{F}, \gamma)) \sim (\mathcal{M}, h) \otimes \operatorname{tr}_{k'/k}(\mathcal{F}, \gamma)$$

for all $\varepsilon$-hermitian spaces $(\mathcal{M}, h)$ over $\mathcal{A} = A \otimes_k \mathcal{O}_X$ and symmetric bilinear spaces $(\mathcal{F}, \gamma)$ over $X'$. Analogously,

$$T(((M, h) \otimes_A (A \otimes_k k')) \otimes (F, \gamma)) \sim (M, h) \otimes \operatorname{tr}_{k'/k}(F, \gamma)$$

for all $\varepsilon$-hermitian spaces $(M, h)$ over $A$ and nonsingular symmetric bilinear spaces $(F, \gamma)$ over $k'$. Since $[k' : k]$ is odd, we get

$$\operatorname{tr}_{k'/k}(\langle 1\rangle_{\mathbb{O}_{X'}}) \sim \langle 1\rangle_{\mathbb{O}_X},$$
$$T((\mathcal{M}, h) \otimes_{\mathcal{A}} (\mathcal{A} \otimes_{\mathbb{O}_X} \mathbb{O}_{X'})) \sim (\mathcal{M}, h) \quad \text{and} \quad T((M, h) \otimes_A (A \otimes_k k')) \sim (M, h)$$

as in [Knus 1991, p. 62]. For an $\varepsilon$-hermitian space $(\mathcal{M}, h)$ over $\mathcal{A}$ it follows that

$$\begin{aligned}(\mathcal{M}, h) &\cong (\mathcal{M}, h) \otimes \langle 1\rangle_{\mathbb{O}_X}\\ &\sim (\mathcal{M}, h) \otimes \operatorname{tr}_{k'/k}(\langle 1\rangle_{\mathbb{O}_{X'}})\\ &\sim T(((\mathcal{M}, h) \otimes_{\mathcal{A}} (\mathcal{A} \otimes_{\mathbb{O}_X} \mathbb{O}_{X'})) \otimes \langle 1\rangle_{\mathbb{O}_{X'}})\\ &\sim T((M', h') \otimes_{A'} (A' \otimes_{k'} \mathbb{O}_{X'})) \sim T(M', h') \otimes_A (A \otimes \mathbb{O}_X)\end{aligned}$$

for a suitable hermitian space $(M', h')$ over $A'$, where the second to last equivalence holds by the assumption that $U_\tau : W^\varepsilon(A') \to W^\varepsilon(A \otimes_k \mathbb{O}_{X'})$ is surjective. $\square$

**Corollary 5.** *Let $X$ be a Brauer–Severi variety of odd index. Let* $\operatorname{Mat}_s(D)$ *be the central simple algebra associated to $X$. If $k'/k$ is a finite separable field extension that is a maximal subfield of $D$ and such that $U_\tau : W^\varepsilon(A') \to W^\varepsilon(A \otimes_k \mathbb{O}_{X'})$ is surjective* ($X' \cong \mathbb{P}_k^{n-1}$)*, then $U_\tau : W^\varepsilon(A) \to W^\varepsilon(A \otimes_k \mathbb{O}_X)$ is surjective.*

## 3. Extension theorem for hermitian spaces

Let $X$ be a scheme such that $2 \in H^0(X, \mathbb{O}_X^\times)$, and let $\mathcal{A}$ be an algebra over $X$ with an $\mathbb{O}_X$-linear involution $\sigma$. An $\varepsilon$-hermitian space $(\mathcal{M}, h)$ with $\varepsilon = 1$ is called a *hermitian space*. For a hermitian space $(\mathcal{M}, h)$, a subbundle $\mathcal{N} \subset \mathcal{M}$ is called *totally isotropic* if $\mathcal{N} \subset \mathcal{N}^\perp$. For a totally isotropic subbundle $\mathcal{N} \subset \mathcal{M}$, we obtain an induced hermitian space $(\overline{\mathcal{M}}, \bar{h})$ by setting $\overline{\mathcal{M}} = \mathcal{N}^\perp/\mathcal{N}$ and writing $\iota : \mathcal{N}^\perp \hookrightarrow \mathcal{M}$ for the inclusion, and $\pi : \mathcal{N}^\perp \to \mathcal{M}$ for the projection. Then $\bar{h}$ is uniquely determined by $\iota^* \circ h \circ \iota = \pi^* \circ \bar{h} \circ \pi$. We get a short exact sequence

$$0 \to \mathcal{N}^\perp \xrightarrow{\kappa} \overline{\mathcal{M}} \oplus \mathcal{M} \xrightarrow{(\kappa^*, \bar{h} \oplus -h)} \mathcal{N}^{\perp *} \to 0$$

with $\kappa = (\pi, \mathrm{id})$ implying that $(\overline{\mathcal{M}}, \bar{h}) \perp (\mathcal{M}, -h)$ is metabolic. Since $(\mathcal{M}, h) \perp (\mathcal{M}, -h)$ is metabolic as well, $(\mathcal{M}, h)$ and $(\overline{\mathcal{M}}, \bar{h})$ are Witt equivalent. We get a short exact sequence of locally free right $\mathcal{A}$-modules of constant finite rank

$$0 \to \mathcal{N} \xrightarrow{\iota} \mathcal{N}^\perp \xrightarrow{\pi} \overline{\mathcal{M}} \to 0.$$

Analogously to what was observed in [Pumplün 1998b, Section 4], we can reverse this construction as follows:

For a locally free right $\mathcal{A}$-module $\mathcal{M}$ of constant finite rank, let $\{\operatorname{Ext}^i(\mathcal{M}, \cdot)\}$ be the right derived functor of the group of $\mathcal{A}$-module homomorphisms $\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \cdot)$, which is a universal contravariant $\delta$-functor from locally free right $\mathcal{A}$-modules of constant finite rank to abelian groups.

**Theorem 6.** *Let $(\mathcal{G}, b)$ be a hermitian space over $\mathcal{A}$ and let*

$$0 \to \mathcal{N} \xrightarrow{\iota} \mathcal{B} \xrightarrow{\pi} \mathcal{G} \to 0 \tag{1}$$

*be a short exact sequence of locally free right $\mathcal{A}$-modules of constant finite rank. Suppose that*

$$\mathrm{Ext}^1(\mathcal{N}^*, \mathcal{N}) = \mathrm{Ext}^2(\mathcal{N}^*, \mathcal{N}) = 0.$$

*Then there exists a hermitian space $(\mathcal{M}, h)$ and identifications of $\mathcal{N}$ and $\mathcal{B}$ in $\mathcal{M}$ such that $\mathcal{B} = \mathcal{N}^\perp$ in $(\mathcal{M}, h)$ and $(\mathcal{G}, b) \cong (\overline{M}, \bar{h})$. In particular, $(\mathcal{G}, b)$ and $(\mathcal{M}, h)$ are Witt equivalent.*

Moreover, for $(\mathcal{M}, h)$ as in Theorem 6, we have $\mathcal{B}^\perp = \mathcal{N}$; hence the sequence

$$0 \to \mathcal{N} \to \mathcal{M} \xrightarrow{\iota^* h} \mathcal{B}^* \to 0 \tag{2}$$

is exact, with $\iota$ being the inclusion $\mathcal{B} \hookrightarrow \mathcal{M}$.

For the proof of this result, we need the following elementary results.

**Lemma 7.** (a) *Let $(P)$ and $(Q)$ be two extensions of locally free right $\mathcal{A}$-modules of constant finite rank such that*

$$\begin{tikzcd}
(P) & 0 \arrow[r] & \mathcal{M}' \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathcal{M} \arrow[r] \arrow[d, "\alpha"'] & \mathcal{M}'' \arrow[r] \arrow[d, "\alpha''"'] & 0 \\
(Q) & 0 \arrow[r] & \mathcal{M}' \arrow[r, "p^*\iota"] & \mathcal{M}_1\kappa^* \arrow[r] & \mathcal{M}_2'' \arrow[r] & 0
\end{tikzcd}$$

*with $\alpha'' : \mathcal{M}'' \to \mathcal{M}_1''$ an $\mathcal{A}$-linear map. If $\xi \in \mathrm{Ext}^1(\mathcal{M}'', \mathcal{M}')$ corresponds to the extension $(P)$, and if $\xi_1 \in \mathrm{Ext}^1(\mathcal{M}_1'', \mathcal{M}')$ corresponds to the extension $(Q)$, then the following statements are equivalent*:

(i) *There exists an $\mathcal{A}$-linear map $\alpha : \mathcal{M} \to \mathcal{M}_1$ that makes the diagram above commutative.*

(ii) $\mathrm{Ext}^1(\alpha'', \mathcal{M}')\xi_1 = \xi$.

