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We consider an annular region $\Omega \subset \mathbb{R}^2$ and analyze the capillary surface $z = u(x, y)$ formed within an annular cylinder $\Omega \times \mathbb{R}$. Assuming identical contact angles γ along the inner and outer boundaries, we determine several qualitative properties of the surface. In particular, we examine the behavior of u in the limiting cases of Ω approaching a disk, a thin ring, and the exterior of a disk.

1. Introduction

The equilibrium liquid-gas interface formed within a capillary tube has been studied extensively over the past two hundred years. The most widely used modern reference is [Finn 1986]. We will consider the related annular geometry in the presence of gravity first examined by Laplace in 1806; see [Laplace 1966, supplements to book X]. Here two concentric circular cylinders define an annular cross section $\Omega \subset \mathbb{R}^2$. If the cylinders are immersed vertically in an infinite reservoir of incompressible fluid, the surface $Z = U(X, Y)$ formed between the tubes will satisfy the boundary value problem

$$\begin{cases} NU = \kappa U & \text{in } \Omega, \\ \hat{\nu} \cdot TU = \cos\gamma & \text{on } \partial\Omega, \end{cases}$$

where $TU = \nabla U/\sqrt{1 + |\nabla U|^2}$, $NU = \nabla \cdot TU$, $\hat{\nu}$ is the exterior unit normal on the boundary $\partial\Omega$ and $\kappa > 0$ is the capillary constant. The contact angle $\gamma \in [0, \pi]$ is defined on the inner and outer boundaries and gives the angle at which the interface meets the bounding wall. For this investigation, γ is assumed to be constant and equal along each cylinder. Such a scenario arises when both tubes are made of the same uniform material.

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The axisymmetric nature of such annular solutions allows us to analyze the boundary value problem for an ordinary differential equation:

$$\text{(1)} \qquad \begin{cases} \dfrac{1}{R}\left(\dfrac{RU_R}{\sqrt{1+U_R^2}}\right)_R = \kappa U \quad \text{for } R_1 < R < R_2, \\ U_R(R_1^+) = -\cot\gamma, \\ U_R(R_2^-) = \cot\gamma, \end{cases}$$

where U is the surface height, R is the radial variable and $(\cdot)_R$ denotes differentiation with respect to R. System (1) is made dimensionless by introducing the variables

$$u = U/R_2 \quad \text{and} \quad r = R/R_2,$$

which gives

$$\text{(2)} \qquad \begin{cases} Nu = \dfrac{(r\sin\psi)_r}{r} = Bu \quad \text{for } a < r < 1 \\ \sin\psi(a) = -\cos\gamma, \\ \sin\psi(1) = \cos\gamma, \end{cases}$$

where B is a positive constant known as the Bond number, and we define $\psi(r)$ as the inclination angle of $u(r)$:

$$\sin\psi(r) = \frac{u_r}{\sqrt{1+u_r^2}}.$$

See Figure 1. The outer radius of the region is now fixed at $r = 1$, while the inner boundary will occur at $r = a$ for $0 < a < 1$. Additionally, we need only consider

$$\text{(3)} \qquad 0 \le \gamma < \pi/2$$

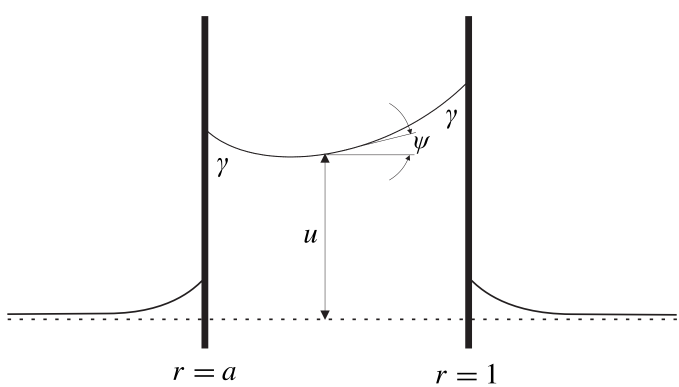

Figure 1. Radial cross section of annular capillary surface.

since the other possibilities are accounted for as follows:

- If $\gamma = \pi/2$, then $u = 0$ is the unique solution.
- For a solution u with $\gamma \in (\pi/2, \pi]$, let $\bar{u} = -u$. We therefore have $N\bar{u} = B\bar{u}$ with $\bar{\gamma} = \pi - \gamma$ or $\bar{\gamma} \in [0, \pi/2)$.

Under (3), the comparison principle [Concus and Finn 1974; Finn 1986] requires u to be positive and bounded for any selection of parameters a and B. Additionally, the volume of u above Ω can be determined by

$$\int_a^1 ru(r)\,dr = \frac{\cos\gamma(1+a)}{B}. \tag{4}$$

Contributions to the annular problem have been made by Elcrat, Kim, and Treinen [2004] and Siegel [2006]; however, this research is still in its fledgling stage. In this paper, the comparison principle is used to provide several qualitative results. We begin in Section 2 by illustrating some general properties of u, the solution to (2); specifically, there exists a unique radius $r = m$ at which u achieves its minimum value, $u(a) < u(1)$, $m \in (a, (1+a)/2)$ and m is monotone increasing with respect to a. Section 3 then explores the behaviour of solutions to the annular problem (1) in the following limiting cases:

- For the dimensionless version of (2), we consider the two cases of $a \to 0$ and $a \to 1$.
- Alternatively, the using the variables

  $$u = U/R_1 \quad \text{and} \quad r = R/R_1$$

  to make (1) dimensionless reformulates it as

  $$\begin{cases} Nu = \dfrac{(r\sin\psi)_r}{r} = Bu & \text{for } 1 < r < b, \\ \sin\psi(1) = -\cos\gamma, \\ \sin\psi(b) = \cos\gamma \end{cases} \tag{5}$$

  The behaviour of u is consequently examined as $b \to \infty$.

