Pacific Journal of Mathematics

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Volume 247 No. 2 October 2010

PACIFIC JOURNAL OF MATHEMATICS
Vol. 247, No. 2, 2010

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Consider an annular region $\Omega \subset \mathbb{R}^2$. We extend the iterative procedure of Siegel to the case of symmetric capillary surfaces $z = u(x, y)$ formed within the annular cylinder $\Omega \times \mathbb{R}$ and having identical contact angles γ along the inner and outer boundaries. We demonstrate convergence under conditions that include $\gamma = 0$, and we recover the interleaving properties noted by Siegel for a particular geometry.

1. Introduction

We continue our examination of annular capillary surfaces of the form described in [Gordon and Siegel 2010]: Two concentric circular cylinders that define an annular cross section $\Omega \subset \mathbb{R}^2$ are immersed vertically in an infinite reservoir of incompressible fluid. Under the influence of gravity, the surface $Z = U(X, Y)$ formed between the tubes will satisfy the boundary value problem

$$
\begin{cases} NU = \kappa U & \text{in } \Omega, \\ \hat{\nu} \cdot TU = \cos\gamma & \text{on } \partial\Omega, \end{cases} \tag{1}
$$

where $TU = \nabla U / \sqrt{1 + |\nabla U|^2}$, $NU = \nabla \cdot TU$, the exterior unit normal on the boundary $\partial\Omega$ is $\hat{\nu}$, and $\kappa > 0$ is the capillary constant. The contact angle $\gamma \in [0, \pi]$ is defined on the inner and outer boundaries and is the angle at which the interface meets the bounding wall. Again, γ is assumed to be constant and equal along each boundary.

The axisymmetric nature of such annular solutions, along with the change to dimensionless variables

$$u = U/R_2 \quad \text{and} \quad r = R/R_2$$

MSC2000: primary 34B15, 35J25, 76B45; secondary 34E10.
Keywords: annular capillary surface, iterative approximations.
Gordon is supported by an Ontario Graduate Scholarship. Siegel is supported by a Natural Sciences and Engineering Research Council of Canada Discovery Grant.

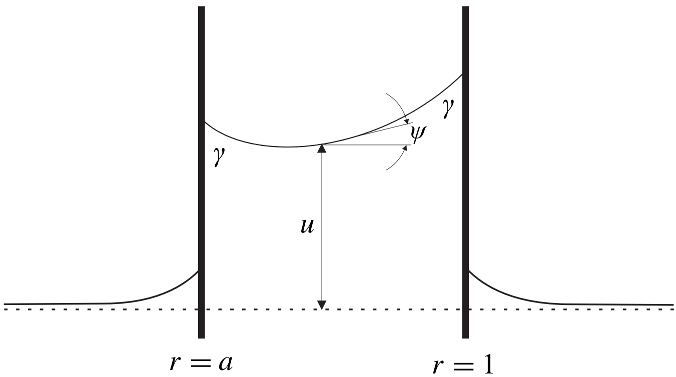

Figure 1. Radial cross section of annular capillary surface.

allows us to convert (1) into a boundary value problem for an ordinary differential equation:

$$
\begin{cases}
Nu = \dfrac{(r \sin \psi)_r}{r} = Bu \quad \text{for } a < r < 1, \\
\sin \psi(a) = -\cos \gamma, \\
\sin \psi(1) = \cos \gamma
\end{cases}
\tag{2}
$$

where $B > 0$ is a positive constant known as the Bond number, $(\cdot)_r$ denotes differentiation with respect to the radial variable r, and we define the inclination angle $\psi(r)$ of $u(r)$ as $\sin \psi(r) = u_r/\sqrt{1+u_r^2}$. See Figure 1. Note also that the outer boundary of Ω is now fixed at $r = 1$, while the inner boundary occurs at $r = a$, where $0 < a < 1$. For reasons described in [Gordon and Siegel 2010], we need only consider $0 \le \gamma < \pi/2$. Additionally, u is positive and bounded by the comparison principle [Concus and Finn 1974; Finn 1986], and the volume lifted can be determined from

$$
\int_a^1 ru(r)\,dr = \frac{\cos \gamma (1+a)}{B}.
\tag{3}
$$

In this paper, we apply the iterative procedure introduced by Siegel [2006] to the boundary value problem considered here. Specifically, Section 2 provides conditions under which the approximate functions generated converge to the solution of (2). In [2006], Siegel demonstrated that the iterates display a highly organized interplay, which he called *interleaving properties*. This allowed for bounds to be placed on the surface height at the inner and outer radii. In Section 3, we prove that, under certain conditions, the interleaving properties may be recovered for (2). Section 4 summarizes a new and more complicated behavior that can also occur.

2. Iterative procedure

The iterative scheme of [Siegel 2006] was employed successfully to approximate annular surfaces with the inner contact angle fixed at $\pi/2$. We intend to extend the procedure to the boundary value problem (2). An outline follows: Consider a function u_1 that satisfies the volume condition (3), and suppose there exists a function u_2 such that $Nu_2 = Bu_1$ or equivalently

$$(r \sin\psi_2)_r = Bru_1,$$

where ψ_2 is the inclination angle of u_2. Requiring $\sin\psi_2(a) = -\cos\gamma$, we arrive at an integral equation for ψ_2:

$$\sin\psi_2(r) = \frac{B}{r}\int_a^r su_1(s)\,ds - \frac{a}{r}\cos\gamma.$$

Since u_1 satisfies the volume requirement, it is easily verified that u_2 also has the correct boundary condition at $r = 1$. Given that $(u_2)_r = \sin\psi_2/\sqrt{1-\sin^2\psi_2}$, we derive an expression for u_2:

$$u_2(r) = u_2(a) + \int_a^r \frac{\sin\psi_2(s)}{\sqrt{1-\sin^2\psi_2(s)}}\,ds, \tag{4}$$

with $u_2(a)$ selected so that u_2 has the correct volume, that is,

$$u_2(a) = \frac{2\cos\gamma}{B(1-a)} - \frac{1}{1-a^2}\int_a^1 (1-s^2)\frac{\sin\psi_2(s)}{\sqrt{1-\sin^2\psi_2(s)}}\,ds. \tag{5}$$

The following theorem solidifies these ideas. Here, a complicating assumption of $\sin\psi_2 \le r\cos\gamma$ arises that did not occur in Siegel's previous analysis.