(b) *Let*

$$\begin{tikzcd}
(P) & 0 \arrow[r] & \mathcal{M}' \arrow[r] \arrow[d] & \mathcal{M} \arrow[r, "\pi"] \arrow[d, "\alpha"'] & \mathcal{M}'' \arrow[r] \arrow[d] & 0 \\
(Q) & 0 \arrow[r, "\iota_1"] & \mathcal{M}' \arrow[r, "p^*\iota"] & \mathcal{M}_1 \arrow[r, "\kappa^*"] & \mathcal{M}_2'' \arrow[r] & 0
\end{tikzcd}$$

*be a commutative diagram of locally free right $\mathcal{A}$-modules of constant finite rank with exact rows. Then an $\mathcal{A}$-linear map $\beta : \mathcal{M} \to \mathcal{M}_1$ makes the diagram commutative as well if and only if there exists an $\mathcal{A}$-linear map $\gamma : \mathcal{M}'' \to \mathcal{M}_1$ such that $\beta = \alpha + \iota_1 \gamma \pi$. In this case $\gamma$ is unique.*

The proof of Theorem 6 is now analogous to the proof in [Arason 1980, Erster Schritt]:

*Proof.* We dualize (1) and replace $\mathscr{G}^*$ by $\mathscr{G}$ via $b$. This yields the short exact sequence

$$0 \to \mathscr{G} \xrightarrow{\pi^* b} \mathscr{B}^* \xrightarrow{\iota^*} \mathscr{N}^* \to 0. \tag{3}$$

By applying $\{\mathrm{Ext}^i(\,\cdot, \mathscr{N})\}$ to (3) we obtain a long exact sequence. In particular,

$$0 = \mathrm{Ext}^1(\mathscr{N}^*, \mathscr{N}) \to \mathrm{Ext}^1(\mathscr{B}^*, \mathscr{N}) \xrightarrow{\mathrm{Ext}^1(\pi^* b, \mathscr{N})} \mathrm{Ext}^1(\mathscr{G}, \mathscr{N}) \to \mathrm{Ext}^2(\mathscr{N}^*, \mathscr{N}) = 0. \tag{4}$$

Therefore the middle map is an isomorphism. Let $\xi \in \mathrm{Ext}^1(\mathscr{G}, \mathscr{N})$ correspond to the isomorphism class of extension (1). Then we thus find a unique $\xi_1 \in \mathrm{Ext}^1(\mathscr{B}^*, \mathscr{N})$ such that $\mathrm{Ext}^1(\pi^* b, \mathscr{N})(\xi_1) = \xi$. This yields an extension of locally free right $\mathscr{A}$-modules of constant finite rank

$$0 \to \mathscr{N} \to \mathscr{M} \xrightarrow{p} \mathscr{B}^* \to 0 \tag{5}$$

over $X$; see (2). Using (1) and (5) we obtain the commutative diagram of Diagram 1 (see Lemma 7(a)): Diagram chasing confirms that the middle column of this is also exact. Using that $b = b^*$, we dualize and obtain Diagram 2. By replacing $\mathscr{G}^*$ with $\mathscr{G}$ via $b$, we replace $b\pi$ by $\pi$ and $\pi^*$ by $\pi^* b$. We obtain Diagram 3. Let $\xi_1^* \in \mathrm{Ext}^1(\mathscr{B}^*, \mathscr{N})$ correspond to the extension

$$0 \to \mathscr{N} \xrightarrow{p^* \iota} \mathscr{M}^* \xrightarrow{\kappa^*} \mathscr{B}^* \to . \tag{6}$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{N} \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"'] & \mathscr{B} \arrow[r, "\pi"] \arrow[d, "\kappa"'] & \mathscr{G} \arrow[r] \arrow[d, "\pi^* b"'] & 0 \\
0 \arrow[r] & \mathscr{N} \arrow[r] & \mathscr{M} \arrow[r, "p"] \arrow[d, "\iota^* p"'] & \mathscr{B}^* \arrow[r] \arrow[d, "\iota^*"'] & 0 \\
 & & \mathscr{N}^* \arrow[r, "\mathrm{id}"] \arrow[d] & \mathscr{N}^* \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

**Diagram 1**

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathcal{G}^* \arrow[u] \arrow[r, "\pi^*"] & \mathcal{B}^* \arrow[u] \arrow[r, "\iota^*"] & \mathcal{N}^* \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{B} \arrow[u, "b\pi"] \arrow[r, "p^*"] & \mathcal{M}^* \arrow[u, "\kappa^*"] \arrow[r] & \mathcal{N}^* \arrow[u, "\mathrm{id}"] \arrow[r] & 0 \\
 & \mathcal{N} \arrow[u, "\iota"] \arrow[r, "\mathrm{id}"] & \mathcal{N} \arrow[u, "p^*\iota"] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

**Diagram 2**

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{N} \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"] & \mathcal{B} \arrow[r, "\pi"] \arrow[d, "p^*"] & \mathcal{G} \arrow[r] \arrow[d, "\pi^* b"] & 0 \\
0 \arrow[r] & \mathcal{N} \arrow[r, "p^*\iota"] & \mathcal{M}^* \arrow[r, "\kappa^*"] \arrow[d, "\iota^* p"] & \mathcal{B}^* \arrow[r] \arrow[d, "\iota^*"] & 0 \\
 & & \mathcal{N}^* \arrow[r, "\mathrm{id}"] \arrow[d] & \mathcal{N}^* \arrow[d] & \\
 & & 0 \arrow[r] & 0 &
\end{tikzcd}$$

**Diagram 3**

Then $\mathrm{Ext}^1(\pi^* b, \mathcal{N})\xi_1^* = \xi$ by Lemma 7(a); thus $\xi_1^* = \xi$ and the extensions (5) and (6) are isomorphic. (This step does not generalize to $\varepsilon$-hermitian forms with $\varepsilon \neq 1$.) Therefore there exists an $\mathcal{A}$-linear map $h : \mathcal{M} \to \mathcal{M}^*$ that makes the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N} \arrow[r, "\kappa\iota"] \arrow[d, "\mathrm{id}"] & \mathcal{M} \arrow[r, "p"] \arrow[d, "h"] & \mathcal{B}^* \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathcal{N} \arrow[r, "p^*\iota"] & \mathcal{M}^* \arrow[r, "\kappa^*"] & \mathcal{B}^* \arrow[r] & 0
\end{tikzcd} \tag{7}$$

$h$ is an isomorphism and is by Lemma 7(b) unique up to summands of the form $p^*\iota\beta p$ with $\beta \in \mathcal{H}om_{\mathcal{A}}(\mathcal{B}^*, \mathcal{N})$. The diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N} \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"'] & \mathcal{B} \arrow[r, "\pi"] \arrow[d, "{p^*, h\kappa}"'] & \mathcal{G} \arrow[r] \arrow[d, "\pi^* b"'] & 0 \\
0 \arrow[r] & \mathcal{N} \arrow[r, "p^*\iota"] & \mathcal{M}^* \arrow[r, "\kappa^*"] & \mathcal{B}^* \arrow[r] & 0
\end{tikzcd}$$

is made commutative by both maps written next to the arrow in the middle, since we have $h\kappa\iota = p^*\iota$ by (7) and $\kappa^* p^* = \pi^* b\pi = p\kappa = \kappa^* h\kappa$ by Diagrams 3 and 1 and Equation (7). Lemma 7(b) implies that there exists a $\gamma \in \mathrm{Ext}^1(\mathcal{G}, \mathcal{N})$ such that $h\kappa = p^* + p^*\iota\gamma\pi$. Since $\mathrm{Ext}^1(\mathcal{N}^*, \mathcal{N}) = 0$, (3) induces the exact sequence

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{N}^*, \mathcal{N}) \xrightarrow{\mathcal{H}om_{\mathcal{A}}(\iota^*, \mathcal{N})} \mathcal{H}om_{\mathcal{A}}(\mathcal{B}^*, \mathcal{N}) \xrightarrow{\mathcal{H}om_{\mathcal{A}}(\pi^* b, \mathcal{N})} \mathcal{H}om_{\mathcal{A}}(\mathcal{G}, \mathcal{N}) \to 0. \tag{8}$$

Therefore $\gamma = \beta\pi^* b$ for some $\beta \in \mathcal{H}om_{\mathcal{A}}(\mathcal{B}^*, \mathcal{N})$, which yields

$$h\kappa = p^* + p^*\iota\beta\pi^* b\pi = p^* + p^*\iota\beta p\kappa,$$

and so $(h - p^*\iota\beta p)\kappa = p^*$. Since $h$ is unique up to certain summands (see above), we may assume that

$$h\kappa = p^*. \tag{9}$$

Moreover, $h$ is uniquely determined by this equation together with (7), up to summands of the form $p^*\iota\beta p$ with $\beta \in \mathcal{H}om_{\mathcal{A}}(\mathcal{B}^*, \mathcal{N})$ such that $p^*\iota\beta p\kappa = 0$. We also have $p^*\iota\beta p\kappa = 0$ if and only if $p^*\iota\beta\pi^* b\pi = 0$ by Diagram 1, if and only if $\beta\pi^* b = 0$ ($p^*\iota$ is injective, and $\pi$ is surjective), if and only if $\beta = \alpha\iota^*$ by (8). Therefore $h$ is uniquely determined up to summands of the form $p^*\iota\alpha\iota^* p$ with $\alpha \in \mathcal{H}om_{\mathcal{A}}(\mathcal{N}^*, \mathcal{N})$.