2. General properties

In this section, the comparison principle will be used to present a number of qualitative results. We start by confirming the uniqueness of the minimum surface height, which is mentioned under more general conditions in [Elcrat et al. 2004].

Theorem 2.1. *Let u be a solution to the boundary value problem (2). There exists a unique radius $r = m$ at which u achieves its minimum value.*

Proof. Since $\sin\psi$ is continuous with

$$\sin\psi(a) = -\cos\gamma < 0 \quad \text{and} \quad \sin\psi(1) = \cos\gamma > 0,$$

there exists at least one point in $(a, 1)$ where $\sin\psi = 0$, which corresponds to an extremum of u. Define $r = m$ as the first zero of $\sin\psi$. Using the first of (2), we note

$$(\sin\psi)_r = Bu - (\sin\psi)/r \tag{6}$$

$$> 0 \quad \text{for } \sin\psi \le 0 \tag{7}$$

and specifically, $\sin\psi$ is increasing at $r = m$. Suppose there exists more than one point where $\sin\psi = 0$ and let m' be the next zero immediately following m. Because $\sin\psi$ is increasing at m, it must be nonincreasing as it touches the r-axis at m':

$$(\sin\psi)_r|_{r=m'} \le 0.$$

However, this is in contradiction to (7), and m must be the unique extremum point of u. Inequality (7) also implies this is a minimum. □

For the next theorem, we compare boundary heights.

Lemma 2.2. *The function* $\sin\psi$ *is monotone increasing on* $[a, 1]$.

Proof. Given that the zero of $\sin\psi$ is unique, we consider $\sin\psi$ on two subintervals. We have $\sin\psi \le 0$ on $[a, m]$, and (6) ensures that $(\sin\psi)_r > 0$. On $(m, 1]$, $\sin\psi > 0$ and thus u is increasing. In this case, we multiply the first of (2) by r and integrate from m to r to obtain

$$\begin{aligned} \sin\psi(r) &= \frac{B}{r}\int_m^r su(s)\,ds \\ &< \frac{Bu(r)}{r}\left(\frac{r^2-m^2}{2}\right) \\ &< \frac{Bru(r)}{2}. \end{aligned} \tag{8}$$

Therefore $Bu - (\sin\psi)/r > 0$. Equation (6) confirms that $(\sin\psi)_r > 0$. □

Remark. Lemma 2.2 also implies that u is convex.

Theorem 2.3. $u(a) < u(1)$.

Proof. The construction of this proof follows the ideas of [Serrin 1971]. Starting with the annular region Ω, we place as in Figure 2 a line T that separates from Ω a cap Γ. Let Γ' be the reflection of Γ with respect to T, and observe that T is positioned so that Γ' is internally tangent to $\partial\Omega$ at p. Finally, let $\hat{n}$ be the exterior unit normal on $\partial\Gamma'$. With the coordinate system (x, y) oriented so that the y-axis

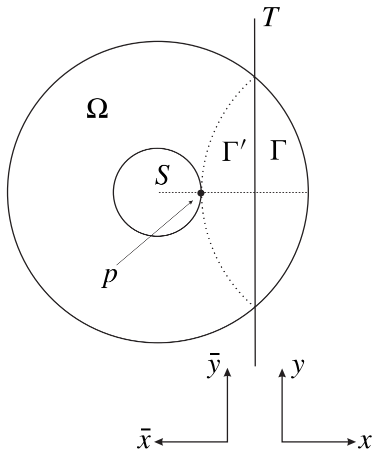

Figure 2. Configuration of reflected region Γ' superimposed onto Ω.

is aligned with T, we define a function $\bar{u}$ on Γ' as

$$\bar{u}(x, y) = u(\bar{x}, \bar{y}) = u(-x, y) \quad \text{for } (x, y) \in \Gamma'.$$

Let $\overline{N}$ be the N operator with respect to the coordinate system $(\bar{x}, \bar{y})$. Clearly, $\overline{N}\bar{u} = B\bar{u}$. However, N is invariant under reflections; thus, $N\bar{u} = \overline{N}\bar{u} = B\bar{u}$ and $\bar{u}$ also satisfies the capillary equation in Γ'. The boundary of Γ' is now decomposed into two pieces, with Σ_α being the portion along T and Σ_β as the remaining curved piece. We subsequently examine how u and $\bar{u}$ compare on each boundary component. It is immediately clear that $u = \bar{u}$ on Σ_α. On Σ_β, note that $\hat{n} \cdot Tu = \sin\psi \; \hat{n} \cdot \hat{r}$, where $\hat{r}$ is the unit vector in the radial direction. Since $\sin\psi$ is increasing, this yields $-\cos\gamma \le \hat{n} \cdot Tu \le \cos\gamma$. Of course, $\hat{n} \cdot T\bar{u} = \cos\gamma$ and hence $\hat{n} \cdot T\bar{u} \ge \hat{n} \cdot Tu$ on Σ_β. As a result, the comparison principle requires

$$\bar{u} \ge u \quad \text{in } \Gamma', \tag{9}$$

which can be extended to the boundary point p by continuity:

$$u(p) \le \bar{u}(p) \quad \text{if and only if} \quad u(a) \le u(1). \tag{10}$$

The possibility of $u(p) = \bar{u}(p)$ is excluded by contradiction. In this case, our attention is restricted to the dashed line S of Figure 2 and both functions are described in terms of the radial variable only. We next assume that $u(a) = u(1)$, which allows the meridional curvature $k_m = (\sin\psi)_r$ of the surface to be compared at $r = a$ and $r = 1$:

$$(\sin\psi)_r|_{r=a} = Bu(a) + (\cos\gamma)/a > Bu(1) - \cos\gamma = (\sin\psi)_r|_{r=1}.$$

Consequently, there exists a $\delta > 0$ such that

$$\min_{r\in[a,a+\delta]}\{(\sin\psi)_r\} > \max_{r\in[1-\delta,1]}\{(\sin\psi)_r\}.$$

