Pacific Journal of Mathematics

GLOBAL CLASSICAL SOLUTIONS TO HYPERBOLIC GEOMETRIC FLOW ON RIEMANN SURFACES

FAGUI LIU AND YUANZHANG ZHANG

Volume 246 No. 2 June 2010

PACIFIC JOURNAL OF MATHEMATICS
Vol. 246, No. 2, 2010

GLOBAL CLASSICAL SOLUTIONS TO HYPERBOLIC GEOMETRIC FLOW ON RIEMANN SURFACES

FAGUI LIU AND YUANZHANG ZHANG

We consider the Cauchy problem for hyperbolic geometric flow equations introduced recently by Kong and Liu motivated by the Einstein equation and Hamilton Ricci flow, and obtain a necessary and sufficient condition for the global existence of classical solutions to this kind of flow on Riemann surfaces. The results show that the scalar curvature of the solution metric g_{ij} converges to one of flat curvature, and the hyperbolic geometric flow has the advantage that the surgery technique may be replaced by choosing a suitable initial velocity tensor.

1. Introduction

Let $\mathcal{M}$ be an n-dimensional complete Riemann manifold with Riemann metric g_{ij}. The general evolution equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathcal{F}\left(g, \frac{\partial g}{\partial t}\right) = 0$$

for the metric g_{ij} has been recently introduced by Kong and Liu [2007] and called the generalized *hyperbolic geometric flow* (denoted by HGF). Here $\mathcal{F}$ are some given smooth functions of the Riemann metric g and its first derivative with respect to t, and R_{ij} are the components of the Ricci curvature tensor. In this paper, we may the HGF the Kong–Liu hyperbolic geometric flow. The local existence and nonlinear stability have been proved by Dai, Kong and Liu [$\geq$ 2010; 2008]. See the celebrated survey paper [Kong 2008] for progress on this new topic.

In this paper, we study the evolution of a Riemann metric g_{ij} on a Riemann surface $\mathcal{M}$ by its Ricci curvature tensor R_{ij} under the HGF equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}.$$

MSC2000: primary 30F45, 58J45; secondary 35L65, 35L45.
Keywords: hyperbolic geometric flow, Riemann surface, quasilinear hyperbolic system, classical solution.
Supported by the NSF of China (grant 10571024) and the Basic Research of Henan Province.

Here, we are interested in the initial metric

$$ds^2 = u_0(x)(dx^2 + dy^2) \quad \text{at } t = 0 \tag{1}$$

on a surface of topological type $\mathbb{R}^2$, where $u_0(x)$ is a C^2 function with bounded C^2 norm and satisfies

$$0 < k \leq u_0(x) \leq M < \infty, \tag{2}$$

where k and M are positive constants.

On this surface, the HGF equation simplifies because all of the information about curvature is contained in the scalar curvature function R. In our notation, $R = 2K$, where K is the Gauss curvature. The Ricci curvature is given by $R_{ij} = \frac{1}{2} R g_{ij}$, so the HGF equation simplifies to

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -R g_{ij}. \tag{3}$$

The metric for a surface can always be written (at least locally) as $g_{ij} = u(t, x, y)\delta_{ij}$, where $u(t, x, y) > 0$, and δ_{ij} is Kronecker's symbol. Therefore, we have

$$R = -\frac{\Delta \ln u}{u}. \tag{4}$$

Thus, Equation (3) reduces to $u_{tt} - \Delta \ln u = 0$.

The initial data $u_0(x)$ depends only on x and not y; therefore, we may consider the Cauchy problem

$$\begin{cases} u_{tt} - (\ln u)_{xx} = 0, \\ \quad u = u_0(x) \quad \text{at } t = 0, \\ u_t = u_1(x) \quad \text{at } t = 0, \end{cases} \tag{5}$$

where $u_1(x) \in C^1$ with bounded C^1 norm.

By using the transformation

$$\phi = \ln u, \tag{6}$$

Kong and Liu have proved the following theorem.