Theorem 2.1. *Let u_1 be a continuous, positive function defined on $[a, 1]$ that satisfies the volume condition (3). Define*

$$\sin\psi_2(r) = \frac{B}{r}\int_a^r su_1(s)\,ds - \frac{a}{r}\cos\gamma$$

and assume $\sin\psi_2 \le r\cos\gamma$.

(i) *We have* $-(a/r)\cos\gamma \le \sin\psi_2$ *on* $[a, 1]$.

(ii) *There exists a function u_2 defined and continuous on* $[a, 1]$ *given as*

$$u_2(r) = u_2(a) + \int_a^r \frac{\sin\psi_2(s)}{\sqrt{1-\sin^2\psi_2(s)}}\,ds$$

with

$$u_2(a) = \frac{2\cos\gamma}{B(1-a)} - \frac{1}{1-a^2}\int_a^1 (1-s^2)\frac{\sin\psi_2(s)}{\sqrt{1-\sin^2\psi_2(s)}}\,ds.$$

As a result, $Nu_2 = Bu_1$.

(iii) *The function u_2 satisfies both the volume condition and the boundary conditions listed in (2).*

(iv) *There is a unique point $r = m_2$ at which u_2 achieves its minimum value.*

(v) *If*

$$B < \frac{2}{(1-a)(\frac{1}{3}\sqrt{1-a^2}+a\log(1+\sqrt{1-a^2})-a\log a)},$$

then u_2 will also be positive.

Proof. (i) Using that u_1 is positive, we note that $(r\sin\psi_2)_r = Bru_1 > 0$ and the function $r\sin\psi_2$ is monotone increasing. The remainder of the proof mirrors [Gordon and Siegel 2010, Lemma 3.2].

(ii) To show u_2 is defined and continuous, it suffices to show that u_2 is bounded. Since the function $p/\sqrt{1-p^2}$ is increasing on $(-1, 1)$ with

$$-(a/r)\cos\gamma \le \sin\psi_2 \le r\cos\gamma,$$

Equations (4) and (5) give

$$\begin{aligned} u_2(r) &= \frac{2\cos\gamma}{B(1-a)} - \frac{1}{1-a^2}\int_a^1 (1-s^2)\frac{\sin\psi_2}{\sqrt{1-\sin^2\psi_2}}\,ds \\ &\qquad\qquad + \int_a^r \frac{\sin\psi_2}{\sqrt{1-\sin^2\psi_2}}\,ds \\ &\ge \frac{2\cos\gamma}{B(1-a)} - \frac{1}{1-a^2}\int_a^1 (1-s^2)\frac{s\cos\gamma}{\sqrt{1-s^2\cos^2\gamma}}\,ds \\ &\qquad\qquad - \int_a^r \frac{a\cos\gamma}{\sqrt{s^2-a^2\cos^2\gamma}}\,ds \\ &\ge \cos\gamma\Big(\frac{2}{B(1-a)} - \tfrac{1}{3}\sqrt{1-a^2} - a\log(1+\sqrt{1-a^2}) + a\log a\Big). \end{aligned} \tag{6}$$

Given that $a\log a \ge -1/e$ on $(0, 1]$, we can bound u_2 below:

$$u_2(r) > \cos\gamma(2/B - 1/3 - \log 2 - 1/e) > -\infty,$$

Similarly, u_2 can be bounded above:

$$\begin{aligned} u_2(r) &\le \frac{2\cos\gamma}{B(1-a)} + \frac{1}{1-a^2}\int_a^1 (1-s^2)\frac{a\cos\gamma}{\sqrt{s^2-a^2\cos^2\gamma}}ds + \int_a^r \frac{s\cos\gamma}{\sqrt{1-s^2\cos^2\gamma}}\,ds \\ &< \cos\gamma\Big(\frac{2}{B(1-a)} + \log 2 + \frac{1}{e} + 1\Big) < \infty. \end{aligned}$$

Finally, the introductory discussion confirms that $Nu_2 = Bu_1$.

(iii) This is easily proven from the volume condition.

(iv) This argument follows the proof of [Gordon and Siegel 2010, Theorem 2.1].

(v) The lower bound given in (6) is required to be positive:

$$u_2(r) \geq \cos\gamma\left(\frac{2}{B(1-a)} - \tfrac{1}{3}\sqrt{1-a^2} - a\log(1+\sqrt{1-a^2}) + a\log a\right) > 0.$$

Solving for B produces the desired result. □

Theorem 2.1 creates the framework needed to generate a sequence of iterates $\{u_n\}$ defined recursively as

$$Nu_{n+1} = Bu_n \quad \text{for } n \geq 0. \tag{7}$$

We take the initial function u_0 to be the constant function that satisfies the volume condition:

$$u_0 = \frac{2\cos\gamma}{B(1-a)}. \tag{8}$$

It can be shown that for suitable restrictions on B, the sequence $\{u_n\}$ is one in which

- (i) through (v) of Theorem 2.1 are satisfied for all $n \geq 1$, and
- $\{u_n\}$ converges to the solution of the boundary value problem (2).

