BI-HAMILTONIAN FLOWS AND THEIR REALIZATIONS AS
CURVES IN REAL SEMISIMPLE HOMOGENEOUS
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We describe a reduction process that allows us to define Hamiltonian structures on the manifold of differential invariants of parametrized curves for any homogeneous manifold of the form G/H, with G semisimple. We also prove that equations that are Hamiltonian with respect to the first of these reduced brackets automatically have a geometric realization as an invariant flow of curves in G/H. This result applies to some well-known completely integrable systems. We study in detail the Hamiltonian structures associated to the sphere $\mathrm{SO}(n+1)/\mathrm{SO}(n)$.

1. Introduction

Completely integrable systems are PDEs for which one can find an infinite family of preserved functionals in involution. Most of these systems are bi-Hamiltonian, that is, they are Hamiltonian with respect to two different but compatible Hamiltonian structures (compatible means that their sum is also a Hamiltonian structure). The Hamiltonian structures are used to generate a recursion operator, an operator that when reiteratively applied to one initial preserved functional generates the entire family — or hierarchy — see [Magri 1978]. In recent years a large number of publications have shown that many completely integrable systems appear linked to the geometric background of curves and surfaces; see for example [Anco 2006; Doliwa and Santini 1994; Ferapontov 1995; Gay-Balmaz and Ratiu 2007; Chou and Qu 2002; 2003; Langer and Perline 1991; 2000; Sanders and Wang 2003; Terng and Thorbergsson 2001; Terng and Uhlenbeck 2006; 2000; Yasui and Sasaki 1998; Marí 2008a; 2006; 2008b; 2007; 2005; 2009; Marí et al. 2002] and references within. Some of this work relates the integrable systems to invariant flows of (in general parametrized) curves in different types of manifolds through geometric realizations, that is, evolutions of curves inducing the integrable system on its curvatures, or differential invariants in general. Perhaps the best known example of

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such a geometric realization is that of the nonlinear Schrödinger equation (NLS) by the vortex filament flow (VF). Hasimoto [1972] showed that VF, viewed as a flow of spacial Euclidean curves, induces the NLS on its curvature and torsion via what it became known as the Hasimoto transformation. The Hasimoto transformation was proved to be a Poisson map between two equivalent bi-Poisson manifolds, that of the standard curvature and torsion and the manifold of natural curvatures; see [Langer and Perline 1991; Marí et al. 2002].

The author of this paper has linked the bi-Hamiltonian structures of many of these integrable systems to a process that allows us to reduce well-known compatible Poisson brackets on the manifold of loops in the dual of a Lie algebra, which we will call $\mathscr{L}\mathfrak{g}^*$, to the manifold of differential invariants. This reduction process was described in [Marí 2008a] for homogeneous manifolds of the form G/H where $\mathfrak{g}$, the Lie algebra associated to G, is |1|-graded. These include $\mathbb{RP}^{n+1}$, the conformal Möbius sphere, the Grassmannian, the Lagrangian Grassmannian and others. The reduction process was also described in [Marí 2006] for the case of affine geometries, that is, homogeneous manifolds of the form $G \ltimes \mathbb{R}^n/G$ with G semisimple. In both cases a well-known Poisson structure (we will refer to it as our *first bracket*) on $\mathscr{L}\mathfrak{g}^*$ can be reduced to the space of differential invariants to produce some of the best known Hamiltonian structures used in the integration of PDEs. This structure is also linked to geometric realizations in the sense that under minimal conditions one can find geometric realizations for any Hamiltonian evolution, and hence for bi-Hamiltonian integrable systems. The reduction of a *second* compatible bracket is not guaranteed, and neither is the existence of an associated integrable system. Indeed it was shown in [Marí 2005] that in the Lagrangian 2-Grassmannian manifold (or Grassmannian of Lagrangian planes in $\mathbb{R}^4$), the second bracket in $\mathscr{L}\mathfrak{sp}(2)^*$ never reduces. No completely integrable systems induced by Lagrangian flows on the differential invariants have been found. On the other hand, the reduction of the second bracket, whenever possible, points at the existence of an associated completely integrable system, or at least it is so in all known examples. Coming from a different direction, Terng, with Thorbergsson in [2001] and with Uhlenbeck in [2000; 2006], started by constructing classical completely integrable systems that are Hamiltonian with respect to the reduction of the second bracket, the bracket defined on coadjoint orbits, and after finding the existence of these systems they link them to our first bracket. These two different approaches have not been clearly bridged yet.

Even in the cases where the second bracket does not reduce, one can at times find integrable systems as level sets of Hamiltonian evolutions: the second bracket might not reduce to the complete manifold of differential invariants, but it might reduce to a submanifold of it defined by some chosen invariants. The geometric realization might exist if initial conditions are restricted to the types of curves

for which the undesired invariants are constant. For example, in the case of the Lagrangian n-Grassmannian the second Poisson bracket does not reduce in general, but it does always reduce to the submanifold defined by the eigenvalues of the so-called *Lagrangian Schwarzian* derivatives, whenever the other invariants vanish. In fact, it has been conjectured (and studies are supportive of this) that the type of Poisson structures/integrable systems and the character of the chosen invariants are closely related. For example, one can usually reduce the second bracket to a submanifold of differential invariants of projective type (as was done in [Marí 2008a; 2008b; 2007]) to obtain Poisson structures and integrable systems of KdV type (for example, the KdV equation or systems of decoupled KdV in [2008a; 2008b; 2007], complexly coupled KdV equations in [2008b], and Adler–Gel'fand–Dikii evolutions in [2008a]). Similarly, one can reduce to a submanifold of curvatures of *natural-type* to obtain modified KdV vector equations and NLS systems, as in [Anco 2006; Marí et al. 2002; Sanders and Wang 2003; Terng and Thorbergsson 2001; Terng and Uhlenbeck 2006].

A last relevant feature of these brackets is the following. Some of the Poisson structures obtained when reducing our first bracket are not truly structures associated to parametrized curves, but trivial extensions of Poisson brackets associated to unparametrized curves and extended trivially to the differential invariant of arc length type, as defined in [Marí 2009]. Except for the case $G = \mathrm{GL}(n, \mathbb{R})$, all classical affine geometries $G \ltimes \mathbb{R}^n / G$ have first reductions for which Hamiltonian evolutions will always preserve the invariant of arc length type [Marí 2009]. On the other hand, all known examples for semisimple parabolic cases (G/P, with P parabolic) have reductions of the first bracket that do not preserve parameters of arc length type. Indeed, geometric realizations of equations of KdV type do not preserve any invariant of arc length type. Thus, having first reductions on parametrized or unparametrized curves seems to be linked to the type of geometry that the manifold has.

In this paper we describe the reduction process for the general case of a homogeneous manifold G/H with G semisimple. Semisimplicity can be trivially assumed for the definition of the bracket; otherwise the bracket will only be defined on the semisimple component of the algebra. The reduction process here is, in fact, a simplification of the process in [Marí 2008a]. We prove in Theorem 4.3 that any system that is Hamiltonian with respect to the first reduced bracket possesses a geometric realization by an invariant flow on G/H. Our running example is that of $\mathrm{SO}(2, 2)/P$ for an appropriate choice of parabolic subgroup P. This manifold is geometrically equivalent to $\mathbb{RP}^1 \times \mathbb{RP}^1$ and we show that both brackets reduce to produce a decoupled system of KdV bi-Hamiltonian structures. The manifold $\mathrm{SO}(3, 1)/P$ (the conformal plane) is known [Marí 2005] to produce a system of two complexly coupled KdV equations. Thus, we show that the exchange

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

in the bilinear form defining the group effectively decouples the KdV system.

Finally, the Hamiltonian structures of the sphere $SO(n+1)/SO(n)$ is studied in Section 5. In [Terng and Thorbergsson 2001; Anco 2006], the authors found a geometric realization on this sphere for a vector system of modified KdV equations. The authors do not study the generation of the mKdV bi-Hamiltonian structures, or their possible definition by reduction (Anco provides a recursion operator that is said to be encoded by the geometry, but he provides no explanation on how the encoding takes place). The case of $SO(n+1)/SO(n)$ is interesting because, being a semisimple case (albeit not a parabolic one), one would think that the arc length does not need to be preserved; said differently, the first reduced Poisson structure should be expected to be a structure on parametrized curves. On the other hand, the mKdV systems found by Anco and Terng and Thorbergsson associated to this geometric background (and found also in the Euclidean case, an affine manifold) are arc length preserving. In Section 5, we show that the first reduced bracket does not preserve arc length, so that the bracket is defined on parametrized curves, in accordance with the manifold being homogeneous and semisimple. But here it is the *second bracket* that always preserves arc length and, hence, forces any bi-Hamiltonian system to be arc length preserving, in accordance with vector mKdV being the associated integrable system. The system of vector mKdVs is shown to be a bi-Hamiltonian system with respect to both reductions.

The reduction method we use is strongly rooted in the use of group-based moving frames. The method is relatively new so we include a description in our first section, together with other background definitions.