Theorem 1.1 [Kong et al. 2009]. *Suppose that $u_1(x) \geq |u_0'(x)|/\sqrt{u_0(x)}$ for all $x \in \mathbb{R}$. Then, the Cauchy problem (5) admits a unique global solution for all $t \in \mathbb{R}$.*

Moreover, if $u_1(x) \equiv u_0'(x)/\sqrt{u_0(x)}$, and there exists a point $x_0 \in \mathbb{R}$ such that $u_0'(x_0) < 0$, then the Cauchy problem (5) admits a unique classical solution only in $[0, T) \times \mathbb{R}$, where

$$T = -\frac{2}{\inf_x \{u_0'(x) u_0^{-3/2}(x)\}}.$$

In this paper, we will prove the following theorems without using (6).

Theorem 1.2. *Suppose that*

$$u_1(x) + \frac{u_0'(x)}{\sqrt{u_0(x)}} \geq 0 \quad \textit{for all } x \in \mathbb{R}, \tag{7}$$

$$u_1(x) - \frac{u_0'(x)}{\sqrt{u_0(x)}} \geq 0 \quad \textit{for all } x \in \mathbb{R}. \tag{8}$$

Then (5) admits a unique global solution for all $t \in \mathbb{R}$.

Theorem 1.3. *If there exists a point $x_0 \in \mathbb{R}$, such that*

$$u_1(x_0) + \frac{u_0'(x_0)}{\sqrt{u_0(x_0)}} < 0, \tag{9}$$

or there exists a point $x_0 \in \mathbb{R}$ such that

$$u_1(x_0) - \frac{u_0'(x_0)}{\sqrt{u_0(x_0)}} < 0, \tag{10}$$

then (5) admits a unique classical solution only in $[0, T) \times \mathbb{R}$.

Remark 1.4. Following Theorem 1.2, it is easy to see that Cauchy problem

$$\begin{cases} \dfrac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij} & \text{for } i, j = 1, 2, \\ g_{ij} = u_0(x)\delta_{ij} & \text{for } t = 0, \\ \dfrac{\partial g_{ij}}{\partial t} = u_1(x)\delta_{ij} & \text{for } t = 0 \text{ and } i, j = 1, 2 \end{cases} \tag{11}$$

has a unique smooth solution for all time $t \in \mathbb{R}$, and the solution metric g_{ij} assumes the form

$$g_{ij} = u(x, t)\delta_{ij} \quad \text{for } i, j = 1, 2. \tag{12}$$

Theorems 1.2 and 1.3 will be proved in Sections 3 and 4. Using Theorem 1.2, we can further prove this:

Theorem 1.5. *Suppose that*

$$\inf_x \left\{ u_1(x) + \frac{u_0'(x)}{\sqrt{u_0(x)}} \right\} > 0 \quad \textit{and} \quad \inf_x \left\{ u_1(x) - \frac{u_0'(x)}{\sqrt{u_0(x)}} \right\} > 0. \tag{13}$$

Then (11) has a unique classical solution of the form (12) for all time. Moreover the scalar curvature $R(x, t)$ corresponding to the solution metric g_{ij} satisfies

$$R(x, t) \to 0 \quad \textit{as } t \to +\infty,$$

and there exists positive constant $k_1 > 0$ independent of t and x such that

$$R(x, t) \leq k_1 \quad \textit{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Theorem 1.5 shows that any metric on a simply connected noncompact surface converges to one of flat curvature by choosing a suitable initial velocity tensor $\partial g_{ij}(x,0)/\partial t$, but the general case of a metric in $\mathbb{R}^2$ is still open. Theorem 1.3 shows that if we do not choose suitable a initial velocity tensor $\partial g_{ij}(x,0)/\partial t$, the solution to Cauchy problem will blow up in finite time.

2. Preliminaries

We need only discuss the classical solution on $t \geq 0$. The result for $t \leq 0$ can be obtained by changing t to $-t$.

Let

$$u_t = v \quad \text{and} \quad u_x = w. \tag{14}$$

Then, it follows from the first equation in (5) and (14) that

$$u_t = v, \quad w_t - v_x = 0, \quad v_t - (1/u)w_x = -w^2/u^2. \tag{15}$$

It is easy to see that the eigenvalues of equations (15) are

$$\lambda_1 = -\lambda, \quad \lambda_2 = 0, \quad \lambda_3 = \lambda, \quad \lambda = 1/\sqrt{u},$$

and the matrix $L(U)$ (where $U = (u, w, v)^T$) of left eigenvectors and the matrix $R(U)$ of right eigenvectors are respectively

$$L(U) = \begin{pmatrix} l_1(U) \\ l_2(U) \\ l_3(U) \end{pmatrix} = \begin{pmatrix} 0 & \lambda & 1 \\ 1 & 0 & 0 \\ 0 & -\lambda & 1 \end{pmatrix},$$

$$R(U) = (r_1(U), r_2(U), r_3(U)) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda & 0 & -\lambda \\ 1 & 0 & 1 \end{pmatrix}.$$

Since $\nabla\lambda_i(U)r_i(U) \equiv 0$ for $i = 1, 2, 3$, (15) is a linear degenerate strict hyperbolic system.

