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We prove that the reduced Hochschild homology of a commutative DG Frobenius algebra has the natural structure of a Batalin–Vilkovisky coalgebra, and the reduced cyclic homology has the natural structure of a gravity coalgebra. As an application, this gives an algebraic model for a Batalin–Vilkovisky coalgebra structure on the reduced homology of the free loop space of a simply connected closed oriented manifold, and a gravity coalgebra structure on the reduced equivariant homology.

1. Introduction

Let M be a simply connected closed oriented m-manifold, and let LM be its free loop space. Félix and Thomas [2008] gave a construction of the Batalin–Vilkovisky algebra structure on the homology of LM in terms of Hochschild homology of a Poincaré duality model of M. The aim of this paper is to show that the reduced Hochschild homology, which gives the homology of LM relative to constant loops, has the structure of a Batalin–Vilkovisky coalgebra. As a consequence it is also shown that the reduced cyclic homology of the Poincaré duality model, which models the equivariant homology of LM relative to the constant loops, has the structure of a gravity coalgebra.

Throughout this paper, we shall work over the field of rational numbers. By $C_*(\cdot)$ and $C^*(\cdot)$, we mean the complex of singular chains and the complex of singular cochains. We shall grade $C^*(\cdot)$ negatively. By applying [Lambrechts and Stanley 2008, Theorem 1.1] to the Sullivan minimal model of M, it follows that there is a commutative differential graded (DG) algebra A such that

- A is connected, finite-dimensional, and quasiisomorphic to the DG algebra $C^*(M)$; and
- there is an A-module isomorphism $A \to A^\vee$ of degree m commuting with the differential and inducing the Poincaré duality isomorphism $H^*(M) \to H_{m+*}(M)$ on homology.

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Following Félix and Thomas [2008], we call A a *Poincaré duality model* for M.

Let $C = A^\vee$, the dual space of A. Since A is a commutative DG algebra, C is a cocommutative DG coalgebra. The linear isomorphism $A \xrightarrow{\cong} C[m]$ induces the structure of a commutative DG algebra on C whose product is of degree $-m$. Moreover, the coproduct

$$\Delta : C \to C \otimes C, \quad x \mapsto x' \otimes x''$$

is a map of C-bimodules. Thus, C forms a commutative DG Frobenius algebra in the following sense, which models the chain complex of M:

Definition 1. Let C be a chain complex over a field k. A *commutative DG Frobenius algebra* of degree m on C is a triple $(C, \cdot, \Delta)$ such that $(C, \cdot)$ is a DG commutative algebra whose product is of degree $-m$, (C, Δ) is a DG cocommutative coalgebra, and

$$(1) \quad (x \cdot y)' \otimes (x \cdot y)'' = (x \cdot y') \otimes y'' = (-1)^{m|x'|} x' \otimes (x'' \cdot y) \quad \text{for any } x, y \in C.$$

In Definition 1, C is not necessarily finite-dimensional.

From now on, we shall denote by C a commutative DG Frobenius algebra of degree m with differential d, counit ε, and a coaugmentation $\mathbb{Q} \hookrightarrow C$. By the Hochschild homology $HH_*(C)$ and cyclic homology $HC_*(C)$ of C, we mean the Hochschild homology and cyclic homology of the underlying DG *coalgebra* structure of C. We recall their definitions:

Definition 2. The *Hochschild homology* $HH_*(C)$ of C is the homology of the normalized cocyclic cobar complex $(CC_*(C), b)$, where

$$CC_*(C) = \prod_{n=0}^{\infty} C \otimes (\Sigma \overline{C})^{\otimes n},$$

and

$$\begin{aligned} b(a_0[a_1|\cdots|a_n]) \\ := da_0[a_1|\cdots|a_n] + \sum_{i=1}^{n} (-1)^{|a_0|+|[a_1|\cdots|a_{i-1}]|} a_0[a_1|\cdots|da_i|\cdots|a_n] \\ + \sum_{i=1}^{n} (-1)^{|a_0|+|[a_1|\cdots|a_{i-1}|a_i']|} a_0[a_1|\cdots|a_i'|a_i''|\cdots|a_n] \\ + (-1)^{|a_0'|} a_0' \big([a_0''|a_1|\cdots|a_n] - (-1)^{(|a_0''|-1)|[a_1|\cdots|a_n]|}[a_1|\cdots|a_n|a_0'']\big). \end{aligned}$$

Here, $\overline{C} := C/\mathbb{Q} \simeq \ker\{\varepsilon : C \to \mathbb{Q}\}$ and Σ is the desuspension functor (shifting the degrees of $\overline{C}$ down by one), and we write the elements of $C \otimes (\Sigma \overline{C})^{\otimes n}$ in the form $a_0[a_1|\cdots|a_n]$. In particular, the degree $|[a_1|\cdots|a_n]|$ is $(|a_1|-1)+\cdots+(|a_n|-1)$.

One easily checks that $b^2 = 0$. Connes' cyclic operator on the normalized co-cyclic cobar complex is given by

$$\begin{aligned} B: \quad CC_*(C) &\to CC_{*+1}(C), \\ a_0[a_1|\cdots|a_n] &\mapsto \sum_{i=1}^{n}(-1)^{|[a_i|\cdots|a_n]||[a_1|\cdots|a_{i-1}]|} \\ &\qquad\qquad \cdot \varepsilon(a_0)a_i[a_{i+1}|\cdots|a_n|a_1|\cdots|a_{i-1}]. \end{aligned}$$

One has $B^2 = 0$ and $bB + Bb = 0$.

Definition 3. The *cyclic homology* $HC_*(C)$ of the coalgebra C is the homology of the chain complex $CC_*(C)[u]$, where u is a parameter of degree 2, with differential $b + u^{-1}B$ defined by

$$(b+u^{-1}B)(\alpha\otimes u^n) = \begin{cases} b\alpha\otimes u^n + B\alpha\otimes u^{n-1} & \text{if } n>0, \\ b\alpha & \text{if } n=0 \end{cases}$$

for $\alpha \in CC_*(C)$.

As in the algebra case, one has Connes' exact sequence:

$$\text{(2)} \qquad \cdots \longrightarrow HH_*(C) \xrightarrow{\mathrm{E}} HC_*(C) \longrightarrow HC_{*-2}(C) \xrightarrow{\mathrm{M}} HH_{*-1}(C) \longrightarrow \cdots;$$

compare with [Chen 2007, Theorem 8.3]. We now recall the Batalin–Vilkovisky algebra structure on the Hochschild homology of a commutative DG Frobenius algebra.

