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We introduce a new probability distribution, which is useful in the study of basic hypergeometric series. As applications, we give probabilistic derivations of the q-binomial theorem, the q-Gauss summation formula, a new multiple identity, and an extension of the Rogers–Ramanujan identities.

1. Introduction

The probabilistic method is a useful tool in the study of basic hypergeometric series [Chapman 2005; Evans 2002; Fulman 2001; Rawlings 1997]. In this paper, we introduce a new probability distribution and then demonstrate the applications of this distribution in q-series. We begin with recall some definitions, notations and known results in [Andrews et al. 1999; Gasper and Rahman 1990; Liu 2003]. Throughout the paper, we suppose that $0 < q < 1$. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We also adopt a compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$

where n is an integer or ∞. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

In 1846, Heine introduced the ${}_{r+1}\phi_r$ basic hypergeometric series, which is defined by

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n x^n}{(q, b_1, b_2, \ldots, b_r; q)_n}.$$

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F. H. Jackson [1910] defined the q-integral by

$$\int_0^d f(t)d_qt = d(1-q)\sum_{n=0}^{\infty} f(dq^n)q^n, \tag{1-1}$$

and

$$\int_c^d f(t)d_qt = \int_0^d f(t)d_qt - \int_0^c f(t)d_qt. \tag{1-2}$$

The q-integrals are important in the theory and application of basic hypergeometric series. For example, the author gives some applications of the q-integral in [Wang 2008; 2009b; 2009a; 2010b; 2010a]. The Andrews–Askey [1981] integral is

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} d_qt = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}, \tag{1-3}$$

which can be derived from Ramanujan's ${}_1\psi_1$ summation provided that no zero factors occur in the denominator of the integral.

The Al-Salam–Carlitz polynomials $\varphi_n^{(a)}(x\,|\,q)$ are defined by

$$\varphi_n^{(a)}(x\,|\,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k (a; q)_k,$$

[Srivastava and Jain 1989] and have the q-integral representation [Wang 2009b]

$$\varphi_n^{(a)}(x\,|\,q) = \frac{(ax, a; q)_\infty}{(1-q)(q, q/x, x; q)_\infty} \int_x^1 \frac{(qt/x, qt; q)_\infty t^n}{(at; q)_\infty} d_qt \tag{1-4}$$

provided that no zero factors occur in the denominator.

We frequently use the following well-known theorems:

Theorem (analytic continuation theorem). *If f and g are analytic at z_0 and agree at infinitely many points, which include z_0 as an accumulation point, then $f = g$.*

Theorem (Lebesgue's dominated convergence theorem). *Suppose that $\{X_n, n \geq 1\}$ is a sequence of random variables such that $X_n \to X$ pointwise almost everywhere as $n \to \infty$, and such that $|X_n| \leq Y$ for all n, where the random variable Y is integrable. Then X is integrable, and*

$$\lim_{n\to\infty} EX_n = EX,$$

where $E(\cdot)$ denotes expected value.

Tannery's theorem is a special case of Lebesgue's dominated convergence theorem on the sequence space L^1.

Theorem [Tannery 1904]. *If* $s(n) = \sum_{k\geq 0} f_k(n)$ *is a finite sum* (*or a convergent series*) *for each* n,

$$\lim_{n\to\infty} f_k(n) = f_k, \quad |f_k(n)| \leq M_k, \quad \text{and} \quad \sum_{k=0}^{\infty} M_k < \infty$$

then

$$\lim_{n\to\infty} s(n) = \sum_{k=0}^{\infty} f_k.$$

2. A new probability distribution

In order to use Lebesgue's dominated convergence theorem to get q-identities, we need to find some special probability distributions. In this section, we introduce a useful probability distribution.

The main method of this paper as follows: First, we define a probability distribution by q-shifted factorials; its expected value can be easily obtained. Then we construct a sequence of random variables with this probability distribution. Finally, we use Lebesgue's dominated convergence theorem to obtain a q-identity.

Lemma 2.1. *Suppose* x *is a real such that* $x < 0$; *then we have*

$$\frac{(-x)^n (x^{n-1}q^{k+1}, x^n q^{k+1}; q)_\infty q^k}{(q, q/x, x; q)_\infty} \geq 0 \tag{2-1}$$

and

$$\sum_{n=0}^{1}\sum_{k=0}^{\infty} \frac{(-x)^n (x^{n-1}q^{k+1}, x^n q^{k+1}; q)_\infty q^k}{(q, q/x, x; q)_\infty} = 1, \tag{2-2}$$

where $n = 0, 1$ *and* $k = 0, 1, 2, \ldots$.

