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We offer a new proof to the classical topological disk theorem of Reifenberg. The novelty of our method is that we construct the approximating surfaces globally, which makes our proof rather simple and direct.

1. Introduction

Definition 1.1 (Reifenberg flat set). We say that a compact set $K \subseteq \bar{B}_1^n \subseteq \mathbb{R}^n$ is an m-dimensional (δ, R)-Reifenberg flat set if for every $a \in K$ and $r \in (0, R]$ with $|a| + r \leq 1$, there exists an m-plane $T_{a,r} \in \mathrm{GL}(m, n)$ such that

$$HD(K \cap B_r^n(a), T_{a,r} \cap B_r^n(a)) \leq \delta r,$$

where $HD(A, B) = \max(\sup\{\mathrm{dist}(a, B) : a \in A\}, \sup\{\mathrm{dist}(b, A) : b \in B\})$ is the Hausdorff distance.

The definition is only meaningful for small $\delta > 0$.

Theorem 1.2 (Reifenberg's topological disk theorem). *For K as above, if δ is sufficiently small, then $K \cap B_{1/2}^n$ is a C^α-topological m-dimensional disk.*

This theorem is adopted from the book [Lin and Yang 2002, Chapter 2, page 58]. The original and its proof is due to Reifenberg [1960, Chapter 4]. In that paper, Reifenberg investigated the higher-dimensional Plateau problem. He started with this kind of very weak "surfaces"; the C^α topological disk property was used in a fundamental way. Both his original proof and our new one are based on a sequence of approximating surfaces converging to K. The first such surface is just a standard m-dimensional disk, and there exist uniform bi-Lipschitz maps between any two adjacent surfaces. The composition of these bi-Lipschitz maps gives a bi-Hölder map from B_1^m to K. What is crucial is the actual construction of these surfaces. Reifenberg did the work locally, using several averaging processes and complicated estimates, which made his proof horribly difficult and messy. Recently, the authors had the idea of constructing these surfaces globally by mollifying the distance function to K, which made it possible to write down a much simpler proof

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of this beautiful theorem. In fact, we define the bi-Lipschitz maps between two adjacent level sets of the mollified distance functions to be the gradient vector of the mollified distance functions. These level sets will converge to K, but they are codimension 1 surfaces. It is easy to extract an m-dimensional surface from the first of these level sets, and we track its images under these bi-Lipschitz maps step by step. Then we obtain a sequence of m-dimensional surfaces converging to K. It is crucial in this process to show that at every scale and every neighborhood the corresponding surface is almost parallel to K. The uniform bound $C(m,n)\delta$ for the error angles guarantees the uniform bi-Lipschitz property of the restricted maps.

In [Hong and Wang 2007], we could only treat the codimension 1 case, when we took the level sets at $1/2$ of the mollified characteristic functions of Ω (here $\partial\Omega$ is the Reifenberg flat set) as the approximating surfaces. The method of this paper works for any codimension.

2. Proof of Theorem 1.2

We will use standard notations. Let $e_1,\dots,e_n$ be the standard basis in $\mathbb{R}^n$. Let O be the origin of $\mathbb{R}^n$. We use the same notation in another coordinate system if no confusion arises. For simplicity, we assume $T_{O,1}=\operatorname{span}\{e_1,\dots,e_m\}$. For fixed $m<n$, we write $x=(x',x'')\in\mathbb{R}^n$ with $x'=(x_1,\dots,x_m)$ and $x''=(x_{m+1},\dots,x_n)$, or $x=(x',x''',x_n)$ with $x'''=(x_{m+1},\dots,x_{n-1})$. Then $T_{O,1}=\{x\in\mathbb{R}^n:|x''|=0\}$. We denote the distance function from K by d and the distance functions from $T_{a,10r}$ by $d_{a,r}$; then $d_{O,0.1}(x)=|x''|$.