Definition 4. A *Batalin–Vilkovisky algebra* is a graded commutative algebra $(V, \bullet)$ together with a linear map $\Delta: V_* \to V_{*+1}$ such that $\Delta\circ\Delta = 0$, and for all $a, b, c \in V$,

$$\text{(3)} \quad \begin{aligned} \Delta(a\bullet b\bullet c) &= \Delta(a\bullet b)\bullet c + (-1)^{|a|}a\bullet\Delta(b\bullet c) + (-1)^{(|a|-1)|b|}b\bullet\Delta(a\bullet c) \\ &\quad - (\Delta a)\bullet b\bullet c - (-1)^{|a|}a\bullet(\Delta b)\bullet c - (-1)^{|a|+|b|}a\bullet b\bullet(\Delta c). \end{aligned}$$

Now for a commutative DG Frobenius algebra C, define a product

$$\bullet: CC_*(C)\otimes CC_*(C) \to CC_*(C)$$

by

$$a_0[a_1|\cdots|a_n]\bullet b_0[b_1|\cdots|b_r] := (-1)^{|b_0||[a_1|\cdots|a_n]|}a_0b_0[a_1|\cdots|a_n|b_1|\cdots|b_r].$$

Theorem 5 [Tradler 2008]. *The Hochschild homology $HH_*(C)[m]$ is a Batalin–Vilkovisky algebra with differential B and product $\bullet$.*

Using the maps E and M in Connes' exact sequence (2), define, for each integer $n \geq 2$, a map c_n of degree $2-n$ by

$$\begin{aligned} c_n : HC_*(C)[m-2]^{\otimes n} &\to HC_*(C)[m-2] \\ \alpha_1 \otimes \cdots \otimes \alpha_n &\to (-1)^{\epsilon}\, \mathrm{E}(\mathrm{M}(\alpha_1) \bullet \cdots \bullet \mathrm{M}(\alpha_n)), \end{aligned}$$

where $\epsilon = (n-1)|\alpha_1| + (n-2)|\alpha_2| + \cdots + |\alpha_{n-1}|$. The corollary, which follows from a general result (see Proposition 24), is this:

Corollary 6. *The cyclic homology* $(HC_*(C)[m-2], \{c_n\})$ *is a gravity algebra.*

Definition 7. A *gravity algebra* is a graded vector space V with a sequence of graded skew-symmetric operators

$$\{x_1, \ldots, x_k\} : V^{\otimes k} \to V \quad \text{for } k = 2, 3, \ldots$$

of degree $2-k$ that satisfy the generalized Jacobi identities

$$(4) \qquad \sum_{1 \leq i < j \leq k} (-1)^{\epsilon(i,j)} \{\{x_i, x_j\}, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_k, y_1, \ldots, y_l\} = \begin{cases} \{\{x_1, \ldots, x_k\}, y_1, \ldots, y_l\} & \text{if } l > 0, \\ 0 & \text{if } l = 0. \end{cases}$$

where $\epsilon(i,j) = (|x_1| + \cdots + |x_{i-1}|)|x_i| + (|x_1| + \cdots + |x_{j-1}|)|x_j| + |x_i||x_j|$.

In this paper, by the *reduced Hochschild homology* $\widetilde{HH}_*(C)$ of C, we mean the homology of

$$\widetilde{CC}_*(C) := CC_*(C)/C = \prod_{n=1}^{\infty} C \otimes (\Sigma \overline{C})^{\otimes n}.$$

By the *reduced cyclic homology* $\widetilde{HC}_*(C)$ of C, we mean the homology of

$$\widetilde{CC}_*(C)[u] = CC_*(C)[u]/C[u].$$

As above, we have $\mathrm{E} : \widetilde{HH}_*(C) \to \widetilde{HC}_*(C)$ and $\mathrm{M} : \widetilde{HC}_*(C) \to \widetilde{HH}_{*+1}(C)$.

Define a coproduct

$$\vee : \widetilde{CC}_*(C) \to \widetilde{CC}_*(C) \otimes \widetilde{CC}_*(C)$$

by

$$(5) \qquad \vee(a_0[a_1 | \cdots | a_n]) := \sum_{i=2}^{n-1} (-1)^{\epsilon(i)}\, (a_0 a_i)'[a_1 | \cdots | a_{i-1}] \otimes (a_0 a_i)''[a_{i+1} | \cdots | a_n],$$

where $\epsilon(i) = |a_0| + (1 + |a_i| + |(a_0 a_i)''|)|[a_1 | \cdots | a_{i-1}]|$.

Our main result is the following.

Theorem 8. *The reduced Hochschild homology $\widetilde{HH}_*(C)[1-m]$ is a Batalin–Vilkovisky coalgebra with differential B and coproduct $\vee$.*

The proof of Theorem 8 uses several identities at the chain level involving certain homotopies which we will give in Section 5.

Similarly to above, define a map $s_n : \widetilde{HC}_*(C)[2-m] \to \widetilde{HC}_*(C)[2-m]^{\otimes n}$ of degree $2-n$ by

$$s_n(\alpha) := (\mathrm{E} \otimes \cdots \otimes \mathrm{E}) \circ (\vee \otimes \mathrm{id}^{\otimes n-2}) \circ \cdots \circ (\vee \otimes \mathrm{id}) \circ \vee \circ \mathrm{M}(\alpha)$$

for any $\alpha \in \widetilde{HC}_*(C)[2-m]$.

Corollary 9. *The reduced cyclic homology $(\widetilde{HC}_*(C)[2-m], \{s_n\})$ is a gravity coalgebra.*

In the two statements above, the Batalin–Vilkovisky coalgebra and gravity coalgebra are defined as dual versions of the corresponding algebras (see Definitions 14 and 15). The Batalin–Vilkovisky algebra and the gravity algebra structures of $HH_*(C)$ and $HC_*(C)$ descend to $\widetilde{HH}_*(C)$ and $\widetilde{HC}_*(C)$, respectively. Thus, we obtain both Batalin–Vilkovisky algebra and coalgebra structures on $\widetilde{HH}_*(C)$, and gravity algebra and coalgebra structures on $\widetilde{HC}_*(C)$.

Let A be a Poincaré duality model for M and $C = A^\vee$. Let LM be the free loop space of M. From [Jones 1987], one has isomorphisms

$$H_*(LM, M) \cong \widetilde{HH}_*(C) \quad \text{and} \quad H_*^{S^1}(LM, M) \cong \widetilde{HC}_*(C).$$

Following [Chas and Sullivan 2004], we call $H_*(LM, M)$ the reduced homology of the free loop space, and $H_*^{S^1}(LM, M)$ the reduced equivariant homology of the free loop space. As a consequence, the choice of a Poincaré duality model for M gives the reduced homology of the free loop space the structure of a Batalin–Vilkovisky coalgebra, and the reduced equivariant homology of the free loop space the structure of a gravity coalgebra. In string topology, the loop product $\bullet$ was first introduced in [Chas and Sullivan 1999]; see also [Cohen and Jones 2002]. The coproduct $\vee$ was introduced in [Sullivan 2004]. The operators c_n and s_n were first introduced in [Chas and Sullivan 2004] and discussed further in [Sullivan 2004]; see also [Westerland 2008].

Getzler [1994a; 1994b; 1995] studied Batalin–Vilkovisky algebras and gravity algebras in his works on topological conformal field theories (TCFT). He showed that a (genus zero) TCFT (respectively, an equivariant TCFT) with one output is the same as a Batalin–Vilkovisky algebra (respectively, a gravity algebra). If we consider multiple inputs and outputs, we then obtain both Batalin–Vilkovisky algebra and coalgebra (respectively, gravity algebra and coalgebra). Our construction gives an algebraic proof that string topology is a part of a (genus zero) TCFT. We expect that the constructions above can be generalized to homotopy versions of commutative DG Frobenius algebras.

Remark 10. Sullivan's coproduct $\vee$ is not the same as the loop coproduct introduced in [Cohen and Godin 2004]; see also [Godin 2007].