2. Background definitions

2a. *Moving frames, differential invariants, Serret–Frenet equations and geometric realizations.* The classical concept of moving frame was developed by Élie Cartan [1935; 1937]. A classical moving frame along a curve in a manifold M is a curve in the frame bundle of the manifold over the curve, invariant under the action of a transformation group under consideration. This method is a very powerful tool, but its explicit application relied on intuitive choices that were not clear in a general setting. Ideas in Cartan's work and later work of Griffiths [1974], Green [1978] and others laid the foundation for the concept of a group-based moving frame, that is, an equivariant map between the jet space of curves in the manifold and the group of transformations. Recent work by Fels and Olver [1999] finally gave the precise

definition of the group-based moving frame and extended its application beyond its original geometric picture. In this section we will describe Fels and Olver's moving frame and its role in our study. From now on we will assume $M = G/H$ with G semisimple and acting on M via left multiplication on representatives of a class. We will also assume that curves in M are parametrized, and that therefore the group G does not act on the parameter.

Definition 2.1. Let $J^k(\mathbb{R}, M)$ be the space of k-jets of curves, that is, the set of equivalence classes of curves in M up to k-th order of contact. If we denote by $u(x)$ a curve in M and by u_r the r-th derivative of u with respect to the parameter x, that is, $u_r = d^r u/dx^r$, then the jet space has local coordinates that can be represented by $u^{(k)} = (x, u, u_1, u_2, \ldots, u_k)$. The group G acts naturally on parametrized curves; therefore it acts naturally on the jet space via the formula

$$g \cdot u^{(k)} = (x, g \cdot u, (g \cdot u)_1, (g \cdot u)_2, \ldots),$$

where by $(g \cdot u)_k$ we mean the formula obtained when one differentiates $g \cdot u$ and then writes the result in terms of g, u, u_1, and so on. This is usually called the *prolonged* action of G on $J^k(\mathbb{R}, M)$.

Definition 2.2. A function $I : J^k(\mathbb{R}, M) \to \mathbb{R}$ is called a k-th order *differential invariant* if it is invariant with respect to the prolonged action of G.

Definition 2.3. A map $\rho : J^k(\mathbb{R}, M) \to G$ is called a left (respectively right) *moving frame* if it is equivariant with respect to the prolonged action of G on $J^k(\mathbb{R}, M)$ and the left (respectively right) action of G on itself.

The group-based moving frame appears in a familiar method for calculating the curvature of a curve $u(s)$ in the Euclidean plane. In this method one uses a translation to take $u(s)$ to the origin and then a rotation to make one of the axes tangent to the curve. The curvature can classically be found as the coefficient of the second order term in the expansion of the curve around $u(s)$. The crucial observation made by Fels and Olver is that the element of the group carrying out the translation and rotation depends on u and its derivatives and so defines a map from the jet space to the group. This map is a right moving frame, and it carries all the geometric information of the curve. In fact, Fels and Olver developed a similar normalization process to find right moving frames.

Theorem 2.4 [Fels and Olver 1999]. *Let $\cdot$ denote the prolonged action of the group on $u^{(k)}$ and assume we have* normalization equations *of the form $g \cdot u^{(k)} = c_k$, where at least some of the entries of c_k are constants (they are called normalization constants). Assume we have enough normalization equations to determine g as a function of $u, u_1, \ldots$. Then $g = \rho$ is a right invariant moving frame.*

Next is the description of the equivalent to the classical Serret–Frenet equations. We are denoting by L_g^* (respectively R_g^*), the map induced on TG by L_g, the left multiplication by g (respectively R_g, the right multiplication). From now on, we will also assume the (local) connection on M is flat, although some modifications can be introduced to assume constant curvature.

Definition 2.5. Consider $K\,dx$ to be the horizontal component of the pullback of the left- (respectively right)-invariant Maurer–Cartan form of the group G via a left (respectively right) moving frame ρ. That is,

$$K = L_{\rho^{-1}}^* \rho_x \in \mathfrak{g} \qquad (\text{respectively} \quad K = R_{\rho^{-1}}^* \rho_x)$$

We call K the *left* (respectively *right*) *Serret–Frenet equations* for the moving frame ρ.

Notice that, if ρ is a left moving frame, then ρ^{-1} is a right moving frame and their Serret–Frenet equations are the negatives of each other. A complete set of generating differential invariants can always be found among the coordinates of group-based Serret–Frenet equations, a crucial difference with the classical picture. The next theorem is a direct consequence of the results in [Fels and Olver 1999]. A more general result can be found in [Hubert 2007].

Theorem 2.6. *Let ρ be a* (*left or right*) *moving frame along a curve u. Let us fix a basis for $\mathfrak{g}$. Then, the coordinates of the* (*left or right*) *Serret–Frenet equations for ρ contain a basis for the space of differential invariants of the curve. That is, any other differential invariant for the curve is a function of the coordinates of K and their derivatives with respect to x.*

If we can find a moving frame using a set of normalization equations as in Theorem 2.4, we can also find algebraically the explicit form of the Serret–Frenet equations of the frame, following a parallel set of *recurrence equations*. Let $K \cdot u$ represent the infinitesimal action of the algebra $\mathfrak{g}$, likewise with $K \cdot u^{(k)}$, which represents the infinitesimal prolonged action. The following theorem is a rewriting of results appearing in [Fels and Olver 1999].

Theorem 2.7. *Let $K = L_{\rho^{-1}}^* \rho_x$ be the left Serret–Frenet equation associated to the left moving frame ρ. Let ρ be determined by normalization equations of the form $\rho^{-1} \cdot u_k = c_k$. Then, K satisfies the equations*

$$K \cdot u_k|_{\mathscr{I}} = c_{k+1} - (c_k)_x,$$

where $K \cdot u_k|_{\mathscr{I}}$ denotes what is usually called the invariantization of $K \cdot u_k$, that is, the expression $K \cdot u_k$ with all u_r substituted by c_r.

Definition 2.8 (geometric realization of an evolution of invariants). Let $\boldsymbol{k}$ denote a vector whose entries form an independent and generating system of differential

invariants for curves. That is, k is a vector whose entries are differential invariants for the curve; the entries of k and their derivatives are functionally independent (no entry can be written as a function of the other entries and their derivatives); and any other differential invariant is a function of the entries of k and their derivatives.

Let

$$k_t = F(k, k_x, k_{xx}, \dots) \tag{1}$$

be an evolution of k. We say that

$$u_t = Q(u, u_x, u_{xx}, u_{xxx}, \dots) \tag{2}$$

is *a geometric realization of* (1) *on* G/H whenever $u(t, x) \in G/H$, (2) is invariant under the action on G (that is, G takes solutions to solutions) and the evolution induced on k by (2) is (1). Equivalently, we say that (1) is the *invariantization* of (2).

2b. *Poisson brackets on $\mathscr{L}\mathfrak{g}^*$.* Consider the group of loops $\mathscr{L}G = C^\infty(S^1, G)$ and its Lie algebra $\mathscr{L}\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$. Let $\hat{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be an ad-invariant non-degenerate bilinear form of the algebra. We can use $\hat{B}$ to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ so that $X^* = \hat{B}(X, \cdot) \in \mathfrak{g}^*$. For example, if $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$, then $\hat{B}$ can be the trace of the matrix product. With this bilinear form, the dual to E_{ij} (the matrix having 1 in place (i, j) and 0 elsewhere) is given by E_{ji}. The bilinear form

$$B(X, Y) = \int_{S^1} \hat{B}(X, Y) dx \tag{3}$$

will give us the analogous form defined on $\mathscr{L}\mathfrak{g}$, and we can identify $\mathscr{L}\mathfrak{g}^*$ (the regular part of $(\mathscr{L}\mathfrak{g})^*$) with $\mathscr{L}\mathfrak{g}$ using B.

One can define two natural Poisson brackets on $\mathscr{L}\mathfrak{g}^*$; for more information see [Pressley and Segal 1989]. If $\mathscr{H}, \mathscr{G} : \mathscr{L}\mathfrak{g}^* \to \mathbb{R}$ are two functionals defined on $\mathscr{L}\mathfrak{g}^*$, then $\delta\mathscr{H}/\delta L$ denotes the variational derivative of $\mathscr{H}$ at L and it can be identified, using (3), with an element of $\mathscr{L}\mathfrak{g}$ so that

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathscr{H}(L + \epsilon V) = \int_{S^1} \hat{B}\left(\frac{\delta\mathscr{H}}{\delta L}, V\right) dx. \tag{4}$$

Likewise with $\mathscr{G}$. If $L \in \mathscr{L}\mathfrak{g}^*$, we define

$$\{\mathscr{H}, \mathscr{G}\}(L) = \int_{S^1} \left\langle \left(\frac{\delta\mathscr{H}}{\delta L}\right)_x + \mathrm{ad}^*\left(\frac{\delta\mathscr{H}}{\delta L}\right)(L), \frac{\delta\mathscr{G}}{\delta L}\right\rangle dx \tag{5}$$

where $\langle \cdot, \cdot \rangle$ is the natural coupling between $\mathfrak{g}^*$ and $\mathfrak{g}$ and where we identify $(\delta\mathscr{H}/\delta L)_x$ with its dual counterpart. If we identify L with its dual, then we have $\mathrm{ad}^*(\delta\mathscr{H}/\delta L)(L) = -\mathrm{ad}(\delta\mathscr{H}/\delta L)(L)$.

One also has a compatible family of second brackets, namely

$$(6) \qquad \{\mathcal{H}, \mathcal{G}\}(L) = \int_{S^1} \Big\langle \Big(\mathrm{ad}^*\Big(\frac{\delta\mathcal{H}}{\delta L}\Big)(L_0), \frac{\delta\mathcal{G}}{\delta L}\Big\rangle dx,$$

where $L_0 \in \mathfrak{g}^*$ is any constant element.

