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REPRESENTATIONS OF THE TWO-FOLD CENTRAL EXTENSION OF $\mathrm{SL}_2(\mathbb{Q}_2)$

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We define a notion of pseudospherical type for smooth representations of the nontrivial two fold central extension of $\mathrm{SL}_2(\mathbb{Q}_2)$. We describe completely the irreducible representations that contain the pseudospherical type. We relate our results to Kohnen's plus and minus spaces of classical modular forms of half integral weight.

1. Introduction

Let $\mathbb{Q}$ be the field of rational numbers. For every place v of $\mathbb{Q}$, let $\mathbb{Q}_v$ denote the corresponding local field. Then $\mathbb{Q}_v = \mathbb{R}$ or $\mathbb{Q}_p$ for a prime p. The group $\mathrm{SL}_2(\mathbb{Q}_v)$ has a nontrivial two-fold central extension

$$1 \to \mu_2 \to G(\mathbb{Q}_v) \to \mathrm{SL}_2(\mathbb{Q}_v) \to 1, \tag{1}$$

where $\mu_2 = \{\pm 1\}$. Recall that an irreducible representation of $G(\mathbb{Q}_v)$ is called genuine if the central subgroup μ_2 acts faithfully on it. Gelbart's book [1976] contains a basic theory of genuine representations of $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$ for $p \neq 2$. Our intent is to develop a theory in the case of $G(\mathbb{Q}_2)$. The main difference between $G(\mathbb{Q}_2)$ and $G(\mathbb{Q}_p)$ for $p \neq 2$ lies in the fact that the central extension splits over $\mathrm{SL}_2(\mathbb{Z}_p)$ when $p \neq 2$. In particular, we have a subgroup $K_p \subseteq G(\mathbb{Q}_p)$ isomorphic to $\mathrm{SL}_2(\mathbb{Z}_p)$ under the natural projection from $G(\mathbb{Q}_p)$ to $\mathrm{SL}_2(\mathbb{Q}_p)$ for every $p \neq 2$. A genuine representation π of $G(\mathbb{Q}_p)$ is called *unramified* if it contains a nonzero K_p-fixed vector.

Assume now that $p = 2$. Let K denote the full inverse image of $\mathrm{SL}_2(\mathbb{Z}_2)$ in $G(\mathbb{Q}_2)$. In this case the central extension splits over a smaller subgroup. More precisely, we have a subgroup $K_1(4) \subseteq K$ isomorphic to the subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$ given by the congruence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod 4$$

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In this paper we completely describe genuine irreducible representations of $G(\mathbb{Q}_2)$ containing nonzero $K_1(4)$-fixed vectors. More precisely, in Section 3, we describe a Hecke algebra $H(\gamma)$ that captures the structure of all representations generated by $K_1(4)$-fixed vectors and with a fixed central character γ. In Section 4, we show that $H(\gamma)$ is isomorphic to the Iwahori–Matsumoto Hecke algebra for $\mathrm{PGL}_2(\mathbb{Q}_2)$. In this way we get a correspondence between (some) representations of $G(\mathbb{Q}_2)$ and representations of $\mathrm{PGL}_2(\mathbb{Q}_2)$. We call this correspondence a local Shimura correspondence.

In Section 5, we show that the compact group K has exactly two irreducible genuine representations, with the fixed central character γ, containing nonzero $K_1(4)$-fixed vectors. These representations are denoted by $V(2)$ and $V(-1)$ and have dimensions 2 and 4, respectively. We show that a representation π of $G(\mathbb{Q}_2)$ has $V(2)$ as a K-type if and only if it corresponds to an unramified representation of $\mathrm{PGL}_2(\mathbb{Q}_2)$, by the local Shimura correspondence. Thus, it is natural to define unramified representations of $G(\mathbb{Q}_2)$ to be those that contain $V(2)$ as a K-type, and we call $V(2)$ a pseudospherical type.

We should point out that the center of $G(\mathbb{Q}_2)$ is a cyclic group of order 4. Thus, we have two different genuine central characters γ and two classes of unramified representations. This is analogous to the case of the real group $G(\mathbb{R})$, where the weights $-1/2$ and $1/2$ are called pseudospherical types.

We apply our local results in a global setting in Section 8. Let $\mathbb{A}$ be the ring of adeles, and let $G(\mathbb{A})$ be the two-fold cover of $\mathrm{SL}_2(\mathbb{A})$. Let $r > 1$ be an odd integer. Let $\pi = \otimes\pi_v$ be a genuine cuspidal automorphic representation such that

- π_∞ is a holomorphic discrete series representation with the lowest weight $r/2$,
- π_p is unramified for all $p \neq 2$, and
- π_2 contains a nonzero $K_1(4)$-fixed vector.

Every such π corresponds to a Hecke eigenspace in $S_{r/2}(\Gamma_0(4))$, the space of cuspidal modular forms of weight $r/2$. Roughly speaking, a function $f = \bigotimes f_v$ in π gives naturally a modular form in $S_{r/2}(\Gamma_0(4))$ if f_∞ is a lowest weight vector in π_∞, f_p is K_p-fixed and f_2 is $K_1(4)$-fixed. Since the space of $K_1(4)$-fixed vectors in π_2 is two-dimensional, unless π_2 is a Steinberg representation, the cuspidal automorphic representation π gives rise to a two-dimensional Hecke eigenspace in $S_{r/2}(\Gamma_0(4))$. We can pick a line in this subspace by taking f_2 to be in the K-type isomorphic to $V(2)$. In this way we get a representation-theoretic description of Kohnen's plus space $S^+_{r/2}(\Gamma_0(4))$ [1980]. We also obtain that "new forms" in Kohnen's minus space $S^-_{r/2}(\Gamma_0(4))$ correspond to automorphic representations π, where the local component π_2 is a Steinberg representation. Finally, we show that the global Shimura correspondence is compatible with our local Shimura correspondence at the place $p = 2$.

Representations of $G(\mathbb{Q}_2)$ have been studied in great detail by Waldspurger [1980; 1981; 1991]. However, his approach does not involve the Hecke algebra $H(\gamma)$. Furthermore, a representation-theoretic description of Kohnen's plus space has already been given in [Baruch and Mao 2007]. That approach relies heavily on the mentioned results of Waldspurger, where needed local results are hard to extract.

