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Let L and H be finite-dimensional restricted Lie algebras over a perfect field $\mathbb{F}$. Suppose $u(L) \cong u(H)$, where $u(L)$ is the restricted enveloping algebra of L. We prove that $L \cong H$ if L is p-nilpotent and abelian, or if L is abelian and $\mathbb{F}$ is algebraically closed. We use these results to prove our main result, that if L is p-nilpotent, then $L/L'^p + \gamma_3(L) \cong H/H'^p + \gamma_3(H)$.

1. Introduction

Let L be a restricted Lie algebra with restricted enveloping algebra $u(L)$. We say that a particular invariant of L is *determined* by $u(L)$ if every restricted Lie algebra H also has this invariant whenever $u(L)$ and $u(H)$ are isomorphic as associative algebras. The restricted isomorphism problem asks whether the isomorphism type of L is determined by $u(L)$. This problem is motivated by the classical isomorphism problem for group rings: Is every finite group G determined by its integral group ring $\mathbb{Z}G$? The survey article [Sandling 1985] contains most of the development in this area. In the late 1980s, Roggenkamp and Scott [1987] and Weiss [1988] independently settled the group ring problem for finite nilpotent groups.

There are close analogies between restricted Lie algebras and finite p-groups. In particular, the restricted isomorphism problem is the Lie analogue of the modular isomorphism problem that asks, Given finite p-groups G and H with the property that $\mathbb{F}_pG \cong \mathbb{F}_pH$, can we deduce that $G \cong H$? Here, $\mathbb{F}_p$ denotes the field of p elements. There has been intensive investigation on the modular isomorphism problem; however the main problem is rather far from being completely answered. Unfortunately not every technique from finite p-groups can be used for restricted Lie algebras. For example, it is known that the class sums form a basis of the center of $\mathbb{F}G$. It then follows that the center of G is determined; see [Sehgal 1978, Theorem 6.6]. Whether or not the center of L is determined by $u(L)$ remains an interesting open question.

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In analogy with finite p-groups, we consider the class $\mathcal{F}_p$ of restricted Lie algebras that are finite-dimensional and p-nilpotent. Let $L \in \mathcal{F}_p$. It follows from Engel's theorem that L is nilpotent. We shall examine the nilpotence class of L in Corollary 2.2. Whether or not the nilpotence class of G is determined by $\mathbb{F}_p G$ has been considered in recent years; however no major result is reported at this time; see [Bagiński and Konovalov 2007].

We start investigating the restricted isomorphism problem by first considering the abelian case. In Proposition 2.5, we prove that if $L \in \mathcal{F}_p$ is an abelian restricted Lie algebra over a perfect field $\mathbb{F}$, then the isomorphism type of L is determined by $u(L)$. Furthermore, if $\mathbb{F}$ is algebraically closed, then every abelian restricted Lie algebra is determined by its enveloping algebra; see Corollary 2.8.

It is not clear what the next step is beyond the abelian case in both the modular isomorphism problem and the restricted isomorphism problem. Nevertheless, we have proved in [Usefi 2008] that if $L \in \mathcal{F}_p$ is a metacyclic restricted Lie algebra over a perfect field, then the isomorphism type of L is determined by $u(L)$. The main result of this paper, which will be proved in Section 3, is another contribution in this direction; a similar result for finite p-groups was proved by Sandling [1989]. For a Lie subalgebra $I \subseteq L$, we denote by I^p the restricted Lie subalgebra of L generated by all x^p for $x \in I$. Also, $\gamma_i(L)$ denotes the i-th term of the lower central series of L. Our main result is as follows:

Theorem. *Let $L \in \mathcal{F}_p$ be a restricted Lie algebra over a perfect field. Then the restricted Lie algebra $L/(L'^p + \gamma_3(L))$ is determined.*

2. Preliminaries

Let L be a restricted Lie algebra with restricted enveloping algebra $u(L)$ over a field $\mathbb{F}$. By the Poincaré–Birkhoff–Witt (PBW) theorem (see [Jacobson 1962]), we can view L as a restricted Lie subalgebra of $u(L)$. Denote by $\omega(L)$ the augmentation ideal of $u(L)$ that is the kernel of the augmentation map $\epsilon_L : u(L) \to \mathbb{F}$ induced by $x \mapsto 0$ for every $x \in L$.