Now $h^* : \mathcal{M} \to \mathcal{M}^*$ satisfies $h^*\kappa = (\kappa^* h)^* = p^*$ by (7), hence (9), and $h^*\kappa\iota = p^*\iota$ and $\kappa^*\mathcal{H}^* = (h\kappa)^* = p$ by (9), hence (7). Therefore the fact that $h$ is uniquely determined up to summands of the form $p^*\iota\alpha\iota^* p$ with $\alpha \in \mathcal{H}om_{\mathcal{A}}(\mathcal{N}^*, \mathcal{N})$ even yields a unique $\alpha \in \mathcal{H}om_{\mathcal{A}}(\mathcal{N}^*, \mathcal{N})$ satisfying $h^* = h + p^*\iota\alpha\iota^* p$, and dualizing implies that $\alpha = -\alpha^*$. Replacing $h$ by $h + \frac{1}{2} p^*\iota\alpha\iota^* p$ if necessary, we may assume in addition that $h = h^*$. We have thus obtained a hermitian space $(\mathcal{M}, h)$ containing $\mathcal{N}$ as a subbundle via $\kappa\iota$ by Diagram 1, such that

$$\mathcal{N}^\perp = \ker(\iota^*\kappa^* h) = \ker(\iota^*(h\kappa)^*) = \ker(\iota^* p) = \mathrm{im}(\kappa)$$

and $\mathcal{N} = \mathrm{im}(\kappa\iota) \subset \mathrm{im}(\kappa)$. We conclude that $\mathcal{N}$ is totally isotropic and that $\mathcal{B}$, viewed as a subbundle of $\mathcal{M}$ via $\kappa$, can be identified with $\mathcal{N}^\perp$. Under these identifications, the diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\kappa"] \arrow[d, "\pi"'] & \mathcal{M} \arrow[r, "h"] & \mathcal{M}^* \arrow[r, "\kappa^*"] & \mathcal{B}^* \arrow[d, "\pi^*"'] \\ \mathcal{G} \arrow[rrr, "b"] & & & \mathcal{G}^* \end{tikzcd}$$

corresponds to the equation displayed in the first paragraph of Section 3, and it commutes because of Diagram 3. Hence $(\mathcal{G}, b)$ and $(\overline{\mathcal{M}}, \bar{h})$ are isometric, as claimed. The last assertion now follows easily. □

Our assumption that $2 \in H^0(X, \mathcal{O}_X^{\times})$ is needed in the proof and cannot be omitted.

## 4. A generalization of Horrocks's theorem

Let $k$ be a field and $D$ be a division algebra over $k$. Let $X = \mathbb{P}_k^n$ and $\tau : \mathbb{P}_k^n \to \operatorname{Spec} k$ be the structure morphism. Let $\mathcal{D} = \tau^* D \cong D \otimes_k \mathcal{O}_X$.

Given a locally free right $\mathcal{D}$-module $\mathcal{E}$, let $\mathcal{E}(m) = \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{E}$ for any integer $m$. For a locally free right $\mathcal{D}$-module $\mathcal{E}$, define

$$\operatorname{Ext}^i(\mathcal{D}, \mathcal{E}(*)) = \bigoplus_{j \in \mathbb{Z}} \operatorname{Ext}^i(\mathcal{D}, \mathcal{E}(j)) \quad \text{for integers } j \geq 0.$$

We generalize Horrocks's theorem [Barth and Hulek 1978, Section 5, Lemma 1], an important ingredient in the proofs of [Arason 1980] and [Pumplün 1998b]:

**Theorem 8.** *A locally free right $\mathcal{D}$-module $\mathcal{E}$ satisfies*

$$\mathcal{E} \cong \mathcal{D}(m_1) \oplus \cdots \oplus \mathcal{D}(m_t)$$

*if and only if*

$$\operatorname{Ext}^i(\mathcal{D}, \mathcal{E}(*)) = 0 \quad \textit{for } i \in \mathbb{Z} \textit{ with } 0 < i < n. \tag{10}$$

*Proof.* By the cohomology of projective space, the condition (10) is necessary.

We prove that it is sufficient by induction on $n$. For $n = 1$, every locally free right $\mathcal{D}$-module $\mathcal{E}$ is of the form

$$\mathcal{E} \cong \mathcal{D}(m_1) \oplus \cdots \oplus \mathcal{D}(m_t)$$

[Knus 1991, page 407, VII (3.1.1)], so there is nothing to prove. So suppose $n > 1$ and assume that the assertion holds for $n-1$ in place of $n$. $Z = \mathbb{P}_k^{n-1}$ is a closed subscheme of $X$ with inclusion $i : Z \hookrightarrow X$. Via identification with the hyperplane $x_n = 0$, we obtain a short exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to i_* \mathcal{O}_Z \to 0.$$

Let $\mathscr{E}$ be a locally free right $\mathscr{D}$-module satisfying (10); then tensor the sequence above with $\mathscr{E}(j)$ to obtain

$$0 \to \mathscr{E}(j-1) \to \mathscr{E}(j) \to i_*[(\mathscr{E}|_Z)(j)] \to 0. \tag{11}$$

By (10) this yields

$$\operatorname{Ext}^i(\mathscr{D}, (\mathscr{E}|_Z)(\cdot\,)) = 0 \quad \text{for } 0 < i < n-1,$$

so, by the induction hypothesis, $\mathscr{E}|_Z$ is a direct sum of locally free right $\mathscr{D}|_Z$-modules of rank one of the kind $\mathscr{D}|_Z(m)$: There are integers $s_1, \dots, s_t$, a sum $\mathscr{F} = \mathscr{D}(s_1) \oplus \dots \mathscr{D}(s_t)$, and an isomorphism $\Psi : \mathscr{F}|_Z \to \mathscr{E}|_Z$ of locally free right $\mathscr{D}|_Z$-modules. Put $j = 0$ in (11); then

$$0 \to \mathscr{E}(-1) \to \mathscr{E} \to i_*(\mathscr{E}|_Z) \to 0.$$

is a short exact sequence of locally free right $\mathscr{D}$-modules, where $\mathscr{E} \to i_*(\mathscr{E}|_Z)$ is the canonical restriction map. Applying $\mathscr{H}om_{\mathscr{D}}(\mathscr{F}, \cdot\,)$ to this yields the exact sequence

$$\mathscr{H}om_{\mathscr{D}}(\mathscr{F}, \mathscr{E}) \to \mathscr{H}om_{\mathscr{D}}(\mathscr{F}, i_*(\mathscr{E}|_Z)) = \mathscr{H}om_{\mathscr{D}}(\mathscr{F}|_Z, \mathscr{E}|_Z) \to \operatorname{Ext}^1(\mathscr{F}, \mathscr{E}(-1))$$

and since we assume that $\mathscr{E}$ satisfies (10),

$$\operatorname{Ext}^1(\mathscr{F}, \mathscr{E}(-1)) \cong \bigoplus_{j=1}^{t} \operatorname{Ext}^1(\mathscr{D}(s_j), \mathscr{E}(-1)) \cong \bigoplus_{j=1}^{t} \operatorname{Ext}^1(\mathscr{D}, \mathscr{E}(-s_j - 1)) = 0.$$

Therefore the natural map $\mathscr{H}om_{\mathscr{D}}(\mathscr{F}, \mathscr{E}) \to \mathscr{H}om_{\mathscr{D}}(\mathscr{F}|_Z, \mathscr{E}|_Z)$ is surjective, so that $\Psi$ extends to a $\mathscr{D}$-linear homomorphism $\varphi : \mathscr{F} \to \mathscr{E}$. Now view $\varphi$ as an $\mathscr{O}_X$-linear map between vector bundles $\mathscr{F}$ and $\mathscr{E}$ over $X$. Then

$$\det\varphi \in \mathscr{H}om_{\mathscr{O}_X}(\det\mathscr{F}, \det\mathscr{E}) \cong H^0(X, (\det\mathscr{F})^\vee \otimes (\det\mathscr{E})) \cong H^0(X, \mathscr{O}_X(m))$$

for some integer $m$. Restricting this to $Z$ shows that $m = 0$ and thus $\det\varphi \in k^\times$. Hence $\varphi$ is an isomorphism. □

## 5. Killing extension groups for $X = \mathbb{P}_k^{n-1}$

The proof of surjectivity of the base change morphism $\tau^* : W(k) \to W(X)$ between the Witt rings of $k$ and a Brauer–Severi variety $X$ in [Arason 1980; Pumplün 1998b; Pumplün 2000] used the killing of cohomology groups. In our setup, this corresponds to the following observations we phrase in terms of extension groups. We phrase the proofs in a general setting in order to see if and where they could be used in a more general setup.

Let $k$ be a field of characteristic not 2, and let $D$ be a division algebra over $k$. Let $X = \mathbb{P}_k^{n-1}$, $\mathscr{D} = \tau^* D \cong D \otimes_k \mathscr{O}_X$ and $\mathscr{F}(m) = \mathscr{O}_X(m) \otimes \mathscr{F}$ for any integer $m$ and any locally free right $\mathscr{D}$-module $\mathscr{F}$.

Every right $D$-module $W$ may be viewed as a right module over Spec $D$, so for a right $\mathcal{D}$-module $\mathcal{F}$, the notation $\mathcal{F} \otimes_D W = \mathcal{F} \otimes_{\mathcal{O}_{\operatorname{Spec} D}} W$ used in the following makes sense and is a $\mathcal{D}$-module.

Let $\Omega = \Omega_{X/k}$ be the sheaf of relative differentials of $X$ over $k$ and $\Omega^l = \Lambda^l \Omega$ the sheaf of $l$-forms over $k$. Define $\Omega^l_D = \Omega^l \otimes_{\mathcal{O}_X} \mathcal{D}$.

**5.1.** Let $X$ be a scheme and $\mathcal{A}$ an algebra over $X$. Let $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ be an $\mathcal{A}$-linear map of right $\mathcal{A}$-modules. For any right $\mathcal{A}$-module $\mathcal{G}$ and $i \in \mathbb{N}_0$ we get an induced homomorphism

$$\operatorname{Ext}^i(\alpha, \mathcal{G}) : \operatorname{Ext}^i(\mathcal{F}_2, \mathcal{G}) \to \operatorname{Ext}^i(\mathcal{F}_1, \mathcal{G}),$$

and

$$\{\operatorname{Ext}^i(\alpha, \cdot)\} : \{\operatorname{Ext}^i(\mathcal{F}_2, \cdot)\} \to \{\operatorname{Ext}^i(\mathcal{F}_1, \cdot)\}$$

is a homomorphism of $\delta$-functors.