2. Double cover of $\mathbf{SL_2}(\mathbb{Q}_v)$

We now describe the double cover $G(\mathbb{Q}_v)$ in (1). A section $\boldsymbol{s} : \mathrm{SL}_2(\mathbb{Q}_v) \to G(\mathbb{Q}_v)$ allows us to identify $G(\mathbb{Q}_v)$ with the set $\mathrm{SL}_2(\mathbb{Q}_v) \times \mu_2$, with group law

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \sigma_v(g_1, g_2)),$$

where $\sigma_v(g_1, g_2)$ is a cocycle that depends on $\boldsymbol{s}$. Following [Gelbart 1976], we make the following choice of the cocycle σ_v. Let $(\cdot\,,\cdot)_v$ be the Hilbert symbol over $\mathbb{Q}_v$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Q}_v)$, we define

$$x(g) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0 \end{cases} \quad \text{and} \quad s(g) = \begin{cases} (c, d)_v & \text{if } v \text{ is a finite prime, } cd \neq 0 \\ & \qquad \text{and } \operatorname{ord}(c) \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Then $\sigma_v(g_1, g_2) = (x(g_1 g_2) x(g_1), x(g_1 g_2) x(g_2))_v s(g_1) s(g_2) s(g_1 g_2)$.

An advantage of this particular section is that $K_p = \boldsymbol{s}(\mathrm{SL}_2(\mathbb{Z}_p))$ is a subgroup in $G(\mathbb{Q}_p)$ if $p \neq 2$. If $p = 2$, we define

$$K_1(4) = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, 1 \right) \in \mathrm{SL}_2(\mathbb{Z}_2) \times \{\pm 1\} : a \in 1 + 4\mathbb{Z}_2, c \in 4\mathbb{Z}_2 \right\}.$$

By [Gelbart 1976, Proposition 2.14], $K_1(4)$ is a compact subgroup of $G(\mathbb{Q}_2)$.

A smooth representation of $G(\mathbb{Q}_v)$ is called *genuine* if μ_2 acts nontrivially. If p is an odd prime number, a smooth genuine representation of $G(\mathbb{Q}_p)$ is called *unramified* if it contains a vector fixed by K_p. A vector fixed by K_p is called a *spherical* vector.

If $p = 2$, a smooth genuine representation is called *tamely ramified* if it contains a vector fixed by $K_1(4)$. Unfortunately $\mathrm{SL}_2(\mathbb{Z}_2)$ does not split in $G(\mathbb{Q}_2)$, so we cannot define spherical vectors in the same manner as those for odd primes. The objective of this paper is to motivate and define spherical vectors of genuine representations of $G(\mathbb{Q}_2)$.

We set up some notation for later. For $u \in \mathbb{Q}_v$ and $t \in \mathbb{Q}_v^\times$, we define these elements in $\mathrm{SL}_2(\mathbb{Q}_v)$:

$$\underline{x}(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \qquad \underline{y}(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \qquad \underline{w}(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}, \qquad \underline{h}(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Let $x(u)=\boldsymbol{s}(\underline{x}(u))$, $y(u)=\boldsymbol{s}(\underline{y}(u))$, $w(t)=\boldsymbol{s}(\underline{w}(t))$ and $h(t)=\boldsymbol{s}(\underline{h}(t))$ in $G(\mathbb{Q}_v)$. Note that $h(t)h(s)=h(ts)(t,s)_v$. Let $N=\{x(u):u\in\mathbb{Q}_v\}$, $\overline{N}=\{y(u):u\in\mathbb{Q}_v\}$ and T be the subgroup of G generated by elements $h(t)$.

3. Hecke algebra at $p=2$

We fix $p=2$ from Sections 3 through 6. We will denote $G(\mathbb{Q}_2)$ by G and $K_1(4)$ by K_1. The objective of these sections is to classify genuine representations of G containing a nonzero vector fixed by K_1.

Let M be the center of G. It is a cyclic group of order 4 generated by $h(-1)$. (Note that $h(-1)h(-1)=(-1,-1)_2=-1\in\mu_2$.) Thus, a genuine central character γ is determined by its value on $h(-1)$; this value is a fourth root of 1. Let K and K_0 be the open compact subgroups in G equal to the inverse images of $\mathrm{SL}_2(\mathbb{Z}_2)$ and

$$\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}_2) : c\in 4\mathbb{Z}_2\right\}$$

respectively. Let $K(4)\subset K_1$ denote the principal congruence subgroup. It is the image under the section $\boldsymbol{s}$ of the subgroup of $\mathrm{SL}_2(\mathbb{Z}_2)$ consisting of matrices congruent to 1 modulo 4. We have $K\supset K_0\supset K_1\supset K(4)$ and $K_0=M\times K_1$. We extend the central character γ to K_0, so that it is trivial on K_1. Given a smooth representation (π,V) of G, we let

$$V^{K_0,\gamma}:=\{v\in V:\pi(k_0)v=\gamma(k_0)v \text{ for all } k_0\in K_0\}.$$

Let $\mathscr{R}(G,\gamma)$ denote the category of admissible smooth (necessarily genuine) representations V of G such that $V^{K_0,\gamma}$ generates V as a G-module.

Next we define the corresponding Hecke algebra. Let $C_c(G)$ denote the set of locally constant, compactly supported functions on G. Let

$$H(\gamma)=\{f:C_c(G): f(k_0gk_0')=\overline{\gamma}(k_0)f(g)\overline{\gamma}(k_0') \text{ for all } k_0,k_0'\in K_0\}.$$

For $f_1,f_2\in H(\gamma)$, we define

$$f_1\cdot f_2(g_0)=\int_G f_1(g)f_2(g^{-1}g_0)dg=\int_G f_1(g_0g)f_2(g^{-1})dg,$$

where dg is the Haar measure on G such that the measure of K_0 is 1. Then $H(\gamma)$ is a $\mathbb{C}$-algebra. For $f\in H(\gamma)$ and $v\in V$, we have

$$\pi(f)v=\int_G f(g)\pi(g)v\,dg\in V^{K_0,\gamma}.$$

In this way $V^{K_0,\gamma}$ is a left $H(\gamma)$-module. Let $\mathscr{R}(H(\gamma))$ denote the category of finite-dimensional left $H(\gamma)$-modules. We have a functor $A:\mathscr{R}(G,\gamma)\to\mathscr{R}(H(\gamma))$

given by $V \mapsto V^{K_0,\gamma}$. Since the group K_0 has a triangular decomposition

$$K_0 = (K_0 \cap \overline{N})(K_0 \cap T)(K_0 \cap N),$$

the functor A is an equivalence of categories. This follows, in essence, from [Casselman 1995, Corollary 3.3.6]; see also [Borel 1976] and [Bernšteĭn and Zelevinskiĭ 1976, Theorem 4.2].