We can then integrate $(\sin\psi)_r$ over these regions, giving

$$\sin\psi(a+r) > -\sin\psi(1-r) \quad \text{for all } r\in(0,\delta],$$

and since the function $p/\sqrt{1-p^2}$ is increasing on $(-1, 1)$, we have

$$\frac{\sin\psi(a+r)}{\sqrt{1-\sin^2\psi(a+r)}} > -\frac{\sin\psi(1-r)}{\sqrt{1-\sin^2\psi(1-r)}}.$$

Thus,

$$\begin{aligned} u(a+\delta) &= u(a) + \int_a^{a+\delta} u_s(s)\,ds \\ &= u(a) + \int_a^{a+\delta} \frac{\sin\psi(s)}{\sqrt{1-\sin^2\psi(s)}}\,ds \qquad (11) \\ &> u(1) - \int_{1-\delta}^{1} \frac{\sin\psi(s)}{\sqrt{1-\sin^2\psi(s)}}\,ds = u(1-\delta). \end{aligned}$$

This that $u(a+\delta) > \bar{u}(a+\delta)$, which is in contradiction to (9) and the inequality of (10) must be strict. □

Theorem 2.4. *The function u achieves its minimum on $(a, (1+a)/2)$.*

Proof. We refer to Figure 2 and again consider u and $\bar{u}$ along S. The proof will be by contradiction; we assume that the minimum of u occurs at $m \in ((1+a)/2, 1)$. If $\overline{m}$ is defined as the location of the minimum of $\bar{u}$, we then have $\overline{m} \in (a, (1+a)/2)$. However, the convexity of u implies that

$$u(m) < u(\overline{m}) \quad \text{if and only if} \quad \bar{u}(\overline{m}) < u(\overline{m})$$

with $\overline{m} \in \Gamma'$, which is in contradiction to (9). Thus, $m \in (a, (1+a)/2]$. Next, assume $m = (1+a)/2$. Given that $(\sin\psi)_{rr} = u_r - (\sin\psi)_r/r + (\sin\psi)/r^2$, Lemma 2.2 provides $(\sin\psi)_{rr}|_{r=m} < 0$ and continuity requires that there exists a $\delta > 0$ such that $(\sin\psi)_{rr} < 0$ on $[m-\delta, m+\delta]$. With $(\sin\psi)_r$ decreasing on the interval, this gives $-\sin\psi(m-r) > \sin\psi(m+r)$ for all $r \in (0, \delta]$. Finally, an argument similar to (11) yields $u(m-\delta) > u(m+\delta)$, and we conclude $u(m-\delta) > \bar{u}(m-\delta)$. This again contradicts (9); therefore the minimum of u occurs on $(a, (1+a)/2)$. □

Theorem 2.5. *The minimum value m is monotone increasing with respect to a.*

Proof. We proceed by contradiction. First, suppose there exist two inner radii $\bar{a}$ and $\hat{a}$ where m decreases with respect to a. This gives rise to the following configuration as shown in Figure 3:

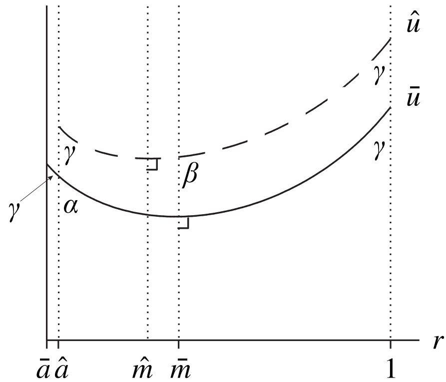

Figure 3. Hypothetical configuration assuming m is decreasing with respect to two values of a.

(i) $\bar{u}$ is the unique solution over $[\bar{a}, 1]$ whose minimum is at $r = \overline{m}$.

(ii) $\hat{u}$ is the unique solution over $[\hat{a}, 1]$ whose minimum is at $r = \hat{m}$.

(iii) $\bar{a} < \hat{a}$.

(iv) $\hat{m} < \overline{m}$.

Consider $\bar{u}$ and $\hat{u}$ on the region $[\hat{a}, 1]$. Here, the contact angle of $\bar{u}$ at $r = \hat{a}$ will be $\alpha > \gamma$, and the comparison principle therefore implies

$$\bar{u} < \hat{u} \quad \text{in } (\hat{a}, 1). \tag{12}$$

Alternatively, we can examine the solutions over $[\overline{m}, 1]$, in which the contact angle of $\hat{u}$ at $r = \overline{m}$ will be $\beta > \pi/2$. Here, the comparison principle would require $\bar{u} > \hat{u}$ in $(\overline{m}, 1)$ which is in disagreement with (12). Consequently, $\overline{m} \le \hat{m}$ for $\bar{a} < \hat{a}$. Now suppose that m is constant for two increasing values of a. Again, $\bar{u}$ and $\hat{u}$ will be configured as before, only with (iv) altered as

(iv)′ $\bar{u}$ and $\hat{u}$ share the same minimum at $r = m$.

Figure 4 depicts this possibility. In like manner, we have

$$\bar{u} < \hat{u} \quad \text{in } (\hat{a}, 1). \tag{13}$$

However, on $[m, 1]$, both $\bar{u}$ and $\hat{u}$ have identical contact angles and uniqueness requires $\bar{u} \equiv \hat{u}$, which contradicts (13), and we conclude $\overline{m} < \hat{m}$ for $\bar{a} < \hat{a}$. □

3. Solutions in limiting cases

Preliminary lemmas.

Lemma 3.1. *The function* $(\sin\psi)/r$ *is monotone increasing on* $[a, 1]$.

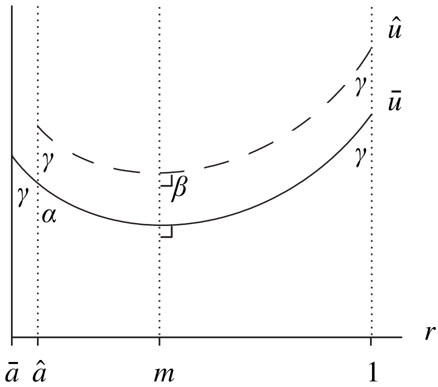

Figure 4. Hypothetical configuration assuming m is constant with respect to two values of a.