2.1. ***Derivative estimates on the mollified distance functions.*** Our mollifier η is defined so that $\eta(x)=C_1\exp(-1/(1-|x|^2))$ for $x\in B_1$ and $\eta(x)=0$ for x outside of B_1, where C_1 is chosen so that $\int\eta=1$. Through the paper, C (with any subscript) denotes a constant only depending on dimensions m and n. Define $\eta_\epsilon(x):=\epsilon^{-n}\eta(\epsilon^{-1}x)$. Then $d_\epsilon:=d*\eta_\epsilon$ are the mollified distance functions.

We define $h_{a,r}:=d-d_{a,r}$; then $HD(K\cap B_{10r}(a),T_{a,10r}\cap B_{10r}(a))\le 10\delta r$ implies $|h_{a,r}|\le 10\delta r$ in $B_{9.9r}(a)$. Thus $d_\epsilon=d_{a,r}*\eta_\epsilon+h_{a,r}*\eta_\epsilon$; in particular

$$d_{0.01r}=d_{a,r}*\eta_{0.01r}+h_{a,r}*\eta_{0.01r}:=\tilde d_{a,r}+\tilde h_{a,r}.$$

The following two lemmas give derivative estimates on $\tilde d_{a,r}$ and $\tilde h_{a,r}$ near point a.

Lemma 2.1. *For any unit vector ν perpendicular to $T_{a,10r}$, any $b\in T_{a,10r}$ and t satisfying $0.4r<t$, we have $\nabla\tilde d_{a,r}(b+t\nu)=\tau\nu$ with $0.95<\tau<1.05$; moreover, if we chose a coordinate system such that $b=O$, $T_{a,10r}=\operatorname{span}\{e_1,\dots,e_m\}$ and $\nu=e_n$, then*

$$|\partial_i\partial_j\tilde d_{a,r}(b+t\nu)|<3/r\quad\text{if } m<i,\ j\le n,$$
$$|\partial_i\partial_j\tilde d_{a,r}(b+t\nu)|=0\qquad\text{if } \min(i,j)\le m.$$

Proof. In the coordinate system above, $d_{a,r}(x) = |x''|$. Then for $x \notin T_{a,r}$, we have $\nabla d_{a,r}(x) = (0, x'')/|x''|$ and $\partial_i \partial_j d_{a,r}(x) = \delta_{ij}/|x''| - x_i x_j/|x''|^3$ when $\min(i, j) > m$ and $\partial_i \partial_j d_{a,r}(x) = 0$ when $\min(i, j) \leq m$. Therefore,

$$\begin{aligned}\nabla \tilde{d}_{a,r}(t\nu) &= \int_{B_{0.01r}} \frac{(0, (te_n - y)'')}{|(te_n - y)''|} \cdot \eta_{0.01r}(y) dy \\ &= \int_{B_1} \frac{(0, (te_n - 0.01ry)'')}{|(te_n - 0.01ry)''|} \cdot \eta(y) dy \\ &= \int_{B_1} \frac{(0, (e_n - 0.01(r/t)y)'')}{|(e_n - 0.01(r/t)y)''|} \cdot \eta(y) dy.\end{aligned}$$

Now $\partial_i \tilde{d}_{a,r}(t\nu) = 0$ for $1 \leq i \leq m$, whereas

$$\partial_i \tilde{d}_{a,r}(t\nu) = \int_{B_1} \frac{-0.01(r/t)y_i}{|(e_n - 0.01(r/t)y)''|} \cdot \eta(y) dy = 0 \quad \text{for } m+1 \leq i \leq n-1$$

by symmetry properties of the functions about the hyperplane $\{x_i = 0\}$; finally

$$\partial_n \tilde{d}_{a,r}(t\nu) = \int_{B_1} \frac{(1 - 0.01(r/t)y_n)}{|(e_n - 0.01(r/t)y)''|} \cdot \eta(y) dy.$$

Both $1 - 0.01(r/t)y_n$ and $|(e_n - 0.01(r/t)y)''|$ are between 0.975 and 1.025 for $y \in B_1$, which implies the first part of the conclusion. The estimates of the second order derivatives can be proved by the same way. □