In the next section we will show how (5) can be always reduced to the space of differential invariants of curves. The compatible bracket (6) can only be reduced sometimes. Recall that the appearance of compatible pairs of Poisson brackets often indicates the existence of completely integrable systems.

3. Geometric Poisson brackets on the space of differential invariants of curves

Since $H \subset G$ is a subgroup, the algebra $\mathfrak{g}$ has a splitting of the form

$$(7) \qquad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

where $\mathfrak{m}$ is a vector subspace complement to the subalgebra $\mathfrak{h}$, but not a subalgebra in general. From now on we will also assume that our curves on homogeneous manifolds have a *group monodromy*, that is, there exists $m \in G$ such that

$$u(t + T) = m \cdot u(t),$$

where T is the period. Under these assumptions, the Serret–Frenet equations will be periodic and will belong to $\mathcal{L}\mathfrak{g}^*$ (under proper identification). Alternatively, one could assume that u is asymptotic at $\pm\infty$, so that the invariants vanish at infinity, and describe a similar situation.

Theorem 3.1. *Let u be a generic curve on the homogeneous manifold G/H. Let ρ be a left moving frame with $\rho \cdot o = u$. Locally, we can find moving frames for curves $\hat{u}$ in a neighborhood of u (with respect to the C^∞ topology) such that $\rho \cdot o = \hat{u}$. Let $\mathcal{H}$ be the submanifold of $\mathcal{L}\mathfrak{g}$ given by the Serret–Frenet equations associated to these left moving frames, in the sense of the previous section. Then, when identified with its dual, $\mathcal{H}$ defines a section of the quotient $\mathcal{L}\mathfrak{g}^*/\mathcal{L}H$, where the subgroup $\mathcal{L}H$ acts on $\mathcal{L}\mathfrak{g}^*$ via the standard gauge action*

$$a(g)(L) = L_{g^{-1}}^* g_x + \mathrm{Ad}^*(g)(L)$$

and where again the element $L_{g^{-1}}^* g_x$ is identified with its dual.

Proof. This theorem is proved using the definition of moving frame. Assume $m \in \mathcal{L}\mathfrak{g}^*$ and identify the element with an element in the algebra. Let η be a (local) solution of the equation $L_{\eta^{-1}}^* \eta_x = m$. We call $u = \eta \cdot o$ and we denote by ρ a left moving frame associated to u, with $\rho \cdot o = u$. The frame ρ has the same

monodromy as u, and u has the same monodromy as η. Hence, ρ and η have the same monodromy.

With these choices we have $\rho = \eta\eta^{-1}\rho = \eta g$ and $g \cdot o = \eta^{-1}\rho \cdot o = \eta^{-1} \cdot u = o$. Since H is the isotropy group of o (which represents the class of H in G/H), we conclude that $g(x) \in H$ for any x. Furthermore, the monodromy of both ρ and η are the same, and therefore $g \in \mathcal{L}H$. The action of $\mathcal{L}G$ on the space of solutions $\eta \to \eta g$ induces the gauge action described in the theorem on the elements of $\mathcal{L}\mathfrak{g}^*$ defining the equations satisfied by η. If identified with $\mathcal{L}\mathfrak{g}$, $\mathrm{Ad}^*(g)(L) = \mathrm{Ad}(g^{-1})(L)$ and the action on $\mathcal{L}\mathfrak{g}$ induced by the gauge action is $L^*_{g^{-1}}g_x + \mathrm{Ad}(g^{-1})L$. □

Example 3.2. Our running example will be the case $G = \mathrm{SO}(2, 2)$ for $H = P$ given by a particular parabolic choice. Assume $\mathrm{SO}(2, 2)$ is the isotropy group of the bilinear form defined by the matrix

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

that is, $\mathfrak{o}(2, 2)$ is the set of matrices that are skew-symmetric with respect to the secondary diagonal. Locally, $g \in \mathrm{SO}(2, 2)$ can be factored as

$$g = g_1(v)g_0(\alpha, \Theta)g_{-1}(y)$$
$$= \begin{pmatrix} 1 & v^1 & v^2 & -v^2v^1 \\ 0 & 1 & 0 & -v^2 \\ 0 & 0 & 1 & -v^1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ y^1 & 1 & 0 & 0 \\ y^2 & 0 & 1 & 0 \\ -y^2y^1 & -y^2 & -y^1 & 1 \end{pmatrix},$$

with $\alpha \in \mathbb{R}$ and $\Theta \in \mathrm{SO}(1, 1)$. This factorization corresponds to the algebra grada-tion $\mathfrak{o}(2, 2) = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ as in the diagram

$$\begin{pmatrix} 0 & +1 & +1 & +1 \\ -1 & 0 & 0 & +1 \\ -1 & 0 & 0 & +1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

Let us choose the parabolic subgroup $H = P = G_1 \cdot G_0$, that is, the subgroup defined by elements g such that $y^1 = y^2 = 0$. Notice that $\mathrm{SO}(3, 1)$ has the exact same description, with one difference, namely $\Theta \in \mathrm{SO}(2)$ (here $-v^1v^2 = -\frac{1}{2}\|v\|_J$ — see below — while for $\mathrm{SO}(3, 1)$ we would have $-\frac{1}{2}\|v\| = -\frac{1}{2}v^T v$ instead).

With this representation, the action of $\mathrm{SO}(2, 2)$ on $\mathrm{SO}(2, 2)/H$ is determined by the relation $gg_{-1}(u) = g(g \cdot u)h$ for some $h \in H$. We will use the section $\varsigma : \mathrm{SO}(2, 2)/H \to \mathrm{SO}(2, 2)$ given by $\varsigma(u) = g_{-1}(u)$ to locally identify $\mathrm{SO}(2, 2)/H$

with G_{-1}. The subgroups G_i are the exponentials of the Lie subalgebras $\mathfrak{g}_i$. One can readily find an explicit formula for the action using this notation:

$$(8) \qquad g \cdot u = \frac{\alpha^{-1}\Theta(\boldsymbol{u}+\boldsymbol{y}) + \frac{1}{2}\alpha^{-2}\|\boldsymbol{u}+\boldsymbol{y}\|_J^2 \boldsymbol{v}^*}{1 + \alpha^{-1}\boldsymbol{v}^T\Theta(\boldsymbol{u}+\boldsymbol{y}) + \frac{1}{4}\alpha^{-2}\|\boldsymbol{v}\|_J^2\|\boldsymbol{u}+\boldsymbol{y}\|_J^2}$$

where $\|x\|_J = \hat{x}^T J\hat{x}$ for $\hat{x} = (0, x, 0)$ and where, if $\boldsymbol{v} = \binom{v^1}{v^2}$, then $\boldsymbol{v}^* = \binom{v^2}{v^1}$. One can check that this action decouples into two projective actions. If $\Theta \in \mathrm{SO}(1, 1)$ with $\Theta = \mathrm{diag}(a, a^{-1})$, the two projective actions are given by $(y^1, a\alpha^{-1}, v^1)$ acting projectively on u^1 and $(y^2, a\alpha^{-1}, v^2)$ acting projectively on u^2. This is because the isomorphism $\mathfrak{o}(2, 2) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ induces a splitting of $\mathrm{SO}(2, 2)$ into $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ and also an equivalence of $O(2, 2)/P$ with $\mathbb{RP}^1 \times \mathbb{RP}^1$. (A choice of Θ on the second connected component of $O(1, 1)$ will simply produce an involution exchanging u^1 and u^2.)

   If g is as in (8), the zero normalization equation is $g \cdot \boldsymbol{u} = 0$, which can be solved with the choice $\boldsymbol{y} = -\boldsymbol{u}$. If $\boldsymbol{u} = \boldsymbol{u}(x)$, the first normalization equation is $g \cdot \boldsymbol{u}_1 = c_1$, obtained by differentiating the action (8) with respect to x and substituting $\boldsymbol{y} = -\boldsymbol{u}$. It is given by

$$\alpha^{-1}\Theta\boldsymbol{u}_1 = c_1.$$

Since $\Theta = \mathrm{diag}(a, a^{-1}) \in O(1, 1)$, we need to choose nonvanishing normalization values for each of the entries of c_3. We choose $c_1 = \binom{1}{1}$, rather than the usual $c_1 = e_1$ favored in normalizations — in this case e_1 would be a singular choice. This choice forces the values

$$\alpha = \|\boldsymbol{u}_x\|_J 2^{-1/2} \quad \text{and} \quad \Theta^{-1}\binom{1}{1} = \frac{\sqrt{2}\boldsymbol{u}_x}{\|\boldsymbol{u}_x\|_J}.$$

This condition completely determines $\Theta = \mathrm{diag}(\alpha(u_x^1)^{-1}, \alpha^{-1}u_x^1)$.

