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We prove in dimension $n > 2$ that a K-quasiconformal harmonic mapping u of the unit ball B^n onto itself is Euclidean bi-Lipschitz if $u(0) = 0$ and $K < 2^{n-1}$. This is an extension of a similar result of Tam and Wan for hyperbolic harmonic mappings with respect to a hyperbolic metric. The proof uses Möbius transformations on the related space and a recent result of the first author, which states that harmonic quasiconformal self-mappings of the unit ball are Lipschitz continuous.

1. Introduction

A twice differentiable function u defined in an open subset Ω of the Euclidean space $\mathbb{R}^n$ is said to be *harmonic* if it satisfies the differential equation

$$\Delta u(x) := D_{11}u(x) + D_{22}u(x) + \cdots + D_{nn}u(x) = 0.$$

Throughout the paper B^n denotes the unit ball in $\mathbb{R}^n$, and S^{n-1} denotes the unit sphere. Also we suppose that $n > 2$ (the case $n = 2$ has already been treated by many authors). Recall that for a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with the usual norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$ and a matrix $A \in M_{n\times n}$, the matrix norm of A is defined as

$$|A| = \sup\{|Ax| : |x| = 1\}.$$

By $\langle \cdot, \cdot \rangle$ we denote the inner product in $\mathbb{R}^n$. Given $k \in \mathbb{N}$ and a normed space $\mathbb{X}$, the norm of a k-linear mapping from the k-fold Cartesian product of $\mathbb{R}^n$ to $\mathbb{X}$ is defined by

$$|P| = \sup\{|P(v_1, \dots, v_k)| : |v_1| = \cdots = |v_k| = 1\}.$$

For $K \geq q$, a homeomorphism $u : \Omega \to \Omega'$ between two open subsets Ω and Ω' of Euclidean $\mathbb{R}^n$ will be called a *K-quasiconformal* or shortly a quasiconformal mapping if the following two conditions are satisfied.

(i) The homeomorphism u is an absolutely continuous function in almost every segment parallel to some of the coordinate axes, and the partial derivatives

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of u exist and are locally L^n integrable functions on Ω. For such a u, we write $u \in \mathrm{ACL}^n$.

(ii) For almost every x in Ω,

$$|Du(x)|^n/K \leq J_u(x) \leq Kl(Du(x))^n,$$

where $l(Du(x)) := \inf\{|Du(x)\zeta| : |\zeta| = 1\}$ and $J_u(x)$ is the Jacobian determinant of u [Rešetnjak 1968].

For a continuous function u, the condition (i) is equivalent to the fact that u belongs to the Sobolev space $W^1_{n,\mathrm{loc}}(\Omega)$.

Let P denote the Poisson kernel, that is,

$$P(x,\eta) = \frac{1-|x|^2}{|x-\eta|^n} \quad \text{for } x \in B^n \text{ and } \eta \in S^{n-1}.$$

Let $f : S^{n-1} \to \mathbb{R}^n$ be a bounded integrable function on the unit sphere S^{n-1}. The solution of the equation $\Delta u = 0$ in the unit ball $\mathbb{B}^n$ satisfying the boundary condition $u|_{S^{n-1}} = f \in L^1(S^{n-1})$ is given by

$$u(x) = P[f](x) = \int_{S^{n-1}} P(x,\eta) f(\eta) d\sigma(\eta) \quad \text{for } |x| < 1.$$

Here $d\sigma$ is the Lebesgue $(n-1)$-dimensional measure of the sphere S^{n-1} satisfying the condition $P[1](x) \equiv 1$. It is well known that if f is continuous in S^{n-1}, then the mapping $u = P[f]$ has a continuous extension $\tilde{u}$ to the boundary, and $\tilde{u} = f$ on S^{n-1}.

We will consider those harmonic mappings, namely, the solutions of the PDE $\Delta u = 0$, that are also quasiconformal.

Martio [1968] was the first to consider harmonic quasiconformal mappings on the complex plane. Recent papers [Kalaj 2004, 2008; Kalaj and Mateljević 2006; Kalaj and Pavlović 2005, 2009; Manojlović 2009; Pavlović 2002] shed much light on this topic.

Proposition 1.1 [Kalaj 2009]. *Let $u : B^n \to \Omega$ for $n \geq 3$ be a twice differentiable quasiconformal mapping of the unit ball onto the bounded domain Ω in $\mathbb{R}^n$ with a C^2 boundary satisfying the differential inequality*

$$|\Delta u| \leq A|Du|^2 + B \quad \textit{for } A, B \geq 0.$$

Then Du (the first derivative of u) is bounded and u is Lipschitz continuous.

Because techniques of complex analysis are not available, the problem in the space $\mathbb{R}^n$ with $n \geq 3$ is much more complicated. For example, any harmonic mapping in a simply connected domain in the plane can be expressed as the sum of an analytic and an antianalytic function. The corresponding representation formula for harmonic mappings in the space is not true. On the other hand, Lewy's theorem and the theorem of Rado, Kneser and Choquet are essentially planar. According to the latter theorem, the harmonic extension (via Poisson integral) of a homeomorphism of the unit circle is always a diffeomorphism of the unit disk. However, in higher dimensions the situation is quite different: Martio [2009] and Melas [1993] constructed a homeomorphism of the unit sphere S^{n-1} for $n \geq 3$ whose harmonic extension fails to be diffeomorphic; see also [Laugesen 1996].