Set

$$p = v + \lambda w \quad \text{and} \quad q = v - \lambda w. \tag{16}$$

Lemma 2.1. *p and q satisfy*

$$p_t - \lambda p_x = \tfrac{1}{4}\lambda^2 p(q-p), \tag{17a}$$

$$u_t = \tfrac{1}{2}(p+q), \tag{17b}$$

$$q_t + \lambda q_x = \tfrac{1}{4}\lambda^2 q(p-q). \tag{17c}$$

Proof. Noting that $\lambda_t = -\frac{1}{2}\lambda^3 v$ and $\lambda_x = -\frac{1}{2}\lambda^3 w$, we calculate

$$\begin{aligned} p_t - \lambda p_x &= (v + \lambda w)_t - \lambda (v + \lambda w)_x \\ &= v_t - \lambda v_x + \lambda (w_t - \lambda w_x) + w(\lambda_t - \lambda \lambda_x) \\ &= -\tfrac{1}{2}\lambda^3 w(v + \lambda w) = \tfrac{1}{4}\lambda^2 (q - p)p. \end{aligned}$$

In a similar way, we can prove (17c), and (17b) is obvious. ☐

Let

$$r = p_x + \tfrac{1}{8}\lambda pq \quad \text{and} \quad s = q_x - \tfrac{1}{8}\lambda pq.$$

Lemma 2.2. *r and s satisfy*

$$r_t - \lambda r_x = \tfrac{1}{4}\lambda^2 (2q - 3p) r + \tfrac{1}{32}\lambda^3 p^2 (4p - 5q), \tag{18}$$

$$s_t + \lambda s_x = \tfrac{1}{4}\lambda^2 (2p - 3q) s + \tfrac{1}{32}\lambda^3 q^2 (5p - 4q). \tag{19}$$

Proof. Let

$$L_1 = \frac{\partial}{\partial t} - \lambda \frac{\partial}{\partial x} \quad \text{and} \quad L_2 = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}.$$

Then,

$$\begin{aligned} L_1 p_x &= \tfrac{1}{4}\lambda^2 ((2q - 3p) p_x + p q_x) + \tfrac{1}{8}\lambda^3 p (p - q)^2, \\ L_2 q_x &= \tfrac{1}{4}\lambda^2 ((2p - 3q) q_x + q p_x) - \tfrac{1}{8}\lambda^3 q (p - q)^2. \end{aligned}$$

Noting that

$$\begin{aligned} L_1(\lambda p) &= -\tfrac{1}{4}\lambda^3 p (q + p), & L_2(\lambda q) &= -\tfrac{1}{4}\lambda^3 q (p + q), \\ L_1 q &= -\tfrac{1}{16}\lambda^3 p q^2 - \tfrac{1}{8} L_1(\lambda p q), & L_2 p &= \tfrac{1}{16}\lambda^3 q p^2 + \tfrac{1}{8} L_2(\lambda p q), \end{aligned}$$

by a direct calculation, we easily prove (18) and (19). ☐

Noting that (5), (14) and (16), we write at $t = 0$

$$\begin{aligned} &p = p_0(x) \equiv u_1(x) + \lambda_0(x) u_0'(x), \quad u = u_0(x), \\ &q = q_0(x) = u_1(x) - \lambda_0(x) u_0'(x), \end{aligned} \tag{20}$$

where $\lambda_0(x) = 1/\sqrt{u_0(x)}$.