Remark 11. Theorem 5 is not new; it is well known that the Hochschild cohomology of a Frobenius algebra has the structure of a Batalin–Vilkovisky algebra; see, for example, [Menichi 2004] and [Tradler 2008]. However, notice that the formulas we give above in terms of the Hochschild homology of a Frobenius coalgebra are really explicit and simple. The proof of Theorem 5 is included in this paper so that the reader can compare it with the proof of Theorem 8. As far as we are aware, Theorem 8 is new. Its statement is not true in general at the chain level, and the homotopy operators that appear in its proof are also new.

The BV coalgebra structure in Theorem 8 also appears to be related to [Eu and Schedler 2009, Question 2.3.72].

The rest of this paper is organized as follows. We recall the definitions of Batalin–Vilkovisky algebras and gravity algebras in Section 2 and the proof of Theorem 5 in Section 3. We give the proof of Corollary 6 in Section 4, the proof of Theorem 8 in Section 5, and the proof of Corollary 9 in Section 6.

Koszul sign rule. All $\pm$ signs in this paper are determined by the Koszul rule for signs. Thus, whenever we switch two elements $a\otimes b\mapsto b\otimes a$, we put $(-1)^{|a||b|}$ in front of $b\otimes a$ and write $\pm b\otimes a$. Also, if f and g are operators of homogeneous degree, then $(f\otimes g)(a\otimes b)=\pm f(a)\otimes g(b)=(-1)^{|g||a|}f(a)\otimes g(b)$. For example, in (5), to see that $\epsilon(i)=|a_0|+(1+|a_i|+|(a_0a_i)''|)|[a_1|\cdots|a_{i-1}]|$ is given by the Koszul sign rule, note that the term $(1+|a_i|)|[a_1|\cdots|a_{i-1}]|$ comes from first moving $[a_i]$ to the left of $[a_1|\cdots|a_{i-1}]$, the term $|a_0|$ comes from moving a suspension operator to the right of a_0 to apply it to $[a_i]$, and $|(a_0a_i)''||[a_1|\cdots|a_{i-1}]|$ comes from moving $(a_0a_i)''$ to the right of $[a_1|\cdots|a_{i-1}]$. Similarly, the signs in the formulas above for b, B, the product $\bullet$, c_n, and so on, are also all given by the Koszul sign rule.

2. Batalin–Vilkovisky algebras and gravity algebras

Lemma 12. *Let $(V,\bullet,\Delta)$ be a Batalin–Vilkovisky algebra. Define*

$$\{\cdot,\cdot\}:V\otimes V\to V$$

by

$$\{a,b\}:=(-1)^{|a|}\Delta(a\bullet b)-(-1)^{|a|}(\Delta a)\bullet b-a\bullet(\Delta b).$$

Then $(V[-1],\{\cdot,\cdot\},\Delta)$ is a DG Lie algebra.

Proof. See [Getzler 1994a, Proposition 1.2]. □

More generally, one has the following result proved by Getzler; see [1994b, Theorem 4.5] and [1995, §3.4].

Theorem 13. *Let* $(V, \bullet, \Delta)$ *be a Batalin–Vilkovisky algebra. For* $k = 2, 3, \ldots,$ *define*

$$\{\cdot, \ldots, \cdot\} : V^{\otimes k} \to V$$

by

$$\{a_1, \ldots, a_k\} := (-1)^{\epsilon}\Big(\Delta(a_1 a_2 \cdots a_k) - \sum_{i=1}^{k} (-1)^{|a_1|+\cdots+|a_{i-1}|} a_1 \cdots (\Delta a_i) \cdots a_k\Big),$$

where $\epsilon = (k-1)|a_1| + (k-2)|a_2| + \cdots + |a_{k-1}|$. *Then* $V[-1]$ *is a DG gravity algebra with differential* Δ *and brackets* $\{a_1, \ldots, a_k\}$.

A DG gravity algebra is a gravity algebra with a differential commuting with all the brackets. Thus, for a Batalin–Vilkovisky algebra $(V, \bullet, \Delta)$, its homology $H(V, \Delta)[-1]$ has a gravity algebra structure. Taking $k = 3$ and $l = 0$ in (4) gives the graded Jacobi identity. Hence, a gravity algebra has a graded Lie algebra structure.

Analogously, we may introduce the notions of a Batalin–Vilkovisky coalgebra and a gravity coalgebra.

Definition 14. A *Batalin–Vilkovisky coalgebra* is a graded cocommutative coalgebra $(V, \vee)$ together with a linear map $\Delta : V_* \to V_{*+1}$ such that $\Delta \circ \Delta = 0$, and

$$\begin{aligned}(\Delta \otimes \mathrm{id}^{\otimes 2} + \mathrm{id} \otimes \Delta \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes \Delta) \circ (\vee \otimes \mathrm{id}) \circ \vee(a) \\ = (\tau^2 + \tau + \mathrm{id}) \circ (\vee \circ \Delta \otimes \mathrm{id}) \circ \vee(a) + (\vee \otimes \mathrm{id}) \circ \vee \circ \Delta(a)\end{aligned}$$

for all $a \in V$, where τ is the cyclic permutation $\tau : a \otimes b \otimes c \mapsto c \otimes a \otimes b$.

Similarly to the Batalin–Vilkovisky algebra case, the chain complex (V, Δ) is a DG gravity coalgebra:

Definition 15. A *gravity coalgebra* is a graded vector space V with a sequence of graded skew-symmetric operators

$$m_k : V \to V^{\otimes k} \quad \text{for } k = 2, 3, 4, \ldots$$

of degree $2 - k$, such that

$$S_{2,k-2} \circ (m_2 \otimes \mathrm{id}^{\otimes k-2}) \circ m_{k-1+l} = (m_k \otimes \mathrm{id}^{\otimes l}) \circ m_{l+1} : V \to V^{k+l}, \tag{6}$$

where the range of the mapping $(m_2 \otimes \mathrm{id}^{\otimes k-2}) \circ m_{k-1+l} : V \to V^{k+l}$ is identified with $V^{\otimes 2} \otimes V^{\otimes k-2} \otimes V^{\otimes l}$ and $S_{2,k-2}$ is the shuffle product $V^{\otimes 2} \otimes V^{\otimes k-2} \to V^{\otimes k}$, and if $l = 0$, we set $m_1 = 0$.

Theorem 16. *Let $(V, \vee, \Delta)$ be a Batalin–Vilkovisky coalgebra. For any $x \in V$, let*

$$\vee_k(x) := (\vee \otimes \mathrm{id}^{\otimes k-2}) \circ \cdots \circ (\vee \otimes \mathrm{id}) \circ \vee(x) = \sum x_1 \otimes x_2 \otimes \cdots \otimes x_k,$$

and let

$$s_k(x) := \sum (-1)^{(k-1)|x_1|+(k-2)|x_2|+\cdots+|x_{k-1}|} \cdot \Big(\vee_k(\Delta x) - \sum_{i=0}^{k-1} (\mathrm{id}^{\otimes i} \otimes \Delta \otimes \mathrm{id}^{\otimes k-i-1}) \circ \vee_k(x)\Big),$$

for $k = 2, 3, \ldots$. Then $V[1]$ is a DG gravity coalgebra with differential Δ and cobrackets $\{s_n\}$. In particular, $(V[1], s_2, \Delta)$ is a DG Lie coalgebra.

The proof of the theorem is completely dual to that of Theorem 13.

3. The Batalin–Vilkovisky algebra

Next we recall the proof of Theorem 5 from [Chen 2007].