Let $K \in [1, 2^{n-1})$. Our main result, Theorem 3.1, states that the norm of the gradient of any K-quasiconformal harmonic self-mapping u of the unit ball with $u(0) = 0$ is bounded from below by a positive constant c_K that depends only on K. In contrast to the planar case, not all conformal mappings in the space are harmonic; only orthogonal transformations are, while other Möbius transformations are not, at least with respect to the Euclidean metric. However, Möbius transformations will play an important role in this paper. In this regard, Lemma 2.4 is of independent interest. In Section 4 we will give some nontrivial examples of quasiconformal self-mappings of the unit ball, and we will show that our result can be considered as a partial extension of Fefferman's theorem [1974] concerning biholomorphisms between smooth domains in the space.

2. Preliminaries and auxiliary results

Quasiconformal maps are locally well behaved with respect to distance distortion. If $f : \Omega \to \Omega'$ is a K-quasiconformal mapping between domains $\Omega, \Omega' \subset \mathbb{R}^n$, then f is locally Hölder continuous with the exponent $\alpha = K^{1/(1-n)}$, that is,

$$|f(x) - f(y)| \leq M|x - y|^{\alpha}, \tag{2-1}$$

whenever x and y lie in a fixed compact subset E of Ω; see [Vuorinen 1988, Theorem 10.11]. Here M is a constant depending only on K and E. Such an M can in general tend to infinity as the distance from E to the boundary of Ω tends to zero. However, if the boundary of Ω is regular enough, then an inequality similar to (2-1) holds uniformly in Ω [Gehring and Martio 1985; Koskela et al. 2001].

See also [Finn and Serrin 1958] for related results about the class of (K, K') planar quasiconformal mappings, which generalizes the class of standard quasiconformal mappings.

The following lemma is nothing but a slight reformulation of a corresponding lemma in [Tam and Wan 1998]. For the sake of completeness, we give its proof here and show that the constant is sharp.

Lemma 2.1. *If $u \in C^{1,1}$ is a K-quasiconformal mapping defined in a domain $\Omega \subset \mathbb{R}^n$ for $n \geq 3$, then $J_u(x) > 0$ for $x \in \Omega$ if $K < 2^{n-1}$. The constant 2^{n-1} is sharp.*

Proof. Assume that $J_u(a) = 0$ for some $a \in \Omega$. This yields $Du(a) = 0$. Without loss of generality we can assume that $a = 0$ and $u(0) = 0$. Choose $r > 0$ with $r < \operatorname{dist}(0, \partial\Omega)$, and let $E = \overline{B^n(0,r)} := \{x \in \mathbb{R}^n : |x| \le r\}$. Applying (2-1) to the mapping $f = u^{-1}$ defined in $\Omega' = u(\Omega)$, we obtain

$$|f(y)| \le M_E |y|^{K^{1/(1-n)}} \quad \text{for } y \in u(E),$$

where M_E is a constant depending on E. This implies

$$M_E^{-K^{1/(n-1)}} |x|^{K^{1/(n-1)}} \le |u(x)| \quad \text{for } x \in E. \tag{2-2}$$

Now since u is twice differentiable, with $Du(0) = 0$ and $u(0) = 0$, it follows from Taylor's formula that there exists a positive constant N such that

$$|u(x)| \le N|x|^2 \quad \text{for } x \in E. \tag{2-3}$$

Combining (2-2) and (2-3), we have

$$M_E^{-K^{1/(n-1)}}/N \le |x|^{2-K^{1/(n-1)}} \quad \text{for } x \in E.$$

This is only possible if $2 - K^{1/(n-1)} \le 0$.

Thus $K \ge 2^{n-1}$, which is a contradiction.

To prove sharpness, consider the mapping $u(x) = |x|^\alpha x$ with $\alpha \ge 1$. Then

$$J_u(x) = (1+\alpha)|x|^{n\alpha} \tag{2-4}$$

and

$$|Du(x)| = (\alpha+1)|x|^\alpha. \tag{2-5}$$

By (2-4) and (2-5) it follows that

$$\frac{|Du(x)|^n}{J_u(x)} = (\alpha+1)^{n-1}.$$

Therefore, u is a twice differentiable $(1+\alpha)^{n-1}$-quasiconformal self-mapping of the unit ball with $J_u(0) = 0$, meaning that the constant 2^{n-1} is the best possible. □

Lemma 2.2. *Let u be a harmonic mapping of the unit ball onto itself such that $u(0) = 0$. Then there exists a positive constant C_n such that*

$$\frac{1-|x|^2}{1-|u(x)|^2} \le C_n \quad \textit{for } x \in B^n. \tag{2-6}$$

Proof. Let S^+ denote the northern hemisphere and let S^- be the southern hemisphere. Let $U = P[\chi_{S^+}] - P[\chi_{S^-}]$ be the Poisson integral of the function χ_S that equals 1 on S^+ and -1 on S^-. Then by the Schwarz lemma [Axler et al. 1992],

$$\langle u(x), u(x_0)/|u(x_0)| \rangle \le \big|U(|x|N)\big|$$

for a fixed x_0, where N is the north pole.

It follows that $|u(x_0)|^2 \le \left|U(|x_0|N)\right|^2$. Thus

$$\frac{1-|x|^2}{1-|u(x)|^2} \le \frac{1-|x|^2}{1-U(|x|N)^2} =: g(r) \quad \text{for } r = |x|.$$

We will need Hopf's boundary point lemma:

Lemma 2.3 [Hopf 1952; Protter and Weinberger 1967]. *Let v satisfy $\Delta v \ge 0$ in an open set $D \subset \mathbb{R}^n$ and suppose $v \le M$ in D and $v(P) = M$ for some $P \in \partial D$. Assume that P lies on the boundary of a ball*

$$B^n(a, r) := \{x : |x-a| < r\} \subset D.$$

If v is continuous on $D \cup P$ and if the outward directional derivative $\partial v/\partial n$ exists at P, then $v \equiv M$ or

$$\partial v(P)/\partial n > 0.$$

Applying this lemma to the function $U(x)$ and taking $h(r) = U(rN)$, we obtain

$$h'(1) = \frac{\partial U(N)}{\partial n} > 0.$$

Thus

$$C_n := \sup_{|x|\le 1} \left\{ \frac{1-|x|^2}{1-U(|x|N)^2} \right\} < \infty,$$

and the proof of Lemma 2.2 is complete. □

Following the book of Ahlfors [1981], for $a, x \in B^n$ we define

$$[x, a]^2 = 1 + |x|^2|a|^2 - 2\langle x, a\rangle$$

and the inversion x^* of $x \ne 0$ by $x^* = x/|x|^2$. Since

$$\begin{aligned}[x, a]^2 &= |x|^2|x^* - a|^2 = |a|^2|x - a^*|^2 \\ &= \left||x|a - |x|x^*\right|^2 = \left||a|x - |a|a^*\right|^2 = \left||x|a - |x|x^*\right| \cdot \left||a|x - |a|a^*\right|,\end{aligned}$$

we have

$$[x, a]^2 \ge (1-|a|)(1-|x|) \quad \text{and} \quad [x, a]^2 \ge (1-|x|)^2. \tag{2-7}$$

Assume that p is a conformal mapping of the unit ball onto itself. Then it is well known that p is a Möbius transformation of the unit ball onto itself. Under the normalization $p(0) = -a \ne 0$ (or $p(a) = 0$), the mapping p is given (up to some orthogonal transformation of the unit ball) by

$$p(x) = \frac{(1-|a|^2)(x-a) - |x-a|^2 a}{[x, a]^2}. \tag{2-8}$$

Lemma 2.4. *If p is a Möbius transformation of the unit ball onto itself, with $p(0)=a$, then for all $k,l\in\mathbb{N}_0$ with $k>l$, there exists a constant $C_{k,l}$ such that*

$$C_{k,l}\geq k\cdot(k-1)\cdots(l+1),$$

$$\frac{k!\,|a|^{k-1}(1-|a|^2)}{[x,a]^{k+1}}\leq|p^{(k)}(x)|\leq\frac{C_{k,0}|a|^{k-1}(1-|a|^2)}{[x,a]^{k+1}}\quad\textit{for } x\in B^n,\tag{2-9}$$

$$\frac{|p^{(k)}(x)|}{|p^{(l)}(x)|}\leq C_{k,l}\frac{1}{(1-|x|)^{k-l}}\quad\textit{for } x\in B^n\quad\textit{and}\quad l>0,\tag{2-10}$$

$$|p^{(k)}(x)|\leq\frac{C_{k,0}}{(1-|p(0)|^2)^{(k-1)/2}}\left(\frac{1-|p(x)|^2}{1-|x|^2}\right)^{(k+1)/2}\textit{for } x\in B^n.\tag{2-11}$$

Proof. It follows from (2-8) that

$$p'(x)=\frac{1-|a|^2}{[x,a]^2}\Delta(x,a),$$

where $\Delta(x,a)=(I-2Q(a))(I-2Q(x-a^*))$, and $Q(y)$ is the matrix whose elements have the form $Q(y)_{i,j}=y_iy_j/|y|^2$. For every $y\in B^n$, we have $K(y):=I-2Q(y)\in O_n$, where O_n is the set of all orthogonal matrices. Thus $\Delta(x,a)$ is an orthogonal matrix as well, and consequently $|\Delta(x,a)|=1$. This means that

$$|p'(x)|=\frac{1-|a|^2}{[x,a]^2}.$$

According to (2-7),

$$|p'(x)|\leq\frac{2}{1-|x|}.\tag{2-12}$$

If we put

$$A=\frac{1-|a|^2}{[x,a]^2}\quad\text{and}\quad B=(I-2Q(a))(I-2Q(x-a^*)),$$

then we have $p'=AB$. Therefore, for the $(k+1)$-st derivative of p, we have

$$p^{(k+1)}(x)=\sum_{j=0}^{k}\binom{k}{j}A^{(j)}B^{(k-j)},\tag{2-13}$$

and $p^{(k+1)}$ can be treated as a k-linear form between the k-fold product of $\mathbb{R}^n$ and and $M_{n\times n}$. We will use the notation

$$Q(y)=\frac{y\otimes y}{|y|^2},\tag{2-14}$$

where $\otimes$ denotes the tensor product of vectors.

Let us prove that for $k \in \mathbb{N}_0$ there exists a $(2k+2)$-linear form from the $(k+2)$-fold product of $\mathbb{R}^n$ to $M_{n\times n}$ such that

$$Q^{(k)}(y)(h_1, h_2, \dots h_k) = \frac{1}{|y|^{2k+2}} P^k(y, \dots, y, h_1, \dots, h_k). \tag{2-15}$$

We proceed by induction on k. It is evident from (2-14) that (2-15) is true for $k=0$.

Assume that (2-15) is true for some k, and prove it for $k+1$. By (2-15), it follows that

$$\begin{aligned} Q^{(k+1)}(y)(h_1, h_2, \dots h_k, h_{k+1}) &= \frac{1}{|y|^{2k+2}} \sum_{j=1}^{k+2} P^k(y, \dots, y, \overset{j\downarrow}{h_{k+1}}, y, \dots, y, h_1, \dots, h_k) \\ &\quad - (k+2)\frac{\langle y, h_{k+1}\rangle}{|y|^{2k+4}} P^k(y, \dots, y, h_1, \dots, h_k), \end{aligned} \tag{2-16}$$

where the j pointing to h_{k+1} denotes that h_{k+1} is in the j-th position. Thus

$$Q^{(k+1)}(y)(h_1, h_2, \dots, h_k, h_{k+1}) = \frac{P^{k+1}(y, \dots, y, h_1, \dots, h_k, h_{k+1})}{|y|^{2(k+1)+2}}, \tag{2-17}$$

where

$$\begin{aligned} P^{k+1}(e_1, \dots, e_{k+3}, f_1, \dots f_{k+1}) &= \sum_{j=1}^{k+2} \langle e_{k+3}, e_j\rangle P^k(e_1, \dots, \overset{j\downarrow}{f_{k+1}}, \dots, e_{k+2}, f_1, \dots, f_k) \\ &\quad - 2(k+1)\langle e_{k+3}, f_{k+2}\rangle P^k(e_1, \dots, e_{k+2}, f_1, \dots, f_k). \end{aligned}$$

We first prove the left side of inequality (2-9).