Let H be another restricted Lie algebra such that $\varphi : u(L) \to u(H)$ is an algebra isomorphism. The map $\eta : L \to u(H)$ defined by $\eta = \varphi - \epsilon_H \varphi$ is a restricted Lie algebra homomorphism. Therefore, η extends to an algebra homomorphism $\bar{\eta} : u(L) \to u(H)$. In fact, $\bar{\eta}$ is an isomorphism preserving the augmentation ideals, that is, $\bar{\eta}(\omega(L)) = \omega(H)$; see [Riley and Usefi 2007] for the proof of a similar fact for Lie algebras. So, without loss of generality, we assume that $\varphi : u(L) \to u(H)$ is an algebra isomorphism that preserves the augmentation ideals.

Recall that L is said to be nilpotent if $\gamma_n(L) = 0$ for some n; the nilpotence class $\mathrm{cl}(L)$ of L is the minimal integer c such that $\gamma_{c+1}(L) = 0$. We denote by L'_p the restricted subalgebra of L generated by $L' = \gamma_2(L)$. The n-th dimension subalgebra of L is

$$D_n(L) = L \cap \omega^n(L) = \sum_{ip^j \geq n} \gamma_i(L)^{p^j};$$

see [Riley and Shalev 1995].

Recall that L is said to be in the class $\mathscr{F}_p$ if L is finite-dimensional and p-nilpotent. The *exponent* of $x \in L$, denoted by $\exp(x)$, is the least integer s such that $x^{p^s} = 0$. Whether or not $L \in \mathscr{F}_p$ is determined by the following lemma; see [Riley and Shalev 1995].

Lemma 2.1. *Let L be a restricted Lie algebra. Then $L \in \mathscr{F}_p$ if and only if $\omega(L)$ is nilpotent.*

Now, consider the graded restricted Lie algebra

$$\operatorname{gr}(L) := \bigoplus_{i \geq 1} D_i(L)/D_{i+1}(L),$$

where the Lie bracket and the p-map are induced from L. It is well known that $u(\operatorname{gr}(L)) \cong \operatorname{gr}(u(L))$ as algebras; see [Usefi 2008]. So we may identify $\operatorname{gr}(L)$ as the graded restricted Lie subalgebra of $\operatorname{gr}(u(L))$ generated by $\omega^1(L)/\omega^2(L)$. Thus, $\operatorname{gr}(L)$ is determined. We can now deduce the following:

Corollary 2.2. *Let L and H be restricted Lie algebras such that $u(L) \cong u(H)$. If $L \in \mathscr{F}_p$, then $|\operatorname{cl}(L) - \operatorname{cl}(H)| \leq 1$.*

Proof. Let $c = \operatorname{cl}(L)$. We note that

$$\gamma_n(\operatorname{gr}(L)) = \bigoplus_{i \geq n} \gamma_i(L) + D_{i+1}(L)/D_{i+1}(L) \quad \text{for every } n \geq 1.$$

Since $\operatorname{gr}(L)$ is determined, $\gamma_{c+1}(\operatorname{gr}(H)) = 0$. Hence, $\gamma_{c+1}(H) \subseteq D_{c+2}(H)$. So, $\gamma_{c+2}(H) = \gamma_{c+3}(H)$. Since H is nilpotent, it follows that $\gamma_{c+2}(H) = 0$. □

Note that $D_n(\operatorname{gr}(L)) = \bigoplus_{i \geq n} D_i(L)/D_{i+1}(L)$. Thus, $D_n(L)/D_{n+1}(L)$ is determined for every $n \geq 1$. The methods of [Ritter and Sehgal 1983] and [Riley and Usefi 2007] can be adapted to prove that $D_n(L)/D_{2n+1}(L)$ and $D_n(L)/D_{n+2}(L)$ are also determined for every $n \geq 1$. In particular, $L/D_3(L)$ is determined. We need the following analogue of [Riley and Usefi 2007, Lemma 5.1].