Lemma 17. *The chain complex $(CC_*(C)[m], b)$ is a DG algebra with product $\bullet$.*

Proof. The proof is by direct verification; see [Chen 2007, Lemma 4.1]. □

The product $\bullet$ on $CC_*(C)[m]$ is not commutative, but homotopy commutative:

Lemma 18. *Define a bilinear operator*

$$* : CC_*(C) \otimes CC_*(C) \to CC_*(C)$$

as follows: for $\alpha = a_0[a_1 | \cdots | a_n]$, $\beta = b_0[b_1 | \cdots | b_r] \in CC_*(C)$,

$$\alpha * \beta := \sum_{i=1}^{n} (-1)^{|b_0|+(|\beta|-1)|[a_{i+1}|\cdots|a_n]|} \cdot \varepsilon(a_i b_0) a_0[a_1 | \cdots | a_{i-1} | b_1 | \cdots | b_r | a_{i+1} | \cdots | a_n]. \tag{7}$$

Then

$$b(\alpha * \beta) = b\alpha * \beta + (-1)^{|\alpha|+1} \alpha * b\beta + (-1)^{|\alpha|} (\alpha \bullet \beta - (-1)^{|\alpha||\beta|} \beta \bullet \alpha). \tag{8}$$

Proof. The proof is by direct verification; see [Chen 2007, Lemma 5.1]. □

It follows from Lemma 17 and Lemma 18 that $(HH_*(C)[m], \bullet)$ is a graded commutative algebra.

Define the binary operator

$$\{\cdot, \cdot\} : CC_*(C) \otimes CC_*(C) \to CC_*(C)$$

to be the commutator of $*$ above, namely

$$\{\alpha, \beta\} := \alpha * \beta - (-1)^{(|\alpha|+1)(|\beta|+1)} \beta * \alpha \quad \text{for } \alpha, \beta \in CC_*(C).$$

Lemma 19. *The chain complex* $(CC_*(C)[m-1], b)$ *is a DG Lie algebra with the Lie bracket* $\{\cdot\,,\cdot\}$.

Proof. The proof is direct; see [Chen 2007, Lemma 5.4 and Corollary 5.5]. □

In particular $HH_*(C)[m-1]$ is a graded Lie algebra. Moreover, $\bullet$ and $\{\cdot\,,\cdot\}$ are compatible in the following sense, which makes $HH_*(C)[m]$ into a Gerstenhaber algebra:

Definition 20 [Gerstenhaber 1963]. Let V be a graded vector space. A *Gerstenhaber algebra* on V is a triple $(V, \cdot, \{\cdot\,,\cdot\})$ such that

(i) $(V, \cdot)$ is a graded commutative algebra;

(ii) $(V, \{\cdot\,,\cdot\})$ is a graded Lie algebra whose Lie bracket is of degree 1;

(iii) for any $\alpha, \beta, \gamma \in V$, one has

$$\{\alpha \bullet \beta, \gamma\} = \alpha \bullet \{\beta, \gamma\} + (-1)^{|\beta|(|\gamma|+1)}\{\alpha, \gamma\} \bullet \beta. \tag{9}$$

Theorem 21. *The Hochschild homology* $HH_*(C)[m]$ *is a Gerstenhaber algebra, with product* $\bullet$ *and bracket* $\{\cdot\,,\cdot\}$.

Proof. From above, $HH_*(C)[m]$ is both a graded commutative algebra and a degree one graded Lie algebra. Equation (9) is immediate from Lemma 22. □

Lemma 22. *For any*

$$\alpha = a_0[a_1 | \cdots | a_n], \quad \beta = b_0[b_1 | \cdots | b_r], \quad \gamma = c_0[c_1 | \cdots | c_l] \in CC_*(C),$$

one has

(i) $(\alpha \bullet \beta) * \gamma = \alpha \bullet (\beta * \gamma) + (-1)^{|\beta|(|\gamma|+1)}(\alpha * \gamma) \bullet \beta$; *and*

(ii) $\gamma * (\alpha \bullet \beta) - (\gamma * \alpha) \bullet \beta - (-1)^{|\alpha|(|\gamma|+1)}\alpha \bullet (\gamma * \beta) = (b \circ \rho - \rho \circ b)(\alpha \otimes \beta \otimes \gamma)$, *where*

$$\rho(\alpha \otimes \beta \otimes \gamma) := \sum_{i<j} (-1)^{\epsilon} \varepsilon(c_i a_0)\varepsilon(c_j b_0) c_0[c_1 | \cdots | c_{i-1} | a_1 | \cdots | a_n | c_{i+1} | \cdots | c_{j-1} | b_1 | \cdots | b_r | c_{j+1} | \cdots | c_l],$$

and $\epsilon = (|\alpha| - 1)|[c_{i+1} | \cdots | c_n]| + (|\beta| - 1)|[c_{j+1} | \cdots | c_n]|$.

Proof. The proof is by direct verification; see [Chen 2007, Lemma 5.8]. □

Theorem 5 follows from [Getzler 1994a, Proposition 1.2], Theorem 21, and the following:

Lemma 23. *For any* $\alpha, \beta \in HH_*(C)[m]$, *one has*

$$\{\alpha, \beta\} = (-1)^{|\alpha|} B(\alpha \bullet \beta) - (-1)^{|\alpha|} B(\alpha) \bullet \beta - \alpha \bullet B(\beta).$$

More precisely, for $\alpha = x[a_1 | \cdots | a_n]$ *and* $\beta = y[b_1 | \cdots | b_r] \in CC_*(C)$, *define*

$$\begin{aligned}&\phi(\alpha, \beta)\\ &\quad := \sum_{i<j} \pm\varepsilon(x)\varepsilon(a_j y) a_i [a_{i+1} | \cdots | a_{j-1} | b_1 | \cdots | b_r | a_{j+1} | \cdots | a_n | a_1 | \cdots | a_{i-1}],\\ &\psi(\alpha, \beta)\\ &\quad := \sum_{k<l} \pm\varepsilon(y)\varepsilon(b_l x) b_k [b_{k+1} | \cdots | b_{l-1} | a_1 | \cdots | a_n | b_{l+1} | \cdots | b_r | b_1 | \cdots | b_{k-1}],\end{aligned}$$

and let $\theta := \phi + \psi$. (*The* $\pm$ *signs are determined by the Koszul sign rule.*) *Then*

$$\begin{aligned}&(b \circ \theta + \theta \circ b)(\alpha \otimes \beta)\\ &\qquad = \{\alpha, \beta\} - (-1)^{|\alpha|} B(\alpha \bullet \beta) - (-1)^{(|\beta|+1)(|\alpha|+1)} \beta \bullet B(\alpha) + \alpha \bullet B(\beta).\end{aligned}$$

Proof. The proof is by a direct verification; see [Chen 2007, Lemma 7.3]. □

4. The gravity algebra

We define the complex $(CC_*(C)[u, u^{-1}], b + u^{-1}B)$ by

$$(b + u^{-1}B)(\alpha \otimes u^n) = b\alpha \otimes u^n + B\alpha \otimes u^{n-1} \quad \text{for all } n.$$

The quotient of $(CC_*(C)[u, u^{-1}], b + u^{-1}B)$ by its subcomplex $CC_*(C)[u^{-1}]u^{-1}$ is the complex $(CC_*(C)[u], b + u^{-1}B)$ in Definition 3. The short exact sequence

$$0 \to CC_*(C) \longrightarrow CC_*(C)[u] \xrightarrow{u^{-1}} CC_*(C)[u] \to 0$$

induces the long exact sequence (2). By diagram chasing, one can see that

$$\mathrm{M} \circ \mathrm{E} = B : HH_*(C) \to HH_{*+1}(C).$$

Corollary 6 is immediate from Theorem 5 and the following general result; see [Chen 2007, Theorem 8.5].