Proof. Equation (6) yields $((\sin\psi)/r)_r = (2/r)(Bu/2 - (\sin\psi)/r)$. As we did in Lemma 2.2, we can examine $((\sin\psi)/r)_r$ over two subintervals. On $[a, m]$, $\sin\psi \le 0$ and $((\sin\psi)/r)_r > 0$. On $(m, 1]$, result (8) can be used to claim that $Bu/2 - (\sin\psi)/r > 0$ and thus $((\sin\psi)/r)_r > 0$ for $r \in [a, 1]$. □

Lemma 3.2. *We have* $-(a\cos\gamma)/r < \sin\psi < r\cos\gamma$ *on* $(a, 1)$.

Proof. For the lower bound, we observe that the first of (2) provides the differential inequality $(r\sin\psi)_r = Bru > 0$, and thus $r\sin\psi$ is monotone increasing:

$$r\sin\psi(r) > a\sin\psi(a) = -a\cos\gamma \quad \text{for } r \in (a, 1].$$

For the upper bound, Lemma 3.1 may be used to show that

$$(\sin\psi(r))/r < \sin\psi(1) = \cos\gamma \quad \text{for } r \in [a, 1).$$ □

Approaching a disk. We now consider solutions to (2) as $a \to 0$. As such, reference will be made to the interior solution u_{int}, which solves

$$\begin{cases} (r\sin\psi)_r = Bru_{\text{int}} & \text{for } r \in (0, 1), \\ \sin\psi(0) = 0, \\ \sin\psi(1) = \cos\gamma. \end{cases} \tag{14}$$

See [Finn 1986] for background. Siegel [2006] examined the problem (14), along with the annular problem

$$\begin{cases} (r\sin\psi)_r = Bru & \text{for } r \in (a, 1), \\ \sin\psi(a) = 0, \\ \sin\psi(1) = \cos\gamma. \end{cases} \tag{15}$$

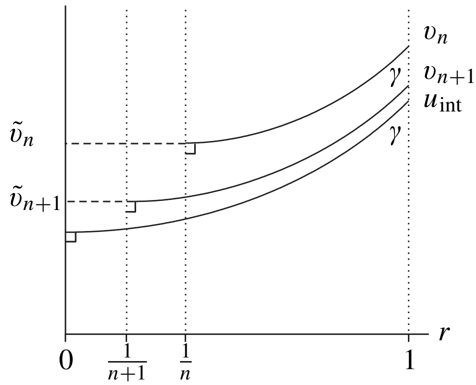

Figure 5. Illustration of $\{v_n\}$ and $\{\tilde{v}_n\}$ compared to u_{int}.

First, it will be shown that the solution of (15) approaches that of (14) as $a \to 0$. Let $\{v_n\}_{n\geq 2}$ be the sequence of functions such that v_n is the unique solution to (15) on the interval $[1/n, 1]$. Thus $\{v_n\}$ is defined on an increasing domain; however, it is desirable to consider also a sequence $\{\tilde{v}_n\}_{n\geq 2}$ of extended functions on $[0, 1]$ by continuing each v_n to $r = 0$ as

$$\tilde{v}_n(r) = \begin{cases} v_n(1/n) & \text{for } r \in [0, 1/n), \\ v_n(r) & \text{for } r \in [1/n, 1]. \end{cases}$$

See Figure 5. Here, $\tilde{v}_n \in C^1[0, 1]$ for all $n \geq 2$. From [Siegel 2006] it can be shown that each function $\tilde{v}_n$, along with the interior solution u_{int}, is increasing and bounded. Siegel also demonstrated that v_n and u_{int} will satisfy the same volume condition:

$$\int_{1/n}^{1} s v_n(s)\, ds = \int_0^1 s u_{\text{int}}(s)\, ds = \frac{\cos\gamma}{B}.$$

Therefore,

$$\int_0^1 s(\tilde{v}_n(s) - u_{\text{int}}(s))\, ds - \int_0^{1/n} s\tilde{v}_n(s)\, ds = 0. \tag{16}$$

Additionally, the comparison principle provides

$$v_{n+1} \leq v_n \quad \text{if and only if} \quad \tilde{v}_{n+1} \leq \tilde{v}_n \text{ for } n \geq 2$$

as well as

$$0 \leq u_{\text{int}} \leq v_n \quad \text{if and only if} \quad 0 \leq u_{\text{int}} \leq \tilde{v}_n \text{ for } n \geq 2.$$

Consequently, we are assured that $\tilde{v}_n \to v$ pointwise on $[0, 1]$ with

$$v \geq u_{\text{int}} \quad \text{on } [0, 1]. \tag{17}$$

Each integral in (16) thus defines a positive decreasing sequence with a defined limit as $n \to \infty$:

$$\lim_{n\to\infty}\int_0^1 s(\tilde{v}_n(s)-u_{\text{int}}(s))\,ds - \lim_{n\to\infty}\int_0^{1/n} s\tilde{v}_n(s)\,ds = 0. \tag{18}$$

The second limit in (18) can be bounded as

$$0 \le \lim_{n\to\infty}\int_0^{1/n} s\tilde{v}_n(s)\,ds \le \tilde{v}_2(1)\cdot\lim_{n\to\infty}\int_0^{1/n} s\,ds = 0,$$

and we conclude $\lim_{n\to\infty}\int_0^{1/n} s\tilde{v}_n(s)\,ds = 0$. The first limit in (18) must now be zero and Lebesgue's dominated convergence theorem can be used to see that

$$0 = \lim_{n\to\infty}\int_0^1 s(\tilde{v}_n(s)-u_{\text{int}}(s))\,ds. = \int_0^1 s(v(s)-u_{\text{int}}(s))\,ds \tag{19}$$