   The second normalization equation is obtained differentiating (8) twice and substituting previously found values. It is given by

$$\alpha^{-2}\Theta\boldsymbol{u}_{xx} - \boldsymbol{v} = c_2 = 0,$$

which is readily resolved choosing $\boldsymbol{v} = \alpha^{-2}\Theta\boldsymbol{u}_{xx}$. This last equation completely determines the right moving frame. Following [Fels and Olver 1999], we have a set of independent and generating invariants given by the entries of c_3; we have two invariants of third order. The interested reader can differentiate once more and find the third normalization equations and the explicit formula for c_3. It is given by $c_3 = \binom{k_1}{k_2}$ with $k_i = S(u^i)$, where $S(f) = f_x^{-1}(f_{xxx} - \frac{3}{2}(f_{xx}/f_x)^2)$ is the Schwarzian derivative of f. The Schwarzian derivative is the generator of projective differential invariants in $\mathbb{RP}^1$.

Let's call ρ the left moving frame, that is, the inverse of the frame we just found:

$$\rho = \begin{pmatrix} 1 & -(\boldsymbol{u}^*)^T & -\frac{1}{2}\|\boldsymbol{u}\|_J^2 \\ 0 & I & \boldsymbol{u} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 & 0 \\ 0 & \Theta^{-1} & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & -\boldsymbol{v}^T & -\frac{1}{2}\|\boldsymbol{v}\|_J \\ 0 & I & \boldsymbol{v}^* \\ 0 & 0 & 1 \end{pmatrix}.$$

In parallel with the normalization equations, we can use recurrence formulas of Theorem 2.7 to determine the matrix $K = \rho^{-1}\rho_x$. If $K = K_1 + K_0 + K_{-1}$ are the gradated components of K and $K_0 = K_\alpha + K_\Theta$ are the two components of K_0, the recurrence formulas are given by

$$K \cdot u|_\mathscr{I} = K_{-1} = c_1 - c_0' = c_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$K \cdot u_1|_\mathscr{I} = K_\Theta c_1 - K_\alpha c_1 = c_2 - c_1' = 0.$$

The last equation implies $K_\Theta = 0$ and $K_\alpha = 0$. The two equations describe K as being of the form

$$(9) \qquad\qquad K = \begin{pmatrix} 0 & k_1 & k_2 & 0 \\ 1 & 0 & 0 & -k_2 \\ 1 & 0 & 0 & -k_1 \\ 0 & -1 & -1 & 0 \end{pmatrix}.$$

The general theory tells us that the entries of K generate all other differential invariants for u, and hence k_1 and k_2 must be generators. If one writes the recurrence equations of Theorem 2.7 for the second prolongation, we also see that K_1 coincides with c_3. This matrix is very similar to the one obtained in the case $G = O(3, 1)$ for which G/H is the conformal plane; see [Marí 2008b]. The only difference is that in the conformal case, $c_1 = e_1$ is a regular value and $K_{-1} = e_1$ was chosen instead. This small difference will create a very significant one for the reduced Poisson brackets and their associated integrable systems.

Our next theorem shows that (5) can be reduced to $\mathscr{K}$, and its proof gives an algebraic method to calculate the reduced bracket explicitly (and also the reduction of (6) whenever possible).

Theorem 3.3. *The Poisson bracket defined on $\mathscr{L}\mathfrak{g}^*$ by (5) is reducible to the submanifold $\mathscr{K}$. We call this the first reduced Poisson bracket associated to curves on G/H.*

Proof. Observe that $\mathscr{K}$ is given locally by the quotient $\mathscr{L}\mathfrak{g}^*/\mathscr{L}H$, where $\mathscr{L}H$ acts in $\mathscr{L}\mathfrak{g}^*$ via the gauge action. The symplectic leaves of the bracket (5) are formed by the orbits of the gauge action itself. For more information on these brackets, see [Pressley and Segal 1989]. Assume we have two functionals $\mathscr{R}$ and $\mathscr{G}$ such that

$\delta\mathcal{R}/\delta L$, $\delta\mathcal{G}/\delta L \in \mathcal{L}\mathfrak{g}$ vanish on the tangent to the $\mathcal{L}H$-leaves. That means

$$(10) \qquad \left(\frac{\delta\mathcal{R}}{\delta L}\right)_x - \mathrm{ad}\left(\frac{\delta\mathcal{R}}{\delta L}\right)(K) \in \mathfrak{h}^0$$

and likewise for $\mathcal{G}$ (we are identifying $\mathcal{L}\mathfrak{g}$ with $\mathcal{L}\mathfrak{g}^*$). Then, the bracket (5) of these two functionals will also vanish on the tangent to the leaves (equivalently, it will be constant on the leaves); one only needs to apply Jacobi's identity for (5) to see this. Hence, the bracket will represent a well-defined functional on the quotient $\mathcal{K}$.

Following the same reasoning as in [Marsden and Ratiu 1986], let $r, g : \mathcal{K} \to \mathbb{R}$ be two functionals, and let $\mathcal{R}$ and $\mathcal{G}$ be two extensions that are constant on the leaves of $\mathcal{L}H$. The bracket

$$(11) \qquad \{r, g\}(K) = \int_{S^1} \left\langle \left(\frac{\delta\mathcal{R}}{\delta L}\right)_x - \mathrm{ad}\left(\frac{\delta\mathcal{R}}{\delta L}\right)(K), \frac{\delta\mathcal{G}}{\delta L} \right\rangle dx$$

describes a well-defined functional on $\mathcal{K}$. It is a Poisson bracket on $\mathcal{K}$ in which Jacobi's identity is given directly by the Jacobi identity of (5). For a complete description of this and other Poisson reductions for finite-dimensional manifolds, see [Marsden and Ratiu 1986]. Our infinite-dimensional case is a straightforward generalization of the results there. $\qquad\square$

Although this bracket seems to be complicated to compute, in all known cases the calculation follows a purely algebraic process that can be done by hand in low dimensions. The essence of the algebraic process is the use of (10).

Example 3.4. We now go back to the case $G = \mathrm{SO}(2, 2)$. In this case $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and so $\mathfrak{h}^0 = \mathfrak{g}_1$. If K is given as in (9), then an extension $\mathcal{R}$ of a functional $r : \mathcal{K} \to \mathbb{R}$ to $\mathcal{L}\mathfrak{o}(2, 2)^*$ will coincide with r in the direction of k_1 and k_2. The variational derivative of $\mathcal{R}$ is defined as in (4), and so

$$(12) \qquad \frac{\delta\mathcal{R}}{\delta L}(K) = \begin{pmatrix} \beta & a & b & 0 \\ \delta r/\delta k_1 & c & 0 & -b \\ \delta r/\delta k_2 & 0 & -c & -a \\ 0 & -\delta r/\delta k_2 & -\delta r/\delta k_1 & -\beta \end{pmatrix}.$$

If we substitute these values in condition (10), we get along $\mathcal{K}$ that

$$\begin{pmatrix} \beta' + k_1\delta r/\delta k_1 + k_2\delta r/\delta k_2 - a - b & a' + ck_1 - \beta k_1 & b' - ck_2 - \beta k_2 & 0 \\ (\delta r/\delta k_1)_x + \beta - c & c' + a + k_2\delta r/\delta k_2 - k_1\delta r/\delta k_1 - b & 0 & * \\ (\delta r/\delta k_2)_x + \beta + c & 0 & * & * \\ 0 & * & * & * \end{pmatrix}$$

$$= \begin{pmatrix} 0 & * & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From here we obtain

$$\beta = -\frac{1}{2}\left(\frac{\delta r}{\delta k_1} + \frac{\delta r}{\delta k_2}\right)', \qquad c = \frac{1}{2}\left(\frac{\delta r}{\delta k_1} - \frac{\delta r}{\delta k_2}\right)'$$

$$a = -\frac{1}{2}\left(\frac{\delta r}{\delta k_1}\right)'' + k_1\frac{\delta r}{\delta k_1}, \quad b = -\frac{1}{2}\left(\frac{\delta r}{\delta k_2}\right)'' + k_2\frac{\delta r}{\delta k_2}$$

The reduced Poisson bracket is defined by (11), where $\mathcal{R}$ and $\mathcal{G}$ are appropriate extensions with variational derivatives as above. After straightforward calculations, these can be written as

$$\{r, g\}(\boldsymbol{k}) = \int_{S^1} \frac{\delta g}{\delta k_1}\left(-\frac{1}{2}D^3 + Dk_1 + k_1 D\right)\frac{\delta r}{\delta k_1} + \frac{\delta g}{\delta k_2}\left(-\frac{1}{2}D^3 + Dk_1 + k_1 D\right)\frac{\delta r}{\delta k_2};$$

therefore the first reduced bracket is defined by two *decoupled second Poisson structures* for KdV equations, one in each k_1 and k_2. We can also check whether or not, for some choice of L_0, the bracket (6) reduces to $\mathcal{K}$ by evaluating (6) in our extensions. If we choose $L_0 = E_{12} - E_{21} + E_{13} - E_{31}$ (that is, the element dual to K_{-1}), the result is

$$\{r, g\}_0(\boldsymbol{k}) = \int_{S^1}\left\langle\frac{\delta \mathcal{G}}{\delta L}(K), \left[L_0, \frac{\delta \mathcal{R}}{\delta L}(K)\right]\right\rangle dx = -2\int_{S^1} \frac{\delta g}{\delta k_1}D\frac{\delta r}{\delta k_1} + \frac{\delta g}{\delta k_2}D\frac{\delta r}{\delta k_2}.$$

Thus, the second reduced bracket is given by two *decoupled first Poisson structures* for KdV equations.