Theorem 2.3 (the existence domain $D(T)$ of the classical solution to (5)). *If there exists a positive constant M_1 such that*

$$0 \le p(x,t) \le M_1 \quad \textit{and} \quad 0 \le q(x,t) \le M_1, \tag{21}$$

then on $D(T)$,

$$\begin{aligned} &|u(x,t)| \le M(T), \quad |u_x(x,t)| \le M(T), \quad |u_t(x,t)| \le M(T), \\ &|r(x,t)| \le M(T), \quad |s(x,t)| \le M(T), \end{aligned}$$

where $M(T)$ is a positive constant, and

$$D(T) = \{(x,t) \mid x \in \mathbb{R},\ 0 \le t \le T,\ T > 0\}.$$

Then, by the local existence theorem of the classical solution to quasilinear hyperbolic systems, (5) *admits a unique global classical solution on $t \ge 0$.*

Remark 2.4. Theorem 2.3 shows that if a singularity occurs in a C^1 solution to Cauchy problem (17) and (20) in finite time, then the solution itself should go to infinity at the starting point of the singularity [Li and Liu 2003].

Proof of Theorem 2.3. Through any point (t,x), let

$$x = x_1(t,\beta_1), \quad x = x_2(t,\beta_2), \quad x = x_3(t,\beta_3)$$

be the λ_1, λ_2 and λ_3 characteristics, respectively, that satisfy

$$\frac{dx_1}{dt} = \lambda_1 = -\lambda, \qquad \frac{dx_2}{dt} = \lambda_2 = 0, \qquad \frac{dx_3}{dt} = \lambda_3 = \lambda,$$
$$x_1(0,\beta_1) = \beta_1, \qquad x_2(0,\beta_2) = \beta_2, \qquad x_3(0,\beta_3) = \beta_3.$$

Along the λ_2 characteristics, integrating (17b) yields

$$u(x,t) = u_0(\beta_2) + \frac{1}{2}\int_0^t (p+q)(x_2(\tau,\beta_2),\tau)\,d\tau. \tag{22}$$

Then, it follows from (17b), (21) and (22) that

$$|u_t| \le M_1(T) \quad \text{and} \quad 0 < \inf_x u_0(x) \le u(x,t) \le M_2(T);$$

hereafter the $M_i(T)$ for $i = 1, 2, \ldots$ denote positive constants.

Along the λ_1 characteristics $x = x_1(t,\beta_1)$, we have by integrating (18) and noting (21) that

$$|r(x,t)| \le M_2(T) + M_3(T)\int_0^t R(\tau)\,d\tau, \quad \text{where } R(t) = \sup_x |r(x,t)|,$$

Thus, by the Bellman lemma, we get $|r(x,t)| \le M_4(T)$. Similarly, we can prove $|s(x,t)| \le M_5(T)$. Noting that $(u_x)_t = \frac{1}{2}(r+s)$, it is easy to see that

$$|u_x(x,t)| \le M_6(T). \qquad \square$$

3. Global classical solution: Proof of Theorem 1.2

According to the local existence and uniqueness theorems of the classical solutions to the quasilinear hyperbolic systems [Li and Yu 1985], to prove Theorem 1.2 it suffices to establish uniform *a priori* estimates of the C^1 norms of p, q and u.

Lemma 3.1 [Hong 1995; Kong 1998]. *Consider*

$$u_t + \lambda_1(x,t)u_x = A(x,t)(u-v),$$
$$v_t + \lambda_2(x,t)v_x = B(x,t)(v-u),$$

where λ_1, λ_2, A and B are continuous functions, and $\lambda_1 \le \lambda_2$. If A and B are both nonpositive, then

$$\min(u_0(x), v_0(x)) \le u(x,t), v(x,t) \le \max(u_0(x), v_0(x)).$$

Lemma 3.2. *On the existence domain of the classical solution to the Cauchy problem (17) and (20), if (7) and (8) hold, then*

$$0 \le p(x,t) \le \sup_{x\in\mathbb{R}} p_0(x), \tag{23}$$
$$0 \le q(x,t) \le \sup_{x\in\mathbb{R}} q_0(x), \tag{24}$$
$$0 < \inf_{x\in\mathbb{R}} u_0(x) \le u(x,t) \le \sup_x u_0(x) + Ct, \tag{25}$$

where $C > 0$ is a constant.