Lemma 2.2. *In $B_{9.8r}(a)$, we have*

$$|\tilde{h}_{a,r}| \leq 10\delta r, \quad |\nabla \tilde{h}_{a,r}| \leq C_2 \delta, \quad |\nabla^2 \tilde{h}_{a,r}| \leq C_2 \delta / r.$$

Proof. Put the derivatives onto $\eta_{0.01r}$ and use $|h_{a,r}| \leq 10\delta r$ in $B_{9.9r}(a)$. □

2.2. ***Construction of approximating surfaces.*** We let $r_k := 0.01 \times 2^{-k}$ and define $L_k := \{x \in B_1 \mid d_{r_k/100} = r_k\}$ for $k = 1, 2, \ldots$. The gradient estimation of $d_{r_k/100}$ and the implicit function theorem tell us that every L_k is a smooth hypersurface. Let $\bar{L}_k := \{x \in B_1 \mid d = r_k\}$. Then $|d - d_{r_k/100}| < r_k/100$ implies $HD(L_k, \bar{L}_k) < r_k/90$, and then $(1 - 1/80)r_k < HD(L_k, K) < (1 + 1/80)r_k$, where we used that δ is small enough.

We define $\pi_k : L_k \to L_{k+1}$ by $\pi_k(P) = P - \sigma(k, P)\nabla d_{r_k/100}(P)$ with a real number $\sigma(k, P)$ chosen so that $\pi_k(P) \in L_{k+1}$. It is easy to show that if δ is small enough, $0.9r_{k+1} < \sigma(k, P) < 1.1r_{k+1}$. We define

$$K_0 := \{x \in \mathbb{R}^n : |x'| < 0.8, |x''| = 0\}$$

to be a standard m-disk, and let

$$K_1 := L_1 \cap \{x \in \mathbb{R}^n : |x'| < 0.8, |x'''| = 0 \text{ and } x_n > 0\}.$$

Then K_1 can be written as $K_1 = \{x + h(x)e_n : x \in K_0\}$ with $0.98r_1 < h(x) < 1.02r_1$ and $|\nabla h(x)| < 0.1$ by the implicit function theorem. Hence $g_0(x) := x + h(x)e_n$ is a bi-Lipschitz map from K_0 to K_1.

Then we define $K_{k+1} := \pi_k(K_k)$.

2.3. ***Uniform bound on the non-tangential angles.*** We need some notation for angles. If T is an m-plane, $\tilde{T}$ is the m-plane parallel to T and containing the origin, and P is a vector in $\mathbb{R}^n$, then we define $\angle(P, T) := \angle(P, \tilde{T}) := \angle(P, \hat{P})$, where $\hat{P}$ is the projection of P onto $\tilde{T}$. If T_1 is another m-plane, then $\angle(T_1, T) := \angle(\tilde{T}_1, \tilde{T}) := \sup\{\angle(P, \tilde{T}) : P \in \tilde{T}_1\}$. Now we are ready to prove the key lemma.

Lemma 2.3. *If δ is small enough, there exists a constant C_3 with $C_3\delta < 10^{-4}$ such that for any k and $P, Q \in L_k \cap B_{0.9}$ with $|P - Q| < 6r_k$, the inequality*

$$\angle(Q - P, \operatorname{span}\{T_{a,10r_k}, P\}) < C_3\delta$$

implies

$$0.9 < |\pi_k(P) - \pi_k(Q)|/|P - Q| < 1.1,$$
$$\angle(\pi_k(Q) - \pi_k(P), \operatorname{span}\{T_{\tilde{a},10r_{k+1}}, \pi_k(P)\}) < C_3\delta.$$

Here a is any of the points in K nearest to P, and $\tilde{a}$ is one of the points in K nearest to $\pi_k(P)$.