Proof. Inequality (2-1) is obvious by the definition of the q-shifted factorials and the assumption that $x < 0$. We only need to prove (2-2).

Since

$$\begin{aligned}(2\text{-}3)\quad &\sum_{n=0}^{1}\sum_{k=0}^{\infty} \frac{(-x)^n (x^{n-1}q^{k+1}, x^n q^{k+1}; q)_\infty q^k}{(q, q/x, x; q)_\infty} \\ &= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \\ &\qquad \times \left((1-q)\sum_{k=0}^{\infty} (q^{k+1}/x, q^{k+1}; q)_\infty q^k - x(1-q)\sum_{k=0}^{\infty} (q^{k+1}, xq^{k+1}; q)_\infty q^k \right),\end{aligned}$$

using the definition of the q-integral gives

$$(1-q)\sum_{k=0}^{\infty}(q^{k+1}/x, q^{k+1}; q)_\infty q^k = \int_0^1 (qt/x, qt; q)_\infty d_q t$$

and

$$x(1-q)\sum_{k=0}^{\infty}(q^{k+1}, xq^{k+1}; q)_\infty q^k = \int_0^x (qt/x, qt; q)_\infty d_q t.$$

Consequently, we have

$$\begin{aligned}(1-q)&\sum_{k=0}^{\infty}(q^{k+1}/x, q^{k+1}; q)_\infty q^k \\ &= \int_0^1 (qt/x, qt; q)_\infty d_q t - \int_0^x (qt/x, qt; q)_\infty d_q t = \int_x^1 (qt/x, qt; q)_\infty d_q t.\end{aligned} \tag{2-4}$$

Employing the Andrews–Askey integral (1-3) gives

$$\int_x^1 (qt/x, qt; q)_\infty d_q t = (1-q)(q, q/x, x; q)_\infty. \tag{2-5}$$

Substituting (2-4) and (2-5) into (2-3) gives (2-2). □

Definition 2.2. A random variable ξ has distribution $W(x; q)$ if

$$P(\xi = x^n q^k) = \frac{(-x)^n (x^{n-1}q^{k+1}, x^n q^{k+1}; q)_\infty q^k}{(q, q/x, x; q)_\infty},$$

where $x < 0$, $0 < q < 1$, $n = 0, 1$ and $k = 0, 1, 2, \dots$.

The distribution $W(x; q)$ has some applications in the study of basic hypergeometric series.

Before giving applications, we need the following lemmas.

Lemma 2.3. *Let $-1 < x < 0$ and $|a| < 1$. Let ξ denote a random variable having with $W(x; q)$. Then we have*

$$E\left(\frac{\xi^m}{(a\xi; q)_\infty}\right) = \frac{1}{(a, ax; q)_\infty}\, \varphi_m^{(a)}(x\,|\,q) \quad \textit{for } m = 0, 1, 2, \dots. \tag{2-6}$$

Proof. Using the definition of the q-integral (1-1), (1-2) and the q-integral representation of the Al-Salam–Carlitz polynomials (1-4), we have

$$\begin{aligned}
&E\left(\frac{\xi^m}{(a\xi;q)_\infty}\right)\\
&\quad=\sum_{n=0}^{1}\sum_{k=0}^{\infty}\frac{(-x)^n(x^{n-1}q^{k+1},x^nq^{k+1};q)_\infty q^k}{(q,q/x,x;q)_\infty}\cdot\frac{x^{nm}q^{km}}{(ax^nq^k;q)_\infty}\\
&\quad=\frac{1}{(1-q)(q,q/x,x;q)_\infty}\Bigg((1-q)\sum_{k=0}^{\infty}(q^{k+1}/x,q^{k+1};q)_\infty\cdot\frac{q^{k(m+1)}}{(aq^k;q)_\infty}\\
&\qquad\qquad-x(1-q)\sum_{k=0}^{\infty}(q^{k+1},xq^{k+1};q)_\infty\cdot\frac{x^mq^{k(m+1)}}{(axq^k;q)_\infty}\Bigg)\\
&\quad=\frac{1}{(1-q)(q,q/x,x;q)_\infty}\\
&\qquad\qquad\times\left(\int_0^1\frac{(qt/x,qt;q)_\infty t^m}{(at;q)_\infty}d_qt-\int_0^x\frac{(qt/x,qt;q)_\infty t^m}{(at;q)_\infty}d_qt\right)\\
&\quad=\frac{1}{(1-q)(q,q/x,x;q)_\infty}\int_x^1\frac{(qt/x,qt;q)_\infty t^m}{(at;q)_\infty}d_qt\\
&\quad=\frac{1}{(a,ax;q)_\infty}\varphi_m^{(a)}(x\,|\,q).
\end{aligned}$$