Lemma 2.3. *If $\varphi : u(L) \to u(H)$ is an isomorphism, then $\varphi(D_n(L) + \omega^{n+1}(L)) = D_n(H) + \omega^{n+1}(H)$ for every positive integer n.*

Now suppose that L is an abelian restricted Lie algebra. The conditions on the p-map reduce to $(x+y)^p = x^p + y^p$ and $(\alpha x)^p = \alpha^p x^p$ for every $x, y \in L$ and $\alpha \in \mathbb{F}$. Thus the p-map is a semilinear transformation. Let σ be an automorphism of $\mathbb{F}$. Consider the skew polynomial ring $\mathbb{F}[t; \sigma]$ that consists of polynomials $f(t) \in \mathbb{F}[t]$ with multiplication given by $\alpha t^i \beta t^j = \alpha \beta^{\sigma^{-i}} t^{i+j}$. It is well known that $\mathbb{F}[t; \sigma]$

is a PID. Now suppose that $\mathbb{F}$ is perfect and let σ be the automorphism given by $\sigma(\alpha) = \alpha^p$. Note that $\mathbb{F}[t;\sigma]$ acts on L by $x \cdot t = x^p$. Then, by the theory of finitely generated modules over a PID, L decomposes as a direct sum of cyclic $\mathbb{F}[t;\sigma]$-modules, with a unique number of the summands. We summarize this in the following; see also [Jacobson 1962] or [Bahturin et al. 1992, Section 4.3]. We denote by $\langle x\rangle_p$ the subalgebra generated by x.

Theorem 2.4. *Let L be a finitely generated abelian restricted Lie algebra over a perfect field $\mathbb{F}$. Then there exist a unique integer n and generators $x_1, \dots, x_n \in L$ such that $L = \langle x_1\rangle_p \oplus \cdots \oplus \langle x_n\rangle_p$.*

Proposition 2.5. *Let $L \in \mathcal{F}_p$ be an abelian restricted Lie algebra over a perfect field $\mathbb{F}$. If H is a restricted Lie algebra such that $u(L) \cong u(H)$, then $L \cong H$.*

Proof. We argue by induction on $\dim_{\mathbb{F}} L$. Let A be the subalgebra of $\omega(L)$ generated by all u^p, where $u \in \omega(L)$. We observe that $A \cong \omega(L^p)$ as algebras. Thus there is an induced isomorphism $\omega(L^p) \cong \omega(H^p)$. Since $L \in \mathcal{F}_p$, we have $\dim_{\mathbb{F}} L^p < \dim_{\mathbb{F}} L$. Thus, by the induction hypothesis, there exists a restricted Lie algebra isomorphism $\varphi : L^p \cong H^p$. We now lift φ to an isomorphism of L and H. By Theorem 2.4, there exist generators $x_1, \dots, x_n \in L$ such that $L = \langle x_1\rangle_p \oplus \cdots \oplus \langle x_n\rangle_p$. Without loss of generality we assume

$$L^p = \langle x_1^p\rangle_p \oplus \cdots \oplus \langle x_m^p\rangle_p \quad \text{for some } m \leq n.$$

Thus, $x_i^p = 0$ for every i in the range $m < i \leq n$. Note that $\dim L = n + \dim L^p$. So, as mentioned in Theorem 2.4, n is determined. Let $y_1, \dots, y_n \in H$ such that $H = \langle y_1\rangle_p \oplus \cdots \oplus \langle y_n\rangle_p$. Then $H^p = \langle y_1^p\rangle_p \oplus \cdots \oplus \langle y_m^p\rangle_p$. So, we can assume that $\varphi(x_i^p) = y_i^p$ for every $1 \leq i \leq m$. We can verify that the map induced by $x_i \mapsto y_i$ for every $1 \leq i \leq n$ is a restricted Lie algebra isomorphism between L and H. □

Corollary 2.6. *Let $L \in \mathcal{F}_p$ be a restricted Lie algebra over a perfect field. Then L/L'_p is determined.*

Proof. Note that $[u(L), u(L)]u(L) = L'_p u(L)$. Also, $u(L/L'_p) \cong u(L)/L'_p u(L)$. Hence, $u(L/L'_p)$ is determined. Since $L/L'_p \in \mathcal{F}_p$, it follows from Proposition 2.5 that L/L'_p is determined. □

It turns out that stronger results hold over an algebraically closed field. Before we state the next result, we recall a well-known theorem; see [Jacobson 1962] or [Bahturin et al. 1992, Section 4.3]. Let $T_L = \langle x \in L \mid x^p = x\rangle_{\mathbb{F}}$, and denote by $\mathrm{Rad}(L)$ the subalgebra of L spanned by all p-nilpotent elements.