From (2-17) and the induction hypothesis, for $y \neq 0$ we obtain

$$Q^{(k+1)}(y)\left(\frac{y}{|y|}, \dots, \frac{y}{|y|}\right) = 0. \tag{2-18}$$

Thus

$$B^{(k)}(x)\left(\frac{x-a^*}{|x-a^*|}, \dots, \frac{x-a^*}{|x-a^*|}\right) = 0. \tag{2-19}$$

Let us prove the left side of (2-9). First, according to (2-13) and (2-19), we obtain

$$\begin{aligned} |p^{(k+1)}(x)| &= \sup_{|h_1|=\cdots=|h_k|=1} |p^{(k+1)}(x)(h_1, \dots, h_k)| \\ &\geq \left| A^{(k)}\left(\frac{x-a^*}{|x-a^*|}, \dots, \frac{x-a^*}{|x-a^*|}\right)\right|. \end{aligned} \tag{2-20}$$

Since P^k is a $(2k+2)$-linear form,

$$|P^k(y,\dots,y,h_1,\dots,h_k)| \le |P^k||y|^{k+2}\prod_{j=1}^{k}|h_j|.$$

Thus $|Q^k(y)| \le |P^k|/|y|^k$, whence we have

$$|Q^k(x-a^*)| \le \frac{|P^k|}{|x-a^*|^k} = \frac{|a|^k|P^k|}{[x,a]^k}.$$

Further, observe that $B(x) = K(a)(I-2Q(x-a^*))$ for $K(a) \in O_n$. Hence

$$B^{(k)}(x) = -2K(a)Q^{(k)}(x-a^*),$$

and using the identity

$$\frac{1-|p(x)|^2}{1-|x|^2} = \frac{1-|a|^2}{[x,a]^2} = \frac{1-|a|^2}{|a|^2|x-a^*|^2},$$

we obtain

$$|B^k(x)| \le \frac{2|a|^k|P^k|}{[x,a]^k} < \frac{2|a|^k|P^k|(1-|p|^2)^{k/2}}{(1-|a|^2)^{k/2}(1-|x|^2)^{k/2}}. \tag{2-21}$$

To estimate the derivatives of $A(x) = (1-|a|^2)/[x,a]^2$, define

$$H(y) = \frac{1}{|y|^2} = \frac{|a|^2}{1-|a|^2}A(x) \quad \text{for } y = x-a^*.$$

Then

$$H'(y)h_1 = -2\frac{\langle y, h_1\rangle}{|y|^4}.$$

Similarly, it can be proved that for every $k \ge 1$ there exists an $\mathbb{R}$-valued $(2k+2)$-linear form G^{k+1} such that

$$H^{(k+1)}(y)(h_1,h_2,\dots h_{k+1}) = \frac{1}{|y|^{2k+4}}G^{k+1}(y,\dots,y,h_1,\dots,h_{k+1}),$$

where

$$\begin{aligned}G^{k+1}&(e_1,\dots,e_{k+1},f_1,\dots f_{k+1})\\ &= \sum_{j=1}^{k}\langle e_{k+1},e_j\rangle G^k(e_1,\dots,\overset{j\downarrow}{f_{k+1}},\dots,e_k,f_1,\dots,f_k)\\ &\qquad -2(k+1)\langle e_{k+1},f_{k+1}\rangle G^k(e_1,\dots,e_k,f_1,\dots,f_k).\end{aligned}$$

Therefore

$$|H^k(y)| \le |G^k|/|y|^{k+2}. \tag{2-22}$$

On the other hand, by the identity $k \cdot (k+1)! - 2(k+1)(k+1)! = -(k+2)!$ and the induction hypothesis, we obtain

$$G^k\left(y, \ldots, y, \frac{y}{|y|}, \ldots, \frac{y}{|y|}\right) = \pm\frac{(k+1)!}{|y|^{k+2}} \quad \text{for } k \in \mathbb{N}.$$

Thus

$$\frac{k!}{|y|^{k+2}} = \left|H^k(y)\left(\frac{y}{|y|}, \ldots, \frac{y}{|y|}\right)\right| \le \left|H^k(y)\right|.$$

In view of the fact

$$\frac{1-|p(x)|^2}{1-|x|^2} = \frac{1-|a|^2}{[x, a]^2},$$

we have

$$\text{(2-23)} \qquad \frac{k!|a|^k(1-|a|^2)}{[x, a]^{k+2}} \le \left|A^k(x)\right| \le \frac{2|G^k||a|^k(1-|p|^2)^{1+k/2}}{(1-|a|^2)^{(k-1)/2}(1-|x|^2)^{1+k/2}}.$$

Now the left side of (2-9) follows from (2-18), (2-13), (2-20) and (2-23).