This result fits well with the equivalence $SO(2, 2)/H \cong \mathbb{RP}^1 \times \mathbb{RP}^1$. Indeed, the two reduced Poisson brackets associated to the geometry of flows in $\mathbb{RP}^1$ are known to be the two KdV Hamiltonian structures. On the other hand, $O(3, 1)/P$ is the conformal plane and the two reduced Poisson brackets were given by the two Hamiltonian structures for a complexly coupled system of KdV equations [Marí 2005]. Thus the change $O(3, 1) \to O(2, 2)$ decouples the Hamiltonian structures.

4. Geometric realizations of Hamiltonian evolutions

Let $\Phi_g : G/H \to G/H$ be defined by the action of $g \in G$ on the quotient, that is, $\Phi_g(x) = \Phi_g([y]) = [gy] = g \cdot x$. Let $\varsigma : G/H \to G$ be a section of the homogeneous quotient such that $\varsigma(o) = e$. The section is compatible with the action of G on G/H, that is,

$$(13) \qquad g\varsigma(x) = \varsigma(\Phi_g(x))h \quad \text{for some } h \in H.$$

This relation in fact determines the action of the group on G/H uniquely, as we saw in our running example. As before, we consider the splitting of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is not in general a Lie subalgebra. Since ς is s section, $d\varsigma(o)$ is an isomorphism between $\mathfrak{m}$ and T_oM.

The following theorem was proved in [Marí 2006] and describes the most general form of invariant evolutions in terms of left moving frames.

Theorem 4.1. *Let $u(t, x) \in G/H$ be a flow, that is, the solution of an invariant evolution of the form*

$$u_t = F(u, u_x, u_{xx}, u_{xxx}, \dots).$$

Assume the evolution is invariant under the action of G, that is, G takes solutions to solutions. Let $\rho(t, x)$ be a family of left moving frames along $u(t, x)$ such that $\rho \cdot o = u$. Then, there exists an invariant family of tangent vectors $\mathbf{r}(t, x)$, that is, a family depending on the differential invariants of u and their derivatives, such that

$$u_t = d\Phi_\rho(o)\mathbf{r}.$$

An interpretation of this theorem is as follows. If we choose coordinates and $d\Phi_\rho(o)$ is considered as an element on $\mathrm{GL}(n, \mathbb{R})$, then its columns $d\Phi_\rho(o) = (T_1, \dots, T_n)$ form a classical moving frame, that is, an invariant curve in the frame bundle. If in those coordinates $\mathbf{r} = (r_1, \dots, r_n)^T$, then $u_t = r_1 T_1 + \cdots + r_n T_n$ for some r_i functions of the differential invariants and their derivatives. Many readers might be more familiar with this writing of an invariant evolution, and it is equivalent to ours.

Before we describe the relation between the evolutions of u and geometric Hamiltonian evolutions, it is convenient to prove this:

Lemma 4.2. *Let $u(t, x)$ be a one-parameter family of curves in G/H. Assume $u(t, x)$ evolves following an evolution invariant under the action of G. Assume the evolution is written as*

(14) $$u_t = d\Phi_\rho(o)\mathbf{r},$$

where ρ is a left moving frame that can be locally factored as $\varsigma(u)\rho_H$ with $\rho_H \in H$, and where $\mathbf{r}$ is some invariant tangent vector.

Let $N = L_{\rho^{-1}}^* \rho_t$ be the left invariant vector field defining the evolution of ρ under (14). Let $N = N_{\mathfrak{m}} + N_{\mathfrak{h}}$ be the splitting of N in its $\mathfrak{m}$ and $\mathfrak{h}$ component. Then $N_{\mathfrak{m}} = d\varsigma(o)\mathbf{r}$.

Note $\rho_H \cdot o = o$ since $\rho_H \in H$. Using (13) we have

$$\varsigma(u)\varsigma(o) = \varsigma(u) = \varsigma(\varsigma(u) \cdot o)h,$$

which is uniquely determined for some value of $h \in H$. The choices $h = e$ and $\varsigma(u) \cdot o = u$ satisfy the equation, so we can conclude that $\varsigma(u) \cdot o = u$.

Proof. Assume $\rho = \varsigma(u)\rho_H$. If we calculate N, we have

$$N = \mathrm{Ad}(\rho_H^{-1})L_{\varsigma(u)^{-1}}^* d\varsigma(u)u_t + L_{\rho_H^{-1}}^* d\rho_H(u)u_t.$$

Since $L^*_{\rho_H^{-1}} d\rho_H(u)u_t \in \mathfrak{h}$ we need to look only for the $\mathfrak{m}$ component of

$$\mathrm{Ad}(\rho_H^{-1}) L^*_{\varsigma(u)^{-1}} d\varsigma(u)u_t.$$

On the other hand, differentiating (13) gives

$$L^*_g d\varsigma(u)u_t = d\varsigma(\Phi_g(u))d\Phi_g(u)u_t h(u,g) + \varsigma(\Phi_g(u))dh(u)u_t.$$

Evaluating this at $g = \rho^{-1}$, we get

$$L^*_{\rho^{-1}} d\varsigma(u)u_t = R^*_{h(u,\rho^{-1})} d\varsigma(o)d\Phi_{\rho^{-1}}(u)u_t + dh(u,\rho^{-1})u_t.$$

Also from (13),

$$\rho^{-1}\varsigma(u) = \rho_H^{-1} = \varsigma(\rho^{-1} \cdot u)h(u,\rho^{-1}) = \varsigma(o)h(u,\rho^{-1}) = h(u,\rho^{-1}),$$

and $(d\Phi_{\rho^{-1}}(u))^{-1} = d\Phi_\rho(\rho^{-1} \cdot u) = d\Phi_\rho(o)$. Therefore

$$R^*_{\rho_H} L^*_{\rho^{-1}} d\varsigma(u)u_t = R^*_{\rho_H} R^*_{h(u,\rho^{-1})} d\varsigma(o)(d\Phi_\rho(o))^{-1}u_t = d\varsigma(o)\boldsymbol{r}$$

whenever u evolves as in (14). This is precisely $N_{\mathfrak{m}}$.   □

In what follows we will assume the manifold to be flat, so its Cartan connection is given by the Maurer–Cartan form. If, for example, the manifold has constant curvature, some modifications can be introduced to adapt the result, much as was done in [Terng and Thorbergsson 2001; Anco 2006; Marí et al. 2002].

Theorem 4.3. *Assume that $\mathcal{K}$ is described by an affine subspace of $\mathcal{L}\mathfrak{g}^*$. Assume that* (14) *is an invariant evolution of curves on G/H and there is a Hamiltonian functional $h : \mathcal{K} \to \mathbb{R}$ such that, if $\mathcal{H} : \mathcal{L}\mathfrak{g}^* \to \mathbb{R}$ is an extension of h satisfying condition* (10), *then*

$$\frac{\delta\mathcal{H}}{\delta L}(\boldsymbol{k})_{\mathfrak{m}} = d\varsigma(o)\boldsymbol{r},$$

where

$$\frac{\delta\mathcal{H}}{\delta L}(\boldsymbol{k}) = \frac{\delta\mathcal{H}}{\delta L}(\boldsymbol{k})_{\mathfrak{m}} + \frac{\delta\mathcal{H}}{\delta L}(\boldsymbol{k})_{\mathfrak{h}}$$

are the components defined by the splitting of the algebra. Then $\mathcal{K}$ induces by (14) *an evolution Hamiltonian with respect to the first reduced Poisson bracket* (11), *with Hamiltonian functional h. In particular, any Hamiltonian evolution in $\boldsymbol{k}$ with respect to the first reduced Poisson bracket* (11) *and Hamiltonian functional $h(\boldsymbol{k})$ has a geometric realization given by*

$$u_t = d\Phi_\rho(o)d\varsigma(o)^{-1}\frac{\delta\mathcal{H}}{\delta L}(\boldsymbol{k})_{\mathfrak{m}}$$

where $\mathcal{H}$ is any extension of h satisfying (10).

Proof. Assume that an evolution of u as in (14) induces a Hamiltonian evolution on $\mathcal{K}$, with Hamiltonian functional $h : \mathcal{K} \to \mathbb{R}$. If $\mathcal{K}$ is an affine subspace of $\mathcal{L}\mathfrak{g}^*$, then K_t is a linear subspace of $\mathcal{L}\mathfrak{g}^*$. Assume $r : \mathcal{K} \to \mathbb{R}$ is any other Hamiltonian functional, and let $\mathcal{R}$ be an extension satisfying (10). Then

$$\int_{S^1} \left\langle K_t, \frac{\delta\mathcal{R}}{\delta L}(K) \right\rangle dx = \{h, r\}(K).$$

On one hand, if $\mathcal{H}$ is an extension of h satisfying (10), then

$$(15) \qquad \{h, r\}(K) = \int_{S^1} \left\langle \left(\frac{\delta\mathcal{H}}{\delta L}(K) \right)_x + \mathrm{ad}^* \left(\frac{\delta\mathcal{H}}{\delta L}(K) \right)(K), \frac{\delta\mathcal{R}}{\delta L}(K) \right\rangle.$$

On the other hand, if $N = L^*_{\rho^{-1}}\rho_t$, then applying the structure equation for the Maurer–Cartan form to the commuting vector fields d/dx and d/dt along ρ results in the compatibility condition $K_t = N_x + \mathrm{ad}(K)(N)$. Therefore, we obtain that

$$\left\langle K_t, \frac{\delta\mathcal{R}}{\delta L}(K) \right\rangle = \left\langle \left(\frac{\delta\mathcal{H}}{\delta L}(K) \right)_x + \mathrm{ad}^* \left(\frac{\delta\mathcal{H}}{\delta L}(K) \right)(K), \frac{\delta\mathcal{R}}{\delta L}(K) \right\rangle$$
$$= \left\langle N_x + \mathrm{ad}^*(N)(K), \frac{\delta\mathcal{R}}{\delta L}(K) \right\rangle,$$

where we are again identifying K with its dual, so that $\mathrm{ad}(K)(N) = \mathrm{ad}^*(N)(K)$. Finally, from (10), the only component involved in (15) is $\delta\mathcal{R}/\delta L_{\mathfrak{m}}$. Likewise for $\delta\mathcal{H}/\delta L$ by skew-symmetry. Therefore, if $\delta\mathcal{H}/\delta L_{\mathfrak{m}} = N_{\mathfrak{m}}$, the evolution induced on $\boldsymbol{k}$ will be Hamiltonian with Hamiltonian functional h. Using the lemma, we arrive to the conclusion of the theorem. $\square$

In general, N and $\delta\mathcal{H}/\delta L$ are different. Only their components tangent to the manifold need to coincide.