**Lemma 9.** *Assume that $D$ is a field extension of $k$. Let $l \in \mathbb{Z}$ with $0 \le l < n-2$ and $\mathcal{F}$ a locally free $\mathcal{D}$-module. Then there exists a finite-dimensional $D$-vector space $W$ as well as an extension*

$$0 \to \mathcal{F} \to \mathcal{P} \to (\Omega^r \otimes_k D) \otimes_D W \to 0.$$

*of locally free $\mathcal{D}$-modules such that the connecting homomorphism*

$$\delta : \mathcal{H}om_{\mathcal{D}}(\Omega^l_D, \Omega^l_D \otimes_D W) \to \operatorname{Ext}^1(\Omega^l_D, \mathcal{F})$$

*is an isomorphism.*

*Proof.* Let $W$ be an arbitrary free $D$-vector space of finite dimension. Multiplication by $x \in W$ yields a $\mathcal{D}$-linear map

$$\tau_x : \Omega^l_D \to \Omega^l_D \otimes_D W, \quad s \to s \otimes x.$$

For a locally free $\mathcal{D}$-module $\mathcal{F}$, the map

$$\theta = \theta_{\mathcal{F}} : \mathcal{H}om_{\mathcal{D}}(\Omega^l_D \otimes_D W, \mathcal{F}) \to \mathcal{H}om_D(W, \mathcal{H}om(\Omega^l_D, \mathcal{F}))$$

defined by $[\theta(\varphi)](x) = \varphi \circ \tau_x$ for $\varphi \in \mathcal{H}om_{\mathcal{D}}(\Omega^l_D \otimes_D W, \mathcal{F})$ and $x \in W$ is an isomorphism with inverse satisfying

$$[\theta^{-1}(\Psi)](s \otimes x) = \Psi(x)s$$

for $\Psi \in \mathcal{H}om_{\mathcal{D}}(W, \mathcal{H}om(\Omega^l_D, \mathcal{F}))$, $s \in \Omega^l_D$, and $x \in W$. The map $\theta$ is functorial in $\mathcal{F}$ and since the functor $\mathcal{H}om_D(W, \cdot)$ is exact for a fixed $D$-vector space $W$, we get an induced isomorphism

$$\{\theta^i\} : \{\operatorname{Ext}^i(\Omega^l_D \otimes_D W, \cdot)\} \to \{\mathcal{H}om_{\mathcal{D}}(W, \operatorname{Ext}^i(\Omega^l_D, \cdot))\}$$

of universal $\delta$-functors.

Now

$$\{\operatorname{Ext}^i(\tau_x,\cdot)\}:\{\operatorname{Ext}^i(\Omega^l_D\otimes_D W,\cdot)\}\to\{\operatorname{Ext}^i(\Omega^l_D,\cdot)\}$$

is a homomorphism of $\delta$-functors by Section 5.1. Let $V$ be a finite-dimensional $D$-vector space; then the evaluation map

$$\varepsilon_x=\varepsilon_{x,Z}:\mathcal{H}om_D(W,V)\to V,\quad \alpha\to\alpha(x)$$

is functorial in $V$. We obtain a diagram of $\delta$-functors

$$\begin{tikzcd}
\{\operatorname{Ext}^i(\Omega^l_D\otimes_D W,\cdot)\} \arrow[rr, "\{\theta^i\}"] \arrow[dr, "\{\operatorname{Ext}^i(\tau_x,\cdot)\}"'] & & \{\mathcal{H}om_{\mathcal{D}}(W,\operatorname{Ext}^i(\Omega^l_D,\cdot))\} \arrow[dl, "\{\varepsilon_{x,\operatorname{Ext}^i(\Omega^l_D,\cdot)}\}"] \\
& \{\operatorname{Ext}^i(\Omega^l_D,\cdot)\}, &
\end{tikzcd}$$

which commutes since it commutes in degree zero:

$$\varepsilon_{x,\mathcal{H}om(\Omega^l_D,\mathcal{F})}\theta(\varphi)=[\theta(\varphi)](x)=\varphi\circ\tau_x=\mathcal{H}om(\tau_x,\mathcal{F})(\varphi).$$

This implies that

$$\text{(A)}\qquad [\theta^i(\zeta)](x)=\varepsilon_x\theta^i(\zeta)=\operatorname{Ext}^i(\tau_x,\varepsilon)(\zeta)$$

for all $i\in\mathbb{N}_0$, $\zeta\in\operatorname{Ext}^i(\Omega^l_D\otimes_D W,\mathcal{F})$ and $x\in W$.

Let $\lambda:W\to\operatorname{Ext}^i(\Omega^l_D,\mathcal{F})$ be a $\mathcal{D}$-linear map, put

$$\zeta=(\theta^1)^{-1}(\lambda)\in\operatorname{Ext}^1(\Omega^l_D\otimes_D W,\mathcal{F}),$$

and write

$$\text{(B)}\qquad 0\to\mathcal{F}\to\mathcal{P}\to(\Omega^l_D)\otimes_D W\to 0.$$

for the extension of $(\Omega^l_D)\otimes_D W$ by $\mathcal{F}$ corresponding to $\zeta$ [Hartshorne 1977, Section III, Example 6.1]. Using the canonical maps

$$\mu:\mathcal{H}om(\Omega^l_D,\Omega^l_D)\otimes_D W\to\mathcal{H}om(\Omega^l_D,\Omega^l_D\otimes_D W),\ \varphi\otimes x\to\tau_x\circ\varphi$$

and

$$\kappa:W\to\mathcal{H}om(\Omega^l_D,\Omega^l_D)\otimes_D W,\ x\to\mathrm{id}\otimes x,$$

the diagram

$$\begin{tikzcd}
W \arrow[r, "\lambda"] \arrow[d, "\kappa"'] & \operatorname{Ext}^1(\Omega^l_D,\mathcal{F}) \arrow[d, "\delta"] \\
\mathcal{H}om(\Omega^l_D,\Omega^l_D)\otimes_D W \arrow[r, "\mu"] & \mathcal{H}om(\Omega^l_D,\Omega^l_D\otimes_D W)
\end{tikzcd}$$

commutes, where $\delta$ is the connecting homomorphism arising from (B). Indeed, given $x \in W$,

$$\{\mathrm{Ext}^i(\tau_x,\,\cdot\,)\} : \{\mathrm{Ext}^i(\Omega^l_D \otimes_D W, \cdot\,)\} \to \{\mathrm{Ext}^i(\Omega^l_D, \cdot\,)\}$$

is a homomorphism of $\delta$-functors by Section 5.1. Thus the diagram

$$\begin{tikzcd}
\mathcal{H}om(\Omega^l_D \otimes_D W, \Omega^l_D \otimes_D W) \arrow[r, "\delta"] \arrow[d, "\mathcal{H}om(\tau_x, \Omega^l_D \otimes_D W)"'] & \mathrm{Ext}^1(\Omega^l_D \otimes_D W, \mathcal{F})VV \arrow[d, "\mathrm{Ext}^1(\tau_x, \mathcal{F})"] \\
\mathcal{H}om(\Omega^l_D, \Omega^l_D \otimes_D W) \arrow[r, "\delta"] & \mathrm{Ext}^1(\Omega^l_D, \mathcal{F})
\end{tikzcd}$$

commutes, yielding

$$\delta\mu\kappa(x) = \delta\mu(\mathrm{id}_{\Omega^l_D} \otimes x) = \delta(\tau_x) = \delta\,\mathcal{H}om(\tau_x, \Omega^l_D \otimes_D W)$$
$$\mathrm{Ext}^1(\tau_x, \mathcal{F})\,\delta\,(\mathrm{id}_{\Omega^l_D \otimes W}) = \mathrm{Ext}^1(\tau_x, \mathcal{F})(\zeta) = [\theta^1(\zeta)](x) = \lambda(x)$$

by (A). The map $\kappa$ is an isomorphism since $\mathcal{H}om(\Omega^l, \Omega^l) \cong k$ (see for instance [Pumplün 1998b, 3.3(b)]), thus $\mathcal{H}om(\Omega^l_D, \Omega^l_D) \cong D$. Because $\mu$ is functorial in $W$ and commutes with direct sums, it is an isomorphism as well since it obviously is so for $V = D$. Taking $W = \mathrm{Ext}^1(\Omega^l_D, \mathcal{F})$ and $\lambda = \mathrm{id}_W$, the map $\delta$ behaves as desired. □

Note that in this last step of the proof, we need $W = \mathrm{Ext}^1(\Omega^l_D, \mathcal{F})$ to be a free $D$-module, which is guaranteed if $D$ is a field extension. It is not clear how to generalize this proof if $D$ is not a field extension, even if we assume that every finitely generated projective right $D$-module is free. The result is needed to prove both Lemma 10 and Proposition 11.