Proposition 24. *Let* $(V, \bullet, \Delta)$ *be a Batalin–Vilkovisky algebra, and let* W *be a graded vector space. Let* $\mathrm{E} : V_* \to W_*$ *and* $\mathrm{M} : W_* \to V_{*+1}$ *be two maps such that* $\mathrm{E} \circ \mathrm{M} = 0$ *and* $\mathrm{M} \circ \mathrm{E} = \Delta$. *Then* $(W[-2], \{c_n\})$ *is a gravity algebra, where*

$$c_n(\alpha_1 \otimes \cdots \otimes \alpha_n) := (-1)^{(n-1)|\alpha_1| + (n-2)|\alpha_2| + \cdots + |\alpha_{n-1}|} \mathrm{E}(\mathrm{M}(\alpha_1) \bullet \cdots \bullet \mathrm{M}(\alpha_n)).$$

Proof. It follows from (3), by induction on n, that

$$\begin{aligned}\Delta(x_1 \bullet x_2 \bullet \cdots \bullet x_n) = &\sum_{i<j} \pm \Delta(x_i \bullet x_j) \bullet x_1 \bullet \cdots \bullet \widehat{x_i} \bullet \cdots \bullet \widehat{x_j} \bullet \cdots \bullet x_n\\ &+ (n-2) \sum_i \pm x_1 \bullet \cdots \bullet \Delta x_i \bullet \cdots \bullet x_n.\end{aligned} \tag{10}$$

Now let $x_i = \mathrm{M}(\alpha_i)$, and apply E to both sides of the above equality; we obtain

$$\begin{aligned}\mathrm{E}\circ\Delta(\mathrm{M}(\alpha_1)\bullet\mathrm{M}(\alpha_2)\bullet\cdots\bullet\mathrm{M}(\alpha_n))\\ =\sum_{i<j}\pm\mathrm{E}\circ\big(\Delta(\mathrm{M}(\alpha_i)\bullet\mathrm{M}(\alpha_j))\bullet\mathrm{M}(\alpha_1)\bullet\cdots\bullet\widehat{\mathrm{M}(\alpha_i)}\bullet\cdots\bullet\widehat{\mathrm{M}(\alpha_j)}\bullet\cdots\bullet\mathrm{M}(\alpha_n)\big)\\ +(n-2)\sum_i\pm\mathrm{E}(\mathrm{M}(\alpha_1)\bullet\cdots\bullet\Delta\circ\mathrm{M}(\alpha_i)\bullet\cdots\bullet\mathrm{M}(\alpha_n)).\end{aligned}$$

Since $\mathrm{E}\circ\Delta=\mathrm{E}\circ\mathrm{M}\circ\mathrm{E}=0$ and $\Delta\circ\mathrm{M}=\mathrm{M}\circ\mathrm{E}\circ\mathrm{M}=0$, we have

$$\sum_{1\le i<j\le n}\pm c_{n-1}(c_2(\alpha_i\otimes\alpha_j)\otimes\alpha_1\otimes\cdots\otimes\widehat{\alpha_i}\otimes\cdots\otimes\widehat{\alpha_j}\otimes\cdots\otimes\alpha_n)=0.$$

Similarly, by multiplying $y_1\bullet\cdots\bullet y_l$ on both sides of (10), letting $y_j=\mathrm{M}(\beta_j)$, and then applying E on both sides, we obtain

$$\begin{aligned}\sum_{1\le i<j\le n}\pm c_{n+l-1}(c_2(\alpha_i\otimes\alpha_j)\otimes\alpha_1\otimes\cdots\otimes\widehat{\alpha_i}\otimes\cdots\otimes\widehat{\alpha_j}\otimes\cdots\otimes\alpha_n\otimes\beta_1\otimes\cdots\otimes\beta_l)\\ =c_{l+1}(c_n(\alpha_1\otimes\cdots\otimes\alpha_n)\otimes\beta_1\otimes\cdots\otimes\beta_l)\end{aligned}$$

for $l>0$. This proves the proposition. □

Proposition 24 can also be applied to the Hochschild homology of a Calabi-Yau algebra (see [Ginzburg 2006, Theorem 3.4.3]) to give a gravity algebra structure on its cyclic homology.

5. The Batalin–Vilkovisky coalgebra

The proof of Theorem 8 is similar to the proof of Theorem 5.

Lemma 25. *The chain complex $(\widetilde{CC}_*(C)[1-m], b)$ is a DG coalgebra with coproduct $\vee$.*

Proof. It is clear that $\vee$ is coassociative. Therefore we only need to check that b is a derivation with respect to $\vee$. Observe that the expressions $b\circ\vee(\alpha)$ and $\vee\circ b(\alpha)$ have two parts, one contains those terms involving the differentials of the entries in α (which we call the *differential part*), the other contains those terms involving the coproducts of the entries in α (which we call the *diagonal part*). It follows directly from the definition of $\vee$ that the differential parts of $b\circ\vee(\alpha)$ and $\vee\circ b(\alpha)$ are equal. For the diagonal parts, omitting the signs determined by the Koszul sign rule from our notation (see 32), we have

$$b\circ\vee(a_0[a_1|\cdots|a_n]) \tag{11}$$

$$=\sum_{1<i<n}b\big((a_0a_i)'[a_1|\cdots|a_{i-1}]\big)\otimes(a_0a_i)''[a_{i+1}|\cdots|a_n] \tag{12}$$

$$\pm \sum_{1<i<n} (a_0a_i)'[a_1|\cdots|a_{i-1}] \otimes b\Big((a_0a_i)''[a_{i+1}|\cdots|a_n]\Big) \tag{13}$$

$$= \sum_{1\le j<i<n} \pm(a_0a_i)'[a_1|\cdots|a'_j|a''_j|\cdots|a_{i-1}] \otimes (a_0a_i)''[a_{i+1}|\cdots|a_n] \tag{14}$$

$$+ \sum_{1<i<n} \pm((a_0a_i)')'[((a_0a_i)')''|a_1|\cdots|a_{i-1}] \otimes (a_0a_i)''[a_{i+1}|\cdots|a_n] \tag{15}$$

$$- \sum_{1<i<n} \pm((a_0a_i)')'[a_1|\cdots|a_{i-1}|((a_0a_i)')''] \otimes (a_0a_i)''[a_{i+1}|\cdots|a_n] \tag{16}$$

$$+ \sum_{1<i<j\le n} \pm(a_0a_i)'[a_1|\cdots|a_{i-1}] \otimes (a_0a_i)''[a_{i+1}|\cdots|a'_j|a''_j|\cdots|a_n] \tag{17}$$

$$+ \sum_{1<i<n} \pm(a_0a_i)'[a_1|\cdots|a_{i-1}] \otimes ((a_0a_i)'')'[((a_0a_i)'')''|a_{i+1}|\cdots|a_n] \tag{18}$$