Example 4.4. Using the data we have on $\mathrm{SO}(2, 2)/H$, one can easily calculate the formula for a general invariant evolution to be

$$u_t = d\Phi_\rho(o)\boldsymbol{r} = \alpha\Theta\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} u_x^1 & 0 \\ 0 & u_x^2 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

which results on the decoupling $u_t^i = u_x^i r_i$ for $i = 1, 2$, where the r_i are any functions depending on k_1, k_2 and their derivatives. The evolutions are not decoupled unless the r_i are decoupled. From the data we obtained in (12) we have

$$\frac{\delta\mathcal{H}}{\delta L}(K)_{\mathfrak{m}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dfrac{\delta h}{\delta k_1} & 1 & 0 & 0 \\ \dfrac{\delta h}{\delta k_2} & 0 & 1 & 0 \\ 0 & -\dfrac{\delta h}{\delta k_2} & -\dfrac{\delta h}{\delta k_1} & 0 \end{pmatrix} \quad \text{and} \quad d\varsigma(o)\boldsymbol{r} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ bfr_1 & 1 & 0 & 0 \\ r_2 & 0 & 1 & 0 \\ 0 & -r_2 & -r_1 & 0 \end{pmatrix},$$

so that the condition for a geometric realization to exist is $\delta h/\delta k_i = r_i$ for $i = 1, 2$. In particular, a pair of decoupled KdV equations is obtained when

$$h(k_1, k_2) = \frac{1}{2} \int_{S^1} (k_1^2 + k_2^2) dx$$

for which $r_i = k_i$ produces a geometric realization. In the conformal case for which $G = O(3, 1)$, these same choices produced a geometric realization for a complexly coupled system of KdV equations. That is, changing from $SO(3, 1)$ to $SO(2, 2)$ effectively decouples the system of coupled KdV equations.

5. The sphere $SO(n + 1)/SO(n)$

In this case $G = SO(n + 1)$ and $H = SO(n)$ is not a parabolic subgroup. We consider the splitting $\mathfrak{o}(n + 1) = \mathfrak{m} \oplus \mathfrak{h}$ of the Lie algebra into subspaces (unlike the previous example, only $\mathfrak{h}$ is a Lie subalgebra here) with

(16)
$$\begin{pmatrix} 0 & y \\ y^T & 0 \end{pmatrix} \in \mathfrak{m} \quad \text{and} \quad \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{h},$$

where $y \in \mathbb{R}^n$ and $A \in \mathfrak{o}(n)$. Associated to this splitting we have a local factorization in the group into factors belonging to $H = SO(n)$ and $\exp(\mathfrak{m})$. This factorization is given by

(17)
$$g = \begin{pmatrix} \Theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I + \cos_y yy^T & \sin_y y \\ -\sin_y y^T & \cos\|y\| \end{pmatrix}$$

where
$$\cos_y = \frac{\cos\|y\| - 1}{\|y\|^2}, \quad \sin_y = \frac{\sin\|y\|}{\|y\|}, \quad \|y\|^2 = y^T y.$$

The factorization exists locally.

Let $\varsigma : M \to G$ be the section defined by the exponential, that is,

$$\varsigma(u) = \begin{pmatrix} I + \cos_u uu^T & \sin_u u \\ -\sin_u u^T & \cos\|u\| \end{pmatrix}.$$

Clearly $d\varsigma(o) : T_oM \to \mathfrak{m}$ is an isomorphism given by $d\varsigma(o)y = \begin{pmatrix} 0 & y \\ y^T & 0 \end{pmatrix}$. The action of $SO(n+1)$ on the sphere — let's denote it by $g \cdot u$ — is determined by the relation $g\varsigma(u) = \varsigma(g \cdot u)h$ for some $h \in SO(n)$ that is also determined by this relation. Let g be as in (17). Straightforward calculations show that if $\eta = g \cdot u$, then

(18)
$$\sin_\eta \eta = \sin_u \Theta u + (\cos_y \sin_u y^T u + \sin_y \cos\|u\|)\Theta y$$

(19)
$$\cos\|\eta\| = \cos\|y\| \cos\|u\| - \sin_u \sin_y y^T u.$$

5a. *Left moving frames, Serret–Frenet equations and geometric Hamiltonian structures for generic curves on the sphere.*

Moving frames. With the factorization above in mind we can use normalization procedures to calculate a right moving frame along a generic curve u. Indeed, if g is as in (17), then the first normalization equation is $g \cdot u = o$, which is resolved by choosing $y = -u$. Notice that, if ς is our section, $\varsigma(u)^{-1} = \varsigma(-u)$, and SO$(n)$ preserves the origin o.

The first normalization equation is given in terms of the prolonged action of the group. The action is an action on parametrized curves. Therefore, its explicit expression is found, as before, by differentiating $g \cdot u$ with respect to the parameter. If we do that and substitute $y = -u$, the first normalization equation is then

$$\sin_u \Theta u_1 + (1 - \sin_u) \frac{\|u\|_1}{\|u\|} \Theta u = s e_1$$

where $s = (\sin_u^2 \|u_x\|^2 + (1 - \sin_u^2) \|u\|_x^2)^{1/2}$ is the spherical *arc length invariant*. The vector e_1 is an arbitrary choice; any other unit vector can be chosen instead. We will not, in general, consider unparametrized curves, so this invariant is not, a priori, constant.

Subsequent normalization equations (up to order n) will determine $\Theta^{-1} e_i$ for $i = 2, \ldots, n$ and with it Θ. The r-th normalization equation will be of the form $\Theta f_r(u^{(r)}) = c_r$ for some function f_r depending on u and its derivatives. The fact that $\Theta \in \mathfrak{o}(n)$ implies that the vector c_r is a function of r differential invariants of order r. Among these r differential invariants, $r - 1$ of them will be functions of lower order differential invariants and their derivatives. Hence, at each step we get a new invariant of order r that is functionally independent from those of lower order. Thus, we have n invariants or increasingly high order, the order increasing by one at each step. According to the theory developed in [Fels and Olver 1999], these would be generators of all differential invariants of the curve u. For the purpose of this example, no more details are needed.

Serret–Frenet equations and natural moving frames. First of all, the $\mathfrak{m}$ component of $\rho(\rho^{-1})_x = \widehat{K}$ is equal to $d\varsigma(0)(e_1)$, as we proved in our previous section when studying the general case.

Indeed, after some straightforward calculations,

$$\rho(\rho^{-1})_x = \begin{pmatrix} \Theta & 0 \\ 0 & 1 \end{pmatrix} s(u)(s(-u))_x \begin{pmatrix} \Theta^{-1} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \Theta(\Theta^{-1})_x & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \Theta(\Theta^{-1})_x + \Theta(\cos_u u_1 u^T - \cos_u u u_1^T)\Theta^{-1} & \sin_u \Theta u_1 + (1 - \sin_u)\frac{\|u\|_1}{\|u\|}\Theta u \\ -\sin_u u_1^T \Theta^T - (1 - \sin_u)\frac{\|u\|_1}{\|u\|}u^T \Theta^T & 0 \end{pmatrix}$$

$$= \begin{pmatrix} K_0 & s e_1 \\ -s e_1^T & 0 \end{pmatrix} = \widehat{K}.$$

Theorem 5.1. *There exists a left moving frame ρ whose associated Serret–Frenet equations are given by*

$$(20) \qquad K = \begin{pmatrix} 0 & -\boldsymbol{v}^T & s \\ \boldsymbol{v} & 0 & 0 \\ -s & 0 & 0 \end{pmatrix},$$

where s is the arc length invariant and $\boldsymbol{v} = (v_i)$ are the natural curvatures. The moving frame will in general be nonlocal, and it is known as the natural moving frame (see [Bishop 1975] for the original definition).