The following theorem demonstrates these results.

Theorem 2.2 (iterate convergence). *For*

$$B < \frac{2a(1-a^2)\cos\gamma}{2(1+a)(1-a)^2C(\gamma,m) + a\pi\cos\gamma},$$

the sequence of iterates $\{u_n\}$ generated via (7) and (8) will be continuous and positive. Furthermore, $B\pi/(2(1-a^2)) < 1$ and $\{u_n\}$ converges to u, the solution of (2), with

$$|u - u_n| < C(\gamma,m)\left(B\frac{\pi}{2(1-a^2)}\right)^n.$$

Here, $C(\gamma,m) = (\sqrt{1-m^2\cos^2\gamma} - \sin\gamma)/\cos\gamma$ with $0 < C(\gamma,m) < C(\gamma,a)$, and $r = m$ is the location of the minimum of u.

Proof. We first prove the case for $n = 0$ and proceed inductively. Since u_0 and u satisfy the volume condition, they must intersect at least once in $(a, 1)$. Using that u is convex and $u(a) < u(1)$ [Gordon and Siegel 2010], $|u - u_0|$ is thus bounded as

$$|u - u_0| \leq \max\{u(1) - u_0, u_0 - u(m)\} < u(1) - u(m) = \int_m^1 \frac{\sin\psi}{\sqrt{1-\sin^2\psi}}\,dr.$$

Additionally, [Gordon and Siegel 2010, Lemma 3.2] provides that

$$\frac{\sin\psi}{\sqrt{1-\sin^2\psi}} \le \frac{r\cos\gamma}{\sqrt{1-r^2\cos^2\gamma}}$$

and consequently

$$|u-u_0| < \int_m^1 \frac{r\cos\gamma}{\sqrt{1-r^2\cos^2\gamma}}\,dr = \frac{\sqrt{1-m^2\cos^2\gamma}-\sin\gamma}{\cos\gamma} := C(\gamma,m).$$

The case $n=0$ is thus proved. Next, assume u_n is continuous, positive, and satisfies the volume condition. Also, let $|u-u_n| < \beta_n := C(\gamma,m)(B\pi/(2(1-a^2)))^n$. The defining equations for $\{u_n\}$ and u yield

$$\sin\psi - \sin\psi_{n+1} = \frac{B}{r}\int_a^r s(u-u_n)\,ds,$$

or equivalently, $\sin\psi - \sin\psi_{n+1} = -(B/r)\int_r^1 s(u-u_n)\,ds$. When used in tandem, these imply

$$|\sin\psi - \sin\psi_{n+1}| \le \frac{B}{2r}\beta_n \min\{r^2-a^2, 1-r^2\},$$

and since $\min\{r^2-a^2, 1-r^2\} \le 2(r^2-a^2)(1-r^2)/(1-a^2)$, this gives

$$|\sin\psi - \sin\psi_{n+1}| \le \frac{\beta_n B}{r}\frac{(r^2-a^2)(1-r^2)}{1-a^2}. \tag{9}$$

We next bound $\sin\psi_{n+1}$. For $n=0$, $\sin\psi_1$ can be written exactly:

$$\sin\psi_1 = \frac{\cos\gamma}{1-a}\Big(r-\frac{a}{r}\Big), \tag{10}$$

and it is easily checked that $-(a/r)\cos\gamma \le \sin\psi_1 \le r\cos\gamma$. For $n \ge 1$, we do not have the luxury of an explicit function and we proceed as follows: To show $\sin\psi_{n+1} \le r\cos\gamma$, consider the distance between $\sin\psi_1$ and $\sin\psi_{n+1}$:

$$\begin{aligned} |\sin\psi_1 - \sin\psi_{n+1}| &\le |\sin\psi - \sin\psi_1| + |\sin\psi - \sin\psi_{n+1}| \\ &\le \frac{B}{r}C(\gamma,m)\frac{(r^2-a^2)(1-r^2)}{1-a^2}\bigg(1+\Big(B\frac{\pi}{2(1-a^2)}\Big)^n\bigg). \end{aligned} \tag{11}$$

The last factor in (11) can be bounded by a geometric series:

$$1+\Big(B\frac{\pi}{2(1-a^2)}\Big)^n < \sum_{k=0}^{\infty}\Big(B\frac{\pi}{2(1-a^2)}\Big)^k,$$

and since

$$B < \frac{2a(1-a^2)\cos\gamma}{2(1+a)(1-a)^2C(\gamma,m)+a\pi\cos\gamma} < \frac{2(1-a^2)}{\pi},$$

the sum is convergent. We now have

$$|\sin\psi_1 - \sin\psi_{n+1}| \le \frac{B}{r} C(\gamma, m) \frac{2(r^2 - a^2)(1 - r^2)}{2(1 - a^2) - B\pi}. \tag{12}$$

The condition on B can be substituted into (12) to obtain

$$|\sin\psi_1 - \sin\psi_{n+1}| \le \frac{a\cos\gamma}{1-a}\Big(\frac{1}{r} - r\Big).$$