In conjuction with (17), this requires

$$v = u_{\text{int}} \quad \text{almost everywhere.} \tag{20}$$

We further comment that v must be nondecreasing and inequalities that occur in (20) are restricted to jump discontinuities in v. However, suppose such a discontinuity of height $\delta > 0$ occurs at a point $c \in [0, 1)$. Here, there will exist a $d > c$ such that u_{int} is continuous on $[c, d]$ with $v - u_{\text{int}} \ge \delta/2$. This is at odds with (19) being 0 and $v \equiv u_{\text{int}}$ on $[0, 1)$. We can also demonstrate that equality holds at $r = 1$. For $n \ge 2$, we shift u_{int} upward to the position of $\bar{u}_{\text{int}}$ so that $\bar{u}_{\text{int}}(1/n) = v_n(1/n)$. In other words,

$$\bar{u}_{\text{int}} = u_{\text{int}} + v_n(1/n) - u_{\text{int}}(1/n).$$

The comparison principle requires

$$u_{\text{int}}(1) \le v_n(1) \le \bar{u}_{\text{int}}(1). \tag{21}$$

Since $v_n(1/n) = \tilde{v}_n(0)$, we get $\lim_{n\to\infty} v_n(1/n) = \lim_{n\to\infty}\tilde{v}_n(0) = u_{\text{int}}(0)$. This with (21) gives $v(1) = u_{\text{int}}(1)$ and $v \equiv u_{\text{int}}$, as required.

Remark. Dini's theorem can be applied at this point to strengthen the convergence claim on $\{\tilde{v}_n\}$ from pointwise to uniform convergence.

Lemma 3.3. *Define u_a and u_{int} as in Theorem 2.5 and consider m as a function of a. If* $\lim_{a\to 0} m(a) = 0$, *then* $\lim_{a\to 0} u_a(m) = u_{\text{int}}(0)$.

Proof. For a given $m(a)$, select the maximum $n \in \mathbb{N}$ such that $m(a) \le 1/n$, which gives $1/(n+1) < m(a) \le 1/n$. With the sequence of functions $\{v_n\}$, the comparison principle produces the following arrangement, shown in Figure 6:

$$v_{n+1}(1/(n+1)) < u_a(m) \le v_n(1/n) \tag{22}$$

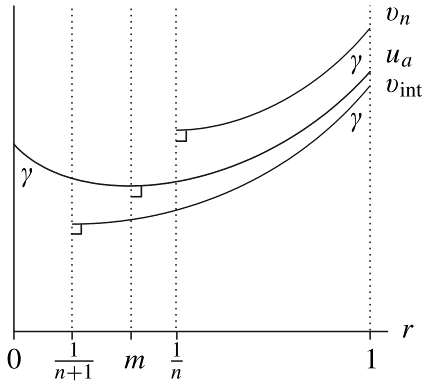

Figure 6. Choosing n so that $v_{n+1}(1/(n+1)) < u_a(m) \leq v_n(1/n)$.

with $\lim_{n\to\infty} v_{n+1}(1/(n+1)) = \lim_{n\to\infty} v_n(1/n) = u_{\text{int}}(0)$. For $\lim_{a\to 0} m(a) = 0$, we have $\lim_{a\to 0} n = \infty$ and (22) requires $\lim_{a\to 0} u_a(m) = u_{\text{int}}(0)$. □

Theorem 3.4. *For* $\gamma \in [0, \pi/2)$, *consider the interior solution* u_{int} *defined on* $[0, 1]$ *together with* u_a, *the solution to* (2) *on* $[a, 1]$. *We have*

$$\lim_{a\to 0} \max_{r\in[a,1]} |u_a(r) - u_{\text{int}}(r)| = 0.$$

Proof. On $[a, 1]$, we compare the contact angles of u_a and u_{int}, noting that the comparison principle requires

$$u_{\text{int}} \leq u_a \quad \text{on } [a, 1]. \tag{23}$$

See Figure 7. Additionally, u_{int} may be shifted upward to the position of $\bar{u}_{\text{int}}$ such that $\bar{u}_{\text{int}}(a) = u_a(a)$. Here again, we use the comparison principle to see that

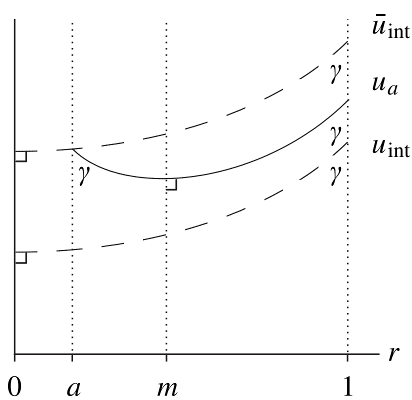

Figure 7. Cross section of comparison surfaces for $a \to 0$.

$u_a \le \bar{u}_{\text{int}}$ on $[a, 1]$. Consequently,

$$\max_{r\in[a,1]} |u_a(r) - u_{\text{int}}(r)| \le [u_a(a) - u_a(m)] + (u_a(m) - u_{\text{int}}(0)), \tag{24}$$

and both bracketed terms of (24) can be bounded. For the first term, we write

$$u_a(a) - u_a(m) = -\int_a^m u_s\, ds = -\int_a^m \frac{\sin\psi}{\sqrt{1-\sin^2\psi}}\, ds,$$

and using Lemma 3.2,

$$u_a(a) - u_a(m) < a\int_a^m \frac{1}{\sqrt{r^2-a^2}}\, ds < a\log(1+\sqrt{1-a^2}) - a\log a$$
$$\to 0 \quad \text{as } a \to 0.$$

For the second term in (24), it is clear that u_a satisifes the boundary value problem (15) on $[m, 1]$. Considering m as a function of a, it is sufficient to show that $\lim_{a\to 0} m(a) = 0$, as Lemma 3.3 would then require $\lim_{a\to 0}(u_a(m) - u_{\text{int}}(0)) = 0$, thus proving the theorem. We proceed by contradiction and assume m does not approach 0. As a result, there exists a $\sigma > 0$ such that

$$m \ge \sigma \quad \text{for all } a \in (0, 1). \tag{25}$$

Suppose that $a < \sigma$. By multiplying the first of (2) by r and integrating from a to m, we have

$$\int_a^m s u_a(s)\, ds = \frac{a\cos\gamma}{B} \quad \text{implies} \quad \lim_{a\to 0}\int_a^m s u_a(s)\, ds = 0.$$