Proof. Let ρ be our previous moving frame. Any other left moving frame will be of the form ρg, where $g \in \mathscr{L}\mathrm{SO}(n+1)$ is an invariant element of the group, that is, a matrix in $\mathrm{SO}(n+1)$ depending on the differential invariants and their derivatives. Since we do not want to change the $\mathfrak{m}$ component of the equation, we will choose $g \in \mathscr{L}H$. If the natural frame (let us call it ρ_n) exists, then $\rho_n = \rho g$ for some invariant g and $K = (\rho g)^{-1}(\rho g)_x = g^{-1}\widehat{K}g + g^{-1}g_x$. If $g = \left(\begin{smallmatrix} \theta & 0 \\ 0 & 1 \end{smallmatrix}\right)$, this relation becomes

$$K = \begin{pmatrix} \theta^T\theta_x + \theta^T K_0\theta & s\theta^T e_1 \\ -se_1^T\theta & 0 \end{pmatrix}.$$

We want the $\mathfrak{m}$ component to remain the same, and so θ should leave e_1 invariant. That is,

$$\theta = \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix} \quad \text{for } \eta \in \mathrm{SO}(n-1).$$

Furthermore, we need

$$\theta^T\theta_x + \theta^T K_0\theta = \begin{pmatrix} 0 & v^T \\ v & 0 \end{pmatrix}, \quad \text{that is,} \quad \eta^T\eta_x + \eta^T K_1\eta = 0 \quad \text{for } K_0 = \begin{pmatrix} 0 & * \\ * & K_1 \end{pmatrix}.$$

In general, the solution of $\eta_x = -K_1\eta$ will be nonlocal. Also, the solution will in general have a monodromy, and it does not need to be periodic. Hence, the calculations that follow are, in that sense, formal. This situation was discussed in [Marí 2006]. $\qquad\square$

When choose a natural moving frame, rather than a classical Riemannian one, the familiar reduced Hamiltonian structures and integrable systems emerge. Any other choice of frame gives an equivalent system, but it will not look familiar to us in general.

5a1. *Geometric Hamiltonian structures.* Finally, we will look into the reduced Poisson bracket defined on the affine subspace $\mathcal{H} \subset \mathscr{L}\mathfrak{o}(n+1)^*$ consisting of matrices of the form (20). For this example we will use as bilinear form the usual $\langle M, N \rangle = \frac{1}{2}\operatorname{tr}(MN)$. As explained in the previous section, we start by considering

a Hamiltonian functional $h : \mathcal{H} \to \mathbb{R}$ and extend it to $\mathcal{H} : \mathcal{L}\mathfrak{o}(n+1)^* \to \mathbb{R}$ so that its variational derivative satisfies

$$(21) \qquad \left(\frac{\delta\mathcal{H}}{\delta L}(K)\right)_x + \left[K, \frac{\delta\mathcal{H}}{\delta L}(K)\right] \in \mathfrak{o}(n)^0.$$

If we write

$$(22) \qquad \frac{\delta\mathcal{H}}{\delta L}(K) = \begin{pmatrix} 0 & \dfrac{\delta h}{\delta v}^T & -\dfrac{\delta h}{\delta s} \\[2mm] -\dfrac{\delta h}{\delta v} & H_0 & v \\[2mm] \dfrac{\delta h}{\delta s} & -v^T & 0 \end{pmatrix}$$

for some $H_0(s, v) \in \mathfrak{o}(n-1)$ and $v(s, v) \in \mathbb{R}^n$, then condition (21) becomes

$$\begin{pmatrix} 0 & \left(\dfrac{\delta h}{\delta v}^T\right)_x - v^T H_0 - s v^T & -\left(\dfrac{\delta h}{\delta s}\right)_x - v^T v \\[2mm] -\left(\dfrac{\delta h}{\delta v}\right)_x - H_0 v + s v & (H_0)_x + v\left(\dfrac{\delta h}{\delta v}\right)^T - \dfrac{\delta h}{\delta v} v^T & v_x - \dfrac{\delta h}{\delta s} v + s \dfrac{\delta h}{\delta v} \\[2mm] \left(\dfrac{\delta h}{\delta s}\right)_x + v^T v & -v_x^T - s \dfrac{\delta h}{\delta v}^T + \dfrac{\delta h}{\delta s} v^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}.$$

This results in

$$v = \frac{1}{s}\left(\left(\frac{\delta h}{\delta v}\right)_x + H_0 v\right) \quad \text{and} \quad H_0 = D^{-1}\left(\frac{\delta h}{\delta v} v^T - v\left(\frac{\delta h}{\delta v}\right)^T\right).$$

If $h, g : \mathcal{H} \to \mathbb{R}$ are two such functionals and the notation is as above, then the reduced bracket defined on $\mathcal{H}$ is given by

$$\{h, g\}_R(s, v) = \int_{S^1} \left\langle \left(\frac{\delta\mathcal{H}}{\delta L}(K)\right)_x + \left[K, \frac{\delta\mathcal{H}}{\delta L}(K)\right], \frac{\delta\mathcal{G}}{\delta L}(K)\right\rangle dx$$

$$= \int_{S^1} \left\langle \begin{pmatrix} 0 & 0 & -\left(\frac{\delta h}{\delta s}\right)_x - v^T v \\[2mm] 0 & 0 & v_x - \frac{\delta h}{\delta s} v + s \frac{\delta h}{\delta v} \\[2mm] \left(\frac{\delta h}{\delta s}\right)_x + v^T v & -v_x^T - s \frac{\delta h}{\delta v}^T + \frac{\delta h}{\delta s} v^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\delta g}{\delta v}^T & -\frac{\delta g}{\delta s} \\[2mm] -\frac{\delta g}{\delta v} & G_0 & v_g \\[2mm] \frac{\delta g}{\delta s} & -v_g^T & 0 \end{pmatrix} \right\rangle dx$$

$$= -\int_{S^1} \frac{\delta g}{\delta s}\left(\left(\frac{\delta h}{\delta s}\right)_x + v^T v\right) + v_g^T\left(v_x + \frac{\delta h}{\delta s} v - s \frac{\delta h}{\delta v}\right) dx.$$

Substituting the known values for v and v_g we get an explicit expression of the first reduced Hamiltonian structure on the sphere. Let

$$\mathcal{D}\left(\frac{\delta h}{\delta v}\right) = H_0 v = D^{-1}\left(\frac{\delta h}{\delta v} v^T - v \frac{\delta h}{\delta v}^T\right) v.$$

It is known (see [Anco 2006; Terng and Thorbergsson 2001] for example) that $D + \mathcal{D}$ defines a Poisson bracket. In terms of this operator, the reduced bracket is

written as

$$(23) \qquad \{h, g\}(s, \boldsymbol{v}) = - \int_{S^1} \left(\frac{\delta g}{\delta s} \; \frac{\delta g}{\delta \boldsymbol{v}} \right) \mathscr{P} \left(\frac{\delta h/\delta s}{\delta h/\delta \boldsymbol{v}} \right) dx$$

where $\mathscr{P}$ is the matrix of differential operators given by

$$\mathscr{P} = \begin{pmatrix} D & \frac{1}{s} \boldsymbol{v}^T D + \frac{1}{s} \boldsymbol{v}^T \mathscr{Q} \\ D\boldsymbol{v}\frac{1}{s} + \mathscr{Q}\frac{1}{s}\boldsymbol{v} & -D\frac{1}{s}D\frac{1}{s}D - D\frac{1}{s}D\frac{1}{s}\mathscr{Q} - \mathscr{Q}\frac{1}{s}D\frac{1}{s}D - \mathscr{Q}\frac{1}{s}D\frac{1}{s}\mathscr{Q} - D + \mathscr{Q} \end{pmatrix}.$$

This bracket does not preserve arc length. In that sense it is a true bracket on parametrized curves. We will come back to this point later.

The companion bracket (6) also reduces to $\mathscr{K}$ for the value $L_0 = E_{1,n+1} - E_{n+1,1}$. Indeed

$$
\begin{aligned}
(24) \qquad \{h, g\}_0(s, \boldsymbol{v}) &= \int_{S^1} \left\langle \frac{\delta \mathscr{G}}{\delta L}, \left[L_0, \frac{\delta \mathscr{H}}{\delta L} \right] \right\rangle dx \\
&= \int_{S^1} \boldsymbol{v}^T \frac{\delta g}{\delta \boldsymbol{v}} - \boldsymbol{v}_g^T \frac{\delta h}{\delta \boldsymbol{v}} = \int_{S^1} \frac{\delta g}{\delta \boldsymbol{v}}^T \mathscr{P}_0 \frac{\delta h}{\delta \boldsymbol{v}} dx,
\end{aligned}
$$

where the Poisson operator $\mathscr{P}_0$ is given by

$$(25) \qquad \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{s}D + D\frac{1}{s} + 2\mathscr{Q} \end{pmatrix}.$$

This operator, in turn, leaves the arc length parameter invariant and hence is a Poisson brackets defined on invariants of unparametrized curves. A discussion about this difference follows in the next subsection. Our last theorem has now been proved:

Theorem 5.2. *The space $\mathscr{K}$ of differential invariants of the Riemannian sphere $\mathrm{SO}(n + 1)/\mathrm{SO}(n)$ is a bi-Poisson manifold with compatible geometric Poisson brackets given by* (23) *and* (24).

5b. *Geometric realizations of Hamiltonian k-evolutions, a geometric realization for a vector modified KdV evolution.* In our final section we will describe the general formula for an invariant evolution of curves u and determine which ones are Hamiltonian with respect to (23).