Proof. Choose a coordinate system such that $T_{a,10r_k} = \{x \in \mathbb{R}^n : |x''| = 0\}$ and $P = |P|e_n$. Then $\operatorname{span}\{T_a, 10r_k, P\} = \{x \in \mathbb{R}^n : |x'''| = 0\}$. Throughout this proof, we restrict our analysis to the inside of the box

$$U := \{x \in \mathbb{R}^n : |(x - P)'| < 8r_k,\ |(x - P)'''| < 8C_3\delta r_k,\ 0.8r_{k+1} < x_n < 1.2r_k\}.$$

Then L_k and L_{k+1} can be represented as the graphs of $(n-1)$-variable functions ϕ and ψ, respectively; for example, $L_k \cap U = \{x \in U : x_n = \phi(x', x''')\}$. By the implicit function theorem, the previous two lemmas tell us information about the derivatives of ϕ and ψ:

$$|\phi_i| = \left|\frac{-\partial_i d_{r_k/100}}{\partial_n d_{r_k/100}}\right| < 1.1C_2\delta, \quad |\psi_i| < 1.1C_2\delta \qquad \text{for } 1 \le i \le m,$$
$$|\phi_i| < 9C_3\delta + 1.1C_2\delta, \qquad |\psi_i| < 18C_3\delta + 1.1C_2\delta \quad \text{for } m+1 \le i \le n-1.$$

Then $|Q_n - |P|| < C_4\delta(|Q'| + |Q'''|)$ by the differential mean value theorem. Our angle assumption reads

$$\arctan \frac{|Q'''|}{\sqrt{|Q'|^2 + (Q_n - |P|)^2}} < C_3\delta,$$

which implies that $|Q'''| < 1.1C_3\delta|Q'|$ if δ is small enough. And $|P - Q| = (|Q'|^2 + |Q'''|^2 + (Q_n - |P|)^2)^{1/2}$, so $|Q'| \le |P - Q| < 1.01|Q'|$ if δ is small enough.

We rewrite $\pi_k(Q) = Q - \sigma(k, Q)\nabla d_{r_k/100}(Q)$ as

$$\begin{aligned}\pi_k(Q) = P &- \sigma(k, P)\nabla d_{r_k/100}(P)\\ &+ (Q', 0, 0) + (0, Q''', Q_n - |P|)\\ &+ \sigma(k, Q)(\nabla d_{r_k/100}(P) - \nabla d_{r_k/100}(Q))\\ &+ (\sigma(k, P) - \sigma(k, Q))\nabla d_{r_k/100}(P).\end{aligned}$$

Because of the estimates of the second order derivatives of $d_{r_k/100}$, the norm of the third row won't exceed $C_5\delta|Q'|$. In fact, along the Q' direction, the second order derivative is as small as $C_2\delta/r_k$ and the step length is $|Q'|$; along the $(Q-P)''$ direction, the second order derivative is bounded by $4/r_k$ but the step length is as small as $(1.1C_3 + 2C_4)\delta|Q'|$. The first row is nothing but $\pi_k(P)$. We denote the first three rows by I_1. We know $d_{r_{k+1}/100}(\pi_k(P)) = r_{k+1}$, hence

$$\begin{aligned}|d_{r_{k+1}/100}(I_1) - r_{k+1}| &< C_2\delta|Q'| + 1.1 \times (1.1C_3 + 2C_4)\delta|Q'| + 1.1 \times C_5\delta|Q'|\\ &= C_6\delta|Q'|.\end{aligned}$$

Here we use $|\partial_i d_{r_{k+1}/100}| = |\partial_i \tilde{h}_{a,r_k}| < C_2\delta$ for $1 \le i \le m$ and $|\nabla d_{r_{k+1}/100}| < 1.1$. But $d_{r_{k+1}/100}(\pi_k(Q)) = r_{k+1}$, and $\partial_\zeta d_{r_{k+1}/100} > 0.8$ near $\pi_k(Q)$ with $\zeta := \nabla d_{r_k/100}(P)$. Therefore, we have $|\sigma(k, P) - \sigma(k, Q)| < 1.25C_6\delta|Q'|$.