Our immediate goal is to understand the structure of $H(\gamma)$. The character γ of the center M extends to a character γ of T that is trivial on $K_1 \cap T$, and $\gamma(h(2^n)) = 1$ for all $n \in \mathbb{Z}$. Let us abbreviate $\gamma(t) = \gamma(h(t))$. We define $\zeta = (1+\gamma(-1))/\sqrt{2}$. Note that ζ is a primitive 8-th root of 1. The character γ of T is invariant under conjugation by $w = w(1)$. We can now extend the character γ from T to the normalizer $N_G(T)$ by defining $\gamma(w) = \zeta$.

We define some functions in $H(\gamma)$. For g in $N_G(T)$, we set X_g to be the function supported on $K_0 g K_0$ such that

$$X_g(k_0 g k_0') = \bar{\gamma}(k_0)\bar{\gamma}(g)\bar{\gamma}(k_0') \quad \text{for all } k_0, k_0' \in K_0.$$

Note that this definition depends only on the image of g in the affine Weyl group $W_a := N_G(T)/(T \cap K_0)$.

Proposition 1. *Functions X_g for g in W_a form a basis of $H(\gamma)$.*

Proof. We need first to determine the K_0-double cosets in G. This can be easily determined in $\mathrm{SL}_2(\mathbb{Q}_2)$ using row-column reduction. In addition to $h(2^n)$ and $w(2^{-n})$ the double coset representatives are

$$y(2),\quad h(2^n)y(2),\quad y(2)h(2^{-n}),\quad y(2)w(2^{-n}),\quad w(2^{-n})y(2),\quad y(2)w(2^{-n})y(2),$$

where $n \geq 1$ in all cases. We claim that the Hecke algebra is not supported on these cosets.

Lemma 2. *The commutator of $x(2)$ and $y(2)$ modulo the principal congruence subgroup $K(4)$ is equal to $-1 \in \mu_2$.*

Proof. This can be easily checked using the multiplication rule. It also follows from applying [Stein 1973, Corollary 2.9] to the ring $A = \mathbb{Z}/4\mathbb{Z}$, □

Now we can easily finish the proof of proposition. Indeed if f is in $H(\gamma)$, then

$$f(y(2)) = f(y(2)x(2)) = -f(x(2)y(2)) = -f(y(2))$$

by the lemma above. This implies that f must vanish on $y(2)$. Other cases are dealt with in the same manner. □

Let $\ell : N_G(T) \to \mathbb{Z}$ be defined by $\ell(g) = \log_2(n)$, where n is the number of left (or right) K_0-cosets in the double coset $K_0 g K_0$. In other words, the volume of $K_0 g K_0$ is $2^{\ell(g)}$. For example, $w(2^{-1})$ normalizes K_0, so $\ell(w(2^{-1})) = 1$.

Proposition 3. *For every integer n, we have*

$$\ell(h(2^n)) = 2|n| \quad \text{and} \quad \ell(w(2^{-n})) = 2|1-n|.$$

More precisely, we have the following decompositions of double cosets:

(i) *If $n \geq 0$,*

$$K_0 h(2^n) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} x(u) h(2^n) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} K_0 h(2^n) y(4u).$$

(ii) *If $n \geq 1$,*

$$K_0 h(2^{-n}) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} y(4u) h(2^{-n}) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n}\mathbb{Z}} K_0 h(2^{-n}) x(u).$$

(iii) *If $n \geq 0$,*

$$K_0 w(2^n) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n+2}\mathbb{Z}} x(u) w(2^n) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n+2}\mathbb{Z}} K_0 w(2^n) x(u).$$

(iv) *If $n \geq 1$,*

$$K_0 w(2^{-n}) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n-2}\mathbb{Z}} y(4u) w(2^{-n}) K_0 = \bigcup_{u \in \mathbb{Z}/2^{2n-2}\mathbb{Z}} K_0 w(2^{-n}) y(4u).$$

Proof. This follows easily from the decomposition $K_0 = (K_0 \cap \overline{N})(K_0 \cap T)(K_0 \cap N)$. Details are left to the reader. □

We record the following tautological lemma:

Lemma 4. *Let g_1 and g_2 be two elements in $N_G(T)$. If $\ell(g_1 g_2) = \ell(g_1) + \ell(g_2)$ then $X_{g_1} \cdot X_{g_2} = X_{g_1 g_2}$.*

Let $T_n = X_{h(2^n)}$ and $U_n = X_{w(2^{-n})}$.

Proposition 5. *Let $T_w = \sqrt{1/2}\, U_0$. We have the following identities, where m, n are any integers unless further specified.*

(i) $(T_w + 1)(T_w - 2) = 0$.

(ii) $U_1 \cdot U_1 = 1$.

(iii) *If $m, n \geq 0$ or $m, n \leq 0$, then $T_m \cdot T_n = T_{m+n}$.*

(iv) $U_1 \cdot T_n = U_{n+1}$ *and* $T_n \cdot U_1 = U_{1-n}$.

(v) $U_1 \cdot U_n = T_{n-1}$ *and* $U_n \cdot U_1 = T_{1-n}$.

Proof. All statements except for (i) follow from Lemma 4. For (i), we need to show $T_w^2 = T_w \cdot T_w = T_w + 2$. Since T_w^2 is supported in K, this is equivalent to $T_w^2(1) = 2$

and $T_w^2(w(1)) = T_w(w(1))$. Suppose $f_1, f_2 \in H(\gamma)$, where f_1 is supported on $K_0 r K_0 = \bigsqcup_{i=1}^{s} r_i K_0$ (disjoint union). Then

$$f_1 \cdot f_2(g) = \sum_{i=1}^{s} f_1(r_i) f_2(r_i^{-1} g).$$

We can apply this observation to $f_1 = f_2 = T_w$. Proposition 3(iii) with $n = 0$ gives a decomposition of $K_0 w(1) K_0$ into single cosets. Hence

$$T_w^2(g) = \sum_{u \bmod 4} T_w(x(u)w(1)) \cdot T_w(w(-1)x(-u)g).$$

If $g = 1$, this gives $T_w^2(1) = 4T_w(w(1)) \cdot T_w(w(-1))$. Since $T_w(w(1)) = 2^{-1/2}\bar{\zeta}$ and $T_w(w(-1)) = 2^{-1/2}\zeta$, we obtain that $T_w^2(1) = 2$. If $g = w(1)$, then

$$T_w^2(w(1)) = T_w(w(1)) \sum_{u \bmod 4} T_w(y(u)).$$

If $u = 0$ or 2, then $y(u)$ is not in $K_0 w(1) K_0$ and $T_w(y(u)) = 0$. If $u = \pm 1$, then $y(u) = x(u)w(-u)x(u)$, and we can rewrite

$$T_w^2(w(1)) = T_w(w(1))[T_w(w(1)) + T_w(w(-1))] = T_w(w(1)). \qquad \square$$

Here is the main result of this section.