$$- \sum_{1<i<n} \pm(a_0a_i)'[a_1|\cdots|a_{i-1}] \otimes ((a_0a_i)'')'[a_{i+1}|\cdots|a_n|((a_0a_i)'')''], \tag{19}$$

while

$$\vee \circ b(a_0[a_1|\cdots|a_n]) \tag{20}$$

$$= \sum_{1\le j<i<n} \pm(a_0a_i)'[a_1|\cdots|a'_j|a''_j|\cdots|a_{i-1}] \otimes (a_0a_i)''[a_{i+1}|\cdots|a_n] \tag{21}$$

$$+ \sum_{1<i<j\le n} \pm(a_0a_i)'[a_1|\cdots|a_{i-1}] \otimes (a_0a_i)''[a_{i+1}|\cdots|a'_j|a''_j|\cdots|a_n] \tag{22}$$

$$+ \sum_{1<i<n} \pm(a_0a'_i)'[a_1|\cdots|a_{i-1}] \otimes (a_0a'_i)''[a''_i|a_{i+1}|\cdots|a_n] \tag{23}$$

$$\pm (a_0a'_n)'[a_1|\cdots|a_{n-1}] \otimes (a_0a'_n)''[a''_n] \tag{24}$$

$$\pm (a_0a''_1)'[a'_1] \otimes (a_0a''_1)''[a_2|\cdots|a_n] \tag{25}$$

$$+ \sum_{1<i<n} \pm(a_0a''_i)'[a_1|\cdots|a_{i-1}|a'_i] \otimes (a_0a''_i)''[a_{i+1}|\cdots|a_n] \tag{26}$$

$$\pm (a'_0a_1)'[a''_0] \otimes (a'_0a_1)''[a_2|\cdots|a_n] \tag{27}$$

$$+ \sum_{1<i<n} \pm(a'_0a_i)'[a''_0|a_1|\cdots|a_{i-1}] \otimes (a'_0a_i)''[a_{i+1}|\cdots|a_n] \tag{28}$$

$$- \sum_{1<i<n} \pm(a'_0a_i)'[a_1|\cdots|a_{i-1}] \otimes (a_0a_i)''[a_{i+1}|\cdots|a_n|a''_0] \tag{29}$$

$$- \pm(a'_0a_n)'[a_1|\cdots|a_{n-1}] \otimes (a'_0a_n)''[a''_0]. \tag{30}$$

Keeping (1) in mind, we see that (14) and (21) are equal; so are (15) and (28), (16) and (26), (17) and (22), (18) and (23), and (19) and (29). Also, (24) and (30) cancel; so do (25) and (27). Hence, (11) = (20). □

Define the permutations τ and σ by

$$\tau : \widetilde{CC}_*(C) \otimes \widetilde{CC}_*(C) \to \widetilde{CC}_*(C) \otimes \widetilde{CC}_*(C)$$
$$\alpha_1 \otimes \alpha_2 \mapsto \pm \alpha_2 \otimes \alpha_1$$

and

$$\sigma : \widetilde{CC}_*(C) \otimes \widetilde{CC}_*(C) \otimes \widetilde{CC}_*(C) \to \widetilde{CC}_*(C) \otimes \widetilde{CC}_*(C) \otimes \widetilde{CC}_*(C)$$
$$\alpha_1 \otimes \alpha_2 \otimes \alpha_3 \mapsto \pm \alpha_2 \otimes \alpha_3 \otimes \alpha_1.$$

The following lemma says that $\vee$ is cocommutative up to homotopy, and therefore $(\widetilde{HH}_*(C)[1-m], \vee)$ is a graded cocommutative, coassociative coalgebra.

Lemma 26. *Let* $h : \widetilde{CC}_*(C) \to \widetilde{CC}_*(C) \otimes \widetilde{CC}_*(C)$ *be defined by*

$$h(\alpha) := \sum_{i<j} \pm a_0[a_1 | \cdots | a_{i-1} | a_{j+1} | \cdots | a_n] \otimes a_i a_j [a_{i+1} | \cdots | a_{j-1}]$$

for any $\alpha = a_0[a_1 | \cdots | a_n] \in \widetilde{CC}_*(C)$. *(The* $\pm$ *sign is determined by the Koszul sign rule on page 32.) Then*

$$b \circ h(\alpha) - h \circ b(\alpha) = \tau \circ \vee(\alpha) - \vee(\alpha). \tag{31}$$

Proof. It is easy to see that the differential parts of the left side of (31) cancel each other, so we only need to consider the diagonal parts. In fact, the diagonal parts of $h(b\alpha)$ are equal to

$$\sum_{i} \pm a_0'[a_{i+1}| \cdots |a_n] \otimes (a_0'' a_i)[a_1| \cdots |a_{i-1}] \tag{32}$$

$$+ \sum_{i<j} \pm a_0'[a_0''|a_1| \cdots |a_{i-1}|a_{j+1}| \cdots |a_n] \otimes (a_i a_j)[a_{i+1}| \cdots |a_{j-1}] \tag{33}$$

$$- \sum_{i} \pm a_0'[a_1| \cdots |a_{i-1}] \otimes (a_i a_0'')[a_{i+1}| \cdots |a_n] \tag{34}$$

$$- \sum_{i<j} \pm a_0'[a_1| \cdots |a_{i-1}|a_{j+1}| \cdots |a_n|a_0''] \otimes (a_i a_j)[a_{i+1}| \cdots |a_{j-1}] \tag{35}$$

$$+ \sum \pm a_0[a_1| \cdots |a_k'|a_k''| \cdots |a_n] \otimes (a_i a_j)[a_{i+1}| \cdots |a_{j-1}] \tag{36}$$

$$+ \sum_{i<k<j} \pm a_0[a_1| \cdots |a_{i-1}|a_{j+1}| \cdots |a_n] \otimes (a_i a_j)[a_{i+1}| \cdots |a_k'|a_k''| \cdots |a_{j-1}] \tag{37}$$

$$+ \sum_{i<j} \pm a_0[a_1| \cdots |a_{i-1}|a_{j+1}| \cdots |a_n] \otimes (a_i a_j)'[(a_i a_j)''|a_{i+1}| \cdots |a_{j-1}] \tag{38}$$

$$- \sum_{i<j} \pm a_0[a_1| \cdots |a_{i-1}|a_{j+1}| \cdots |a_n] \otimes (a_i a_j)'[a_{i+1}| \cdots |a_{j-1}|(a_i a_j)''], \tag{39}$$

(where the sum in (36) is taken over all $k < i < j$ and $i < j < k$).

Now (33)+(35)+(36)+(37)+(38)+(39) is exactly $b(h\alpha)$, while the remaining terms (32) + (34) are exactly $-\tau \circ \vee(\alpha) + \vee(\alpha)$. □

Lemma 27. *Let h be as in Lemma 26. Define $S : \widetilde{CC}_*(C) \to \widetilde{CC}_*(C) \otimes \widetilde{CC}_*(C)$ by*

$$S(\alpha) := h(\alpha) - \tau \circ h(\alpha) \quad \text{for any } \alpha \in \widetilde{CC}_*(C).$$

Then the chain complex $(\widetilde{CC}_*(C)[2-m], b)$ is a DG Lie coalgebra with the cobracket S.