With this in hand, we are able to show that $\sin\psi_{n+1} \le r\cos\gamma$:

$$\begin{aligned} r\cos\gamma - \sin\psi_{n+1} &\ge (r\cos\gamma - \sin\psi_1) - |\sin\psi_1 - \sin\psi_{n+1}| \\ &\ge \bigg(r\cos\gamma - \frac{\cos\gamma}{1-a}\Big(r - \frac{a}{r}\Big)\bigg) - \bigg(\frac{a\cos\gamma}{1-a}\Big(\frac{1}{r} - r\Big)\bigg) = 0, \end{aligned}$$

and conditions (i)–(iv) of Theorem 2.1 apply to u_{n+1}. In addition, the bound on B required here satisfies

$$\begin{aligned} B &< \frac{2a(1-a^2)\cos\gamma}{2(1+a)(1-a)^2 C(\gamma, m) + a\pi\cos\gamma} \\ &< \frac{2}{(1-a)(\frac{1}{3}\sqrt{1-a^2} + a\log(1+\sqrt{1-a^2}) - a\log a)}, \end{aligned}$$

and consequently, u_{n+1} also exhibits property (v) of Theorem 2.1. To summarize, u_{n+1} will be continuous, positive and will satisfy the volume condition.

Next, we bound $|u - u_{n+1}|$. Since both u and u_{n+1} have the correct volume, they must intersect at least once in $(a, 1)$. This allows us to state that

$$|u - u_{n+1}| \le \int_a^1 |u_r - (u_{n+1})_r|\, dr. \tag{13}$$

To estimate the integrand of (13), we use the mean value theorem on the function $f(p) = p/\sqrt{1-p^2}$, so that

$$\frac{f(\sin\psi) - f(\sin\psi_{n+1})}{\sin\psi - \sin\psi_{n+1}} = f'(\xi),$$

where ξ lies between $\sin\psi$ and $\sin\psi_{n+1}$. This can be rewritten as

$$|u_r - (u_{n+1})_r| = \frac{|\sin\psi - \sin\psi_{n+1}|}{(1-\xi^2)^{3/2}}. \tag{14}$$

The numerator of (14) has an upper bound given in (9). For the denominator, [Gordon and Siegel 2010, Lemma 3.2] provides bounds on $\sin\psi$ that are identical to those derived above for $\sin\psi_{n+1}$:

$$-\frac{a\cos\gamma}{r} \le \sin\psi \quad \text{and} \quad \sin\psi_{n+1} \le r\cos\gamma,$$

and ξ is bounded as

$$-\frac{a}{r} \leq -\frac{a\cos\gamma}{r} < \xi < r\cos\gamma \leq r,$$

with $\xi^2 < \max\{a^2/r^2, r^2\}$. The denominator of (14) can thus be estimated using $1-\xi^2 > (1-a^2/r^2)(1-r^2)$, and an upper bound on $|u-u_{n+1}|$ is now possible:

$$\begin{aligned} |u-u_{n+1}| &\leq \int_a^1 \frac{|\sin\psi - \sin\psi_{n+1}|}{(1-\xi^2)^{3/2}}\,dr \\ &< \frac{\beta_n B}{1-a^2}\int_a^1 \frac{r^2}{\sqrt{r^2-a^2}\sqrt{1-r^2}}\,dr = \frac{\beta_n B}{1-a^2}\int_0^1 \frac{\sqrt{1-(1-a^2)t^2}}{\sqrt{1-t^2}}\,dt, \end{aligned}$$

where the change of variables $t=\sqrt{(r^2-1)/(a^2-1)}$ is used in the last equality. This integral is always less than $\pi/2$. Hence,

$$|u-u_{n+1}| < \frac{\beta_n B}{1-a^2}\frac{\pi}{2} = \beta_{n+1}$$

and the inductive step is complete. □

3. Single intersection case: interleaving properties

The iterates of [Siegel 2006] were shown there to exhibit the following structure:

(a) $\psi_0 < \psi_2 < \cdots < \psi < \cdots < \psi_3 < \psi_1$ for $r \in (a,1)$.

(b) $u_1(a) < u_3(a) < \cdots < u(a) < \cdots < u_2(a) < u_0$.

(c) $u_0 < u_2(1) < \cdots < u(1) < \cdots < u_3(1) < u_1(1)$.

These properties were defined collectively by Siegel as the *interleaving properties* of the iterates, with (b) and (c) providing under- and over-estimates for the boundary values of u. Here, the behavior between iterates is more complex, being sensitive to the values of the parameters a, γ and B. However, we are able to recover these interleaving properties under certain conditions. It so happens that it will be necessary to find selections of a, γ and B such that u, u_0 and u_2 will be configured as noted in Figure 2. In other words, there exist unique points b_0 and c_0 in $(a,1)$ such that

$$\begin{cases} u < u_0 & \text{if } r \in [a, b_0), \\ u > u_0 & \text{if } r \in (b_0, 1], \end{cases} \tag{15}$$

and

$$\begin{cases} u_2 < u_0 & \text{if } r \in [a, c_0), \\ u_2 > u_0 & \text{if } r \in (c_0, 1]. \end{cases} \tag{16}$$

It turns out that if (15) and (16) occur, the interleaving properties are a consequence. We examine the conditions necessary for each configuration below.

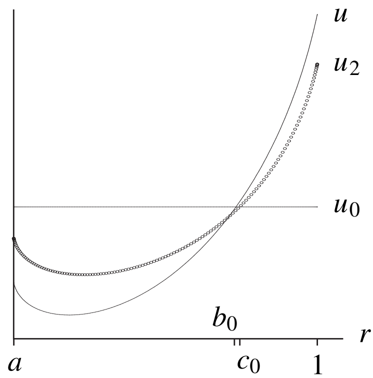

Figure 2. The configuration required for interleaving properties. Whereas u and u_2 must be arranged as shown with respect to u_0, the relative configuration of u and u_2 is not important.