Theorem 6. *The Hecke algebra $H(\gamma)$ is generated by T_w and U_1 as an abstract $\mathbb{C}$-algebra modulo the relations*

(a) $(T_w - 2)(T_w + 1) = 0$ *and*

(b) $U_1^2 = 1$.

Proof. Suppose H is the abstract algebra generated by $U_0 = \sqrt{2}T_w$ and U_1 modulo the relations (a) and (b). We have a natural homomorphism $B : H \to H(\gamma)$ of $\mathbb{C}$-algebras. By Proposition 1, $H(\gamma)$ is spanned by T_n and U_n and by Proposition 5, these elements are generated by U_0 and U_1. This shows that B is surjective. To show that it is injective, suppose $h \in H$ is in the kernel of B. Since U_0 and U_1 satisfy quadratic relations, $h = \sum_i c_i u_i$, where $c_i \in \mathbb{C}$ and $u_i \in H$ is of the form $U_1 U_0 U_1 U_0 \cdots$ or $U_0 U_1 U_0 U_1 \cdots$. Because $U_0 U_1 = T_1$, $B(u_i)$ is either T_n, $T_n U_1 = U_{1-n}$, $U_1 T_n = U_{n+1}$, or $U_1 T_n U_1 = T_{-n}$. These elements have disjoint support as functions in $H(\gamma)$. Therefore $B(h) = \sum_i c_i B(u_i) = 0$ implies that $c_i = 0$ and $h = 0$. This proves that B is an injection and Theorem 6. $\square$

We now give two consequences of Theorem 6:

Proposition 7. *The element $Z := T_1/2 + (T_1/2)^{-1}$ belongs to the center of $H(\gamma)$.*

Proof. By Proposition 5, T_1 and U_1 generate $H(\gamma)$. Clearly Z commutes with T_1. It suffices to show that Z commutes with U_1. Since $T_1 = U_0 U_1$, we can use quadratic relations satisfied by U_0 and U_1 to write

$$2Z = U_0 U_1 + U_1 U_0 - 2^{1/2} U_1. \tag{2}$$

Hence Z commutes with U_1. □

Proposition 8. *For $n \geq 0$, T_n is an invertible element in the algebra $H(\gamma)$.*

Proof. The quadratic relations satisfied by U_0 and U_1 imply that U_0 and U_1 are invertible, and so is T_1, since $T_1 = U_0 U_1$. Hence $T_n = T_1^n$ is invertible. □

Suppose (π, V) is a representation in $\mathcal{R}(G, \gamma)$. Let $V(N)$ denote the span of $\pi(n)v - v$ for all $v \in V$ and $n \in N$, and let $V_N = V/V(N)$ be the Jacquet module. Let

$$(V_N)^{K_0 \cap T, \gamma} = \{v \in V_N : \pi_{V_N}(t)v = \gamma(t)v \text{ for all } t \in K_0 \cap T\}.$$

The invertibility of T_n implies the following; see [Borel 1976, Lemma 4.7].

Corollary 9. *Suppose (π, V) is a representation in $\mathcal{R}(G, \gamma)$. Then the canonical map $V^{K_0, \gamma} \to (V_N)^{K_0 \cap T, \gamma}$ is a bijection. In particular V_N is nonzero, and V cannot be a supercuspidal representation.* □

4. Local Shimura correspondence

Let $G' = \mathrm{PGL}_2(\mathbb{Q}_2)$. Let I be its Iwahori subgroup and H' be its Iwahori–Hecke algebra. Let T'_w and U'_1 denote the characteristic functions of

$$I \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} I \quad \text{and} \quad I \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} I$$

respectively. Then H' is the abstract $\mathbb{C}$-algebra generated by T'_w and U'_1 satisfying the same relations as (a) and (b) of Theorem 6; see [Matsumoto 1977]. This gives the next corollary.

Corollary 10. *The Hecke algebras $H(\gamma)$ and H' are isomorphic $\mathbb{C}$-algebras.*

Let $\mathcal{R}(H')$ denote the category of finite-dimensional representations of H'. Let $\mathcal{R}(G', I)$ denote the category of admissible smooth representations V of G' such that V^I generates V as a G'-module. By [Borel 1976; Bernšteĭn and Zelevinskiĭ 1976], the functor $V \mapsto V^I$ is an equivalence of categories from $\mathcal{R}(G', I)$ to $\mathcal{R}(H)$. The isomorphism in Corollary 10 establishes an equivalence of categories between $\mathcal{R}(H(\gamma))$ and $\mathcal{R}(H')$. Hence the following four categories are equivalent:

$$\mathcal{R}(G, \gamma) \simeq \mathcal{R}(H(\gamma)) \simeq \mathcal{R}(H') \simeq \mathcal{R}(G', I).$$

If V is a representation in $\mathcal{R}(G, \gamma)$, then we call the corresponding representation in $\mathcal{R}(G', I)$ the *local Shimura lift* of V. We denote it by $\mathrm{Sh}(V)$.

Proposition 11. *Let V be a representation in $\mathcal{R}(G,\gamma)$. Then the following are equivalent.*

(i) *The local Shimura lift* $\mathrm{Sh}(V)$ *is a spherical representation of G'.*

(ii) *The action of T'_w on* $\mathrm{Sh}(V)^I$ *has an eigenvalue* 2.

(iii) *The action of T_w on $V^{K_0,\gamma}$ has an eigenvalue* 2.

Proof. The projection map to $G'(\mathbb{Z}_2)$-fixed vectors in $\mathrm{Sh}(V)$ is given by $\frac{1}{3}(T'_w+1)$, since T'_w+1 is the characteristic function of $G'(\mathbb{Z}_2)$ and the volume of $G'(\mathbb{Z}_2)$ is 3. It follows that a $G'(\mathbb{Z}_2)$-fixed vector is an eigenvector of T'_w with eigenvalue 2. This proves the equivalence of (i) and (ii). The equivalence of (ii) and (iii) follows from Corollary 10. □

This proposition motivates the following definition.

Definition. Let V be a smooth representation of G. An eigenvector of T_w in $V^{K_0,\gamma}$ with an eigenvalue 2 is called a γ*-spherical* vector. The representation is called a γ*-unramified* or γ*-spherical* representation if it contains a γ-spherical vector.