□

Lemma 2.4. *Let $-1<x<0$ and $|a|<1$. Let ξ denote a random variable having distribution $W(x;q)$. Then we have*

$$E\left(\frac{1}{(a\xi,b\xi;q)_\infty}\right)=\frac{(abx,;q)_\infty}{(a,b,ax,bx;q)_\infty}.$$

Proof. Using the definition of the q-integral (1-1), (1-2) and the Andrews–Askey integral (1-3), we have

$$\begin{aligned}
&E\left(\frac{1}{(a\xi,b\xi;q)_\infty}\right)\\
&\quad=\sum_{n=0}^{1}\sum_{k=0}^{\infty}\frac{(-x)^n(x^{n-1}q^{k+1},x^nq^{k+1};q)_\infty q^k}{(q,q/x,x;q)_\infty}\cdot\frac{1}{(ax^nq^k,bx^nq^k;q)_\infty}\\
&\quad=\frac{1}{(1-q)(q,q/x,x;q)_\infty}\Bigg((1-q)\sum_{k=0}^{\infty}(q^{k+1}/x,q^{k+1};q)_\infty\cdot\frac{q^k}{(aq^k,bq^k;q)_\infty}\\
&\qquad\qquad-x(1-q)\sum_{k=0}^{\infty}(q^{k+1},xq^{k+1};q)_\infty\cdot\frac{q^k}{(axq^k,bx^nq^k;q)_\infty}\Bigg)
\end{aligned}$$

$$= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \left(\int_0^1 \frac{(qt/x, qt; q)_\infty}{(at, bt; q)_\infty} d_q t - \int_0^x \frac{(qt/x, qt; q)_\infty}{(at, bt; q)_\infty} d_q t \right)$$
$$= \frac{1}{(1-q)(q, q/x, x; q)_\infty} \int_x^1 \frac{(qt/x, qt; q)_\infty}{(at, bt; q)_\infty} d_q t = \frac{(abx, ; q)_\infty}{(a, b, ax, bx; q)_\infty},$$

which completes the proof. □

Lemma 2.5. *Let* $|x| < 1$. *Then*

$$\lim_{n\to\infty} \varphi_n^{(a)}(x\,|\,q) = \sum_{k=0}^{\infty} \frac{(a;q)_k x^k}{(q;q)_k}. \tag{2-7}$$

Proof. Let $f_k(n) = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] x^k (a;q)_k$ if $k \le n$ and $f_k(n) = 0$ if $k > n$. We have

$$\varphi_n^{(a)}(x\,|\,q) = \sum_{k=0}^{\infty} f_k(n).$$

Since

$$\lim_{n\to\infty} f_k(n) = \frac{(a;q)_k x^k}{(q;q)_k}, \quad |f_k(n)| \le \frac{|(a;q)_k x^k|}{(q;q)_\infty}, \quad \sum_{k=0}^{\infty} \frac{|(a;q)_k x^k|}{(q;q)_\infty} < \infty,$$

by Tannery's theorem we know (2-7) holds. □

3. The q-binomial theorem

One of the most important summation formulas for basic hypergeometric series is the q-binomial theorem, which was derived by Cauchy in 1843, Heine in 1847, and by other mathematicians. There are many proofs. By using the probability distribution $W(x; q)$ and the Lebesgue dominated convergence theorem, we give a probabilistic derivation; see also [Andrews et al. 1999; Gasper and Rahman 1990].

Theorem 3.1. $\displaystyle\sum_{n=0}^{\infty} \frac{(a;q)_n x^n}{(q;q)_n} = \frac{(ax;q)_\infty}{(x;q)_\infty}$ *for* $|x| < 1$.