**5.2.** In the ensuing lemma we will use the following property of Ext-functors: Let

$$0 \to \mathcal{M}_j \to \mathcal{F}_j \to \mathcal{N}_j \to 0$$

for $j = 1, 2$ be two short exact sequences of locally free right $\mathcal{D}$-modules. Then the diagram

$$\begin{tikzcd}
\mathrm{Ext}^i(\mathcal{M}_2, \mathcal{N}_1) \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathrm{Ext}^{i+1}(\mathcal{M}_2, \mathcal{M}_1)VV \arrow[d, "-\delta"] \\
\mathrm{Ext}^{i+1}(\mathcal{N}_2, \mathcal{N}_1) \arrow[r, "\delta"] & \mathrm{Ext}^{i+2}(\mathcal{N}_2, \mathcal{M}_1)
\end{tikzcd}$$

commutes for all $i \geq 0$; see [Hilton and Stammbach 1971, IV.9.9], or just adapt [Neukirch 1969, Satz 3.6].

**Lemma 10.** *Assume that $D$ is a field extension of $k$. Let $l \in \mathbb{Z}$ with $0 \le l < n-2$ and $m = 0$, and let $\mathcal{F}$ be a locally free $\mathcal{D}$-module such that*

$$\begin{aligned} \operatorname{Ext}^i(\mathcal{D}, \mathcal{F}(*)) &= 0 \quad \textit{for } 0 < i < l+1, \\ \operatorname{Ext}^{l+1}(\mathcal{D}, \mathcal{F}(j)) &= 0 \quad \textit{for } j > m. \end{aligned}$$

*Then, in the situation of Lemma 9, the connecting homomorphism*

$$\delta : \operatorname{Ext}^l(\mathcal{D}, \Omega_D^l \otimes W, \mathcal{F}) \to \operatorname{Ext}^{l+1}(\mathcal{D}, \mathcal{F})$$

*is an isomorphism.*

*Proof.* For $l = 0$, this is shown in Lemma 9; thus we assume $l > 0$. Let $\mathcal{E}$ be a $\mathcal{D}$-module, let $i, j, p \in \mathbb{Z}$ be such that $i \ge 0$ and $1 \le p \le l$, and let

$$0 \to \Omega^l \to \mathcal{O}_X(-l)^{\binom{n}{l}} \to \Omega^{l-1} \to 0$$

be the extended Euler sequence of $X$ [Pumplün 1998b, 3.1]. Tensoring by $\mathcal{D}$ yields the short exact sequence

$$0 \to \Omega_D^l \to \mathcal{D}(-l)^{\binom{n}{l}} \to \Omega_D^{l-1} \to 0$$

of $\mathcal{D}$-modules and twisting it by $\mathcal{O}_X(j)$ the short exact sequence

$$0 \to \Omega_D^l(j) \to \mathcal{D}(-l+j)^{\binom{n}{l}} \to \Omega_D^{l-1}(j) \to 0$$

of $\mathcal{D}$-modules.

This induces a long exact Ext-sequence, part of it looking as follows:

$$\begin{aligned} &\operatorname{Ext}^{i+l-p}(\mathcal{D}(-p+j), \mathcal{E})^{\binom{n}{p}} \longrightarrow \operatorname{Ext}^{i+l-p}(\Omega_D^p(j), \mathcal{E})^{\binom{n}{p}} \\ &\qquad \xrightarrow{\delta_p} \operatorname{Ext}^{i+l-(p-1)}(\Omega_D^{p-1}(j), \mathcal{E}) \longrightarrow \operatorname{Ext}^{i+l-p+1}(\Omega_D^p(j), \mathcal{E}). \end{aligned}$$

Combining for $p = 1, \ldots, l$, we get a homomorphism

$$\bar{\delta} : \delta_1 \ldots \delta_l : \operatorname{Ext}^i(\Omega_D^l(j), \mathcal{E}) \to \operatorname{Ext}^{i+l}(\mathcal{D}(j), \mathcal{E})$$

which is injective if each $\delta_p$ is, and surjective if each $\delta_p$ is. This is the case if

$$\operatorname{Ext}^{i+l-p}(\mathcal{D}(-p+j), \mathcal{E}) = \operatorname{Ext}^{i+l-p}(\mathcal{D}, \mathcal{E}(-p+j)) = 0$$

and respectively

$$\operatorname{Ext}^{i+l-p+1}(\mathcal{D}(-p+j), \mathcal{E}) = \operatorname{Ext}^{i+l-p+1}(\mathcal{D}, \mathcal{E}(-p+j)) = 0$$

for $1 \le p \le l$.

Applying this to the special cases $i = j = 0$, $\mathscr{E} = \Omega^l_D \otimes_D W$ and $i = 1$, $j = 0$, $\mathscr{E} = \mathscr{F}$, we obtain that the diagram

$$\begin{tikzcd} \mathscr{H}om(\Omega^l_D, \Omega^l_D \otimes_D W) \arrow[r, "\delta"] \arrow[d, "\delta"] & \mathrm{Ext}^l(\mathscr{D}, \Omega^l_D \otimes_D W) \arrow[d, "\delta"] \\ \mathrm{Ext}^1(\Omega^l_D, \mathscr{F}) \arrow[r, "\bar{\delta}"] & \mathrm{Ext}^{l+1}(\mathscr{D}, \mathscr{F}) \end{tikzcd}$$

commutes up to a sign by Section 5.2. For $1 \le p \le l$, we have

$$\mathrm{Ext}^{l-p+1}(\mathscr{D}, \Omega^l_D \otimes_D W \otimes \mathbb{O}_X(p)) = \mathrm{Ext}^{l-p+1}(\mathscr{D}, \Omega^l_D \otimes \mathbb{O}_X(p))^{\mathrm{rank}\, W} = 0$$

since $\mathrm{Ext}^{l-p+1}(\mathbb{O}_X, \Omega^l(p)) = 0$ by [Pumplün 1998b, 3.3(c)], and also

$$\mathrm{Ext}^{l-p+2}(\mathscr{D}, \Omega^l_D \otimes_D W \otimes \mathbb{O}_Z(p)) = \mathrm{Ext}^{l-p+2}(\mathscr{D}, \Omega^l_D \otimes \mathbb{O}_X(p))^{\mathrm{rank}\, W} = 0$$

since $\mathrm{Ext}^{l-p+2}(\mathbb{O}_X, \Omega^l(p)) = 0$ by [Pumplün 1998b, 3.3(d)], so the upper map $\bar{\delta}$ is injective.

For $1 \le p \le l$, we have

$$\mathrm{Ext}^{l-p+1}(\mathscr{D}, \mathscr{F}(p)) = 0$$

and also

$$\mathrm{Ext}^{l-p+2}(\mathscr{D}, \mathscr{F}(p)) = 0$$

by the hypothesis on $\mathscr{F}$, so the lower map $\bar{\delta}$ is an isomorphism. The left map $\delta$ is an isomorphism by Lemma 9; therefore also the right side map $\delta$ must be an isomorphism, as desired. □

**Proposition 11.** *Assume that $D$ is a field extension of $k$.*

*Let $l, m \in \mathbb{Z}$ with $0 \le l \le n-2$ and let $\mathscr{F}$ be a locally free $\mathscr{D}$-module such that*

$$\mathrm{Ext}^i(\mathscr{D}, \mathscr{F}(*)) = 0 \quad \textit{for } 0 < i < l+1,$$
$$\mathrm{Ext}^{l+1}(\mathscr{D}, \mathscr{F}(j)) = 0 \quad \textit{for } j > m.$$

*Then there is a finite-dimensional $D$-vector space $W$ and an extension*

$$0 \to \mathscr{F} \to \mathscr{P} \to \Omega^l_D(-m) \otimes_D W \to 0.$$

*such that*

$$\begin{aligned} \mathrm{Ext}^i(\mathscr{D}, \mathscr{P}(j)) &= 0 && \textit{for } i, j \in \mathbb{Z} \textit{ with } 0 < i < l+1, \\ \mathrm{Ext}^{l+1}(\mathscr{D}, \mathscr{P}(j)) &= 0 && \textit{for } j \in \mathbb{Z} \textit{ with } j \ge m, \\ \mathrm{Ext}^{l+1}(\mathscr{D}, \mathscr{P}(j)) &\cong \mathrm{Ext}^{l+1}(\mathscr{D}, \mathscr{F}(j)) && \textit{for } j \in \mathbb{Z} \textit{ with } j < m. \end{aligned}$$

*Proof.* By twisting by $\mathcal{O}_X(m)$, we may assume that $m=0$.

Twisting the short exact sequence of Lemma 9 by $\mathcal{O}_X(j)$ yields the short exact sequence

$$0 \to \mathcal{F}(j) \to \mathcal{P}(j) \to (\Omega^r_D \otimes_D W)(j) \to 0.$$

By the hypotheses on $\mathcal{F}$, $\mathrm{Ext}^i(\mathcal{D}, \mathcal{F}(*)) = 0$ for $0 < i < l+1$. Moreover, unless $i = l$ and $j = 0$, we have $\mathrm{Ext}^i(\mathcal{D}, \Omega^l_D(j) \otimes_D W) = 0$ [Pumplün 1998b, 3.3(d)], forcing $\mathrm{Ext}^i(\mathcal{D}, \mathcal{P}(j)) = 0$ for $0 < i < l+1$, $i \neq l$ or $j \neq 0$. In case $i = l$ and $j = 0$, the exact sequence

$$0 \to \mathrm{Ext}^l(\mathcal{D}, \mathcal{P}) \to \mathrm{Ext}^l(\mathcal{D}, \Omega^r_D \otimes_D W) \xrightarrow{\delta} \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{F})$$

together with Lemma 10 implies that $\mathrm{Ext}^l(\mathcal{D}, \mathcal{P}) = 0$. Summing up, we have $\mathrm{Ext}^i(\mathcal{D}, \mathcal{P}(j)) = 0$ for $0 < i < l+1$.