Now we consider $E := P - \sigma(k, P)\nabla\tilde{d}_{a,r_k}(P)$, which equals $|P|e_n - \sigma(k, P)\tau_1 e_n$ for $0.95 < \tau_1 < 1.05$ by Lemma 2.1, so $E := \tau_2 r_k e_n$ with $0.4 < \tau_2 < 0.6$ if one notes that $|P|$ and $\sigma(k, P)$ are approximately r_k and $r_k/2$ respectively. Now

$$\begin{aligned}F &:= Q - \sigma(k, Q)\nabla\tilde{d}_{a,r_k}(Q)\\ &= (Q', 0) + (0, Q'') - \sigma(k, Q)\nabla\tilde{d}_{a,r_k}((Q', 0) + (0, Q'')).\end{aligned}$$

Since $1.05r_k > |Q''| > 0.95r_k$, by Lemma 2.1 we have

$$\tilde{d}_{a,r_k}((Q', 0) + (0, Q'')) = \tau_3(0, Q'')/|Q''| \quad \text{for } 0.95 < \tau_3 < 1.05.$$

So $F = (Q', 0) + \tau_4(0, Q'')$ with $0.4 < \tau_4 < 0.6$. Therefore

$$\tan\angle(F - E, \operatorname{span}\{T_{a,10r_k}, P\}) = \frac{|(F-E)'''|}{\sqrt{|(F-E)'|^2 + (F-E)_n^2}} = \frac{|\tau_4 Q'''|}{\sqrt{|Q'|^2 + (F-E)_n^2}}.$$

But

$$(F - E)_n = (Q_n - |P|) + (\sigma(k, P) - \sigma(k, Q))\tau_1 + \sigma(k, Q)(\partial_n\tilde{d}_{a,r_k}(P) - \partial_n\tilde{d}_{a,r_k}(Q)),$$

so $|(F-E)_n| < (2C_4 + 2C_6 + C_5)\delta|Q'|$. Then if δ is small enough, we have

$$\tan\angle(F - E, \operatorname{span}\{T_{a,10r_k}, P\}) < 0.61|Q'''|/|Q'| < 0.7C_3\delta,$$

and $|Q'| \le |F - E| < 1.01|Q'|$.

We know

$$\pi_k(P) = E - \sigma(k, P)\nabla\tilde{h}_{a,r_k}(P) \quad \text{and} \quad \pi_k(Q) = F - \sigma(k, Q)\nabla\tilde{h}_{a,r_k}(Q).$$

So we define

$$G := F - \sigma(k, P)\nabla\tilde{h}_{a,r_k}(P) \quad \text{and} \quad H := G + (\sigma(k, P) - \sigma(k, Q))\nabla\tilde{h}_{a,r_k}(P).$$

Then

$$\pi_k(Q) = H + \sigma(k, Q)(\nabla\tilde{h}_{a,r_k}(P) - \nabla\tilde{h}_{a,r_k}(Q)) \quad \text{and} \quad G - \pi_k(P) = F - E.$$

However, $|H - G| < 1.25C_2C_6\delta^2|Q'| < C_2\delta|Q'|$ (if $C_6\delta < 0.8$) implies

$$\angle(H - \pi_k(P), G - \pi_k(P)) < 2C_2\delta,$$

and $0.99|Q'| < |H - \pi_k(P)| < 1.01|Q'|$ (if $C_2\delta < 10^{-3}$). Moreover

$$|\pi_k(Q) - H| < 0.55r_k \cdot C_2\delta/r_k \cdot 1.01|Q'| < C_2\delta|Q'|$$

implies

$$\angle(\pi_k(Q) - \pi_k(P), H - \pi_k(P)) < 2C_2\delta,$$

and $0.97|Q'| < |\pi_k(Q) - \pi_k(P)| < 1.03|Q'|$, which means

$$0.9 < \frac{|\pi_k(P) - \pi_k(Q)|}{|P - Q|} < 1.1.$$

The three angle inequalities above give

$$\angle(\pi_k(Q) - \pi_k(P), \operatorname{span}\{T_{a,10r_k}, P\}) < (0.7C_3 + 4C_2)\delta.$$