Using (25) and that u_a is decreasing on $[a, m)$, the above integral could also be bounded as $\int_a^m s u_a(s) ds \ge u_a(\sigma)\int_a^\sigma s\, ds$. By (23), $u_a(\sigma) \ge u_{\text{int}}(\sigma)$ so that

$$\lim_{a\to 0}\int_a^m s u_a(s)\, ds \ge u_{\text{int}}(\sigma)\frac{\sigma^2}{2} > 0.$$

This is an impossible situation and m must approach 0 as $a \to 0$. As a result,

$$\max_{r\in[a,1]} |u_a(r) - u_{\text{int}}(r)| \to 0 \quad \text{as } a \to 0. \qquad \square$$

Approaching a thin ring. We next examine solutions to (2) as $a \to 1$. For this, let $u_0 = 2\cos\gamma/(B(1-a))$, the constant function that satisfies the volume condition (4). Also, define the function u_1 by

$$u_1(r) = u_1(a) + \int_a^r \frac{\sin\psi_1(s)}{\sqrt{1-\sin^2\psi_1(s)}}\, ds,$$

with

$$\sin\psi_1(r) = \frac{B}{r}\int_a^r s u_0\,ds - \frac{a}{r}\cos\gamma = \frac{\cos\gamma}{1-a}\Big(r - \frac{a}{r}\Big)$$

and

$$u_1(a) = \frac{2\cos\gamma}{B(1-a)} - \frac{1}{1-a^2}\int_a^1 (1-s^2)\frac{\sin\psi_1(s)}{\sqrt{1-\sin^2\psi_1(s)}}.$$

Here, ψ_1 denotes the inclination angle of u_1. Since

$$-(a/r)\cos\gamma \le \sin\psi_1 \le r\cos\gamma, \tag{26}$$

it is easily checked that u_1 is defined and continuous; the choice of $u_1(a)$ ensures that u_1 also satisfies the volume condition. Note that u_1 is a Delaunay surface (that is, a surface of revolution having constant mean curvature) satisfying the differential equation

$$Nu_1 = Bu_0 \quad \text{if and only if} \quad (r\sin\psi_1(r))_r = Bru_0. \tag{27}$$

For $\gamma \ne 0$, it so happens that u_1 will act as a limiting surface as $a \to 1$.

Theorem 3.5. *Define u_a as in Theorem 3.4 and consider the function u_1 described above. For $\gamma \ne 0$, we have $|u_a - u_1| = O((1-a)^3)$ as $a \to 1$.*

Proof. We first bound $|u_a - u_0|$. Using that u_a is convex and $u_a(a) < u_a(1)$, we have

$$\begin{aligned}|u_a - u_0| \le \max\{u_a(1) - u_0, u_0 - u_a(m)\} < u_a(1) - u_a(m)\\ = \int_m^1 \frac{\sin\psi_a}{\sqrt{1-\sin^2\psi_a}}\,dr,\end{aligned}$$

where ψ_a is the inclination angle of u_a. Lemma 3.2 provides that

$$\frac{\sin\psi}{\sqrt{1-\sin^2\psi}} \le \frac{r\cos\gamma}{\sqrt{1-r^2\cos^2\gamma}}$$

and consequently

$$\begin{aligned}|u - u_0| &< \int_m^1 \frac{r\cos\gamma}{\sqrt{1-r^2\cos^2\gamma}}\,dr\\ &= \frac{\sqrt{1-m^2\cos^2\gamma} - \sin\gamma}{\cos\gamma} := C(\gamma, m) < C(\gamma, a).\end{aligned}$$

Using the first of (2) and (27), we write

$$\sin\psi_a - \sin\psi_1 = \frac{B}{r}\int_a^r s(u_a - u_0)\,ds,$$

or equivalently, $\sin\psi_a - \sin\psi_1 = -(B/r)\int_r^1 s(u_a - u_0)\,ds$. Taken together, these yield

$$|\sin\psi_a - \sin\psi_1| \le \frac{B}{2r}C(\gamma, a)\min\{r^2 - a^2, 1 - r^2\},$$

and given that $\min\{r^2 - a^2, 1 - r^2\} \le 2(r^2 - a^2)(1 - r^2)/(1 - a^2)$, we have

$$|\sin\psi_a - \sin\psi_1| \le \frac{B}{r}\,C(\gamma, a)\frac{(r^2 - a^2)(1 - r^2)}{1 - a^2}.$$

Continuing, we bound $|u_a - u_1|$ by first noting that u_a and u_1 have the correct volume; therefore they must intersect at least once in $(a, 1)$, with

$$|u_a - u_1| \le \int_a^1 |(u_a)_r - (u_1)_r|\,dr. \tag{28}$$

To estimate the integrand of (28), we apply the mean value theorem to the function $f(p) = p/\sqrt{1 - p^2}$, so that

$$|u_r - (u_{n+1})_r| = \frac{|\sin\psi - \sin\psi_{n+1}|}{(1 - \xi^2)^{3/2}} \quad \text{implies} \quad |u_a - u_1| \le \int_a^1 \frac{|\sin\psi_a - \sin\psi_1|}{(1 - \xi^2)^{3/2}}\,dr,$$

where ξ lies between $\sin\psi_a$ and $\sin\psi_1$. By Lemma 3.2 and (26), we have $-\cos\gamma < \xi < \cos\gamma$, so that $1 - \xi^2 > \sin^2\gamma > 0$ for $\gamma \ne 0$. We may bound $|u_a - u_1|$ further:

$$|u_a - u_1| < \int_a^1 \frac{\dfrac{BC(\gamma, a)}{r}\,\dfrac{(r^2 - a^2)(1 - r^2)}{1 - a^2}}{\sin^3\gamma}\,dr < \frac{B}{a\sin^3\gamma}C(\gamma, a)(1 - a^2)(1 - a).$$