Proof. Let ξ be a random variable having distribution $W(x; q)$, where $-1 < x < 0$. We consider the sequence

$$\left\{ \frac{\xi^n}{(a\xi;q)_\infty} \right\}_{n=1}^{\infty} \quad \text{for } |a| < 1$$

of random variables (on a probability space). It is easy to see that ξ^n converges to $I_{(\xi=1)}$, which has Binomial distribution $B(1, 1/(x;q)_\infty)$ and

$$\lim_{n\to\infty} \frac{\xi^n}{(a\xi;q)_\infty} = \frac{I_{(\xi=1)}}{(a;q)_\infty},$$

where I_Ω is the indicator function defined by

$$I_\Omega(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Since

$$\left| \frac{\xi^n}{(a\xi; q)_\infty} \right| \le \frac{1}{(|a|; q)_\infty},$$

using Lebesgue's dominated convergence theorem gives

$$\lim_{n\to\infty} E\left(\frac{\xi^n}{(a\xi; q)_\infty}\right) = E\left(\frac{I_{(\xi=1)}}{(a; q)_\infty}\right). \tag{3-1}$$

Employing (1-4) and using Tannery's theorem gives

$$\begin{aligned} \lim_{m\to\infty} E\left(\frac{\xi^m}{(a\xi; q)_\infty}\right) &= \frac{1}{(a, ax; q)_\infty} \lim_{m\to\infty} \varphi_m^{(a)}(x\,|\,q) \\ &= \frac{1}{(a, ax; q)_\infty} \sum_{m=0}^{\infty} \frac{(a; q)_m x^m}{(q; q)_m}. \end{aligned} \tag{3-2}$$

By direct calculation,

$$E\left(\frac{I_{(\xi=1)}}{(a; q)_\infty}\right) = \frac{1}{(a, x; q)_\infty}. \tag{3-3}$$

Substituting (3-2) and (3-3) into (3-1) gives

$$\sum_{n=0}^{\infty} \frac{(a; q)_n x^n}{(q; q)_n} = \frac{(ax; q)_\infty}{(x; q)_\infty},$$

where $-1 < x < 0$ and $|a| < 1$. By analytic continuation, we may replace the assumptions $-1 < x < 0$ by $|a| < 1$ by $|x| < 1$. Thus, we get Theorem 3.1. □

4. The q-Gauss summation formula

In 1847, Heine derived a q-analogue of Gauss's summation formula. We show that this result can be recovered with the probability distribution $W(x; q)$.

Theorem 4.1. ${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}$ *for* $|c/(ab)| < 1$.

Proof. Let ξ and η denote two independent random variables having distributions $W(x; q)$ and $W(y; q)$, respectively, where we set $-1 < x, y < 0$. We consider the following sequence of random variables (on a probability space):

$$\left\{ \frac{\eta^n}{(a\xi\eta; q)_\infty} \right\}_{n=1}^{\infty} \quad \text{for } |a| < 1.$$

Clearly η^n converges to $I_{(\eta=1)}$ having binomial distribution $B(1, 1/((y;q)_\infty))$ and

$$\lim_{n\to\infty} \frac{\eta^n}{(a\xi\eta;q)_\infty} = \frac{I_{(\eta=1)}}{(a\xi;q)_\infty},$$

where I_Ω is the indicator function.

Since

$$\left|\frac{\eta^n}{(a\xi\eta;q)_\infty}\right| \le \frac{1}{(|a|;q)_\infty},$$

using Lebesgue's dominated convergence theorem gives

$$\lim_{n\to\infty} E\left(\frac{\eta^n}{(a\xi\eta;q)_\infty}\right) = E\left(\frac{I_{(\eta=1)}}{(a\xi;q)_\infty}\right). \tag{4-1}$$

Observe that

$$\begin{aligned} E\left(\frac{\eta^n}{(a\xi\eta;q)_\infty}\right) &= E\left(E\left(\frac{\eta^n}{(a\xi\eta;q)_\infty}\Big|\xi\right)\right) \\ &= E\left(\frac{1}{(a\xi, ay\xi;q)_\infty}\,\varphi_n^{(a\xi)}(x\,|\,q)\right) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^k \cdot E\left(\frac{1}{(a\xi q^k, ay\xi;q)_\infty}\right) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} y^k \cdot \frac{(a^2xyq^k;q)_\infty}{(aq^k, axq^k, ay, axy;q)_\infty} \\ &= \frac{(a^2xy;q)_\infty}{(a, ax, ay, axy;q)_\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \cdot \frac{(a, ax;q)_\infty y^k}{(a^2xy;q)_\infty}. \end{aligned}$$

Hence, we get the left hand side of (4-1):

$$\lim_{n\to\infty} E\left(\frac{\eta^n}{(a\xi\eta;q)_\infty}\right) = \frac{(a^2xy;q)_\infty}{(a, ax, ay, axy;q)_\infty} \sum_{k=0}^{\infty} \frac{(a, ax;q)_\infty y^k}{(q, a^2xy;q)_\infty}. \tag{4-2}$$