Combining (2-13), (2-21), (2-22) and (2-7) for $k \ge 1$, we obtain

$$|p^{(k)}(x)| \le C_{k,0}\frac{|a|^k(1-|a|^2)}{[x, a]^{k+1}} < 2C_{k,0}\frac{1}{(1-|x|)^k},$$

where

$$C_{k,0} = |G^{k-1}| + 2\sum_{j=1}^{k-1}\binom{k-1}{j}|P^j||G^{(k-1-j)}|.$$

This proves (2-9). Inequalities (2-10) and (2-11) immediately follow from (2-9) and (2-23), and the proof is complete. □

Remark 2.5. For fixed $k, l \in \mathbb{N}_0$, we denote by $C^*_{k,l}$ the infimum taken over all constants $C_{k,l}$ satisfying the previous lemma. In the complex plane $\mathbb{C}$ there holds $C^*_{k,l} = k \cdot (k-1) \cdots (l+1)$, and therefore (2-9) reduces to the equality. According to (2-12), this occurs in the higher dimensions as well for $k = 1$. We believe that $C^*_{k,l} = k \cdot (k-1) \cdots (l+1)$ for arbitrary n, k and l.

3. The main result

Theorem 3.1. *Let $K < 2^{n-1}$ and let u be a K-quasiconformal harmonic mapping of the unit ball onto itself satisfying the normalization $u(0) = 0$. Then there exists a positive constant $c_K > 0$ such that*

$$|Du(x)| \ge c_K \quad \textit{for each } x \in B^n.$$

The first step of the proof is similar to that in [Tam and Wan 1998]. However here the problem is more complicated, because Möbius transformations are harmonic with respect to the hyperbolic metric, but not with respect to the Euclidean metric.

Proof. We will prove by contradiction that the function $|Du|$ is uniformly bounded below away from 0. Suppose that there exists a sequence $\{x_i\}$ in B^n such that $Du(x_i) \to 0$ as $i \to \infty$. We will use Proposition 1.1 together with the following lemma.

Lemma 3.2. *Let u be a harmonic Lipschitz mapping of the unit ball onto itself. Let $\{x_i\}$ be a sequence in B^n. For arbitrary $i \in \mathbb{N}$, let p_i and q_i be two Möbius transformations of B^n such that $q_i(0) = x_i$ and $p_i(u(x_i)) = 0$. Take $u_i = p_i \circ u \circ q_i$. Then*

$$|D^{(k)} u_i(x)| \le c_n^k \frac{1}{(1-|x|^2)^k} \quad \textit{for } k \in \mathbb{N},$$

where c_n^k is independent of x and i.

Proof. To simplify calculations in this proof, sometimes we will omit the arguments of functions.

Since

$$|p_i'(u)| = \frac{1-|p_i(u)|^2}{1-|u|^2} \tag{3-1}$$

and

$$|q_i'(x)| = \frac{1-|q_i(x)|^2}{1-|x|^2}, \tag{3-2}$$

according to (2-6) it follows that

$$\begin{aligned} |Du_i| &\le |p_i'||Du||q_i'| \\ &\le \frac{1-|p_i(u(q_i(x)))|^2}{1-|u(q_i(x))|^2} \frac{1-|q_i(x)|^2}{1-|x|^2} |Du| \\ &\le C_n |Du|_\infty \frac{1-|p_i(u(q_i(x)))|^2}{1-|x|^2}. \end{aligned}$$

Thus

$$|Du_i| \le C_n |Du|_\infty \frac{1}{1-|x|^2}.$$

For $m \in \mathbb{N}$, we make use of the Cauchy estimates [Axler et al. 1992, pp. 33–35]:

$$|D^m(u)(q_i(x))| \le A_m \frac{|Du|_\infty}{(1-|q_i(x)|)^{m-1}}. \tag{3-3}$$

To estimate the norm of $D^k u_i$ for $k > 1$, we use induction. Obviously, it is complicated to compute $D^k u_i$ for large k. However, it is clear that it can be written as a sum of products:

$$D^k u_i = \sum \Big(p_i^{(\tau)} \prod D^{j_t} u q_i^{(s_{t1})} \cdot \dots \cdot q_i^{(s_{tl_t})} \Big), \tag{3-4}$$

where $\prod$ and $\sum$ denote the corresponding finite product and sum of linear operators. The indices t and τ range at most from 1 to k; the indices j_t, s_{t1}, ..., s_{tl_t} satisfy similar bounds.

Because of (3-3) we have

$$|p_i^{(\tau)}| \prod |D^{j_t} u| |q_i^{(s_{t1})}| \cdots |q_i^{(s_{tl_t})}| \\ \le \mathrm{const} |p_i^{(\tau)}| \prod \frac{|Du|_\infty}{(1-|q_i(x)|)^{j_t-1}} |q_i^{(s_{t1})}| \cdots |q_i^{(s_{tl_t})}|. \tag{3-5}$$

Therefore, it is enough to prove that

$$|p_i^{(\tau)}| \prod \frac{|Du|_\infty}{(1-|q_i(x)|)^{j_t-1}} |q_i^{(s_{t1})}| \dots |q_i^{(s_{tl_t})}| \le \mathrm{const} \frac{1}{(1-|x|)^k}. \tag{3-6}$$

For $k = 1$, inequality (3-6) is satisfied. Assume that (3-6) is true for some k, and therefore (3-5) is true as well. We will prove (3-6) for $k+1$.