Now consider the exact sequence

$$\mathrm{Ext}^l(\mathcal{D}, \Omega^r_D \otimes_D W(j)) \xrightarrow{\delta} \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{F}(j)) \\ \longrightarrow \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{P}(j)) \longrightarrow \mathrm{Ext}^{l+1}(\mathcal{D}, (\Omega^l_D \otimes_D W)(j)) = 0,$$

the last extension group being zero by [Pumplün 1998b, 3.3(d)]. If $j \neq 0$, also $\mathrm{Ext}^l(\mathcal{D}, (\Omega^r_D \otimes_D W)(j)) = 0$ which implies $\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{P}(j)) \cong \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{F}(j)) = 0$ for $j > 0$. If $j = 0$, $\delta$ is an isomorphism by Lemma 10, which shows that $\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{P}) = 0$ and completes the proof. □

**Lemma 12.** *Let $D$ be a field extension of $k$. Let $l, m \in \mathbb{Z}$ with $0 \le l \le \frac{1}{2}(n-1) - 1$ and $m \ge -l-1$ for $l = \frac{1}{2}(n-1) - 1$. Given any finite-dimensional $D$-vector space $W$, put $\mathcal{R} = \Omega^l_D(-m) \otimes_D W$. Then*

$$\mathrm{Ext}^1(\mathcal{R}, \mathcal{R}^*) = \mathrm{Ext}^2(\mathcal{R}, \mathcal{R}^*) = 0.$$

*Proof.* $\mathrm{Ext}^i$ is additive in both variables, so we may assume that $W = D$ and have to show that

$$\mathrm{Ext}^i(\Omega^l_D(-m), (\Omega^l_D(-m))^*) = 0 \quad \text{for } i = 1, 2.$$

Put $j = -m$ and $\mathcal{F} = \Omega^l_D(-m)^*$; then as in the proof of Lemma 10 we obtain a homomorphism

$$\bar{\delta} : \mathrm{Ext}^i(\Omega^l_D(-m), (\Omega^l_D(-m))^*) \to \mathrm{Ext}^{i+l}(\mathcal{D}, (\Omega^l_D(-2m))^*).$$

Since $n-1 \ge 2l+1$ we have $l < n-1-i-l+p$ for all $p = 1, \dots, l$. By Serre duality, we know that $H^{i+l-p}(X, \Omega^l(-m)^\vee(p+m)) = 0$; therefore $\mathrm{Ext}^{i+l-p}(\mathcal{D}, \Omega^l_D(-m)^*(p+m)) = 0$, which together with [Pumplün 1998b, 3.3(d)], proves injectivity of $\bar{\delta}$ as in the proof of Lemma 10. Using Serre duality and [Pumplün 1998b, 3.3(d)], we can check that $H^{i+l}(X, \Omega^l(-2m)^\vee) = 0$; therefore $\mathrm{Ext}^{i+l}(\mathcal{D}, \Omega^l_D(-2m)^*) = 0$ for $i = 1, 2$. Hence $\bar{\delta}$ is surjective as well. □

## 6. Hermitian forms over field extensions of $k$

**6.1.** Let $k$ be a field of characteristic not 2 and $X$ a Brauer–Severi variety over $k$. Let $l/k$ be a separable field extension with a $k$-linear involution $\sigma$. Put $Y = X \times_k l$ and $X_s = X \times_k k_s \cong \mathbb{P}^{n-1}_{k_s}$, where $k_s$ is a separable closure of $k$. Recall from [Pumplün 1998b, 5.2] that every line bundle $\mathcal{O}_{X_s}(m)$ has a $G$-invariant isomorphism class, where $G = \mathrm{Gal}\,(k_s/k)$ is the Galois group of $k_s/k$. Thus $\mathcal{O}_{X_s}(m) \cong \mathcal{O}_{X_s}(m)^\tau$ for all $\tau \in G$. We look at $(\sigma)$-hermitian spaces $(\mathcal{M}, h)$ over $\mathfrak{D} = l \otimes_k \mathcal{O}_X$, pointing out that $l \otimes_k \mathcal{O}_X = \pi_* \mathcal{O}_Y$; see Section 1.5. In other words, $Y$ is affine over $X$ and defined by the sheaf of $\mathcal{O}_X$-algebras $l \otimes_k \mathcal{O}_X$:

$$Y = \underline{\mathrm{Spec}}_X (l \otimes_k \mathcal{O}_X);$$

see [Hartshorne 1977, II, Example 5.17].

Let $\mathcal{M}$ be a right $\mathfrak{D}$-module that is locally free of finite rank. Then $\mathcal{M}$ canonically is an $\mathcal{O}_X$-module, and we denote the associated $\mathcal{O}_Y$-module by $\tilde{\mathcal{M}}$ as in [Hartshorne 1977, II, Example 5.17].

**Proposition 13.** *Suppose $Y = \mathbb{P}^{n-1}_l$. Then every hermitian space $(\mathcal{M}, h)$ over $l \otimes_k \mathcal{O}_X$ such that $\tilde{\mathcal{M}}$ splits into a direct sum of line bundles is Witt equivalent to a hermitian space extended from $l$.*

*Proof.* If $\tilde{\mathcal{M}}$ splits into the direct sum of line bundles, then

$$\tilde{\mathcal{M}} \cong \bigoplus_{i=1}^{t} \mathcal{O}_Y(s_i) \quad \text{as } \mathcal{O}_Y\text{-module.}$$

We have $\mathcal{O}_Y(m) \cong \mathcal{O}_Y(m)^*$ if and only if $m = 0$ [Pumplün 1998b, 5.2]. Hence there is no nontrivial line bundle over $Y$ that is selfdual with respect to $^*$. By the Krull–Schmidt theorem for hermitian spaces [Knus 1991, (6.3.1), page 98], and by [Knus 1991, (6.4.1), page 99],

$$(\mathcal{M}, h) \cong (M_0, h_0) \otimes_l (l \otimes \mathcal{O}_Y) \perp \text{ a hyperbolic space.} \qquad \square$$

**Theorem 14.** *Suppose $X = \mathbb{P}^{n-1}_k$. Then $U_\tau : W^1(l) \to W^1(l \otimes_k \mathcal{O}_X)$ is surjective.*

The proof is similar to the one given in [Arason 1980] or [Pumplün 1998b, 5.1]:

*Proof.* In the case $n = 2$, for every hermitian space $(\mathcal{M}, h)$ over $l \otimes_k \mathcal{O}_X$, the vector bundle $\mathcal{M}$ splits into a direct sum of line bundles; hence surjectivity follows from Proposition 13, and we may assume $n \geq 3$.

We show by induction on $a \geq 0$: If $(\mathcal{M}, h)$ is a hermitian space over $\mathfrak{D} = l \otimes_k \mathcal{O}_X$ such that

$$a = \max\{i \in \mathbb{Z} \mid 0 \leq i < n-1,\ \mathrm{Ext}^{n-i-1}(\mathfrak{D}, \mathcal{M}(*)) \neq 0\},$$

then $(\mathcal{M}, h)$, up to Witt equivalence, is a hermitian space extended from $l$. Note that the set on the right hand side is not empty here.

If $a = 0$, then $\mathrm{Ext}^{n-i-1}(\mathcal{D}, \mathcal{M}(*)) = 0$ for $0 < i < n-1$, so by the generalization of Horrocks's theorem, $\mathcal{M} \cong \mathcal{D}(s_1) \oplus \cdots \oplus \mathcal{D}(s_t)$ for some of the $s_i$ and, by Proposition 13, $(\mathcal{M}, h)$ is Witt equivalent to some hermitian space extended from $l$. This settles the induction beginning.

In the induction step, let $a > 0$ and suppose the induction hypothesis holds for all nonnegative integers $a' < a$.

There is no harm in assuming $a < n-1$, so if $l = n-2-a$, then

$$0 \le l < n-2. \tag{12}$$

It suffices to show that a hermitian space $(\mathcal{M}, h)$ with

$$\mathrm{Ext}^i(\mathcal{D}, \mathcal{M}(*)) = 0 \quad \text{for } 0 < i < l+1$$

is, up to Witt equivalence, extended from $l$. This will be done by induction on

$$s = \dim \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(*)).$$

If $s = 0$, then $\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(*)) = \mathrm{Ext}^{n-a-1}(\mathcal{D}, \mathcal{M}(*)) = 0$; therefore

$$\max\{i \in \mathbb{Z} \mid 0 \le i < n-1,\ \mathrm{Ext}^{n-i-1}(\mathcal{D}, \mathcal{M}(*)) \ne 0\} < a$$

and we are done by induction hypothesis on $a$. If $s > 0$, then

$$l \text{ is the least nonnegative integer such that } \mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(*)) \ne 0. \tag{13}$$

By [Hartshorne 1977, III, Example 6.10], $\mathrm{Ext}^q(\mathcal{D}, \mathcal{M}(j)) = \mathrm{Ext}^q(\mathcal{O}_Y, \tilde{\mathcal{M}}(j)) = H^q(Y, \tilde{\mathcal{M}}(j))$. Thus (13) is equivalent to saying that

$$l \text{ is the least nonnegative integer such that } H^{l+1}(Y, \tilde{\mathcal{M}}(*)) \ne 0. \tag{15$'$}$$