Finally, we rewrite $C(\gamma, a)$ as

$$C(\gamma, a) = \frac{\cos\gamma\,(1 - a^2)}{\sqrt{1 - a^2\cos^2\gamma} + \sin\gamma} < \frac{\cos\gamma\,(1 - a^2)}{2\sin\gamma},$$

and thus

$$|u_a - u_1| < \frac{B\cos\gamma}{2a\sin^4\gamma}(1 - a^2)^2(1 - a) = O((1 - a)^3) \quad \text{as } a \to 1. \qquad \square$$

For $\gamma = 0$, the term $(1 - \xi^2)$ can no longer be assigned a positive lower bound and the argument above does not yield the asymptotic behaviour of u_a as $a \to 1$. Further work is needed to understand this special case.

Finally, we add to Theorem 3.5 by showing that the limiting surface u_1 will in turn approach the lower portion of a torus as $a \to 1$.

Theorem 3.6. *Consider the function*

$$t(r) = -\sqrt{\left(\frac{1-a}{2}\right)^2\sec^2\gamma - \left(r - \frac{1+a}{2}\right)^2} + b(a, \gamma, B),$$

where

$$b(a,\gamma,B)=\frac{2\cos\gamma}{B(1-a)}+\frac{1-a}{8}\sec^2\gamma(\pi-2\gamma-\sin 2\gamma)+\left(\frac{1-a}{2}\right)\tan\gamma.$$

On $[a,1]$, *the function* $t(r)$ *describes the lower portion of a torus that satisfies the boundary conditions of* (2) *and the volume condition* (4). *For* $\gamma\neq 0$, *we have*

$$|u_1-t|=O((1-a)^2)\quad \text{as } a\to 1.$$

Proof. It can be shown that the inclination angle of $t(r)$ is given as

$$\sin\omega(r)=\frac{\cos\gamma}{1-a}(2r-1-a),$$

with $|\sin\psi_1-\sin\omega|$ being maximized on $[a,1]$ at $r=\sqrt{a}$ such that

$$|\sin\psi_1-\sin\omega|\leq\frac{\cos\gamma}{(1+\sqrt{a})^2}(1-a).$$

We argue analagously to the previous theorem that

$$|u_1-t|\leq\int_a^1\frac{|\sin\psi_1-\sin\omega|}{(1-\xi^2)^{3/2}}\,dr,$$

where $-\cos\gamma\leq\sin\omega<\xi<\sin\psi_1\leq\cos\gamma$. For $\gamma\neq 0$, $|u_1-t|$ is then bounded as

$$|u_1-t|<\int_a^1\frac{\frac{\cos\gamma}{(1+\sqrt{a})^2}(1-a)}{\sin^3\gamma}=O((1-a)^2)\quad\text{as } a\to 1.\qquad\square$$

When considered together, Theorems 3.5 and 3.6 allow us to conclude that for $\gamma\neq 0$, the solution surface u_a approaches the torus portion $t(r)$ as $O((1-a)^2)$:

$$|u_a-t|\leq|u_a-u_1|+|u_1-t|=O((1-a)^2)\quad\text{as } a\to 1.$$

Approaching the exterior of a disk. Finally, consider solutions to (5) where $b\to\infty$. Here, we will make use of the exterior solution u_{ext} that solves

$$\begin{cases}(r\sin\psi)_r=Bru_{\text{ext}} & \text{for } r\in(1,\infty),\\ \sin\psi(1)=-\cos\gamma,\\ \lim\limits_{r\to\infty}u(r)=0.\end{cases}$$

See [Siegel 1980] for background. As well, define a sequence of functions $\{w_n\}_{n\geq 2}$ such that w_n is the solution to the boundary value problem

$$\begin{cases}(r\sin\psi)_r=Brw_n & \text{for } r\in(1,n),\\ \sin\psi(1)=-\cos\gamma,\\ \sin\psi(n)=0.\end{cases}$$

We start by demonstrating that $w_n \to u_{\text{ext}}$ as $n \to \infty$. It can be verified that each function w_n, as well as u_{ext}, is decreasing. Also, the comparison principle requires that $w_{n+1} \leq w_n$ and $0 < u_{\text{ext}} \leq w_n$ for $n \geq 2$. Furthermore, u_{ext} can be shifted vertically to the position of $\bar{u}_{\text{ext}}$ such that

$$\bar{u}_{\text{ext}} = u_{\text{ext}} + w_n(n) - u_{\text{ext}}(n), \tag{29}$$

and the comparison principle gives $u_{\text{ext}} \leq w_n \leq \bar{u}_{\text{ext}}$ on $[1, n]$. We consider the limit of (29) as $n \to \infty$. Clearly $\lim_{n\to\infty} u_{\text{ext}}(n) = 0$ and we will prove by contradiction that $\lim_{n\to\infty} w_n(n) = 0$. Assume there exists a $\delta > 0$ such that $w_n(n) \geq \delta$ for all $n \geq 2$. This would imply

$$\int_1^n s w_n \, ds > \delta \int_1^n s \, ds \to \infty \quad \text{as } n \to \infty. \tag{30}$$

However, each w_n obeys the volume condition $\int_1^n s w_n \, ds = (\cos\gamma)/B$, which contradicts (30) and necessarily $\lim_{n\to\infty} w_n(n) = 0$. Therefore, (29) provides that $\lim_{n\to\infty} \bar{u}_{\text{ext}} = u_{\text{ext}}$ and $w_n \to u_{\text{ext}}$ and $n \to \infty$.

The behaviour of u as $b \to \infty$ is divided into the following two theorems, with each considering u on the stated subinterval of $[1, b]$.