Theorem 2.7. *Let L be a finite-dimensional abelian restricted Lie algebra over an algebraically closed field $\mathbb{F}$. Then $L = T_L \oplus \mathrm{Rad}(L)$.*

Corollary 2.8. *Let L be a finite-dimensional abelian restricted Lie algebra over an algebraically closed field $\mathbb{F}$. If H be a restricted Lie algebra such that $u(L)\cong u(H)$, then $L\cong H$.*

Proof. Note that for every $k\geq 1$,

$$\dim_{\mathbb{F}} L/D_{p^k}(L)=\dim_{\mathbb{F}} L/D_p(L)+\cdots+\dim_{\mathbb{F}} D_{p^{k-1}}(L)/D_{p^k}(L)$$

is determined. So $\dim_{\mathbb{F}} D_{p^k}(L)$ is determined for every $k\geq 1$. Let t be the least integer such that $\mathrm{Rad}(L)^{p^t}=0$. It follows that $D_{p^t}(L)=T_L$. Hence, $\dim_{\mathbb{F}}\mathrm{Rad}(L)=\dim_{\mathbb{F}}\mathrm{Rad}(H)$ by Theorem 2.7. Note that $L/T_L\cong\mathrm{Rad}(L)$ as restricted Lie algebras. We claim that $\varphi(u(T_L))=u(T_H)$. If so, then $\varphi(T_L u(L))=T_H u(H)$. So

$$u(L/T_L)\cong u(L)/T_L u(L)\cong u(H)/T_H u(H)\cong u(H/T_H).$$

Thus, $u(\mathrm{Rad}(L))\cong u(\mathrm{Rad}(H))$. Since $\mathrm{Rad}(L),\mathrm{Rad}(H)\in\mathscr{F}_p$, Proposition 2.5 then implies that there exists an isomorphism $\varphi:\mathrm{Rad}(L)\to\mathrm{Rad}(H)$. Clearly, φ can be extended to an isomorphism of L and H.

Now, we prove the claim. Let $z_1,\dots,z_n$ be a basis of $\mathrm{Rad}(H)$ and $y_1,\dots,y_s$ be a basis of T_H, and assume that every y_i is less than every z_j. Let $x\in T_L$ and express $\varphi(x)$ in terms of PBW monomials in the y_i and z_j. So we have

$$\varphi(x)=u+\sum\alpha y_1^{a_1}\cdots y_s^{a_s}z_1^{b_1}\cdots z_n^{b_n},$$

where u is a linear combination of PBW monomials in the y_i only and each term in the sum has the property that $b_1+\cdots+b_n\neq 0$. Note that for a large k we have $\varphi(x)^{p^k}=u^{p^k}\in u(T_H)$. But $\varphi(x)=\varphi(x)^{p^k}$. So, $\varphi(x)\in u(T_H)$. Since $u(T_L)$ is generated by L and φ is an algebra homomorphism, we can get $\varphi(u(T_L))\subseteq u(T_H)$. But $u(T_L)$ and $u(T_H)$ are finite-dimensional. So we get $\varphi(u(T_L))=u(T_H)$. This proves the claim, completing the proof. □

3. The quotient $L/L'^p+\gamma_3(L)$

We first record a couple of easy statements.

Lemma 3.1. *Let N be a restricted subalgebra of L. Then*

$$\omega(L)N+N\omega(L)=[N,L]+N\omega(L)$$

Lemma 3.2. *For every restricted subalgebra N of L, we have*

- $L\cap([N,L]+N\omega(L))=[N,L]+N^p$ *and*
- $Nu(L)/\omega(L)N+N\omega(L)\cong N/([N,L]+N^p)$.

Now write $J_L=\omega(L)L'+L'\omega(L)=\omega(L)L'_p+L'_p\omega(L)$. Since both $\omega(L)L'$ and $L'\omega(L)$ are determined, it follows that J_L is determined.