Proof. It follows from the definition that S is skew-symmetric, and b commutes with S by (31). Now, for any $\alpha = a_0[a_1|\cdots|a_n]$,

$$\begin{aligned}
&(h\otimes 1)h(\alpha) - (1\otimes h)h(\alpha)\\
&\quad = \sum_{k<l<i<j} \pm a_0[a_1|\cdots|a_{k-1}|a_{l+1}|\cdots|a_{i-1}|a_{j+1}|\cdots|a_n]\\
&\qquad\qquad\qquad\qquad \otimes a_k a_l[a_{k+1}|\cdots|a_{l-1}] \otimes a_i a_j[a_{i+1}|\cdots|a_{j-1}]\\
&\qquad + \sum_{i<j<k<l} \pm a_0[a_1|\cdots|a_{i-1}|a_{j+1}|\cdots|a_{k-1}|a_{l+1}|\cdots|a_n]\\
&\qquad\qquad\qquad\qquad \otimes a_k a_l[a_{k+1}|\cdots|a_{l-1}] \otimes a_i a_j[a_{i+1}|\cdots|a_{j-1}]\\
&\quad = (1\otimes\tau)((h\otimes 1)h(\alpha) - (1\otimes h)h(\alpha)).
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1+\sigma+\sigma^2)(S\otimes 1)S\\
&\quad = (1+\sigma+\sigma^2)\big((h\otimes 1)h - (1\otimes h)h - (1\otimes\tau)((h\otimes 1)h - (1\otimes h)h)\big) = 0,
\end{aligned}$$

so the co-Jacobi identity holds. □

It follows that $(\widetilde{HH}_*(C)[2-m], S)$ is a graded Lie coalgebra. The Lie cobracket S and the cocommutative coproduct $\vee$ are compatible in the following sense:

Definition 28. Let V be a graded vector space. A *Gerstenhaber coalgebra* on V is a triple $(V, \vee, S)$ such that

(i) $(V, \vee)$ is a graded cocommutative coalgebra;

(ii) (V, S) is a graded Lie coalgebra whose Lie cobracket is of degree 1; and

(iii) $S : V \to V \otimes V$ is a coderivation with respect to $\vee$, that is, the following diagram commutes:

$$\begin{array}{ccc}
V & \xrightarrow{\quad\vee\quad} & V\otimes V\\
{\scriptstyle S}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}\otimes\tau)\circ(S\otimes\mathrm{id})+\mathrm{id}\otimes S}\\
V\otimes V & \xrightarrow{\ \vee\otimes\mathrm{id}\ } & V\otimes V\otimes V
\end{array}$$

Theorem 29. *The reduced Hochschild homology* $(\widetilde{HH}_*(C)[1-m], \vee, S)$ *is a Gerstenhaber coalgebra.*

Proof. From the definition of h in Lemma 26, the diagram

$$\begin{array}{ccc} \widetilde{CC} & \xrightarrow{\quad\vee\quad} & \widetilde{CC}\otimes\widetilde{CC} \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle (\mathrm{id}\otimes\tau)\circ(h\otimes\mathrm{id})+\mathrm{id}\otimes h} \\ \widetilde{CC}\otimes\widetilde{CC} & \xrightarrow{\vee\otimes\mathrm{id}} & \widetilde{CC}\otimes\widetilde{CC}\otimes\widetilde{CC} \end{array}$$

commutes. We next show that

$$\begin{array}{ccc} \widetilde{CC} & \xrightarrow{\quad\vee\quad} & \widetilde{CC}\otimes\widetilde{CC} \\ \downarrow{\scriptstyle \tau\circ h} & & \downarrow{\scriptstyle (\mathrm{id}\otimes\tau)\circ(\tau\circ h\otimes\mathrm{id})+\mathrm{id}\otimes\tau\circ h} \\ \widetilde{CC}\otimes\widetilde{CC} & \xrightarrow{\vee\otimes\mathrm{id}} & \widetilde{CC}\otimes\widetilde{CC}\otimes\widetilde{CC}, \end{array} \tag{40}$$

commutes up to homotopy, and therefore, from $S = h - \tau\circ h$, the diagram

$$\begin{array}{ccc} \widetilde{HH} & \xrightarrow{\quad\vee\quad} & \widetilde{HH}\otimes\widetilde{HH} \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle (\mathrm{id}\otimes\tau)\circ(S\otimes\mathrm{id})+\mathrm{id}\otimes S} \\ \widetilde{HH}\otimes\widetilde{HH} & \xrightarrow{\vee\otimes\mathrm{id}} & \widetilde{HH}\otimes\widetilde{HH}\otimes\widetilde{HH} \end{array}$$

commutes. Let $\varrho : \widetilde{CC} \to \widetilde{CC}\otimes\widetilde{CC}\otimes\widetilde{CC}$ be the map defined by

$$\varrho(\alpha) := \sum_{i<j<k<l} \pm a_0[a_1|\cdots|a_{i-1}|a_{j+1}|\cdots|a_{k-1}|a_{l+1}|\cdots|a_n] \otimes a_ia_j[a_{i+1}|\cdots|a_{j-1}]\otimes a_ka_l[a_{k+1}|\cdots|a_{l-1}],$$

for any $\alpha = a_0[a_1|\cdots|a_n]$. (The $\pm$ sign is determined by the Koszul sign rule.) Let $\rho := \sigma\circ\varrho$. Then

$$(b\circ\rho-\rho\circ b)(\alpha) = ((\vee\otimes\mathrm{id})\circ(\tau\circ h)-((\mathrm{id}\otimes\tau)\circ(\tau\circ h\otimes\mathrm{id})+\mathrm{id}\otimes\tau\circ h)\circ\vee)(\alpha) \tag{41}$$

for any $\alpha\in\widetilde{CC}$. Indeed, one has

$$\varrho\circ b(\alpha) - b\circ\varrho(\alpha) =$$

$$\sum_{i<j<k} \pm(a_0a_i)'[a_{i+1}|\cdots|a_{j-1}|a_{k+1}|\cdots|a_n] \otimes(a_0a_i)''[a_1|\cdots|a_{i-1}]\otimes(a_ja_k)[a_{j+1}|\cdots|a_{k-1}] \tag{42}$$

$$+\sum_{j<i<k} \pm a_0[a_1|\cdots|a_{j-1}|a_{k+1}|\cdots|a_n] \otimes(a_ja_ka_i)'[a_{j+1}|\cdots|a_{i-1}]\otimes(a_ja_ka_i)''[a_{i+1}|\cdots|a_{k-1}] \tag{43}$$

$$+\sum_{j<k<i} \pm(a_0a_i)'[a_1|\cdots|a_{j-1}|a_{k+1}|\cdots|a_{i-1}] \otimes(a_ja_k)[a_{j+1}|\cdots|a_{k-1}]\otimes(a_0a_i)''[a_{i+1}|\cdots|a_n]. \tag{44}$$

After applying σ, (42) becomes $(\mathrm{id}\otimes\tau\circ h)\circ\vee(\alpha)$, (43) becomes $(\vee\otimes\mathrm{id})\circ(\tau\circ h)(\alpha)$, and (44) becomes $(\mathrm{id}\otimes\tau)\circ(\tau\circ h\otimes\mathrm{id})\circ\vee(\alpha)$. This proves the identity (41), and hence (40) is proved. □

Theorem 8 follows from the dual version of [Getzler 1994a, Proposition 1.2], Theorem 29, and the following lemma.