Since $D^{k+1}u_i = DD^k u_i$, the first factor in the corresponding formula (3-4) for $D^{k+1}u_i$, instead of $p_i^{(\tau)} D^{j_t} u q_i^{(s_{t1})} \cdots q_i^{(s_{tl_t})}$, contains the term

$$\big(p_i^{(\tau+1)} Du q_i' D^{j_t} u q_i^{(s_{t1})} \cdots q_i^{(s_{tl_t})} + p_i^{(\tau)} D^{j_t+1} u q_i' \cdot q_i^{(s_{t1})} \cdots q_i^{(s_{tl_t})} \\ + p_i^{(\tau)} D^{j_t} u q_i^{(s_{t1+1})} \cdots q_i^{(s_{tl_t})} + \dots + p_i^{(\tau)} D^{j_t} u q_i^{(s_{t1})} \dots q_i^{(s_{tl_t+1})} \big),$$

and consequently the corresponding formulas (3-6) and (3-5), instead of

$$|p_i^{(\tau)}| \frac{|Du|_\infty}{(1-|q_i(x)|)^{j_t-1}} |q_i^{(s_{t1})}| \dots |q_i^{(s_{tl_t})}|,$$

contain

$$\Big(|p_i^{(\tau+1)}| |Du| |q_i'| |D^{j_t} u| |q_i^{(s_{t1})}| \cdots |q_i^{(s_{tl_t})}| \\ + |p_i^{(\tau)}| \frac{|Du|_\infty}{(1-|q_i(x)|)^{j_t}} |q_i'| \cdot |q_i^{(s_{t1})}| \cdots |q_i^{(s_{tl_t})}| \\ + |p_i^{(\tau)}| \frac{|Du|_\infty}{(1-|q_i(x)|)^{j_t-1}} |q_i^{(s_{t1+1})}| \dots |q_i^{(s_{tl_t})}| + \cdots \\ + |p_i^{(\tau)}| \frac{|Du|_\infty}{(1-|q_i(x)|)^{j_t-1}} |q_i^{(s_{t1})}| \cdots |q_i^{(s_{tl_t+1})}| \Big).$$

The other factors in (3-4) can be treated similarly.

Applying (3-2), we get

$$
\begin{aligned}
\frac{|Du|_\infty}{(1-|q_i(x)|)^{j_t}}|q_i'| &= \frac{|Du|_\infty}{(1-|q_i(x)|)^{j_t}}\frac{1-|q_i(x)|^2}{1-|x|^2} \\
&\le \frac{|Du|_\infty}{(1-|q_i(x)|)^{j_t-1}}\frac{2}{1-|x|}.
\end{aligned} \tag{3-7}
$$

Next, by applying (2-6) and (2-10), we obtain

$$
\left|p_i^{(\tau+1)}Duq_i'\right| \le \frac{\mathrm{const}|p_i^{(\tau)}|}{1-|u(q_i(x))|}\frac{1-|q_i(x)|}{1-|x|} \le \mathrm{const}\frac{|p_i^{(\tau)}|}{1-|x|}. \tag{3-8}
$$

On the other hand, according to (2-10) we have

$$
|q_i^{(j+1)}| \le \mathrm{const}\frac{|q_i^{(j)}|}{1-|x|}. \tag{3-9}
$$

By induction, (3-6) is true for k. The last fact and the estimates (3-7), (3-8) and (3-9) imply that (3-6) is also true for $k+1$. Consequently,

$$
|D^{(k)}u_i(x)| \le c_n^k\frac{1}{(1-|x|^2)^k} \quad \text{for } k\in\mathbb{N}. \tag{3-10}
$$

□

We are now ready to finish the proof of Theorem 3.1. According to the notations of the previous lemma, $u_i = p_i\circ u\circ q_i$ is a C^∞ K-quasiconformal mapping of the unit ball onto itself, satisfying the condition $u_i(0)=0$. By (2-6) we have

$$
|Du_i(0)| = \frac{1-|x_i|^2}{1-|u(x_i)|^2}|Du(x_i)| \to 0 \quad \text{as } i\to\infty. \tag{3-11}
$$

For example, by [Fehlmann and Vuorinen 1988], a subsequence of u_i, also denoted u_i, converges uniformly to a K-quasiconformal map u_0 on the closed unit ball B^n. According to Lemma 3.2 together with Proposition 1.1, u_0 is in $C^\infty(B^n;B^n)$ with $u_0(0)=0$. The relation (3-11) implies $D(u_0)(0)=0$. This obviously contradicts the statement of Lemma 2.1. Hence, the proof of Theorem 3.1 is complete. □

Remark 3.3. Let us estimate $Du_i = p_i' Duq_i'$ more precisely. From

$$
q_i(x) = \frac{(1-|x_i|^2)(x+x_i)+|x+x_i|^2x_i}{[x,-x_i]^2},
$$

according to (3-1), it follows that

$$
|q_i'(x)| = \frac{1-|x_i|^2}{|x_i|^2|x+x_i^*|^2}
$$

and

$$
1-|q_i(x)|^2 = |q_i'(x)|(1-|x|^2).
$$

Similarly, since

$$p_i(u) = \frac{(1-|u(x_i)|^2)(u-u(x_i)) + |u-u(x_i)|^2 u(x_i)}{[u, u(x_i)]^2},$$

it follows that

$$|p_i'(u)| = \frac{1-|u(x_i)|^2}{|u(x_i)|^2 |u(q_i(x)) - u(x_i)^*|^2}.$$

Thus

$$|Du_i| \le \frac{1-|u(x_i)|^2}{|u(x_i)|^2 |u(q_i(x)) - u(x_i)^*|^2} |Du|_\infty \frac{1-|x_i|^2}{|x_i|^2 |x + x_i^*|^2}.$$

From

$$|x_i|^2 |x + x_i^*|^2 \ge (1-|x_i|)^2,$$
$$1 - |u(x_i)| \le |Du|_\infty (1-|x_i|),$$
$$|u(x_i)|^2 |u(q_i(x)) - u(x_i)^*|^2 \ge (1-|u(x_i)|)^2,$$

and (2-6) we obtain

$$|Du_i| \le 4 \min\left\{ \frac{|Du|_\infty^2}{[u(q_i(x)), u(x_i)]^2}, \frac{C_n |Du|_\infty}{[x, -x_i]^2} \right\}. \tag{3-12}$$

Assume that $t = \lim_{i\to\infty} x_i \in S^{n-1}$. It follows from (3-12) that

$$|Du_0| = \lim_{i\to\infty} |Du_i|$$

is uniformly bounded in $B^n \setminus B^n(-t, \varepsilon)$ for $\varepsilon > 0$. Is $|Du_0|$ bounded uniformly in B^n?