Single intersection of u with u_0. We will show that the configuration of (15) is a result of $u(a) < u_0$, and it is indeed possible to find conditions under which this is true through a comparison with $u_1(a)$. We thus begin by investigating the conditions necessary for $u_1(a) < u_0$. Consider the difference

$$u_0 - u_1(a) = \frac{1}{1-a^2} \int_a^1 (1-s^2) \frac{\sin \psi_1}{\sqrt{1-\sin^2 \psi_1}} \, ds. \tag{17}$$

The integrand can be bounded from below by estimating $\sin \psi_1 / \sqrt{1-\sin^2 \psi_1}$. For $a \le r \le \sqrt{a}$, we use that $\sin \psi_1$ is concave to see $(r-\sqrt{a}) \cos \gamma / (\sqrt{a}-a) \le \sin \psi_1$, which gives

$$\frac{\sin \psi_1}{\sqrt{1-\sin^2 \psi_1}} \ge \frac{(r-\sqrt{a}) \dfrac{\cos \gamma}{\sqrt{a}-a}}{\sqrt{1-\left((r-\sqrt{a}) \dfrac{\cos \gamma}{\sqrt{a}-a}\right)^2}},$$

and for $\sqrt{a} \le r < 1$, it is easily seen that $\sin \psi_1 / \sqrt{1-\sin^2 \psi_1} \ge \sin \psi_1$. Using these bounds in (17), we have

$$u_0 - u_1(a) > \sqrt{\frac{(\sqrt{a}-a)^2}{\cos^2 \gamma} - (\sqrt{a}-a)^2} - \frac{\sqrt{a}-a}{\cos \gamma} + \frac{\cos \gamma (1-a^2+2a \log a)}{4(1-a)(1-a^2)},$$

and since $a \log a > (a-1) + \frac{1}{2}(a-1)^2 - \frac{1}{6}(a-1)^3$ on $(0, 1)$, we can arrive at

$$u_0 - u_1(a) > \cos \gamma \left[\frac{1-a}{12(1+a)} - (\sqrt{a}-a) \right].$$

To ensure that $u_1(a) < u_0$, the expression in square brackets must be nonnegative, and after some mechanics, this is shown to be true for $a \in (0, \Lambda^2]$ where

$$\Lambda := \tfrac{1}{6}(9+2\sqrt{353})^{1/3} - \frac{11}{6(9+2\sqrt{353})^{1/3}} \doteq 0.09.$$

To compare $u(a)$ with u_0, we employ the same technique used in the proof of Theorem 2.2, namely, find B so that $u(a)$ lies close enough to $u_1(a)$ and necessarily $u(a) < u_0$.

Theorem 3.1. *For any $\gamma \in [0, \pi/2)$, select $a \leq \Lambda^2$ and*

$$B \leq \frac{2(1-a^2)\cos\gamma}{\pi}\Big(\frac{1-a}{12(1+a)} - (\sqrt{a} - a)\Big).$$

Under these conditions, there exists a unique $b_0 \in (a, 1)$ such that $u(b_0) = u_0$ with

$$\begin{cases} u < u_0 & \text{if } r \in [a, b_0), \\ u > u_0 & \text{if } r \in (b_0, 1]. \end{cases}$$

Proof. As mentioned, we will require u_1 as a comparison function and it must be confirmed that u_1 is defined and continuous. In fact, since $\sin\psi_1 \leq r\cos\gamma$ (see (10)) and since

$$\begin{aligned} B &\leq \frac{2(1-a^2)\cos\gamma}{\pi}\Big(\frac{1-a}{12(1+a)} - (\sqrt{a} - a)\Big) \\ &< \frac{2}{(1-a)(\frac{1}{3}\sqrt{1-a^2}+a\log(1+\sqrt{1-a^2})-a\log a)}, \end{aligned}$$

u_1 exhibits all properties of Theorem 2.1. To continue, we write

$$u_0 - u(a) \geq (u_0 - u_1(a)) - |u(a) - u_1(a)|. \tag{18}$$

The selection of a ensures the first term of (18) is positive. For the second term, the proof of Theorem 2.2 for the specific case of u_1 yields

$$|u - u_1| < C(\gamma, m)B\frac{\pi}{2(1-a^2)} < B\frac{\pi}{2(1-a^2)}.$$

with $C(\gamma, m) < 1$. The difference of (18) can now be bounded as

$$u_0 - u(a) > \cos\gamma\Big(\frac{1-a}{12(1+a)} - (\sqrt{a} - a)\Big) - B\frac{\pi}{2(1-a^2)}. \tag{19}$$

Substituting the condition on B produces the desired result,

$$u(a) < u_0. \tag{20}$$

With both u and u_0 having the correct volume, at least one intersection occurs between these functions. The convexity of u, in conjunction with (20), limits this to a unique intersection occurring at a point $b_0 \in (a, 1)$. ☐

Single intersection of u_2 with u_0. In like manner, we are able to find conditions for $u_2(a) < u_0$ that will result in (16). Before turning to this, it should be verified that under the hypotheses of the previous theorem, u_2 is defined and continuous.