Using that $\mathcal{M} \cong \mathcal{M}^*$ and Serre duality, we obtain

$$H^i(Y, \tilde{\mathcal{M}}(j)) \cong H^i(Y, \tilde{\mathcal{M}}^\vee(j)) \cong H^{n-1-i}(Y, \tilde{\mathcal{M}}(-n-2-j))^\vee$$

and may conclude that

$$l+1 \le \tfrac{1}{2}(n-1). \tag{14}$$

Picking $m \in \mathbb{Z}$ maximal such that $\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(m)) \ne 0$, we obtain

$$m \ge -l-1 \quad \text{if } l+1 = \tfrac{1}{2}(n-1). \tag{15}$$

(This is because $m \ge -n-2-m$ for $l+1 = \frac{1}{2}(n-1)$; thus $2m \ge -n-2 = -2l-4$, implying $m \ge -l-1$.) In particular,

$$\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(j)) = 0 \quad \text{for } j > m. \tag{16}$$

Because of (12), (13) and (16), $\mathcal{M}$ satisfies the hypothesis of Proposition 11 and there exists a locally free $\mathcal{D}$-module $\mathcal{E}$ and an extension

$$0 \to \mathcal{M} \to \mathcal{P} \to \mathcal{E} \to 0. \tag{17}$$

such that $\mathcal{P}$ satisfies the conditions listed in Proposition 11. Hence

$$\operatorname{Ext}^1(\mathcal{E}, \mathcal{E}^*) = \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}^*) = 0.$$

Thus we may apply the extension theorem to the dual of (17) with $\mathcal{M}$ replaced by $\mathcal{M}^*$ via $h$, that is, to

$$0 \to \mathcal{E}^* \to \mathcal{P}^* \to \mathcal{M} \to 0. \tag{18}$$

This way we obtain a hermitian space $(\mathcal{S}, b)$ Witt equivalent to $(\mathcal{M}, h)$ and an exact sequence of $\mathcal{D}$-modules

$$0 \to \mathcal{E}^* \to \mathcal{S} \to \mathcal{P} \to 0. \tag{19}$$

Since $\operatorname{Ext}^i(\mathcal{D}, \Omega^l_D(-m)^*(j)) = 0$ for $0 < i \le l+1$, this together with (18) yields

$$\operatorname{Ext}^i(\mathcal{D}, \mathcal{S}(*)) = 0 \quad \text{for } 0 < i < l+1$$

and that $\operatorname{Ext}^{l+1}(\mathcal{D}, \mathcal{S}(j)) \to \operatorname{Ext}^{l+1}(\mathcal{D}, \mathcal{P}(j))$ is injective for all $j \in \mathbb{Z}$. The latter shows by using Proposition 11 that

$$\begin{aligned} \operatorname{Ext}^{l+1}(\mathcal{D}, \mathcal{S}(j)) &= 0 && \text{for } j \ge m', \\ \dim \operatorname{Ext}^{l+1}(\mathcal{D}, \mathcal{S}(j)) &\le \dim \operatorname{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(j)) && \text{for } j < m'. \end{aligned}$$

Together this yields $\dim \operatorname{Ext}^{l+1}(\mathcal{D}, \mathcal{S}(*)) \le s$. Applying the induction hypothesis yields the assertion that $(\mathcal{S}, b)$ and hence also $(\mathcal{M}, h)$ is up to Witt equivalence extended from $l$. □

By Theorems 1, 2 and 4, this settles the case where $X$ is associated to a central simple algebra of odd index:

**Corollary 15.** (i) *Let $X = \mathbb{P}^{n-1}_k$. Then $U_\tau : W^1(l) \to W^1(l \otimes_k \mathcal{O}_X)$ is bijective.*

(ii) *Let $X$ be a Brauer–Severi variety associated to a central simple algebra of odd index. Then $U_\tau : W^1(l) \to W^1(l \otimes_k \mathcal{O}_X)$ is bijective.*

**6.2.** Let $X$ be a Brauer–Severi variety over $k$ with associated central simple algebra $\operatorname{Mat}_s(E)$, where $E$ is a division algebra over $k$.

**Proposition 16.** (i) *Suppose there is a separable maximal subfield $k'$ of $E$ containing $l$. Let $X' = X \times_k k'$. Then every hermitian space $(\mathcal{M}, h)$ over $l \otimes_k \mathcal{O}_X$ such that $\tilde{\mathcal{M}} \otimes \mathcal{O}_{X'}$ splits into the direct sum of line bundles is Witt equivalent to a hermitian space $(M_0, h_0)$ over $l$.*

(ii) *Suppose there is a separable maximal subfield $k'$ of $E$ such that $l$ and $k'$ are linearly disjoint. Let $Y' = X \times_k l'$ with $l' = l \otimes_k k'$. Then every hermitian space $(\mathcal{M}, h)$ over $l \otimes_k \mathcal{O}_X$ such that $\tilde{\mathcal{M}} \otimes \mathcal{O}_{Y'}$ splits into the direct sum of line bundles is Witt equivalent to a hermitian space $(M_0, h_0)$ over $l$.*

*Proof.* (i) Obviously, $X' \cong \mathbb{P}^{n-1}_{k'}$. If $\tilde{\mathcal{M}} \otimes_{\mathcal{O}_Y} \mathcal{O}_{X'}$ splits into the direct sum of line bundles

$$\tilde{\mathcal{M}} \otimes_{\mathcal{O}_Y} \mathcal{O}_{X'} \cong \bigoplus_{i=1}^{t} \mathcal{O}_{X'}(s_i),$$

then, by the theory developed in [Arason et al. 1992],

$$\tilde{\mathcal{M}} \cong \bigoplus_{i=1}^{t} \mathcal{L}(s_i) \oplus \bigoplus_{j=1}^{h} \mathrm{tr}_{l_j/l}(\mathcal{N}_j)$$

as $\mathcal{O}_Y$-module, where the $\mathcal{N}_j$ are line bundles over $Y_j = Y \times_l l_j$ that are not already defined over $Y$, the $l_j/l$ are proper field extensions, and the $\mathrm{tr}_{l_j/l}(\mathcal{N}_j)$ are indecomposable. Now $\mathcal{O}_{X'}(m) \cong \mathcal{O}_{X'}(m)^*$ if and only if $m=0$ [Pumplün 1998b, 5.2]. Hence there is no nontrivial line bundle over $Y$ that is selfdual with respect to $^*$. Consider an indecomposable vector bundle $\mathrm{tr}_{l_j/l}(\mathcal{N}_j)$. Then $\mathrm{tr}_{l_j/l}(\mathcal{N}_j) \cong \mathrm{tr}_{l_j/l}(\mathcal{N}_j)^*$ implies that $\mathcal{O}_{X'}(m) \oplus \cdots \oplus \mathcal{O}_{X'}(m) \cong \mathcal{O}_{X'}(-m) \oplus \cdots \oplus \mathcal{O}_{X'}(-m)$ [Pumplün 1998b, 5.2]; hence $m = 0$ and so there are no indecomposable $\mathcal{O}_Y$-modules of rank $> 1$ that are selfdual with respect to $^*$. By the Krull–Schmidt theorem for hermitian spaces [Knus 1991, (6.3.1), page 98],

$$(\tilde{\mathcal{M}}, \tilde{h}) \cong (M, h) \otimes_l (l \otimes \mathcal{O}_Y) \oplus \text{ a hyperbolic space}$$

and thus also $(\mathcal{M}, h)$ is Witt equivalent to a hermitian space $(M_0, h_0)$ over $l$ (we canonically identify hermitian forms over $l \otimes_k \mathcal{O}_X$ with hermitian forms over $\mathfrak{D}$).

Part (ii) is proved analogously. □

**Theorem 17.** *Let $E$ have even index. Then $U_\tau : W^1(l) \to W^1(l \otimes_k \mathcal{O}_X)$ is surjective.*

*Proof.* If $l$ is a finite field extension of $k$ with an involution $\sigma$ and invariant field $l^\sigma$, for a Brauer–Severi variety $X$ over $k$, we may identify $l^\sigma$-algebras and modules over $X$ with $\mathcal{O}_{X_{l^\sigma}}$-algebras and modules over $X_{l^\sigma}$, analogous to the $\mathcal{M} \to \tilde{\mathcal{M}}$ construction. In particular, we may identify hermitian forms over $l \otimes_k \mathcal{O}_X$ with hermitian forms over $l \otimes_{l^\sigma} \mathcal{O}_{X_{l^\sigma}}$. This way we may restrict without loss of generality to the case that $[l : k] = 2$. In this case, either there is a separable maximal subfield $k'$ of $E$ containing $l$, or there is a maximal separable subfield $k'$ of $E$ such that $k'$ is linearly disjoint with $l$ over $k$.

(i) Suppose that there is a separable maximal subfield $k'$ of $E$ containing $l$. For $n = 2$, we have $Y' \cong \mathbb{P}^1_{l'}$ and the assertion is proved in Proposition 16(i). So we may assume $n \geq 3$. Let $\mathcal{M}' = \tilde{\mathcal{M}} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'}$.

We show by induction on $a \geq 0$: If $(\mathcal{M}, h)$ is a hermitian space such that

$$a = \max\{i \in \mathbb{Z} \mid 0 \leq i \leq n-1,\ \ H^{n-i-1}(Y', \mathcal{M}') \neq 0\}$$

then $(\mathcal{M}, h)$ is, up to Witt equivalence, a hermitian space $(M_0, h_0)$ over $l$.