Corollary 3.3. *If $L \in \mathscr{F}_p$, then* $\dim_{\mathbb{F}}(L/L'^p + \gamma_3(L))$ *is determined.*

Proof. Since $L'_p u(L)$ and J_L are determined, $\dim_{\mathbb{F}}(L'_p/L'^p + \gamma_3(L))$ is determined, by Lemma 3.2. The result follows since L/L'_p is determined by Corollary 2.6. □

From now on we assume that $L \in \mathscr{F}_p$ and $\mathbb{F}$ is perfect. By Theorem 2.4, there exist $e_1, \dots, e_n \in L$ such that $L/L'_p = \langle e_1 + L'_p \rangle_p \oplus \cdots \oplus \langle e_n + L'_p \rangle_p$. Let $\overline{X}$ be a basis of L/L'_p consisting of $\bar{e}_i^{p^j}$, where $\bar{e}_i = e_i + L'_p$ and $1 \le i \le n$. Fix a set X of representatives of $\overline{X}$. So the elements of X are linearly independent modulo L'_p.

We define the *height* $\nu(x)$ of an element $x \in L$ to be the largest integer n such that $x \in D_n(L)$ if n exists and to be infinite otherwise. The *weight* of a PBW monomial $x_1^{a_1} \dots x_t^{a_t}$ is defined to be $\sum_{i=1}^t a_i \nu(x_i)$. We observe that $\nu(e_i^{p^j}) = p^j$ for every $1 \le i \le n$ and every $1 \le j < \exp(\bar{e}_i)$. Indeed, if $e_i^{p^j} \in D_m(L)$ for some $m > p^j$, then

$$e_i^{p^j} = \textstyle\sum_{k>j} \alpha_k e_i^{p^k} \mod L'_p.$$

It follows then that $e_i^{\hat{p}} \in L'_p$, where $\hat{p} = p^{\exp(\bar{e}_i)-1}$, which is a contradiction. Let Y be a linearly independent subset of L'_p such that $Z = X \cup Y$ is a basis of L and the set $\{z + D_{\nu(z)+1} \mid z \in Z\}$ is a basis of $\mathrm{gr}(L)$. One way to construct such a subset Y is to take coset representatives of a basis for

$$\bigoplus_{i \ge 1} D_i(L) \cap (L'_p + \langle X \rangle_{\mathbb{F}})/D_{i+1}(L).$$

We need the following variant of [Riley and Shalev 1995, Theorem 2.1].

Lemma 3.4. *Let $L \in \mathscr{F}_p$. Let $\bar{Z}$ be a homogeneous basis of* $\mathrm{gr}(L)$ *with a fixed set of representatives Z. Then the set of all PBW monomials in Z of weight at least k forms a basis for $\omega^k(L)$ for every $k \ge 1$.*

Note that J_L is linearly independent with the set of all PBW monomials in X. Let E denote the vector space spanned by J_L and all PBW monomials in X of degree at least two. The following lemma is easy to see, so we omit the proof.

Lemma 3.5. (1) $\omega(L) = L + E$.

(2) $(L + J_L) \cap E = J_L = E \cap L'_p u(L)$.

(3) $\omega(L)/J_L = L + J_L/J_L \oplus E/J_L$.

Lemma 3.6. *If $L \in \mathscr{F}_p$ then E/J_L is a central restricted Lie ideal of $\omega(L)/J_L$.*

Proof. The fact that E/J_L is a central Lie ideal of $\omega(L)/J_L$ easily follows from the identity $[ab, c] = a[b, c] + [a, c]b$, which holds in any associative algebra. So we have to prove that E/J_L is closed under the p-map. Since J_L is an associative ideal of $\omega(L)$, it is enough to prove that $u^p \in E$ for every PBW monomial u in E. Let $u = e_1^{a_1} \cdots e_n^{a_n}$, where each a_i is in the range $0 \le a_i < p^{\exp \bar{e}_i}$. It is not hard to see that $u^p = e_1^{pa_1} \cdots e_n^{pa_n}$ modulo J_L. Since $L \in \mathscr{F}_p$, each $\bar{e}_i$ is p-nilpotent. If

$pa_i < p^{\exp(\bar{e}_i)}$ for every $1 \le i \le n$, then u^p is a PBW monomial of degree at least two. Now suppose that $pa_i \ge p^{\exp(\bar{e}_i)}$ for some i. If $pa_i = p^{\exp(\bar{e}_i)}$, then a_i is a power of p. Since u has degree at least two, there exists $j \ne i$ such that $a_j \ne 0$. It now follows that $u^p \in J_L$. If $pa_i > p^{\exp(\bar{e}_i)}$ then $e_i^{pa_i} \in J_L$, and so $u^p \in E$. □

Lemma 3.7. *We have* $H \cap \varphi(E) \subseteq J_H$.