5. Pseudospherical representation of K at $p=2$

We retain the notations in Sections 3 and 4 where $p=2$. In the previous section we defined a representation V of G to be unramified if $V^{K_0,\gamma}\neq 0$ and T_w has an eigenvalue 2 on $V^{K_0,\gamma}$. In this section, we will reinterpret this condition in terms of representations of K, and see that K has only two irreducible representations E such that $E^{K_0,\gamma}\neq 0$. For both representations, $E^{K_0,\gamma}$ is one-dimensional and they are distinguished by the action of T_w on $E^{K_0,\gamma}$. That eigenvalue can be either 2 or -1, so we use the eigenvalue to denote the representations by $V(2)$ and $V(-1)$. Their dimensions are 2 and 4, respectively. Thus, a representation of G is unramified if and only if it contains the two-dimensional K-type $V(2)$, which we may call a pseudospherical type.

If $E^{K_0,\gamma}\neq 0$, then, by Frobenius reciprocity, the K-type E is a summand of a six-dimensional induced representation

$$I_K(\gamma):=\mathrm{Ind}_{K_0}^{K}\gamma=\{\phi:K\to\mathbb{C}:\phi(k_0k)=\gamma(k_0)\phi(k)\text{ for all }k\in K,\ k_0\in K_0\}.$$

Here the group K acts on it by right translation, denoted π_R. Let $H_K(\gamma)$ denote the subalgebra of $H(\gamma)$ consisting of functions supported on K. We have the action of $H_K(\gamma)$ on $I_K(\gamma)^{K_0,\gamma}$, also denoted by π_R. By Proposition 1, $H_K(\gamma)=\mathbb{C}1\oplus\mathbb{C}T_w$ and it is a commutative subalgebra. The algebra $H_K(\gamma)$ is antiisomorphic to the algebra $H_K(\bar\gamma)$ via the map $f\mapsto\hat f$, where $\hat f(g)=f(g^{-1})$.

For $f\in H_K(\bar\gamma)$ and $\phi\in I_K(\gamma)$, we set

$$(\pi_L(f)\phi)(g)=\int_K f(k)\phi(k^{-1}g)dk\quad\text{for all }g\in K.$$

This action commutes with the right action π_R of K on $I_K(\gamma)$ and

$$H_K(\bar{\gamma}) = \mathrm{End}_K(I_K(\gamma)).$$

Note that $I_K(\gamma)^{K_0,\gamma} = H(\bar{\gamma})$. The actions π_L and π_R of $H(\bar{\gamma})$ and $H(\gamma)$ on $I_K(\gamma)^{K_0,\gamma} = H(\bar{\gamma})$ are related by $\pi_L(\hat{f}) = \pi_R(f)$.

We define the functions $F_{-1} := \frac{1}{3}(2 - T_w)$ and $F_2 := \frac{1}{3}(T_w + 1)$ in $H_K(\gamma)$. Then $\{F_{-1}, F_2\}$ is a basis of idempotents of $H_K(\gamma)$.

For $j = -1, 2$, let $V(j) = \pi_L(\hat{F}_j) I_K(\gamma)$. In other words $V(j)$ is the eigenspace of $\pi_L(\hat{T}_w)$ on $I_K(\gamma)$ corresponding to the eigenvalue j. Note that $\hat{F}_j \in V(j)$ and $\pi_R(T_w)\hat{F}_j = j\hat{F}_j$. In particular $\hat{F}_2$ is a γ-spherical vector.

Proposition 12. (i) *We have* $I_K(\gamma) = V(-1) \oplus V(2)$, *where each summand is an irreducible representation of* K.

(ii) *We have* $\dim V(-1) = 4$ *and* $\dim V(2) = 2$.

(iii) *The* K*-submodule* $V(2)$ *contains a* γ*-spherical vector* $\hat{F}_2$. *The space of* γ*-spherical vectors is one-dimensional.*

(iv) *The* K*-submodule* $V(-1)$ *does not have any* γ*-spherical vector.*

Proof. Since $\dim \mathrm{End}(I_K(\gamma)) = 2$, both $V(-1)$ and $V(2)$ are irreducible K-modules. This proves (i).

To compute the dimensions of $V(-1)$ and $V(2)$ we need a lemma.

Lemma 13. *The operator* $\pi_L(\hat{T}_w)$ *as an element in* $\mathrm{End}_K(I_K(\gamma))$ *has trace* 0.

Proof. For $g \in K$, let ϕ_g be an element of $I_K(\gamma)$ such that ϕ_g is supported on $K_0 g$ and $\phi_g(k_0 g) = \gamma(k_0)$. Let S be a set of representatives of $K_0\backslash K$. Then $\{\phi_g : g \in S\}$ is a basis of $I_K(\gamma)$. To prove the lemma, it suffices to show that $(\pi_L(\hat{T}_w)\phi_g)(g) = 0$. Indeed, this shows that the matrix of $\pi_L(\hat{T}_w)$ in the basis ϕ_g has vanishing diagonal entries. Note that $\pi_L(T_w)\phi_g$ is supported on $K_0 w(1) K_0 g$. If $(\pi_L(\hat{T}_w)\phi_g)(g) \neq 0$, then $g \in K_0 w(1) K_0 g$ and $1 \in K_0 w(1) K_0$. This is a contradiction since K_0 is not equal to $K_0 w(1) K_0$. □

We have $\dim V(2) + \dim V(-1) = \dim I_K(\gamma) = [K : K_0] = 6$. By the lemma, $2\dim V(2) - \dim V(-1) = 0$. This implies $\dim V(-1) = 4$ and $\dim V(2) = 2$ and proves Proposition 12(ii). We have $I_K(\gamma)^{K_0,\gamma} = H_K(\bar{\gamma})$ and $\pi_R(F_j) I_K(\gamma) = \mathbb{C}\hat{F}_j$ for $j = -1, 2$. The vector $\hat{F}_2$ is γ-spherical while $\hat{F}_{-1}$ is not. This proves parts (iii) and (iv). □

Theorem 14. *A smooth representation* V *of* G *with central character* γ *is* γ*-unramified if and only if there is a nontrivial* K*-module homomorphism* l *from* $V(2)$ *to* V. *A vector in* V *proportional to* $l(\hat{F}_2)$ *is a* γ*-spherical vector of* V.

Proof. A γ-spherical vector in V generates a representation of K in which every irreducible K-submodule is isomorphic to an irreducible submodule of $I_K(\gamma)$. Now the theorem follows from Proposition 12. □

6. Unramified principal series representations at $p = 2$

In this section, we continue to assume $p = 2$ and use notation of Sections 3 to 5. We will show that γ-unramified representations appear as submodules of principal series representations.