On the other hand, the right hand side of (4-1) equals

$$E\left(\frac{I_{(\eta=1)}}{(a\xi;q)_\infty}\right) = p(\eta=1)E\left(\frac{1}{(a\xi;q)_\infty}\right) = \frac{1}{(a, ax, y;q)_\infty}. \tag{4-3}$$

Substituting (4-2) and (4-3) into (4-1) gives

$$\sum_{k=0}^{\infty} \frac{(a, ax;q)_\infty y^k}{(q, a^2xy;q)_\infty} = \frac{(ay, axy;q)_\infty}{(a^2xy, y;q)_\infty},$$

which is equivalent to the q-Gauss theorem, Theorem 4.1, by analytic continuation.

□

5. A multiple identity

Multiple basic hypergeometric series have been investigated by various authors [Milne 1997; Wang 2009a; Zhang 2006; Zhang and Liu 2006]. We will use the distribution $W(x;q)$ to prove the following multiple identity.

Theorem 5.1. *Let $|a|<1$. Then for any positive integers m and n, we have*

$$\sum_{y_1+\cdots+y_m\ge n}\begin{bmatrix}y_1+\cdots+y_m\\ n\end{bmatrix}q^{y_2+2y_3+\cdots+(m-1)y_m}a^{y_1+\cdots+y_m} = \frac{a^n}{(a;q)_{n+m}}\begin{bmatrix}n+m-1\\ n\end{bmatrix}. \tag{5-1}$$

Proof. Let ξ denote a random variable with distribution $W(x;q)$, where $-1<x<0$. For any positive integer m, we consider the sequence

$$\left\{\frac{(1-(a\xi)^n)(1-(aq\xi)^n)\cdots(1-(aq^{m-1}\xi)^n)}{(a\xi;q)_\infty}\right\}_{n=1}^{\infty}\quad\text{for } |a|<1$$

of random variables (on a probability space). It is easy to see that

$$\lim_{n\to\infty}\frac{(1-(a\xi)^n)(1-(aq\xi)^n)\cdots(1-(aq^{m-1}\xi)^n)}{(a\xi;q)_\infty}=\frac{1}{(a\xi;q)_\infty}.$$

Since $|(1-(a\xi)^n)(1-(aq\xi)^n)\cdots(1-(aq^{m-1}\xi)^n)/(a\xi;q)_\infty|\le 1/(|a|;q)_\infty$, using Lebesgue's dominated convergence theorem gives

$$\lim_{n\to\infty}E\left(\frac{(1-(a\xi)^n)(1-(aq\xi)^n)\cdots(1-(aq^{m-1}\xi)^n)}{(a\xi;q)_\infty}\right)=E\left(\frac{1}{(a\xi;q)_\infty}\right). \tag{5-2}$$

Employing (2-6), we get the right hand side of (5-2):

$$E\left(\frac{1}{(a\xi;q)_\infty}\right)=\frac{1}{(a,ax;q)_\infty}. \tag{5-3}$$

On the other hand, observing that

$$\begin{aligned}\frac{(1-(a\xi)^n)(1-(aq\xi)^n)\cdots(1-(aq^{m-1}\xi)^n)}{(a\xi;q)_\infty} &= \frac{1-(a\xi)^n}{1-a\xi}\cdot\frac{1-(aq\xi)^n}{1-aq\xi}\cdots\frac{1-(aq^{m-1}\xi)^n}{1-aq^{m-1}\xi}\cdot\frac{1}{(aq^m\xi;q)_\infty}\\ &= \sum_{y_1=0}^{n-1}(a\xi)^{y_1}\cdot\sum_{y_2=0}^{n-1}(aq\xi)^{y_2}\cdots\sum_{y_m=0}^{n-1}(aq^{m-1}\xi)^{y_m}\cdot\frac{1}{(aq^m\xi;q)_\infty}\\ &= \sum_{0\le y_1,\ldots,y_m\le n-1}q^{y_2+2y_3+\cdots+(m-1)y_m}a^{y_1+\cdots+y_m}\cdot\frac{\xi^{y_1+\cdots+y_m}}{(aq^m\xi;q)_\infty},\end{aligned}$$

we have

$$E\left(\frac{[1-(a\xi)^n][1-(aq\xi)^n]\cdots[1-(aq^{m-1}\xi)^n]}{(a\xi;q)_\infty}\right)$$
$$=\sum_{0\le y_1,\dots,y_m\le n-1} q^{y_2+2y_3+\cdots+(m-1)y_m} a^{y_1+\cdots+y_m} E\left(\frac{\xi^{y_1+\cdots+y_m}}{(aq^m\xi;q)_\infty}\right)$$
$$=\frac{1}{(aq^m,axq^m;q)_\infty} \times \sum_{0\le y_1,\dots,y_m\le n-1} q^{y_2+2y_3+\cdots+(m-1)y_m} a^{y_1+\cdots+y_m}\, \varphi^{(aq^m)}_{y_1+\cdots+y_m}(x\,|\,q).$$