Proof. We suppose $J_H = 0$ and prove that $H \cap \varphi(E) = 0$. Let $v \in H \cap \varphi(E) \subseteq \omega^2(H)$. Let $u \in E$ such that $\varphi(u) = v$. So, $u \in \omega^2(L)$. We prove by induction that $u \in \omega^n(L)$ for every n. But $\omega(L)$ is nilpotent by Lemma 2.1, and so $u = 0$. Supposing now by induction that $u \in w^n(L)$, we prove that $u \in w^{n+1}(L)$. So, $v \in H \cap w^n(H) = D_n(H)$. Thus, by Lemma 2.3, $u \in (D_n(L) + \omega^{n+1}(L)) \cap E$. But

$$(D_n(L) + \omega^{n+1}(L)) \cap E \subseteq \omega^{n+1}(L).$$

Indeed, let $u = \sum \alpha_i z_i + w$, where each $z_i \in Z$ has height n and $w \in \omega^{n+1}(L)$. By Lemma 3.4, w is a linear combination of PBW monomials in Z of weight at least $n+1$. Since $u \in E$, it follows by the PBW Theorem that $\alpha_i = 0$ for every i. So $u = w \in \omega^{n+1}(L)$, as required. □

Lemma 3.8. *We have* $\omega(H)/J_H = H + J_H/J_H \oplus \varphi(E)/J_H$.

Proof. By Lemma 3.7, it is enough to prove $\omega(H)/J_H \subseteq H + J_H/J_H \oplus \varphi(E)/J_H$. Note that both $\omega(H)/J_H$ and $\varphi(E)/J_H$ are determined. Since $\dim_{\mathbb{F}}(H + J_H/J_H) = \dim_{\mathbb{F}}(H/(H')^p + \gamma_3(H))$ is determined by Corollary 3.3, the result follows from Lemma 3.5. □

Noting that $L + J_L/J_L \cong L/L'^p + \gamma_3(L)$ by Lemma 3.2, we can now finish the proof of our main result.

Lemma 3.9. *The restriction of the natural isomorphism* $\omega(L)/J_L \to \omega(H)/J_H$ *to* $L + J_L/J_L$ *induces an isomorphism of* $L + J_L/J_L$ *and* $H + J_H/J_H$.

Proof. We denote by φ the induced isomorphism $\omega(L)/J_L \to \omega(H)/J_H$. Let $\varphi|_{L+J_L/J_L} = \varphi_1 + \varphi_2$ denote the restriction of φ to $L + J_L/J_L$, where φ_1 maps from $L + J_L/J_L$ to $H + J_H/J_H$. It is enough to prove φ_1 is a restricted Lie algebra isomorphism. Since E/L is a central Lie ideal of $\omega(L)/J_L$ by Lemma 3.6, $\varphi(E)/J_H$ is a central Lie ideal of $\omega(H)/J_H$. So, for every $x, z \in L$, we have

$$\varphi([x, z] + J_L) = [\varphi(x) + J_H, \varphi(z) + J_H] = [\varphi_1(x), \varphi_1(z)] + J_H.$$

So, φ_1 preserves the Lie brackets. Also,

$$\varphi(x^p + J_L) = \varphi(x)^p + J_H = (\varphi_1(x))^p + (\varphi_2(x))^p + J_H$$

Since $(\varphi_2(x))^p + J_H \in \varphi(E)/J_H$, it follows that φ_1 preserves the p-powers. Also, φ_1 is injective by Lemma 3.5. Since $L + J_L/J_L$ and $H + J_H/J_H$ have the same dimension by Corollary 3.3, it follows that φ_1 is an isomorphism. □

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