On the other hand,

$$|(\pi_k(P))'''| < |\sigma(k, P)\nabla\tilde{h}_{a,r_k}(P)| < C_2\delta r_k \quad \text{and} \quad (\pi_k(P))_n > 0.4r_k$$

show that $\angle(\operatorname{span}\{T_{a,10r_k}, \pi_k(P)\}, \operatorname{span}\{T_{a,10r_k}, P\}) < 3C_2\delta$. We also know that $\angle(T_{\tilde{a},10r_{k+1}}, T_{a,10r_k}) < 4\delta$, because K is close to either of these two m-planes. So finally, we can conclude

$$\angle(\pi_k(Q) - \pi_k(P), \operatorname{span}\{T_{\tilde{a},10r_{k+1}}, \pi_k(P)\}) < (0.7C_3 + 7C_2 + 4)\delta < C_3\delta$$

if we let $0.3C_3 > 7C_2 + 4$. □

K_1 satisfies the angle condition in the lemma above if $C_3 > 21$, so all the of K_k satisfy the angle condition and all of the $\pi_k|_{K_k}$ are bi-Lipschitz by induction.

2.4. ***Bi-Hölder parametrization.*** Define $g_k : K_1 \to K_{k+1},\ x \mapsto = \pi_k \circ \cdots \circ \pi_1(x)$ and $f : K_1 \to K,\ x \mapsto \lim_{k\to\infty} g_k(x)$. We are ready to prove f is bi-Hölder continuous. Given x and $y \in K_1$, we write $\ell := |f(x) - f(y)|$. First notice that $|x-y| > 0$ implies $\ell > 0$ since

$$0.9^k |x-y| \le |g_k(x) - g_k(y)| \le 3r_k + \ell \quad \text{for all } k.$$

Now we choose k with $r_{k+1} < 0.01\ell \le r_k$, which implies $-\log_2 \ell - 1 < k \le -\log_2 \ell$. The triangle inequality gives

$$0.9\ell < |g_k(x) - g_k(y)| < 1.1\ell.$$

However $0.9^k|x-y| \le |g_k(x)-g_k(y)| \le 1.1^k|x-y|$, so we obtain

$$0.5|x-y|^\beta < \ell < 2|x-y|^\alpha \quad \text{with } \alpha = \frac{1}{1+\log_2 1.1} \text{ and } \beta = \frac{1}{1+\log_2 0.9}.$$

2.5. ***Completeness of the approximating surfaces.*** To show $f(K_1) \supset K \cap B^n_{1/2}$, we have to show K_k is complete enough so that any point of $K \cap B^n_{1/2}$ is close to it. A short topological argument suffices.

Lemma 2.4. *For any k and a given point $P \in K_k$, there exists an m-plane $\tilde{T}_P$ going through P with $\angle(\tilde{T}_P, T_{a,10r_k}) < 20\delta$, where a is the point in K nearest to P, such that $K_k \cap B^n_{5.1r_k}(P)$ contains the set $\{x + \xi(x) : x \in \tilde{T}_P \cap B^n_{5r_k}(P)\}$, where $\xi(x)$ is a vector-valued function defined on $\tilde{T}_P \cap B^n_{5r_k}(P)\}$ with $\xi(x) \perp \tilde{T}_P$ and $|\xi(x)| < 5(C_3 + 1.1C_2)\delta r_k$.*