Hence, we get the left hand side of (5-2):

$$\lim_{n\to\infty} E\left(\frac{[1-(a\xi)^n][1-(aq\xi)^n]\cdots[1-(aq^{m-1}\xi)^n]}{(a\xi;q)_\infty}\right) = \frac{1}{(aq^m,axq^m;q)_\infty} \times \sum_{y_1,\dots,y_m\ge 0} q^{y_2+2y_3+\cdots+(m-1)y_m} a^{y_1+\cdots+y_m}\, \varphi^{(aq^m)}_{y_1+\cdots+y_m}(x\,|\,q). \tag{5-4}$$

Substituting (5-3) and (5-4) into (5-2) gives

$$\sum_{y_1,\dots,y_m\ge 0} q^{y_2+2y_3+\cdots+(m-1)y_m} a^{y_1+\cdots+y_m}\, \varphi^{(aq^m)}_{y_1+\cdots+y_m}(x\,|\,q) = \frac{1}{(a,ax;q)_m}. \tag{5-5}$$

Using Theorem 3.1 with $a=q^m$ and $x=ax$ gives

$$\sum_{k=0}^{\infty} \begin{bmatrix} m+k-1 \\ k \end{bmatrix} a^k x^k = \frac{1}{(ax;q)_m}. \tag{5-6}$$

Substituting (5-6) into (5-5) and comparing the coefficients of x^n gives (5-1). □

6. An extension of the Rogers–Ramanujan identities

The well-known Rogers–Ramanujan identities are

$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q;q)_m} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{6-1}$$

$$\sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q;q)_m} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{6-2}$$

There are many proofs of this beautiful pair of identities. Baxter's [1982] is based on the statistical mechanics, and the proof of Lepowsky and Milne [1978] uses the character formula on an infinite dimensional Lie algebra. We use our probability distribution to derive an extension of the Rogers–Ramanujan identities.

Theorem 6.1. *We have*

$$\sum_{m=n}^{\infty}\frac{q^{m^2}}{(q;q)_{m-n}}=\frac{1}{(q;q)_\infty}$$
$$\times\Biggl(q^n+\sum_{k=1}^{\infty}\sum_{l=0}^{n}\begin{bmatrix}n\\ l\end{bmatrix}\frac{(-1)^k(q^k;q)_l(q;q)_{2k}}{(1-q^k)(q;q)_{2k+1-l}}\,q^{5\binom{k}{2}+k(n+2-l)}(1-q^{2k}+q^{4k+1})$$
$$-\sum_{k=1}^{\infty}\sum_{l=0}^{n}\begin{bmatrix}n\\ l\end{bmatrix}\frac{(-1)^k(q^k;q)_l(q;q)_{2k}}{(1-q^k)(q;q)_{2k+1-l}}\,q^{5\binom{k}{2}+k(n+4-l)+1-l}\Biggr).$$

Proof. By Watson's q-Whipple transformation formula [Watson 1929],

$${}_8\phi_7\left(\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,d,e,q^{-n}\\ \sqrt{a},-\sqrt{a},qa/b,qa/c,qa/d,qa/e,q^{n+1}a\end{matrix};q,\frac{q^{2+n}a^2}{bcde}\right)$$
$$=\frac{(qa,qa/bc;q)_n}{(qa/b,qa/c)_n}{}_4\phi_3\left(\begin{matrix}q^{-n},b,c,qa/de\\ qa/d,qa/e,q^{-n}bc/a\end{matrix};q,q\right).$$

Letting $b,c,d,e,n\to\infty$ in this equation gives

$$\sum_{m=0}^{\infty}\frac{q^{m^2}a^m}{(q;q)_m}=\frac{1}{(aq;q)_\infty}+\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+2k}}{(q;q)_k}\cdot\frac{a^{2k}}{(aq^k;q)_\infty}$$
$$-\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+4k}}{(q;q)_k}\cdot\frac{a^{2k+1}}{(aq^k;q)_\infty}\quad\text{for } |a|\le 1.$$