Proof. Along the λ_1 characteristics, we have

$$p(x,t) = p_0(\beta_1)\exp\Big(\int_0^t \tfrac{1}{4}\lambda^2(q-p)(x_1(\tau,\beta_1),\tau)\,d\tau\Big).$$

By (7) and (20), we have $p_0(x) \ge 0$ for all $x \in \mathbb{R}$. Thus, we obtain $p(x,t) \ge 0$. Similarly, we can prove $q(x,t) \ge 0$. By these two inequalities, we have

$$\tfrac{1}{4}\lambda^2 p \ge 0 \quad \text{and} \quad \tfrac{1}{4}\lambda^2 q \ge 0.$$

Thus, following Lemma 3.1, it is easy to see that

$$p(x,t) \le \sup_x p_0(x) \quad \text{and} \quad q(x,t) \le \sup_x q_0(x).$$

Integrating (17b), we get

$$u(x,t) = u_0(\beta_2) + \frac{1}{2}\int_0^t (p+q)(x_2(\tau,\beta_2),\tau)\,d\tau.$$

Thus, because $p(x,t) \ge 0$ and $q(x,t) \ge 0$, we obtain (25). □

Proof of Theorem 1.2. It follows easily now from Lemma 3.2 and Theorem 2.3. □

Remark 3.3. By (4) and (25), and under the assumptions of Theorem 1.2, we have

$$|R(x,t)| \le M_7(T).$$

4. Blow-up of classical solutions: Proof of Theorem 1.3

We now investigate the blow-up phenomena of the hyperbolic geometric flow.

Let

$$m = \sqrt{\lambda} p \quad \text{and} \quad n = \sqrt{\lambda} q. \tag{26}$$

Noting that

$$\tfrac{1}{4}\lambda^2 q = \tfrac{1}{4}((\ln u)_t - \lambda(\ln u)_x) \quad \text{and} \quad \tfrac{1}{4}\lambda^2 p = \tfrac{1}{4}((\ln u)_t + \lambda(\ln u)_x),$$

then, following (14), (16), and (17), we can prove the following lemma.

Lemma 4.1. *m and n satisfy*

$$m_t - \lambda m_x = -\tfrac{1}{4}\lambda^{3/2} m^2, \tag{27}$$

$$n_t + \lambda n_x = -\tfrac{1}{4}\lambda^{3/2} n^2. \tag{28}$$

At $t = 0$, set

$$\begin{aligned} m = m_0(x) &= \frac{1}{\sqrt[4]{u_0(x)}}\left(u_1(x) + \frac{u_0'(x)}{\sqrt{u_0(x)}}\right), \\ n = n_0(x) &= \frac{1}{\sqrt[4]{u_0(x)}}\left(u_1(x) - \frac{u_0'(x)}{\sqrt{u_0(x)}}\right). \end{aligned} \tag{29}$$

Proof of Theorem 1.3. Without loss of generality, we suppose that (9) holds; if (10) holds, we proceed similarly.

It follows from (27) and (28) that $m_t - \lambda m_x \le 0$ and $n_t + \lambda n_x \le 0$. Thus, it is easy to see that

$$m(x,t) + n(x,t) \le M_0 \quad \text{and} \quad M_0 \equiv \sup m_0(x) + \sup n_0(x). \tag{30}$$

Noting that $u_0(x) \ge k > 0$, it follows from (9) and (29) that $m_0(x_0) < 0$.

Integrating (27), along λ_1 characteristics, we get

$$m(x_0,t) = m_0(x_0)/F(t,x_0), \tag{31}$$

where

$$F(t,x_0) = 1 + m_0(x_0)/4 \int_0^t \lambda^{3/2}(x_1(x_0,\tau),\tau)\,d\tau \quad \text{and} \quad \lambda^{3/2} = u^{-3/4}. \tag{32}$$

By (17b) and (26), it is easy to see that $(u^{3/4})_t = \frac{3}{8}(m+n)$. Therefore, we have

$$u^{3/4}(x,t) = u_0^{3/4}(x_0) + \frac{3}{8}\int_0^t (m+n)(x_2(x_0,\tau),\tau)\,d\tau. \tag{33}$$

By (2), (30) and (33), we have

$$u^{3/4}(x,0) \ge k^{3/4} \quad \text{and} \quad u^{3/4}(x,t) \le M^{3/4} + \tfrac{3}{8} M_0 t. \tag{34}$$

Case (i). If $M_0 < 0$, then there exists $\tau_0 = 8M^{3/4}/(3(-M_0)) > 0$, such that

$$u(x,t) \le 0 \quad \text{and} \quad t \ge \tau_0.$$

This implies the system in (5) is meaningless for $t \ge \tau_0$, that is, it admits a unique local classical solution.