Lemma 30. *For any* $\alpha = a_0[a_1|\cdots|a_n] \in \widetilde{CC}_*(C)$, *let*

$$\phi(\alpha) := \sum_{i<j<k} \pm\varepsilon(a_0)a_i[a_{i+1}|\cdots|a_{j-1}|a_{k+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] \otimes a_j a_k[a_{j+1}|\cdots|a_{k-1}], \tag{45}$$

$$\psi(\alpha) := \sum_{j<k<i} \pm\varepsilon(a_0)a_j a_k[a_{j+1}|\cdots|a_{k-1}] \otimes a_i[a_{i+1}|\cdots|a_n|a_1|\cdots|a_{j-1}|a_{k+1}|\cdots|a_{i-1}], \tag{46}$$

and let $\theta = \phi + \psi$. *(The* $\pm$ *signs are determined by the Koszul sign rule.) Then*

$$b\circ\theta + \theta\circ b = \vee\circ B - B\circ\vee - S,$$

where S *is as defined in Lemma 27.*

Proof. The proof is similar to that of Lemma 23. For any $\alpha = a_0[a_1|\cdots|a_n]$, the terms on the right hand side of the desired equation are

$$\vee\circ B(\alpha) =$$

$$\sum_{i>j} \pm\varepsilon(a_0)(a_i a_j)'[a_{i+1}|\cdots|a_n|a_1|\cdots|a_{j-1}] \otimes (a_i a_j)''[a_{j+1}|\cdots|a_{i-1}] \tag{47}$$

$$+\sum_{i<j} \pm\varepsilon(a_0)(a_i a_j)'[a_{i+1}|\cdots|a_{j-1}] \otimes (a_i a_j)''[a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}], \tag{48}$$

$$B\circ\vee(\alpha) =$$

$$\sum_{i>k} \pm a_k[a_{k+1}|\cdots|a_{i-1}|a_1|\cdots|a_{k-1}] \otimes a_0 a_i[a_{i+1}|\cdots|a_n] \tag{49}$$

$$+\sum_{i<k} \pm a_0 a_i[a_1|\cdots|a_{i-1}] \otimes a_k[a_{k+1}|\cdots|a_n|a_{i+1}|\cdots|a_{k-1}], \tag{50}$$

$$S(\alpha) =$$

$$\sum_{i<j} \pm a_0[a_1|\cdots|a_{i-1}|a_{j+1}|\cdots|a_n] \otimes a_i a_j[a_{i+1}|\cdots|a_{j-1}] \tag{51}$$

$$+\sum_{i<j} \pm a_i a_j[a_{i+1}|\cdots|a_{j-1}] \otimes a_0[a_1|\cdots|a_{i-1}|a_{j+1}|\cdots|a_n]. \tag{52}$$

It follows that

$$\phi \circ b(\alpha) = -b \circ \phi(\alpha) + (47) - (49) - (51),$$

while

$$\psi \circ b(\alpha) = -b \circ \psi(\alpha) + (48) - (50) - (52). \qquad \square$$

6. The gravity coalgebra

Corollary 9 is immediate from Theorem 8 and the following result.

Proposition 31. *Let* $(V, \vee, \Delta)$ *be a Batalin–Vilkovisky coalgebra, and let* W *be a graded vector space. Let* $\mathrm{E} : V_* \to W_*$ *and* $\mathrm{M} : W_* \to V_{*+1}$ *be two maps such that* $\mathrm{E} \circ \mathrm{M} = 0$ *and* $\mathrm{M} \circ \mathrm{E} = \Delta$. *Define* $s_n : W \to W^{\otimes n}$ *for* $n \geq 2$ *by*

$$s_n(\alpha) := (\mathrm{E} \otimes \cdots \otimes \mathrm{E}) \circ (\vee \otimes \mathrm{id}^{\otimes n-2}) \circ \cdots \circ \vee \circ \mathrm{M}(\alpha)$$

for any $\alpha \in W$. *Then* $(W[1], \{s_n\})$ *is a gravity coalgebra.*

Proof. The proof is analogous to that of Proposition 24. By induction on n, we deduce from the identity in Definition 14 that

$$\text{(53)} \quad \vee_n \circ \Delta(x) - (n-2)\left(\sum_{i=1}^{n-1} \mathrm{id}^{\otimes i} \otimes \Delta \otimes \mathrm{id}^{\otimes n-i-1}\right) \circ \vee_n(x) = S_{2,n-2} \circ (\vee \circ \Delta \otimes \mathrm{id}^{\otimes n-2}) \circ \vee_{n-1}(x),$$

for all $x \in V$, where we set $\vee_n := (\vee \otimes \mathrm{id}^{\otimes n-2}) \circ \cdots \circ \vee : V \to V^{\otimes n}$ as before.

Let $x = \mathrm{M}(\alpha)$ where $\alpha \in W$. Applying $\mathrm{E}^{\otimes n}$ to both sides of (53), we get

$$\mathrm{E}^{\otimes n} \circ \left(\vee_n \circ \Delta(\mathrm{M}(\alpha)) - (n-2)\sum_{i=1}^{n-1}(\mathrm{id}^{\otimes i} \otimes \Delta \otimes \mathrm{id}^{\otimes n-i-1}) \circ \vee_n(\mathrm{M}(\alpha))\right) = \mathrm{E}^{\otimes n} \circ S_{2,n-2} \circ (\vee \circ \Delta \otimes \mathrm{id}^{\otimes n-2}) \circ \vee_{n-1}(\mathrm{M}(\alpha)),$$

where the left side vanishes since $\Delta = \mathrm{M} \circ \mathrm{E}$ and $\mathrm{E} \circ \mathrm{M} = 0$. Hence, we have

$$\begin{aligned} 0 &= \mathrm{E}^{\otimes n} \circ S_{2,n-2} \circ (\vee \circ \Delta \otimes \mathrm{id}^{\otimes n-2}) \circ \vee_{n-1}(\mathrm{M}(\alpha)) \\ &= \mathrm{E}^{\otimes n} \circ S_{2,n-2} \circ (\vee \circ \mathrm{M} \circ \mathrm{E} \otimes \mathrm{id}^{\otimes n-2}) \circ \vee_{n-1}(\mathrm{M}(\alpha)) \\ &= S_{2,n-2} \circ (\mathrm{E}^{\otimes 2} \circ \vee \circ (\mathrm{M} \circ \mathrm{E}) \otimes \mathrm{E}^{\otimes n-2}) \circ \vee_{n-1}(\mathrm{M}(\alpha)) \\ &= S_{2,n-2} \circ (s_2 \otimes \mathrm{id}^{\otimes n-2}) \circ s_{n-1}(\alpha). \end{aligned}$$

This proves the identity (6) in the definition of a gravity coalgebra for the case $l = 0$.

Now let $l > 0$. Let $x = \mathrm{M}(\alpha)$ where $\alpha \in W$ and suppose

$$\vee_{l+1}(x) = x_1 \otimes \cdots \otimes x_{l+1}.$$

Applying the identity (53) to the first component on both sides, by the same argument as above, we obtain

$$S_{2,n-2} \circ (s_2 \otimes \mathrm{id}^{\otimes n-2}) \circ s_{n-1+l}(\alpha) = (s_n \otimes \mathrm{id}^{\otimes l}) \circ s_{l+1}(\alpha).$$

This proves the identity (6) for the case $l > 0$. □

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