Proof. We prove the lemma by induction. We know it is true for K_1, and assume it is for K_k. Given $Q \in K_{k+1}$, define $\tilde{T}_Q := T_{\tilde{a},10r_{k+1}} + (Q - \tilde{a})$ with $\tilde{a}$ being the point in K nearest to Q. For any point $y \in \tilde{T}_Q \cap B^n_{5r_{k+1}}(Q)$, we want to show that the $(n-m)$-plane $N_y := \{x \in \mathbb{R}^n : x - y \perp \tilde{T}_Q\}$ must contain a point of K_{k+1}. Define $P := (\pi_k)^{-1}(Q) \in K_k$. Then we have $\angle(\tilde{T}_Q, \tilde{T}_P) < 23\delta$. Let $\tilde{y}$ be the point in $\tilde{T}_P$ nearest to y, and define $S_{\tilde{y}} := \{x \in \tilde{T}_P : |x - \tilde{y}| = r_k\}$ and $S_y := S_{\tilde{y}} + (y - \tilde{y})$. From Lemma 2.3, we know, for all $x \in S_{\tilde{y}}$, that

$$\begin{aligned} \operatorname{dist}(\pi_k(x+\xi(x)), S_y) &\le |\pi_k(x+\xi(x)) - (x + y - \tilde{y})| \\ &< |\xi(x)| + |\sigma(k, x+\xi(x))\nabla d_{r_k/100}(x+\xi(x)) - (y-\tilde{y})| \\ &< 0.3r_k. \end{aligned}$$

The $(m-1)$-sphere S_y cannot contract to a point without passing through N_y, and neither can the topological $(m-1)$-sphere $\{\pi_k(x+\xi(x)) : x \in S_{\tilde{y}}\}$. So if $N_y \cap K_{k+1} = \varnothing$, then $\{\pi_k(x+\xi(x)) : x \in S_{\tilde{y}}\}$ cannot contract to a point within K_{k+1}, which contradicts that K_{k+1} is a topological m-disk. Once we have a point $z \in N_y \cap K_{k+1}$, it is easy to see z is unique and $|z - y| < 5(C_3 + 1.1C_2)\delta r_{k+1}$ from Lemma 2.3. $\square$

Lemma 2.5. *For any point $b \in K \cap B^n_{1/2}$, we have* $\operatorname{dist}(b, K_k) < 1.1r_k$ *for all k.*

Proof. We prove the lemma by induction. It is true for K_1, and assume it is for K_k. Then we have $P \in K_k$ with $|b-P| < 1.1r_k$. Let a be the point in K nearest to P. Choose a local coordinate system such that $a = O$ and such that $T_{a,10r_k} = \{x \in \mathbb{R}^n : |x''| = 0\}$. Then $|b'| < r_k$ and $|b''| < 10\delta r_k$. Write $Q := \pi_k(P)$ and $E := (b', Q'')$, and let F be the point in $\tilde{T}_Q$ nearest to E. Then $|Q''| < 1.02r_{k+1}$ and $|Q'| < r_{k+1}$. So $E - Q \in T_{a,10r_k}$ and $\angle(\tilde{T}_Q, T_{a,10r_k}) < 23\delta$ implies $|E-F| < 30\delta r_k$. Therefore,

$$\begin{aligned}\operatorname{dist}(b, K_{k+1}) &\le |b-(F+\xi(F))| \\ &\le |b-E| + |E-F| + |\xi(F)| \\ &\le |b''| + |Q''| + 30\delta r_k + 5(C_3 + 1.1C_2)\delta r_{k+1} < 1.1r_{k+1}. \qquad \square\end{aligned}$$

Corollary 2.6. $K \cap B^n_{1/2} \subset f(K_1)$.

Proof. Given $b \in K \cap B^n_{1/2}$, for every k we have

$$\operatorname{dist}(b, K_k) < 1.1r_k \quad \text{and} \quad HD(K_k, f(K_1)) < 1.1r_k.$$

Then $\operatorname{dist}(b, f(K_1)) < 2.2r_k$. But $r_k \to 0$ and $f(K_1) \cap \bar{B}^n_{1/2}$ is closed. $\square$

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