Lemma 3.2 [Siegel 2006]. *Consider two functions v and w defined on $[a, 1]$ with inclination angles given by ψ_v and ψ_w respectively. If $\psi_v < \psi_w$ on $(a, 1)$ and $\int_a^1 rv\,dr = \int_a^1 rw\,dr$, then there exists a unique $b \in (a, 1)$ where $v(b) = w(b)$ and*

$$\begin{cases} w < v & \text{if } r \in [a, b), \\ w > v & \text{if } r \in (b, 1]. \end{cases}$$

To show u_2 is defined and continuous, we first write the difference function

$$r \sin\psi - r \sin\psi_1 = B \int_a^r s(u - u_0)\,ds,$$

which is zero at $r = a$ and $r = 1$. Theorem 3.1 also ensures the function contains a unique extremum at $r = b_0$. Thus, $r \sin\psi - r \sin\psi_1$ must be either positive or negative on $(a, 1)$. It follows from $u(a) < u_0$ that

$$r \sin\psi - r \sin\psi_1 < 0 \quad \text{implies} \quad \psi < \psi_1 \quad \text{for } r \in (a, 1),$$

and Lemma 3.2 requires that there exist a unique $b_1 \in (a, 1)$ where $u_1(b_1) = u(b_1)$ and

$$\begin{cases} u_1 < u & \text{if } r \in [a, b_1), \\ u_1 > u & \text{if } r \in (b_1, 1]. \end{cases}$$

With this in hand, we consider the difference $r \sin\psi - r \sin\psi_2 = B \int_a^r s(u - u_1)\,ds$ and reason accordingly that $\sin\psi_2 < \sin\psi < r \cos\gamma$ for $r \in (a, 1)$. With B bounded as in Theorem 3.1, u_2 will exhibit all properties of Theorem 2.1.

Conditions can now be stated so that $u_2(a) < u_0$. In this case, B will be restricted further than in Theorem 3.1.

Theorem 3.3. *For any $\gamma \in [0, \pi/2)$, select $a \le \Lambda^2$ and*

$$B \le \frac{(1 - a^2)\cos\gamma}{\pi} \left(\frac{1-a}{12(1+a)} - (\sqrt{a} - a) \right).$$

Under these conditions, u_2 satisfies the properties of Theorem 2.1 and there exists a unique $c_0 \in (a, 1)$ such that $u_2(c_0) = u_0$, with

$$\begin{cases} u_2 < u_0 & \text{if } r \in [a, c_0), \\ u_2 > u_0 & \text{if } r \in (c_0, 1]. \end{cases}$$

Proof. Similarly, we write $u_0 - u_2(a) \ge (u_0 - u(a)) - |u(a) - u_2(a)|$. The first term can be bounded as in (19), and an argument identical to the proof of Theorem 2.2

specifically for u_2 estimates the second term. We thus have

$$u_0 - u_2(a) > \cos\gamma\left(\frac{1-a}{12(1+a)} - (\sqrt{a}-a)\right) - B\frac{\pi}{2(1-a^2)}\left(1 + B\frac{\pi}{2(1-a^2)}\right),$$

and substituting for B gives $u_2(a) < u_0$. As before, the volume condition guarantees an intersection between the two functions. Theorem 2.1(iv) implies that u_2 is monotone decreasing on $[a, m_2)$ and monotone increasing on $(m_2, 1]$. Hence, the intersection is unique and the arrangement between functions easily follows. □

Interleaving properties. We can now prove the interleaving properties for $\{u_n\}$. Here, a and B are restricted so that the single intersection case is guaranteed to occur.

Theorem 3.4. *For any $\gamma \in [0, \pi/2)$, select $a \leq \Lambda^2$ and*

$$B \leq \frac{(1-a^2)\cos\gamma}{\pi}\left(\frac{1-a}{12(1+a)} - (\sqrt{a}-a)\right).$$

Under these conditions, the sequence of iterates $\{u_n\}$ defined by (7) and (8) satisfies (i) through (v) of Theorem 2.1. The iterates exhibit the following properties:

(1) $\psi_2 < \psi_4 < \cdots < \psi < \cdots < \psi_3 < \psi_1$ *for $r \in (a, 1)$.*

(2) $u_1(a) < u_3(a) < \cdots < u(a) < \cdots < u_4(a) < u_2(a)$.

(3) $u_2(1) < u_4(1) < \cdots < u(1) < \cdots < u_3(1) < u_1(1)$.

Proof. We first go through a cycle of recursive arguments and proceed to show that the base case, which is known to be true, sets the cycle in motion. To start, assume that for a certain $k \geq 0$,

(a) u_{2k}, u_{2k+1}, and u_{2k+2} satisfy (i) through (v) of Theorem 2.1 (although u_{2k} does not have to satisfy the boundary conditions);

(b) $\psi < \psi_{2k+1}$ with $\sin\psi_{2k+1} < r\cos\gamma$ for $r \in (a, 1)$.

(c) there exists a unique $c_{2k} \in (a, 1)$ such that

$$\begin{cases} u_{2k+2} < u_{2k} & \text{if } r \in [a, c_{2k}), \\ u_{2k+2} > u_{2k} & \text{if } r \in (c_{2k}, 1]. \end{cases}$$

From (b), Lemma 3.2 requires that there exists a unique $b_{2k+1} \in (a, 1)$ with

$$\begin{cases} u > u_{2k+1} & \text{if } r \in [a, b_{2k+1}), \\ u < u_{2k+1} & \text{if } r \in (b_{2k+1}, 1]. \end{cases}$$