Theorem 3.4. *Let $K < 2^{n-1}$ and assume that u is a K-quasiconformal harmonic mapping of the unit ball onto B^2 itself satisfying the normalization $u(0) = 0$. Then u is a bi-Lipschitz mapping.*

Proof. According to Proposition 1.1 and Theorem 3.1, there exists a constant $c \ge 1$ such that

$$c^{-1} \le |Du(x)| \le c \quad \text{for } x \in B^n. \tag{3-13}$$

By using (3-13) and the fact that u is quasiconformal, we obtain

$$|D(u^{-1})(u(x))| = \frac{1}{\inf_h |Du(x)h|} = \frac{1}{l(Du(x))} \le \frac{K^{2/n}}{Du(x)} \le cK^{2/n}.$$

Therefore

$$|D(u^{-1})| \le c_1. \tag{3-14}$$

From (3-13) and (3-14) it follows that u is bi-Lipschitz. □

4. Examples of quasiconformal harmonic mappings

We now give examples of nontrivial harmonic quasiconformal self-mappings of the unit ball.

4.1. ***Holomorphic self-mappings of the unit ball.*** Let $B^{2n} \subset \mathbb{C}^n$ and let f be a holomorphic automorphism of the unit ball B^{2n} onto itself. Then f is a quasiconformal harmonic mapping. To prove this fact, observe that $\bar{\partial} f = 0$ implies $\partial\bar{\partial} f = 0$. Also f has a holomorphic extension up to the boundary. This means that it is bi-Lipschitz. Therefore f is a quasiconformal harmonic mapping. It is interesting to note that the composition of a harmonic and holomorphic mapping is itself harmonic, because $\partial f \circ h = f_h \partial h + f_{\bar{h}} \partial \bar{h}$, and therefore

$$\bar{\partial}\partial f \circ h = f_{hh} \bar{\partial} h \partial h + f_{\bar{h}h} \bar{\partial}\bar{h} \partial h + f_{h\bar{h}} \bar{\partial} h \partial \bar{h} + f_{\bar{h}\bar{h}} \partial \bar{h} \bar{\partial} h.$$

According to Fefferman's theorem [1974], every biholomorphism between two smooth domains has a C^∞ extension to the boundary, which means that these mappings are bi-Lipschitz. Hence, the class of biholomorphic mappings between smooth domains is contained in the class of harmonic quasiconformal mappings. Thus our results can be considered as partial extensions of Fefferman's theorem.

4.2. ***Perturbation of the identity.*** Let us show that small smooth perturbations of the boundary value of a holomorphic automorphism $\phi \in C^2(\overline{B^{2n}})$ of the unit ball onto itself induce harmonic quasiconformal mappings.

Since the composition of a harmonic mapping with a holomorphic automorphism is itself a harmonic mapping, it is enough to perturb the identity map, and after that take the corresponding composition.

Define $I_\delta(x) = x + \delta(x)$, where $x \in B^n$ and $\delta(x) \in B^n$, and take $\phi_\delta = I_\delta / |J_\delta|$, where $|J_\delta|^2 = 1 + 2\langle x, \delta(x)\rangle + |\delta(x)|^2$. Thus

$$|I_\delta(x)| < 1 \quad \text{for } x \in B^n \quad \text{and} \quad |I_\delta(x)| = 1 \quad \text{for } x \in S^{n-1}.$$

We also have

$$|J_\delta(x)|^2 \geq (1 - |\delta(x)|)^2 \quad \text{for } x \in B^n. \tag{4-1}$$

Here $\delta(x)$ is a twice differentiable mapping satisfying

$$|\delta'(x)| < 1 \quad \text{for } x \in S^{n-1}.$$

This condition guarantees the injectivity of $\phi_\delta(x)$ in S^{n-1}. To continue, we use the following result of Gilbarg and Hörmander [1980, Theorem 6.1 and Lemma 2.1].

Proposition 4.1. *The Dirichlet problem $\Delta u = f$ in Ω for $u = u_0$ on $\partial\Omega \in C^1$ has a unique solution $u \in C^{1,\alpha}$ for every $f \in C^{0,\alpha}$ and $u_0 \in C^{1,\alpha}$, and we have*

$$\|u\|_{1,\alpha} \leq C(\|u_0\|_{1,\alpha,\partial\Omega} + \|f\|_{0,\alpha}), \tag{4-2}$$

where C is a constant.

To guarantee the injectivity of the harmonic extension $\Phi_\delta(x) = P[\phi_\delta](x)$ of $\phi_\delta(x)$ in the unit ball, we estimate $|D\phi_\delta(x) - \mathrm{Id}|$ and $|D^2\phi_\delta(x)|$, and use (4-2) to conclude that

$$|DP[\phi_\delta](x) - P[\mathrm{Id}]| \le C\big(|D\phi_\delta(x) - \mathrm{Id}| + |D^2\phi_\delta(x)|\big). \tag{4-3}$$

First,

$$D\phi_\delta(x)h = \frac{h+\delta'(x)h}{|J_\delta|} - \frac{I_\delta(x)}{|J_\delta(x)|^3}\big(\langle\delta(x), h\rangle + \langle\delta'(x)h, x+\delta(x)\rangle\big).$$