Theorem 5.3. *Let $u_t = F(u, u_x, u_{xx}, \ldots)$ be an invariant evolution of curves on the sphere $\mathrm{SO}(n + 1)/\mathrm{SO}(n)$. Let Θ be given by our right moving frame under the factorization in* (17). *Then*

$$(26) \qquad u_t = \left(\sin_u^{-1} \left(I - \frac{uu^T}{\|u\|^2} \right) + \frac{uu^T}{\|u\|^2} \right) \Theta^{-1} \boldsymbol{r}$$

for some invariant vector $\boldsymbol{r}$ depending on s, $\boldsymbol{v}$ and their derivatives.

Proof. First, the action of H on the manifold is linear $(\alpha, \Theta) \cdot u = \alpha^{-1} \Theta u$. On the other hand, $\rho_H = (1, \Theta^{-1})$ and so $d\Phi_{\rho_H}(o)u = \Theta^{-1}u$. The action of $\varsigma(u)$ is slightly more complicated; we can calculate directly that

$$d\Phi_{\varsigma(u)}(o) = \sin_u^{-1}\left(I - \frac{uu^T}{\|u\|^2}\right) + \frac{uu^T}{\|u\|^2}$$

Following Theorem 4.1 we can straightforwardly calculate the most general form for an invariant evolution to be given by

$$u_t = \left(\sin_u^{-1}\left(I - \frac{uu^T}{\|u\|^2}\right) + \frac{uu^T}{\|u\|^2}\right)\Theta^{-1}r$$

for some invariant vector r depending on v, s and their derivatives. $\qquad\square$

Theorem 5.4. *If $u(t, x)$ evolves following (26), then the differential invariants (s, v) evolve following the equations*

$$s_t = (r_1)_x - v^T \hat{r}$$
$$v_t = \frac{1}{s}\left(\hat{r}_{xx} + (r_1 v)_x - D^{-1}\frac{1}{s}(v\hat{r}_x^T - \hat{r}_x v^T)\right), \quad \text{where } r = \begin{pmatrix} r_1 \\ \hat{r} \end{pmatrix}.$$

Proof. We want to calculate $N = \rho^{-1}\rho_t$ whenever $\rho(x, t)$ is the natural left moving frame along the flow $u(x, t)$. Lemma 4.2 tells us that N is of the form

$$N = \rho^{-1}\rho_t = \begin{pmatrix} N_0 & r \\ -r & 0 \end{pmatrix}.$$

Evaluating the Maurer–Cartan structure equations along $\frac{d}{dx}$ and $\frac{d}{dt}$ implies

$$K_t = N_x + [K, N],$$

that is, with $\Upsilon = \begin{pmatrix} 0 & -v^T \\ v & 0 \end{pmatrix}$,

$$\begin{pmatrix} \Upsilon & se_1 \\ -se_1^T & 0 \end{pmatrix}_t = \begin{pmatrix} N_0 & r \\ -r^T & 0 \end{pmatrix}_x + \begin{pmatrix} [\Upsilon, N_0] - s(e_1 r^T - re_1^T) & \Upsilon r - sN_0 e_1 \\ -se_1^T N_0 + r^T \Upsilon & 0 \end{pmatrix}.$$

The $\mathfrak{m}$ component of the equation gives $s_t e_1 = r_x + \Upsilon r - sN_0 e_1$ and implies

$$N_0 e_1 = \frac{1}{s}\begin{pmatrix} 0 \\ \hat{r} + r_1 v \end{pmatrix}$$

where $r = (r_i)$ and $\hat{r} = (r_2, r_3, \ldots, r_{n-1})^T$, and

$$(27) \qquad\qquad\qquad s_t = (r_1)_x - v^T \hat{r}.$$

The evolution $\Upsilon_t = (N_0)_x + [\Upsilon, N_0] - s(e_1 \boldsymbol{r}^T - \boldsymbol{r} e_1^T)$ in the $\mathfrak{o}(n)$ block imposes conditions on N_0. Namely, if

$$N_0 = \begin{pmatrix} 0 & -\hat{\boldsymbol{r}}^T - r_1 v^T \\ \hat{\boldsymbol{r}} + r_1 v & \widehat{N_0} \end{pmatrix}, \quad \text{then } \widehat{N_0} = D^{-1} \frac{1}{s}(v \hat{\boldsymbol{r}}_x^T - \hat{\boldsymbol{r}}_x v^T).$$

We also get directly the evolution of v:

$$(28) \qquad v_t = \frac{1}{s}\Big(\hat{\boldsymbol{r}}_{xx} + (r_1 v)_x - D^{-1} \frac{1}{s}(v \hat{\boldsymbol{r}}_x^T - \hat{\boldsymbol{r}}_x v^T)v\Big). \qquad \square$$

Finally, our last theorem is the direct translation of Theorem 4.3, having in mind the description in (22).

Theorem 5.5. *Let* (26) *be an invariant evolution such that*

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ \hat{\boldsymbol{r}} \end{pmatrix} = \begin{pmatrix} -\dfrac{\delta h}{\delta s} \\ \dfrac{1}{s}(D + \mathcal{D})\Big(\dfrac{\delta h}{\delta v}\Big) \end{pmatrix}$$

for some Hamiltonian functional $h(s, v)$. Then (26) *induces an evolution on (s, v) that is Hamiltonian with respect to* (23), *with Hamiltonian functional h.*

As was pointed out in [Terng and Thorbergsson 2001] and [Anco 2006], the choice of invariant vector $r_1 = \frac{1}{2}\|v\|^2$ and $\hat{\boldsymbol{r}} = v_x$ results in an arc length preserving evolution ($s_t = 0$, we will assume $s = 1$) given by

$$v_t = v_{xxx} + \tfrac{3}{2}\|v\|^2 v_x$$

that is, the vector modified KdV equation.

The final question is whether or not the modified KdV equation is bi-Hamiltonian with respect to the two compatible Poisson brackets we found. Our previous general theorem 4.3 states that the condition for the evolution to be Hamiltonian is the existence of a Hamiltonian $h : \mathcal{K} \to \mathbb{R}$ and an extension $\mathcal{H} : \mathcal{L}\mathfrak{g}^* \to \mathbb{R}$ such that $\delta \mathcal{H}/\delta L_{\mathfrak{m}} = d\varsigma(o)\boldsymbol{r}$. Using (22), this condition is equivalent to

$$-\frac{\delta h}{\delta s} = r_1 = \tfrac{1}{2}\|v\|^2 \quad \text{and} \quad v = \Big(\frac{\delta h}{\delta v}\Big)_x + \mathcal{D}\Big(\frac{\delta h}{\delta v}\Big) = \hat{\boldsymbol{r}} = v_x.$$

Notice that the second relation is satisfied by $\delta h/\delta v = v$.

Consider the Hamiltonian functional

$$h(s, v) = \int_{S^1} -\tfrac{1}{2}\|v\|^2 s + \|v\|^2.$$

Clearly,

$$\frac{\delta h}{\delta s} = -\tfrac{1}{2}\|v\|^2 \quad \text{and} \quad \frac{\delta h}{\delta v} = (2 - s)v.$$

On the preserved level set $s = 1$, the Hamiltonian has the desired properties.

Finally, the vector modified KdV equation is also Hamiltonian with respect to our second reduced Poisson bracket. If we consider as Hamiltonian the operator $h_0 : \mathcal{K} \to \mathbb{R}$ given by

$$h_0(v) = \tfrac{1}{2} \int_{S^1} -\|v_x\|^2 + \tfrac{1}{4} \|v\|^4,$$

then

$$v_t = v_{xxx} + \tfrac{3}{2}\|v\|^2 = \mathcal{P}_0(v_{xx} + \tfrac{1}{2}\|v\|^2 v) = \mathcal{P}_0 \frac{\delta h_0}{\delta v}.$$

Therefore, the modified KdV vector equation is bi-Hamiltonian with respect to both brackets as long as we assume the parameter to be the spherical arc length. This condition is forced upon the equations if we want the equations to be Hamiltonian with respect to the second reduced bracket (24). The second bracket, but not the first, appeared already in [Terng and Thorbergsson 2001; Anco 2006].

The role of invariants of arc length type was studied in [Marí 2007] in the case of affine geometries, which are manifolds of the form $G \ltimes \mathbb{R}^n / G$. Among the classical Lie groups, all manifolds except $G = \mathrm{GL}(n)$ have a common feature: their first geometric Poisson bracket (11) always preserves an invariant of arc length type — they are brackets associated to unparametrized curves. Therefore, any Hamiltonian evolution will have geometric realizations by evolutions that preserve arc length type parameters. This is not a choice, but is imposed by the background geometry. On the other hand, homogeneous manifolds of the form G/H in general do not have this property. All known examples have a geometric Poisson bracket defined as in (11) that does not preserve a parameter of arc length type as defined in [Marí 2009]. On the other hand, the modified KdV equation is usually associated to Riemannian manifolds in general, and to natural frames in particular; it is always the invariantization of a curve evolution parametrized by arc length. Thus, it seemed contradictory that it appears on manifolds of the form G/H, with G semisimple; at the very least, it seemed counterintuitive. As we saw in our example, the imposition of arc length preservation does not come from the first geometric bracket, but from the second. The first bracket does not preserve arc length, in agreement with all other examples of the type G/H, but the second does, in agreement with modified KdV being an evolution associated to evolutions that do so.

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