Using the difference function

$$r\sin\psi - r\sin\psi_{2k+2} = B\int_a^r s(u - u_{2k+1})\,ds,$$

we can show that $\sin\psi_{2k+2} < \sin\psi < r\cos\gamma$ for $r \in (a,1)$. This implies that $\psi_{2k+2} < \psi$ on $(a,1)$. Lemma 3.2 can be used again:

$$\begin{cases} u < u_{2k+2} & \text{if } r \in [a, b_{2k+2}), \\ u > u_{2k+2} & \text{if } r \in (b_{2k+2}, 1]. \end{cases}$$

and a new difference function

$$r\sin\psi - r\sin\psi_{2k+3} = B\int_a^r s(u - u_{2k+2})\,ds$$

produces $\psi < \psi_{2k+3}$ on $(a,1)$. It must now be verified that u_{2k+3} is defined. For this, we use

$$r\sin\psi_{2k+1} - r\sin\psi_{2k+3} = B\int_a^r s(u_{2k} - u_{2k+2})\,ds$$

along with (b) and (c) to reason that $\sin\psi_{2k+3} < \sin\psi_{2k+1} < r\cos\gamma$ for $r \in (a,1)$. With B restricted as hypothesized, u_{2k+3} now obeys Theorem 2.1, and Lemma 3.2 ensures

$$\begin{cases} u_{2k+3} > u_{2k+1} & \text{if } r \in [a, c_{2k+1}), \\ u_{2k+3} < u_{2k+1} & \text{if } r \in (c_{2k+1}, 1]. \end{cases}$$

As a final step, we can similarly argue that u_{2k+4} satisfies Theorem 2.1 since

$$\sin\psi_{2k+2} < \sin\psi_{2k+4} < \sin\psi < r\cos\gamma$$

with

$$\begin{cases} u_{2k+4} < u_{2k+2} & \text{if } r \in [a, c_{2k+2}), \\ u_{2k+4} > u_{2k+2} & \text{if } r \in (c_{2k+2}, 1]. \end{cases}$$

The cycle is now complete as (a), (b) and (c) are proved for the next increment of k. In addition, the discussion above provides the summary

$$\psi_{2k+2} < \psi_{2k+4} < \psi < \psi_{2k+3} < \psi_{2k+1} \quad \text{for } r \in (a,1). \tag{21}$$

It remains to verify that (a), (b) and (c) are true for the base case $k = 0$. However, these were shown as a result of Theorems 3.1 and 3.3, making (21) true for all $k \geq 0$:

$$\psi_2 < \psi_4 < \cdots < \psi < \cdots < \psi_3 < \psi_1 \quad \text{for } r \in (a,1). \tag{22}$$

For parts (2) and (3), apply Lemma 3.2 to each adjacent pair of angles in (22) to arrive at

$$u_1(a) < u_3(a) < \cdots < u(a) < \cdots < u_4(a) < u_2(a).$$

and

$$u_2(1) < u_4(1) < \cdots < u(1) < \cdots < u_3(1) < u_1(1). \qquad \square$$

Remark. Examining the proof of Theorem 3.4, we see that the interleaving properties will hold when instead of the restrictions on a and B we assume that the iterates are defined, continuous and positive, $\psi < \psi_1$, and u_2 satisfies (16).

4. Double intersection case

In contrast to the iterates of Siegel (which consistently intersect once with u_0 and result in interleaving properties throughout), here it is also possible to find selections of a, γ and B where u and u_2 intersect twice with u_0. In this case, there exist exactly two points b_{01} and b_{02} in $(a, 1)$ such that $u(b_{01}) = u(b_{02}) = u_0$ with

$$\begin{cases} u > u_0 & \text{if } r \in [a, b_{01}), \\ u < u_0 & \text{if } r \in (b_{01}, b_{02}), \\ u > u_0 & \text{if } r \in (b_{02}, 1]. \end{cases} \tag{23}$$

As well, there exist exactly two points c_{01} and c_{02} in $(a, 1)$ such that $u_2(c_{01}) = u_2(c_{02}) = u_0$ and

$$\begin{cases} u_2 > u_0 & \text{if } r \in [a, c_{01}), \\ u_2 < u_0 & \text{if } r \in (c_{01}, c_{02}), \\ u_2 > u_0 & \text{if } r \in (c_{02}, 1]. \end{cases} \tag{24}$$

Figure 3 demonstrates these configurations. The effect of (23) and (24) on subsequent iterates is far more varied and less understood. For the sake of brevity, we summarize the conditions necessary for (23) and (24) to occur.

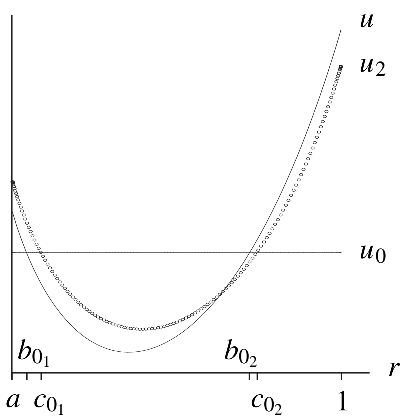

Figure 3. Configuration considered for double intersection case. Whereas u and u_2 must be arranged as shown with respect to u_0, the relative configuration of u and u_2 is not important.