Then letting $a=\xi$ gives

$$\sum_{m=0}^{\infty}\frac{q^{m^2}\xi^m}{(q;q)_m}=\frac{1}{(\xi q;q)_\infty}+\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+2k}}{(q;q)_k}\cdot\frac{\xi^{2k}}{(\xi q^k;q)_\infty}$$
$$-\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+4k}}{(q;q)_k}\cdot\frac{\xi^{2k+1}}{(\xi q^k;q)_\infty}.$$

where ξ is a random variable with distribution $W(x;q)$ and $-1<x<0$. Applying the expectation operator E to both sides of the above, we get

$$E\left(\sum_{m=0}^{\infty}\frac{q^{m^2}\xi^m}{(q;q)_m}\right)=E\left(\frac{1}{(\xi q;q)_\infty}\right)+E\left(\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+2k}}{(q;q)_k}\cdot\frac{\xi^{2k}}{(\xi q^k;q)_\infty}\right)$$
$$-E\left(\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+4k}}{(q;q)_k}\cdot\frac{\xi^{2k+1}}{(\xi q^k;q)_\infty}\right).\tag{6-3}$$

Since $|q^{m^2}\xi^m/(q;q)_m| \le q^{m^2}/(q;q)_m$ and the series $\sum_{m=0}^{\infty} q^{m^2}/(q;q)_m$ converges absolutely, using Lebesgue's dominated convergence theorem and (2-6) gives the left hand side of (6-3):

$$E\biggl(\sum_{m=0}^{\infty}\frac{q^{m^2}\xi^m}{(q;q)_m}\biggr)=\sum_{m=0}^{\infty}\frac{q^{m^2}E\{\xi^m\}}{(q;q)_m}=\sum_{m=0}^{\infty}\frac{q^{m^2}h_m(x\,|\,q)}{(q;q)_m}. \tag{6-4}$$

On the other hand, using (2-6) gives

$$E\biggl(\frac{1}{(\xi q;q)_\infty}\biggr)=\frac{1}{(q,qx;q)_\infty}, \tag{6-5}$$

$$E\biggl(\frac{\xi^{2k}}{(\xi q^k;q)_\infty}\biggr)=\frac{1}{(q^k,q^kx;q)_\infty}\,\varphi_{2k}^{(q^k)}(x\,|\,q), \tag{6-6}$$

$$E\biggl(\frac{\xi^{2k+1}}{(\xi q^k;q)_\infty}\biggr)=\frac{1}{(q^k,q^kx;q)_\infty}\,\varphi_{2k+1}^{(q^k)}(x\,|\,q). \tag{6-7}$$

It is easy to see that

$$\left|\frac{(-1)^kq^{5\binom{k}{2}+4k}}{(q;q)_k}\cdot\frac{\xi^{2k+1}}{(\xi q^k;q)_\infty}\right|\le\left|\frac{(-1)^kq^{5\binom{k}{2}+2k}}{(q;q)_k}\cdot\frac{\xi^{2k}}{(\xi q^k;q)_\infty}\right|\le\frac{q^{5\binom{k}{2}+2k}}{(q;q)_k(q;q)_\infty},$$

and the series $\sum_{k=0}^{\infty} q^{5\binom{k}{2}+2k}/((q;q)_k(q;q)_\infty)$ is converges absolutely. Using Lebesgue's dominated convergence theorem and (6-5), (6-6) and (6-7) gives the right hand side of (6-3):

$$\begin{aligned}
&E\biggl(\frac{1}{(\xi q;q)_\infty}\biggr)+E\biggl(\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+2k}}{(q;q)_k}\cdot\frac{\xi^{2k}}{(\xi q^k;q)_\infty}\biggr)\\
&\qquad\qquad-E\biggl(\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+4k}}{(q;q)_k}\cdot\frac{\xi^{2k+1}}{(\xi q^k;q)_\infty}\biggr)\\
&=E\biggl(\frac{1}{(\xi q;q)_\infty}\biggr)+\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+2k}}{(q;q)_k}E\biggl(\frac{\xi^{2k}}{(\xi q^k;q)_\infty}\biggr)\\
&\qquad\qquad-\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+4k}}{(q;q)_k}E\biggl(\frac{\xi^{2k+1}}{(\xi q^k;q)_\infty}\biggr)\\
&=\frac{1}{(q,qx;q)_\infty}+\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+2k}}{(q;q)_k}\frac{1}{(q^k,q^kx;q)_\infty}\,\varphi_{2k}^{(q^k)}(x\,|\,q)\\
&\qquad\qquad-\sum_{k=1}^{\infty}\frac{(-1)^kq^{5\binom{k}{2}+4k}}{(q;q)_k}\frac{1}{(q^k,q^kx;q)_\infty}\,\varphi_{2k+1}^{(q^k)}(x\,|\,q)
\end{aligned}$$