Case (ii). If $M_0 = 0$, then, by (32) and (34), it is easy to find that

$$F(x_0,t) \le 1 + \tfrac{1}{4} m_0(x_0) M^{-3/4} t.$$

Since $F(x_0,0) = 1 > 0$ and $m_0(x_0) < 0$, there exists $t_0 = 4M^{3/4}/(-m_0(x_0)) > 0$, such that

$$F(x_0,t) \to 0^+ \quad \text{as } t \to t_0^-. \tag{35}$$

So that there exists finite time $T = T(x_0) > 0$, such that

$$m(x_0,t) \to -\infty \quad \text{as } t \to T^-. \tag{36}$$

Case (iii). If $M_0 > 0$, then, it follows from (32) and (34) that

$$F(x_0,t) \le 1 + \frac{2m_0(x_0)}{3M_0} \ln\left(1 + \frac{3M_0}{8M^{3/4}} t\right).$$

Thus, noting that $F(x_0,0) = 1 > 0$ and $m_0(x_0) < 0$, there exists $t_* > 0$ such that $F(x_0,t) \to 0^+$ as $t \to t_*^-$, and then (36) follows. ☐

5. Asymptotic behavior: Proof of Theorem 1.5

We next will give the asymptotic behavior of the scalar curvature $R(x,t)$.

Proof of Theorem 1.5. If (13) holds, then, by Lemmas 3.1 and 3.2, we have

$$C_1 \le p(x,t) \le C_2 \quad \text{and} \quad C_1 \le q(x,t) \le C_2,$$

where here and below C_i for $i = 1, 2, \dots$ denote positive constants independent of t and x. Thus, by (2) and (22), one can get

$$C_3(1+t) \le u(x,t) \le C_4(1+t), \tag{37}$$

and then

$$C_3^{3/4}(1+t)^{3/4} \le u^{3/4}(x,t) \le C_4^{3/4}(1+t)^{3/4} \le C_4^{3/4}(1+t).$$

It follows from (32) and (37) that

$$C_5(1+\ln(1+t)) \le F(x,t) \le C_6(1+(1+t)^{1/4}).$$

Therefore, by (13) and (31), we obtain

$$0 \le \frac{C_7}{1+(1+t)^{1/4}} \le m(x,t) \le \frac{C_8}{1+\ln(1+t)}. \tag{38}$$

Similarly, we have

$$0 \le \frac{C_7}{1+(1+t)^{1/4}} \le n(x,t) \le \frac{C_8}{1+\ln(1+t)}. \tag{39}$$

Thus, $m(x,t) \to 0$ and $n(x,t) \to 0$ as $t \to +\infty$.

Noting (37), (38) and (39), we have

$$p = \frac{1}{\sqrt{\lambda}} m = u^{1/4} m, \quad q = \frac{1}{\sqrt{\lambda}} n = u^{1/4} n, \quad u_x = \frac{p-q}{2\lambda} = \tfrac{1}{2} u^{3/4}(m-n),$$

and

$$-\frac{C_8}{1+\ln(1+t)} \le m(x,t) - n(x,t) \le \frac{C_8}{1+\ln(1+t)}. \tag{40}$$

Thus, it is easy to see that

$$|u_x| \le C_9 \frac{(1+t)^{3/4}}{1+\ln(1+t)}. \tag{41}$$

It is easy to derive that

$$u_{xx} = \frac{p_x - q_x}{2\lambda} + \tfrac{1}{2}\lambda^2 u_x^2 = \tfrac{1}{2} u^{1/2}(p_x - q_x) + \frac{1}{2u} u_x^2. \tag{42}$$

Let $\bar{p} = p/u$ and $\bar{q} = q/u$.Then, following [Kong et al. 2009], we have

$$L_1 \bar{p}_x = -A_1 \bar{p}_x - B_1 \bar{q}_x, \tag{43}$$

$$L_2 \bar{q}_x = -A_2 \bar{p}_x - B_2 \bar{q}_x, \tag{44}$$

where

$$A_1 = \tfrac{1}{4}(2\bar{q} + 3\bar{q}), \quad B_1 = \tfrac{3}{4}\bar{p}, \quad A_2 = \tfrac{3}{4}\bar{q}, \quad B_2 = \tfrac{1}{4}(2\bar{p} + 3\bar{q}).$$