Therefore, using (4-1) we infer that

$$|D\phi_\delta(x)h - h| \le \left|\frac{2|\delta| + 2|\delta'| + |\delta||\delta'|}{1-|\delta(x)|^2}h\right|,$$

that is,

$$|D\phi_\delta(x) - \mathrm{Id}| \le \frac{2|\delta| + 2|\delta'| + |\delta||\delta'|}{1-|\delta(x)|^2}.$$

Next we find

$$\begin{aligned}
&D^2\phi_\delta(x)(h,k)\\
&= \frac{\delta''(x)(h,k)}{|J_\delta|}\\
&\quad - \frac{I_\delta(x)}{|J_\delta(x)|^3}\big(\langle\delta'(x)k, h\rangle + \langle\delta''(x)(h,k), x+\delta(x)\rangle + \langle\delta'(x)(h), k+\delta'(x)k\rangle\big)\\
&\quad - \frac{k+\delta'(x)k}{|J_\delta(x)|^3}\big(\langle\delta(x), h\rangle + \langle\delta'(x)h, x+\delta(x)\rangle\big)\\
&\quad + 3\frac{I_\delta(x)}{|J_\delta(x)|^5}\big(\langle\delta(x), h\rangle + \langle\delta'(x)h, x+\delta(x)\rangle\big)\big(\langle\delta(x), k\rangle + \langle\delta'(x)k, x+\delta(x)\rangle\big).
\end{aligned}$$

Thus

$$\begin{aligned}
&|D^2\phi_\delta(x)|\\
&\qquad \le \frac{3(|\delta|+|\delta'|+|\delta||\delta'|)^2+(1+|\delta'|)(|\delta|+|\delta'|+|\delta||\delta'|)+2|\delta'|^2+|\delta'|+|\delta''|(2+|\delta|)}{(1-|\delta(x)|^2)^2}.
\end{aligned}$$

Choosing δ such that

$$|D^2\phi_\delta(x)| < \frac{1}{2C} \quad \text{and} \quad |D\phi_\delta(x) - \mathrm{Id}| < \frac{1}{2C},$$

according to (4-3) we obtain

$$|D\Phi_\delta(x) - \mathrm{Id}| < \frac{C}{2C} + \frac{C}{2C} = 1 \quad \text{for } x \in B^n.$$

Thus $|\Phi_\delta(x)-\Phi_\delta(y)+y-x|<|x-y|$, and therefore $0<|\Phi_\delta(x)-\Phi_\delta(y)|$. This implies that Φ_δ is injective. Hence, Φ_δ is a quasiconformal harmonic diffeomorphism of the unit ball onto itself.

In particular, let $I_\varepsilon(x)=(x_1+\varepsilon, x_2, x_3)$ and take $j_\varepsilon=(1+2\varepsilon x_1+\varepsilon^2)^{1/2}$. Define

$$\phi_\varepsilon(x)=I_\varepsilon(x)/j_\varepsilon.$$

Now take $\Phi_\varepsilon=P[\phi_\varepsilon]$. Then for sufficiently small ε, the extension Φ_ε is a diffeomorphism of the unit ball onto itself having a diffeomorphic extension to the boundary. This means that Φ_ε is quasiconformal harmonic mapping.

Direct calculations yield

$$\begin{aligned}\frac{1}{C}\big|D\Phi_\varepsilon(x)-\mathrm{Id}\big| &= \frac{1}{C}\big|P[\phi_\varepsilon-\mathrm{Id}](x)\big|\\
&\le \sup_{|x|=1}\big(|D\phi_\varepsilon(x)-\mathrm{Id}|+|D^2\phi_\varepsilon(x)|\big)\\
&< \sup_{|x|=1}\Bigg\{\Bigg(\bigg(-1+\frac{\varepsilon x_1+1}{j_\varepsilon^3}\bigg)^2+2\bigg(-1+\frac{1}{j_\varepsilon}\bigg)^2+\frac{\varepsilon^2x_2^2}{j_\varepsilon^6}-\frac{\varepsilon x_3}{j_\varepsilon^3}\Bigg)^{1/2}\\
&\qquad+\Bigg(2\bigg(-1+\frac{1}{j_\varepsilon}\bigg)^2+\bigg(-1-\frac{\varepsilon(\varepsilon+x_1)}{j_\varepsilon^3}+\frac{1}{j_\varepsilon}\bigg)^2+\frac{\varepsilon^2x_2^2}{j_\varepsilon^6}+\frac{\varepsilon^2x_3^2}{j_\varepsilon^6}\Bigg)^{1/2}\Bigg\}.\end{aligned}$$

Therefore

$$\lim_{\varepsilon\to 0}|D\Phi_\varepsilon(x)-\mathrm{Id}|=0$$

uniformly on B^n. It follows that there exists $\varepsilon>0$ such that

$$\sup_{|x|\le 1}|D\Phi_\varepsilon(x)-\mathrm{Id}|<1.$$

The inequality $|\Phi_\varepsilon(x)-\Phi_\varepsilon(y)+y-x|<|x-y|$ yields $0<|\Phi_\varepsilon(x)-\Phi_\varepsilon(y)|$. This implies that Φ_ε is injective.

4.3. ***A question.*** Lewy's theorem fails in higher dimensions, as shown by Wood [1991], who constructed the harmonic homeomorphism

$$u(x,y,z)=(x^3-3xz^2+yz,\ y-3xz,\ z),$$

which is not a diffeomorphism. The Jacobian of u is $J_u(x,y,z)=3x^2-3z^2$. This means that u is neither a diffeomorphism nor a quasiconformal mapping. Do there exist quasiconformal harmonic mappings which are not diffeomorphisms? If they exist, then of course $K\ge 2^{n-1}$, as shown above (and in [Tam and Wan 1998]).

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