We recall the character γ of T in Section 3. Let $(\pi_s, I(\gamma, s))$ be the normalized induced principal series representation, where $I(\gamma, s)$ is the set of smooth functions $\phi : G \to \mathbb{C}$ satisfying

$$\phi(\epsilon x(u)h(t)g) = \epsilon\gamma(t)|t|^{s+1}\phi(g) \quad \text{for all } \epsilon \in \mu_2,\, u \in \mathbb{Q}_2 \text{ and } t \in \mathbb{Q}_2^\times.$$

The group G acts by left translation: $(\pi_s(g)\phi)(g') = \phi(g'g)$.

Proposition 15. *An irreducible γ-unramified representation V is isomorphic to a submodule of some $I(\gamma, s)$.*

Proof. By Corollary 9, $(V_N)^{K_0\cap T,\gamma}$ is nonzero. Hence there is a nontrivial T-homomorphism $V_N \to \gamma\nu^{s+1}$ for some $s \in \mathbb{C}$. Here ν is the character $\nu(\underline{h}(t)) = |t|$. By Frobenius reciprocity, there is a nontrivial map $V \to I(\gamma, s)$ that is an injection because V is irreducible. □

We recall that $K(4)$ is the principal congruence subgroup in K_1. Restricting functions ϕ in $I(\gamma, s)$ to K gives a natural isomorphism $l : I_K(\gamma) \to I(\gamma, s)^{K(4)}$ of K-modules.

Theorem 16. *The K-types $V(2)$ and $V(-1)$ are of multiplicity one in $I(\gamma, s)$. The space $I(\gamma, s)^{K_0,\gamma}$ is 2-dimensional and is spanned by $l(\hat{F}_2)$ and $l(\hat{F}_{-1})$.*

Waldspurger [1981, Chapter VI] describes an explicit basis of $I(\gamma, s)^{K_0,\gamma}$ and calculates the action of the operator T_1.

We will describe a scalar multiple ϕ_j of $l(\hat{F}_j) \in V_j$ that is more convenient for later calculations. Let $d_2 = 1$ and $d_{-1} = -2$, and define ϕ_j to be the unique vector in $I(\gamma, s)$ whose restriction to K is given by

$$\phi_j(k) = \begin{cases} d_j\gamma(k) & \text{if } k \in K_0, \\ 2^{-1/2}\zeta\gamma(k_0k_0') & \text{if } k = k_0w(1)k_0' \in K_0w(1)K_0, \\ 0 & \text{otherwise.} \end{cases}$$

We define an intertwining map $M(s) : I(\gamma, s) \to I(\gamma, -s)$ by

$$(M(s)\phi)(g) = \int_{\mathbb{Q}_2} \phi(w(1)x(u)g)du,$$

where $g \in G$ and du is the Haar measure on $\mathbb{Q}_2$ such that the measure of $\mathbb{Z}_2$ is 1.

Proposition 17. *We have*

$$M(s)\phi_2 = \frac{\zeta}{\sqrt{2}}\left(\frac{1-\frac{1}{2}(2^{-2s})}{1-2^{-2s}}\right)\phi_2 \quad \textit{and} \quad M(s)\phi_{-1} = -\frac{\zeta}{2\sqrt{2}}\left(\frac{1-2(2^{-2s})}{1-2^{-2s}}\right)\phi_{-1}.$$

Proof. Since the vector ϕ_j is unique up to a scalar in $I_K(\gamma)$, we have $M(s)\phi_j = c\phi_j$ for some $c \in \mathbb{C}$. It remains to determine $c = d_j^{-1} M(s)\phi_j(1)$.

If $u \notin \mathbb{Z}_2$, we have $w(1)x(u) = (-1,u)_2 \cdot x(-u^{-1})h(u^{-1})y(u^{-1})$. We write $u^{-1} = 2^m v$, where $v \in \mathbb{Z}_2^\times$ and $m \geq 1$. Recall that $\gamma(t) = \gamma(h(t))$. Then

$$\begin{aligned} &M(s)\phi_j(1) \\ &\quad = \int_{\mathbb{Z}_2} \phi_j(w(1)x(u))du + \sum_{m=1}^{\infty} 2^{m-1} \int_{\mathbb{Z}_2^\times} (-1, 2^m v)_2 \phi_j(h(2^m v)y(2^m v))d^\times v \\ &\quad = 2^{-1/2}\zeta + \sum_{m=1}^{\infty} 2^{-ms-1} \int_{\mathbb{Z}_2^\times} (-1, v)_2 \gamma(2^m v)\phi_j(y(2^m v))d^\times v, \end{aligned}$$

where $d^\times v$ is the Haar measure of $\mathbb{Z}_2^\times$ with total measure 1. Now $\phi_j(y(2^m v)) = 0$ if $m = 1$ and it is equal to 1 if $m \geq 2$. Since $\gamma(2^m v) = \gamma(2^m)\gamma(v)(2^m, v)_2$ and $\gamma(2^m) = 1$, we can rewrite

$$\begin{aligned} M(s)\phi_j(1) &= 2^{-1/2}\zeta + d_j \sum_{m=2}^{\infty} 2^{-ms-1} \int_{\mathbb{Z}_2^\times} (2, v)_2^m (-1, v)_2 \gamma(v) d^\times v \\ &= 2^{-1/2}\zeta + d_j \sum_{m=2}^{\infty} 2^{-ms-1} \frac{1}{4} \sum_{v \in (\mathbb{Z}/8\mathbb{Z})^\times} (2, v)_2^m (-1, v)_2 \gamma(v). \end{aligned}$$

The sum $\sum_{v \in (\mathbb{Z}/8\mathbb{Z})^\times}$ on the right is zero if m is odd, and equals $\sqrt{2}\zeta$ if m is even. Finally adding up all the terms gives the constant c and the lemma. □

Let $s_0 = 1/2$ or $1/2 + i\pi/\log 2$. From Proposition 17, ϕ_{-1} lies in the kernel of $M(s_0)$, so $I(\gamma, s_0)$ is reducible. Indeed $I(\gamma, s_0)$ has a unique irreducible quotient that is an even Weil representation.

Definition. Let $s_0 = 1/2$ or $1/2 + i\pi/\log 2$. The kernel of $M(s_0)$ is called the *Steinberg* representation of $G(\mathbb{Q}_2)$. We shall denote this representation by $St(\epsilon)$, where $\epsilon = \pm 1$ such that $2^{s_0} = \epsilon\sqrt{2}$.