Theorem 3.7. *For $\gamma \in [0, \pi/2)$, consider the exterior solution u_{ext} defined on $[1, \infty)$ together with u_b, the solution to (5) on $[1, b]$. Let m be the location of the minimum of u_b. On $[1, m]$, we have*

$$\lim_{b\to\infty} \max_{r\in[1,m]} |u_b(r) - u_{\text{ext}}(r)| = 0.$$

Furthermore, $m = m(b)$ is monotone increasing and $m(b) \to \infty$ as $b \to \infty$.

Proof. We compare the three functions u_{ext}, u_b and $\hat{u}_{\text{ext}}$ on $[1, m]$, where

$$\hat{u}_{\text{ext}} = u_{\text{ext}} + u_b(m) - u_{\text{ext}}(m),$$

with $u_{\text{ext}} \leq u_b \leq \hat{u}_{\text{ext}}$ on $[1, m]$ by the comparison principle. Similar to Lemma 3.3, we have

$$\lim_{b\to\infty} m(b) = \infty \quad \text{implies} \quad \lim_{b\to\infty} u_b(m) = u_{\text{ext}}(m),$$

and it is sufficient to show that $\lim_{b\to\infty} m(b) = \infty$, since this would require

$$\max_{r\in[1,m]} |u_b(r) - u_{\text{ext}}(r)| \leq \hat{u}_{\text{ext}} - u_{\text{ext}} = u_b(m) - u_{\text{ext}}(m) \to 0 \quad \text{as } b \to \infty.$$

An argument nearly identical to that proving Theorem 2.5 yields that $m(b)$ is monotone increasing. Furthermore, m increases without bound as $b \to \infty$, which can be shown by contradiction: Assume there exists an $M \in \mathbb{N}$ such that $m(b) \leq M$. The volume condition on u can be used to show that

$$\int_M^b su\,ds \leq \int_m^b su\,ds = \frac{b\cos\gamma}{B}. \tag{31}$$

Additionally, select the function $w_M \in \{w_n\}$ as a lower bound of u on $[1, M]$, so that $w_M(M) \leq w_M \leq u$ on $[1, M]$ by the comparison principle. With (31), this produces

$$w_M(M)(\tfrac{1}{2}(b^2 - M^2)) < \int_M^b su\,ds \leq \frac{b\cos\gamma}{B}. \tag{32}$$

For large enough b, however, (32) cannot hold, and $m \to \infty$ as $b \to \infty$. □

The examination of u on the remaining interval $[m, b]$ will refer to the one-dimensional solution $z(x)$ that solves

$$\begin{cases} \left(\dfrac{z_x}{\sqrt{1 + {z_x}^2}}\right)_x = Bz \quad \text{for } x \in (0, \infty), \\ \sin\phi(0) = -\cos\gamma, \\ \lim\limits_{x\to\infty} z(x) = 0, \end{cases} \tag{33}$$

where $\phi(x)$ denotes the inclination angle of $z(x)$. This problem was first considered by Laplace [1966]; a modern treatment is offered by Siegel [1980]. Physically, z represents the height of a capillary surface on one side of an infinite vertical plate.

Theorem 3.8. *Let $\gamma \in [0, \pi/2)$ and define u_b and m as in the previous theorem. Consider the one-dimensional solution z that satisfies (33). On $[m, b]$, we have*

$$\lim_{b\to\infty} \max_{s\in[0,b-m]} |u_b(b - s) - z(s)| = 0.$$

Proof. We employ the functions $z(s)$ and $u_b(b - s)$ for $s \in [0, b - m]$. This amounts to comparing the annular solution with the capillary surface generated by an infinte plate placed tangentially to the outer boundary of Ω. We also introduce the function $\hat{z}$ defined as $\hat{z}(s) = z(s) + u_b(m) - z(b - m)$. Our comparisons will be largely based upon the results of Siegel [1980], where a similar geometry was used to compare the surface z with the interior solution. In our case, the comparison principle requires $z \leq u_b \leq \hat{z}$ and more specifically, $z(s) \leq u_b(b - s) \leq \hat{z}(s)$ for $s \in [0, b - m]$. Thus

$$\max_{s\in[0,b-m]} |u_b(b - s) - z(s)| \leq \hat{z} - z = u_b(m) - z(b - m).$$

From Theorem 3.7, it is clear that $u_b(m) \to u_{\text{ext}}(m) \to 0$ as $b \to \infty$. Additionally, since $m < (1+b)/2$, we have

$$b - m > (b-1)/2 \to \infty \quad \text{as } b \to \infty \quad \text{and} \quad \lim_{b\to\infty} z(b-m) = 0.$$

Therefore, $\max_{s\in[0,b-m]} |u_b(b-s) - z(s)| \to 0$ as $b \to \infty$. □

References

[Concus and Finn 1974] P. Concus and R. Finn, "On capillary free surfaces in the absence of gravity", *Acta Math.* **132** (1974), 177–198. MR 58 #32327a Zbl 0382.76003

[Elcrat et al. 2004] A. Elcrat, T.-E. Kim, and R. Treinen, "Annular capillary surfaces", *Arch. Math. (Basel)* **82**:5 (2004), 449–467. MR 2006e:76055 Zbl 1136.76353

[Finn 1986] R. Finn, *Equilibrium capillary surfaces*, Grundlehren der Mathematischen Wissenschaften **284**, Springer, New York, 1986. MR 88f:49001 Zbl 0583.35002

[Laplace 1966] P. S. Laplace, *Celestial mechanics*, vol. 4, Chelsea, New York, 1966. MR 42 #28 Zbl 1047.01531

[Serrin 1971] J. Serrin, "A symmetry problem in potential theory", *Arch. Rational Mech. Anal.* **43** (1971), 304–318. MR 48 #11545 Zbl 0222.31007

[Siegel 1980] D. Siegel, "Height estimates for capillary surfaces", *Pacific J. Math.* **88**:2 (1980), 471–515. MR 82h:35037 Zbl 0411.35043

[Siegel 2006] D. Siegel, "Approximating symmetric capillary surfaces", *Pacific J. Math.* **224**:2 (2006), 355–365. MR 2007a:76022 Zbl 1116.76015

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