Thus, by [Kong et al. 2009], one can get

$$|\bar{p}_x(x,t)|, |\bar{q}_x(x,t)| \le C_{10}. \tag{45}$$

Noting that $p_x - q_x = u_x(p-q)/u + u(\bar{p}_x - \bar{q}_x)$, and using (4) and (42), we have

$$R = \frac{1}{u^3}(u_x^2 - u u_{xx}) = \frac{u_x^2}{2u^3} - \frac{p_x - q_x}{2u^{3/2}} = \frac{u_x^2}{2u^3} - \frac{\bar{p}_x - \bar{q}_x}{2u^{1/2}} - \frac{u_x(m-n)}{2u^3}. \tag{46}$$

Therefore, it follows from (34), (40), (41), (45) and (46) that

$$|R(x,t)| \le \frac{C_{11}}{(1+\ln(1+t))^2 (1+t)^{3/2}} + \frac{C_{12}}{(1+t)^{1/2}} + \frac{C_{13}(1+t)^{3/4}}{(1+t)^3 (1+\ln(1+t))^2}.$$

Therefore, $R(x, t) \to 0$ as $t \to +\infty$. □

Acknowledgments

We are particularly grateful to Professor Kong Dexing for helpful discussions, and would like to thank the referee for pertinent comments and valuable suggestions.

References

[Dai et al. 2008] W.-R. Dai, D.-X. Kong, and K. Liu, "Dissipative hyperbolic geometric flow", *Asian J. Math.* **12**:3 (2008), 345–364. MR 2009k:58059 Zbl 1165.58013

[Dai et al. $\geq$ 2010] W.-R. Dai, D.-X. Kong, and K. Liu, "Hyperbolic geometric flow, I: Short-time existence and nonlinear stability", preprint. To appear in *Pure and Applied Mathematics Quarterly*.

[Hong 1995] J. X. Hong, "The global smooth solutions of Cauchy problems for hyperbolic equation of Monge–Ampère type", *Nonlinear Anal.* **24** (1995), 1649–1663. MR 96c:35119 Zbl 0830.35082

[Kong 1998] D.-X. Kong, "Maximum principle in nonlinear hyperbolic systems and its applications", *Nonlinear Anal.* **32**:7 (1998), 871–880. MR 99f:35123 Zbl 0962.35031

[Kong 2008] D.-X. Kong, "Hyperbolic geometric flow", pp. 95–110 in *Proceedings of the 4th International Conference of Chinese Mathematicians* (Hangzhou, 2007), vol. 2, Advanced Lectures in Mathematics **6**, Higher Education Press, Beijing, 2008.

[Kong and Liu 2007] D.-X. Kong and K. Liu, "Wave character of metrics and hyperbolic geometric flow", *J. Math. Phys.* **48**:10 (2007), 103508. MR 2008m:58064 Zbl 1152.81515

[Kong et al. 2009] D.-X. Kong, K. Liu, and D.-L. Xu, "The hyperbolic geometric flow on Riemann surfaces", *Comm. Partial Differential Equations* **34**:4-6 (2009), 553–580. MR 2530709 Zbl 05579317

[Li and Liu 2003] T.-T. Li and F.-G. Liu, "Singularity caused by eigenvalues for quasilinear hyperbolic systems", *Comm. Partial Differential Equations* **28**:3-4 (2003), 477–503. MR 2004b:35212

[Li and Yu 1985] T. T. Li and W. C. Yu, *Boundary value problems for quasilinear hyperbolic systems*, Duke University Math. Series **5**, Duke University Press, Durham, NC, 1985. MR 88g:35115 Zbl 0627.35001

Received December 15, 2008.

FAGUI LIU
NORTH CHINA INSTITUTE OF WATER CONSERVANCY AND HYDROELECTRIC POWER
ZHENGZHOU 450011
CHINA
liufagui@ncwu.edu.cn

YUANZHANG ZHANG
NORTH CHINA INSTITUTE OF WATER CONSERVANCY AND HYDROELECTRIC POWER
ZHENGZHOU 450011
CHINA
zhangyuanzhang@ncwu.edu.cn

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or www.pjmath.org for submission instructions.

The subscription price for 2010 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer-review and production is managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 246 No. 2 June 2010