We claim that $St(\epsilon)$ is an irreducible representation of $G(\mathbb{Q}_2)$. Indeed by [Loke and Savin 2010, Section 6], we have

$$I(\gamma, s)_N^{\mathrm{ss}} \cong \gamma|\cdot|^{s+1} \oplus \gamma|\cdot|^{-s+1} \quad \text{for every } s \in \mathbb{C}, \tag{3}$$

where $I(\gamma, s)_N^{\mathrm{ss}}$ is the semisimplification of $I(\gamma, s)_N$ as a T-module. Hence $I(\gamma, s)$ has at most length 2. The claim now follows because $St(\epsilon)$ is a proper submodule of $I(\gamma, s_0)$. Also see [Savin 2004, Section 7].

Corollary 18. *The even Weil representation contains the irreducible K-module $V(2)$. It is a γ-unramified representation. The Steinberg representation contains the irreducible K-module $V(-1)$.* □

Proposition 19. *Let $Z = T_1/2 + (T_1/2)^{-1}$ be the central element in the Hecke algebra $H(\gamma)$ as in Proposition 7. Then $\pi_s(Z)$ acts on $I(\gamma, s)^{K_0,\gamma}$ as the scalar $2^s + 2^{-s}$.*

Proof. By Corollary 9, the natural projection of $I(\gamma, s)$ onto $I(\gamma, s)_N$ gives an isomorphism of $I(\gamma, s)^{K_0,\gamma}$ and $I(\gamma, s)_N$. From Proposition 3(i)'s decomposition of $K_0 h(2) K_0$ into single K_0-cosets, it follows that the action of T_1 on $I(\gamma, s)^{K_0,\gamma}$ corresponds to the action of $4 \cdot \pi_{s,N}(h(2))$ on $I(\gamma, s)_N$. By (3), the eigenvalues of $T_1/2$ are 2^s and 2^{-s}. □

Corollary 20. *An irreducible γ-unramified representation is uniquely determined by the eigenvalue of the action of Z on its γ-spherical vector.*

Proof. Suppose the irreducible γ-unramified representation is a subquotient of both $I(\gamma, s)$ and $I(\gamma, s')$. Then by Proposition 19, $2^s + 2^{-s} = 2^{s'} + 2^{-s'}$, which implies $2^s = 2^{s'}$ or $2^s = 2^{-s'}$. By Proposition 17 both $I(\gamma, s)$ and $I(\gamma, -s)$ have the same irreducible γ-unramified subquotient. □

Corollary 21. *The Steinberg representation $St(\epsilon)$ corresponds to the one-dimensional representation of $H(\gamma)$ given by $T_w = -1$ and $U_1 = -\epsilon$.*

Proof. We know that $T_w = -1$ on $St(\epsilon)^{K_0,\gamma}$. It remains to compute the action of U_1. Since $St(\epsilon)$ is a subquotient of $I(\gamma, s_0)$, where $2^{s_0} = \epsilon\sqrt{2}$, the central element Z acts on $St(\epsilon)$ by the scalar $\epsilon(2^{1/2} + 2^{-1/2})$. By (2) we have $2^{1/2} Z = T_w U_1 + U_1 T_w - U_1$. Hence $U_1 = -\epsilon$ as claimed. □

Let V be an irreducible γ-unramified representation. By Proposition 15, we may assume that V is the unique γ-unramified subquotient of $I(\gamma, s)$ for some $s \in \mathbb{C}$. By Proposition 11, its local Shimura lift $V' = \mathrm{Sh}(V)$ is an unramified irreducible representation of $G' = \mathrm{PGL}_2(\mathbb{Q}_2)$. Let B' be the Borel subgroup of G'. We may realize V' as the unramified irreducible subquotient of the normalized induced principal series representation $(\pi'_s, I'(t))$ with trivial central character. Here $I'(t) = \mathrm{Ind}_{B'}^{G'} \omega^t$ (normalized induction), where ω is the character

$$\omega \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = |a_1/a_2|.$$

The next theorem is similar to [Waldspurger 1991, Proposition 4]. There the local correspondence is defined by restricting the oscillator representation to the dual pair $G(\mathbb{Q}_2) \times \mathrm{PGL}_2(\mathbb{Q}_2)$, while the local Shimura lift used here is defined by the Hecke algebra isomorphism.

Theorem 22. *If V is the unique γ-unramified irreducible subquotient of $I(\gamma, s)$, then its local Shimura lift* $\mathrm{Sh}(V)$ *is the unique unramified irreducible subquotient of $I'(s)$.*

Proof. Assume that $\mathrm{Sh}(V)$ is a subquotient of $I'(t)$. By Proposition 19 the central operator Z in $H(\gamma)$ acts on $I(\gamma, s)^{K_0,\gamma}$ by the scalar $2^s + 2^{-s}$. The corresponding operator Z' in the algebra H' acts on $I'(t)$ by $2^t + 2^{-t}$. Thus, $2^s + 2^{-s} = 2^t + 2^{-t}$. Solving the equation gives $2^s = 2^t$ or $2^s = 2^{-t}$. Both $I'(t)$ and $I'(-t)$ have the same irreducible subquotients, so we may set $s = t$. □

We deduce a corollary that is a part of [Waldspurger 1980, Propositions 1 and 2].

Corollary 23. *The principal series representation $I(\gamma, s)$ is reducible if and only if $s = 1/2$ or $1/2 + i\pi/\log 2$.*

Proof. Let V be the γ-unramified irreducible subquotient of $I(\gamma, s)$. Let W be the unramified irreducible subquotient of $I'(s)$. Then $V = I(\gamma, s)$ if and only if $\dim V^{K_0,\gamma} = 2$. By Theorem 22, $\dim V^{K_0,\gamma} = \dim W^I$. Now $\dim W^I = 2$ if and only if $I'(s)$ is irreducible. Finally $I'(s)$ is irreducible if and only if $s \neq 1/2$ and $1/2 + i\pi/\log 2$. □

7. Automorphic forms

In this section we review a connection between automorphic forms and classical modular forms of half integral weight. This is mostly well known material that can be found in [Gelbart 1976, Chapters 2 and 3] and in [Waldspurger 1981]. We then transfer the action of the Hecke algebra $H(\gamma)$ to the setting of classical modular forms.