Theorem 4.1. *For a given* $\gamma \in (\arcsin(1/5), \pi/2)$, *select* $a \geq 3/(5\sin\gamma + 2)$ *and*

$$B \leq \frac{\cos\gamma(1-a)^2}{6\pi}\Big(5a - \frac{3-2a}{\sin\gamma}\Big).$$

Under these conditions, there exist exactly two points $b_{01}, b_{02} \in (a, 1)$ *such that* $u(b_{01}) = u(b_{02}) = u_0$ *with*

$$\begin{cases} u > u_0 & \text{if } r \in [a, b_{01}), \\ u < u_0 & \text{if } r \in (b_{01}, b_{02}), \\ u > u_0 & \text{if } r \in (b_{02}, 1]. \end{cases}$$

Theorem 4.2. *For a given* $\gamma \in (\arcsin(1/5), \pi/2)$, *select* $a \geq 3/(5\sin\gamma + 2)$ *and*

$$B \leq \frac{\cos\gamma(1-a)^2}{12\pi}\Big(5a - \frac{3-2a}{\sin\gamma}\Big).$$

Furthermore, assume $\sin\psi_2 < r\cos\gamma$ *on* $(a, 1)$. *Here, there exist exactly two points* $c_{01}, c_{02} \in (a, 1)$ *such that* $u_2(c_{01}) = u_2(c_{02}) = u_0$ *and*

$$\begin{cases} u_2 > u_0 & \text{if } r \in [a, c_{01}), \\ u_2 < u_0 & \text{if } r \in (c_{01}, c_{02}), \\ u_2 > u_0 & \text{if } r \in (c_{02}, 1]. \end{cases}$$

For the most part, the proofs of Theorems 4.1 and 4.2 are analogous to their single intersection counterpart. Note, however, that under the hypotheses of Theorem 4.2, we are unable to prove that $\sin\psi_2 < r\cos\gamma$ on $(a, 1)$ and verify the existence of u_2. This assumption is consequently added to Theorem 4.2.

We briefly outline the resultant behaviors of the double intersection case. Here, it is assumed that all iterates are defined and continuous. Given (23), the behavior of u_1 with respect to u can be examined by considering the difference function

$$r\sin\psi - r\sin\psi_1 = B\int_a^r s(u - u_0)\,ds. \tag{25}$$

In addition to (25) being zero at $r = a$ and $r = 1$, there exist extrema at $r = b_{01}$ and $r = b_{02}$. Since $u(a) > u_0$, ψ and ψ_1 must be arranged as

$$\begin{cases} \psi > \psi_1 & \text{if } r \in (a, \xi_0), \\ \psi < \psi_1 & \text{if } r \in (\xi_0, 1), \end{cases} \tag{26}$$

where $\xi_0 \in (b_{01}, b_{02})$. When (26) is considered in conjunction with the volume condition, three configurations of u with u_1 are possible as illustrated in the first row of Figure 4. Using an analysis similar to the previous section's, we find for configurations A and C (where u and u_1 intersect once) that subsequent iterates will intersect only once with u. Specifically, for configuration A,

$$u_{2n+1}(a) < u(a) < u_{2n+2}(a) \quad \text{and} \quad u_{2n+2}(1) < u(1) < u_{2n+1}(1) \tag{27}$$

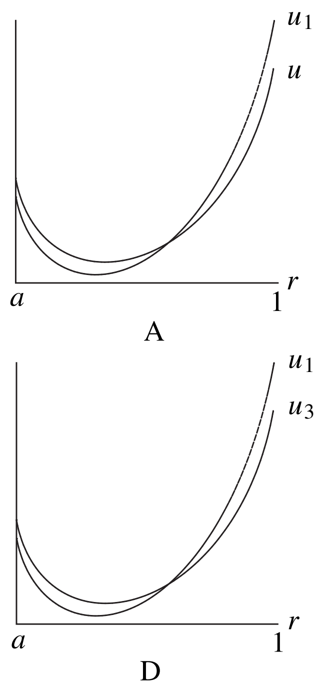

Figure 4. In the first row, potential configurations of u_1 with u_3, assuming (24) holds. In the second, potential configurations of u with u_1, assuming (23) holds.

or for configuration C,

$$u_{2n+2}(a) < u(a) < u_{2n+1}(a) \quad \text{and} \quad u_{2n+1}(1) < u(1) < u_{2n+2}(1)$$

with $n \geq 0$. In configuration B (where u and u_1 intersect twice) the behavior is potentially more diverse. Indeed, three arrangements identical to A, B and C are possible between u_2 and u, and a similar study can be applied here as was done for u versus u_1.

Additionally, (24) can be used to comment on the behavior of u_3 versus u_1. The difference function

$$r \sin \psi_1 - r \sin \psi_3 = B \int_a^r s(u_0 - u_2)\, ds$$

implies three arrangements of u_1 with u_3 are possible as in the second row of Figure 4. Configuration D leads to the predictable behavior

$$\begin{aligned} &\{u_{2n+2}(a)\} \text{ and } \{u_{2n+1}(1)\} \text{ are decreasing for } n \geq 0,\\ &\{u_{2n+1}(a)\} \text{ and } \{u_{2n+2}(1)\} \text{ are increasing for } n \geq 0. \end{aligned} \tag{28}$$

and likewise configuration F gives

$$\{u_{2n+1}(a)\} \text{ and } \{u_{2n+2}(1)\} \text{ are decreasing for } n \geq 0,$$
$$\{u_{2n+2}(a)\} \text{ and } \{u_{2n+1}(1)\} \text{ are increasing for } n \geq 0.$$

Configuration E will itself split into three possible arrangements between u_2 and u_4 that are identical to D, E and F. As one might expect, any configuration of the first row of Figure 4 could conceivably pair with any arrangement of the second row, leading to a far more complex behavior than in the single intersection case. Nevertheless, some pairings will again lead to interleaving iterates. This occurs, for example, when configuration A is paired with configuration D and properties (27) and (28) are matched. The same can be said of pairing configuration C with configuration F. However, if these couples were cross-matched (that is, A–F or C–D), the combined properties would result in diverging iterates. Further research is needed to fully understand the properties of $\{u_n\}$ over the complete parameter space.

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Received January 8, 2009.

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PACIFIC JOURNAL OF MATHEMATICS

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Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 247 No. 2 October 2010