If $a = 0$, then $H^{n-i-1}(Y', \mathcal{M}'(j)) = 0$ for all $j$ and for $0 \leq i \leq n-1$. Then by Horrocks's [Barth and Hulek 1978, Section 5, Lemma 1], $\mathcal{M}'$ splits into the direct sum of line bundles and, by Proposition 16(i), $(\mathcal{M}, h)$ is Witt equivalent to some hermitian space $(M_0, h_0)$ over $l$. This settles the induction beginning. In the induction step, let $a > 0$ and suppose the induction hypothesis holds for all non-negative integers $a' < a$. Then the assertion is proved analogously as Theorem 14 was using [Pumplün 2000, Proposition 4.1] (there use $X_l$ instead of $X$), [Pumplün 2000, Lemma 4.4], Lemma 10 and Theorem 6.

(ii) Suppose there is a maximal separable subfield $k'$ of $E$ such that $k'$ is linearly disjoint with $l$ over $k$. Then the proof is analogous to the one in (i), but works over $\mathbb{P}^{n-1}_{l'}$ instead. □

**Remark 18.** Let $X = \mathbb{P}^{n-1}_{\mathbb{R}}$, and let $\sigma$ be the standard involution on $\mathbb{C}$. Then $W^1(\mathbb{C}) \cong W^1(\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}_X)$. Since $W^{-1}(\mathbb{C}) \cong W^1(\mathbb{C}) \cong \mathbb{Z}$ [Knus 1991, page 63], this implies

$$W^{\pm 1}(\mathbb{C}) \cong W^1(\mathbb{C} \otimes_{\mathbb{R}} \mathcal{O}_X) \cong \mathbb{Z}.$$

## 7. Hermitian spaces over division algebras

**7.1.** Let $k$ be a field of characteristic not 2, and let $D$ be a division algebra over $k$ with a $k$-linear involution $\sigma$. Let $X = \mathbb{P}^1_k$, and let $\mathfrak{D} = \tau^* D \cong D \otimes_k \mathcal{O}_X$ and $\mathfrak{D}(m) = \mathcal{O}_X(m) \otimes \mathfrak{D}$ for any integer $m$.

The theorem of Krull and Schmidt holds for locally free right $\mathfrak{D}$-modules by [Knus 1991, page 96], and for $\varepsilon$-hermitian spaces over $\mathfrak{D}$ if we restrict to $\varepsilon = \pm 1$, by [Knus 1991, page 99, (6.5.1)].

**Proposition 19.** *Let $X = \mathbb{P}^1_k$ and $\varepsilon = \pm 1$. Every $\varepsilon$-hermitian space $(\mathcal{M}, h)$ over $D \otimes_k \mathcal{O}_X$ such that*

$$\mathcal{M} \cong \mathfrak{D}(m_1) \oplus \cdots \oplus \mathfrak{D}(m_t)$$

*is Witt equivalent to an $\varepsilon$-hermitian space $(M_0, h_0) \otimes_D (D \otimes \mathcal{O}_X)$, where $(M_0, h_0)$ is an $\varepsilon$-hermitian space over $D$.*

Note that for the possible extension of these results described in Section 7.2, we would need this proposition also for $X = \mathbb{P}^n_k$ (and $\varepsilon = 1$).

*Proof.* We have $\mathfrak{D}(m) \cong \mathfrak{D}(m)^*$ if and only if $m = 0$ [Knus 1991, page 96, (5.4.1)]. Hence $\mathfrak{D}$ itself is the only locally free right $\mathfrak{D}$-module of rank 1 that is selfdual with respect to $^*$. Any $\varepsilon$-hermitian space with underlying vector bundle of type $\{\mathfrak{D}(m), \mathfrak{D}(m)^*\}$ with $m \neq 0$ is isometric to a hyperbolic space [Knus 1991, page 99, (6.4.2)]. By the Krull–Schmidt theorem for $\varepsilon$-hermitian spaces ([Scharlau 1985, page 272] or [Knus 1991, pages 96 and 99]), $\mathcal{M} \cong \mathcal{D}(m_1) \oplus \cdots \oplus \mathcal{D}(m_t)$ implies that

$$(\mathcal{M}, h) \cong (M_0, h_0) \otimes_D (D \otimes \mathcal{O}_X) \perp \perp_{j=1}^{h} (\mathcal{D}(m_j) \oplus \mathcal{D}(-m_j), \left[\begin{smallmatrix} 0 & 1 \\ \varepsilon & 0 \end{smallmatrix}\right])$$

for a suitable $\varepsilon$-hermitian space over $D$ and suitable $m_j \neq 0$. We conclude that

$$(\mathcal{M}, h) \cong (M_0, h_0) \otimes_D (D \otimes \mathcal{O}_X) \perp \text{ a hyperbolic space.} \qquad \square$$

**Corollary 20.** *Let $X = \mathbb{P}_k^1$ and $\varepsilon = \pm 1$. Every $\varepsilon$-hermitian space $(\mathcal{M}, h)$ over $D \otimes_k \mathcal{O}_X$ is Witt equivalent to an $\varepsilon$-hermitian space $(M_0, h_0) \otimes_D (D \otimes \mathcal{O}_X)$ with $(M_0, h_0)$ an $\varepsilon$-hermitian space over $D$. In particular, $U_\tau : W^\varepsilon(D) \to W^\varepsilon(D \otimes \mathcal{O}_X)$ is bijective.*

*Proof.* For $X = \mathbb{P}_k^1$, every $\varepsilon$-hermitian space $(\mathcal{M}, h)$ over $D \otimes_k \mathcal{O}_X$ satisfies $\mathcal{M} \cong \mathcal{D}(m_1) \oplus \cdots \oplus \mathcal{D}(m_t)$ [Knus 1991, page 407, VII.(3.1.1)]. $\square$

**7.2.** Let $X = \mathbb{P}_k^n$ and $\operatorname{char} k \neq 2$. It would be desirable to prove that for a division algebra $D$ with a $k$-linear involution $\sigma$, the group homomorphism

$$U_\tau : W^1(D) \to W^1(D \otimes \mathcal{O}_X)$$

is surjective. However, we will leave this open for now and only briefly discuss the problems arising in a possible proof.

First, we need to assume that $\operatorname{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(*))$ is of *finite rank* as a right $D$-module and that $\operatorname{Ext}^{n-1}(\mathcal{D}, \mathcal{M}(*)) \neq 0$.

For $n = 1$ the assertion has been proved in Corollary 20, so let $n \geq 2$. Imitating the proofs in [Pumplün 1998b] or [Arason 1980], we proceed as follows: it would suffice to show by induction on $a \geq 0$ that if $(\mathcal{M}, h)$ is a hermitian space satisfying

$$a = \max\{\, i \in \mathbb{Z} \mid 0 \leq i < n,\ \operatorname{Ext}^{n-i-1}(\mathcal{D}, \mathcal{M}(*)) \neq 0\},$$

then $(\mathcal{M}, h)$ is Witt equivalent to a hermitian space $(M_0, h_0) \otimes_D (D \otimes \mathcal{O}_X)$.

If $a = 0$, then $\operatorname{Ext}^{n-i}(\mathcal{D}, \mathcal{M}(j)) = 0$ for all $j$ and $0 < i < n$ and by Theorem 8 (the generalized Horrocks theorem), $\mathcal{M} \cong \mathcal{D}(m_1) \oplus \cdots \oplus \mathcal{D}(m_t)$. If Proposition 19 can be generalized to $\mathbb{P}_k^n$, this would imply that $(\mathcal{M}, h)$ is Witt equivalent to a hermitian space $(M_0, h_0) \otimes_D (D \otimes \mathcal{O}_X)$ and settle the induction beginning.

In the induction step, let $a > 0$ and suppose that the induction hypothesis holds for all nonnegative integers $a' < a$. There is no harm in assuming $a < n$; thus we have $0 \leq l = n - 1 - a < n - 1$. It suffices to show that a hermitian space $(\mathcal{M}, h)$ with $\operatorname{Ext}^i(\mathcal{D}, \mathcal{M}(*)) = 0$ for $0 < i < l + 1$ is Witt equivalent to a hermitian space that is extended from $D$. This is done by induction on $s = \dim \operatorname{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(*))$. If $s = 0$, then we are done by the induction hypothesis on $a$. If $s > 0$, then $l$ is the least nonnegative integer such that $\operatorname{Ext}^{l+1}(X, \mathcal{M}(*)) \neq 0$. We next would have to be able to conclude that $l+1 \leq n/2$. It is not clear if we can show it by using Serre duality. Let $m \in \mathbb{Z}$ be maximal such that $\mathrm{Ext}^{l+1}(\mathcal{D}, \mathcal{M}(m)) \neq 0$. If $l+1 = n/2$, $m \geq -m-1-m$; hence $2m \geq -n-1 = -2l-3$, forcing $m \geq -l-1$ if $l+1 = n/2$.

We would now need to apply a similar result as Proposition 11 in our setting here if we want to proceed with our proof along the same lines as in Theorem 14. However, it is not clear how to prove a statement like this; see Section 5. At this point, all we can say is that $(\mathcal{M}, q_h)$, the quadratic space over $X$ induced by $(\mathcal{M}, h)$, is Witt equivalent to a quadratic space defined over $k$. We leave it open if it is possible to fix these gaps in the proof.

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SUSANNE PUMPLÜN
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF NOTTINGHAM
UNIVERSITY PARK
NOTTINGHAM NG7 2RD
UNITED KINGDOM

susanne.pumpluen@nottingham.ac.uk