$$= \frac{1}{(q, qx; q)_\infty} + \frac{1}{(q, x; q)_\infty} \sum_{k=1}^{\infty} \frac{(-1)^k (x; q)_k}{1-q^k} q^{5\binom{k}{2}+2k} \bigl(\varphi_{2k}^{(q^k)}(x\,|\,q) - q^{2k} \varphi_{2k+1}^{(q^k)}(x\,|\,q)\bigr).$$

Substituting this and (6-4) into (6-3) gives

$$\sum_{m=0}^{\infty} \frac{q^{m^2} h_m(x\,|\,q)}{(q;q)_m} = \frac{1}{(q, qx; q)_\infty} + \frac{1}{(q, x; q)_\infty} \sum_{k=1}^{\infty} \frac{(-1)^k (x; q)_k}{1-q^k} q^{5\binom{k}{2}+2k} \bigl(\varphi_{2k}^{(q^k)}(x\,|\,q) - q^{2k} \varphi_{2k+1}^{(q^k)}(x\,|\,q)\bigr),$$

where $-1 < x < 0$. By analytic continuation, we may replace the assumption $-1 < x < 0$ by $|x| < 1$.

Substituting the expansion

$$\frac{1}{(z;q)_\infty} = \sum_{l=0}^{\infty} \frac{z^l}{(q;q)_l}$$

into the last, we have

$$\sum_{m=0}^{\infty} \frac{q^{m^2} h_m(x\,|\,q)}{(q;q)_m} = \frac{1}{(q;q)_\infty} \sum_{l=0}^{\infty} \frac{q^l x^l}{(q;q)_l} + \frac{1}{(q;q)_\infty} \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \left(\frac{q^{kl} x^l}{(q;q)_l} \cdot \frac{(-1)^k q^{5\binom{k}{2}+2k}}{1-q^k} \bigl(\varphi_{2k}^{(q^k)}(x\,|\,q) - q^{2k} \varphi_{2k+1}^{(q^k)}(x\,|\,q)\bigr) \right).$$

Comparing the coefficients of x^n in this identity gives

$$\sum_{m=n}^{\infty} \frac{q^{m^2} \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]}{(q;q)_m} = \frac{q^n}{(q;q)_\infty (q;q)_n} + \frac{1}{(q;q)_\infty} \sum_{k=1}^{\infty} \sum_{l=0}^{n} \frac{(-1)^k (q^k;q)_l}{(1-q^k)(q;q)_{n-l}} q^{5\binom{k}{2}+k(n+2-l)} \left(\begin{bmatrix} 2k \\ l \end{bmatrix} + q^{2k} \begin{bmatrix} 2k+1 \\ l \end{bmatrix} \right),$$

which can be written as Theorem 6.1. □

The Rogers–Ramanujan identities are special cases of Theorem 6.1. Letting $n = 0$ and then applying the Jacobi triple product identity [Andrews et al. 1999]

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n = (q, x, q/x; q)_\infty$$

leads to the Rogers–Ramanujan identity (6-1). In fact, when $n=0$, we have

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}&=\frac{1}{(q;q)_\infty}\biggl(1+\sum_{k=1}^{\infty}(-1)^k(1+q^k)q^{5\binom{k}{2}+2k}\biggr)\\&=\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^kq^{5\binom{k}{2}+2k}\\&=\frac{(q^5,q^2,q^3;q^5)_\infty}{(q;q)_\infty}=\frac{1}{(q,q^4;q^5)_\infty}.\end{aligned}$$

Similarly, the case $n=1$ of Theorem 6.1 results in another identity due to Rogers and Ramanujan:

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{q^{m^2+m}}{(q;q)_m}&=\sum_{m=0}^{\infty}\frac{q^{m^2}}{(q;q)_m}-\sum_{m=1}^{\infty}\frac{q^{m^2}}{(q;q)_{m-1}}\\&=\frac{1}{(q;q)_\infty}\biggl(1+\sum_{k=1}^{\infty}(1-q^{2k+1})q^{5\binom{k}{2}+4k}\biggr)\\&=\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{\infty}(-1)^kq^{5\binom{k}{2}+4k}\\&=\frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty}=\frac{1}{(q^2,q^3;q^5)_\infty}.\end{aligned}$$

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