Let $\mathbb{A} = \prod_v \mathbb{Q}_v$ be the ring of adeles over $\mathbb{Q}$. We recall K_p, $s(g)$ and the cocycle σ_v defined in Section 2. Let $G(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A}) \times \{\pm 1\}$ as a set. For $g_1 = (g_{1,v})$, $g_2 = (g_{2,v}) \in \mathrm{SL}_2(\mathbb{A})$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, the group law on $G(\mathbb{A})$ is given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \sigma(g_1, g_2)),$$

where $\sigma(g_1, g_2) = \prod_v \sigma_v(g_{1,v}, g_{2,v})$. Then $\mathrm{pr} : G(\mathbb{A}) \to \mathrm{SL}_2(\mathbb{A})$, $(g, \epsilon) \mapsto g$ is a twofold cover that splits over the subgroup $\mathrm{SL}_2(\mathbb{Q})$. Since $\mathrm{SL}_2(\mathbb{Q})$ is perfect, this splitting is unique and given by

$$s_{\mathbb{Q}} : \mathrm{SL}_2(\mathbb{Q}) \to G(\mathbb{A}), \quad g \mapsto (g, s_{\mathbb{A}}(g)), \quad \text{where } s_{\mathbb{A}}(g) = \textstyle\prod_v s(g_v).$$

We also need a description of a maximal compact subgroup in $G(\mathbb{R})$. Let

$$\underline{k}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \quad \text{for } -\pi < \theta \leq \pi.$$

Then $\underline{K}_\infty := \{\underline{k}(\theta) : -\pi < \theta \le \pi\}$ is a maximal compact subgroup in $\mathrm{SL}_2(\mathbb{R})$. Let $K_\infty = \{k(\theta) : -2\pi < \theta \le 2\pi\}$, where

$$k(\theta) = \begin{cases} (\underline{k}(\theta), 1) & \text{if } -\pi < \theta \le \pi, \\ (\underline{k}(\theta), -1) & \text{if } -2\pi < \theta \le -\pi \text{ or } \pi < \theta \le 2\pi. \end{cases}$$

Then K_∞ is a maximal compact subgroup of $G(\mathbb{R})$ and $\mathrm{pr}(K_\infty) = \underline{K}_\infty$. If r is an odd integer, then $k(\theta) \mapsto e^{ir\theta/2}$ defines a genuine character of K_∞

Let $A_{r/2}(4)$ denote the set of functions φ in $L^2(\mathrm{SL}_2(\mathbb{Q})\backslash G(\mathbb{A}))$ satisfying the following properties:

(1) $\varphi(gk_1) = \varphi(g)$ for all $k_1 \in K_1(4) \prod_{p\neq 2,\infty} K_p$;

(2) $\varphi(gk_0) = \gamma(k_0)\varphi(g)$ for all $k_0 \in K_0$ in $G(\mathbb{Q}_2)$, where $\gamma(-1) = -i^r$;

(3) $\varphi(gk(\theta)) = e^{i\frac{r}{2}\theta}\varphi(g)$;

(4) φ is smooth as a function on $G(\mathbb{R})$ and satisfies $\triangle\varphi = -\frac{1}{4}r(\frac{1}{4}r - 1)\varphi$, where $\triangle$ is the Casimir operator; and

(5) φ is cuspidal, that is, $\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(x(u)g)du = 0$ for all $g \in G(\mathbb{A})$.

A basis of $A_{r/2}(4)$ arises from cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A})$ such that π_∞ is a holomorphic discrete series representation with the lowest weight $r/2$, π_p is unramified for all $p \neq 2$, and π_2 contains a $K_1(4)$-fixed vectors. In particular, $\pi_2^{K_0,\gamma} \neq 0$ for some central character γ. Note that γ is determined by r. Indeed, since the local components of $\boldsymbol{s}_{\mathbb{Q}}(h(-1))$ for $v \neq \infty, 2$ are contained in K_p, we have $\varphi(1) = \varphi(\boldsymbol{s}_{\mathbb{Q}}(h(-1))) = \gamma(-1)e^{i\pi r/2}\varphi(1)$, and therefore $\gamma(-1) = -i^r$.

Let $\mathcal{H}$ be the complex upper half plane. For elements $\underline{g} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, $g = (\underline{g}, \epsilon) \in G(\mathbb{R})$ and $z \in \mathcal{H}$, we define

$$gz = \underline{g}z = \frac{az+b}{cz+d}.$$

We define a holomorphic function on $\mathcal{H}$ by

$$J(g, z) = J((\underline{g}, \epsilon), z) := \epsilon\,(cz+d)^{1/2}.$$

Here we choose $w^{1/2}$ so that $-\pi/2 < \arg(w^{1/2}) \le \pi/2$. We call $J(g, z)$ a *factor of automorphy*. By [Gelbart 1976, Lemma 3.3], it has $J(gg', z) = J(g, g'z)J(g', z)$ for any two g and g' in $G(\mathbb{R})$. Define a congruence subgroup $\Gamma_0(4)$ by

$$\Gamma_0(4) := G(\mathbb{R}) \cap \Big(\boldsymbol{s}_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q})) \cdot K_0(4) \cdot \prod_{p\neq 2} K_p\Big).$$

Similarly, define $\Gamma_1(4) \subseteq \Gamma_0(4)$ by replacing $K_0(4)$ with $K_1(4)$. Let $S_{r/2}(\Gamma_0(4))$ and $S_{r/2}(\Gamma_1(4))$ be the spaces of classical modular forms of weight $r/2$. By

[Koblitz 1984, page 183], we have $S_{r/2}(\Gamma_0(4)) = S_{r/2}(\Gamma_1(4))$. We will denote this space by $S_{r/2}(4)$.

By [Gelbart 1976, Proposition 3.1], there is a bijection $Q : A_{r/2}(4) \to S_{r/2}(4)$, which we recall: If $\varphi \in A_{r/2}(4)$, then

$$(Q\varphi)(z) = \varphi(g_\infty)J(g_\infty, i)^r, \quad \text{where } z = g_\infty i \in \mathcal{H}.$$

Conversely, given $f \in S_{r/2}(4)$, let $g \in G(\mathbb{A})$. By [Gelbart 1976, Lemma 3.2], $g = g_{\mathbb{Q}} g_\infty k$ for some $g_{\mathbb{Q}} \in \boldsymbol{s}_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q}))$, $g_\infty \in G(\mathbb{R})$ and $k \in K_1(4)\prod_{p\neq 2,\infty} K_p$. Then $(Q^{-1}f)(g) = f(g_\infty(i))J(g_\infty, i)^{-r}$.

Using the bijection Q, we define another bijection between the spaces of operators by

$$\boldsymbol{q} : \mathrm{End}_{\mathbb{C}}(A_{r/2}(4)) \to \mathrm{End}_{\mathbb{C}}(S_{r/2}(4)), \quad L \mapsto QLQ^{-1}.$$

Since the Hecke algebra $H(\gamma)$ defined in Section 3 acts on $A_{r/2}(4)$, it is of interest to reinterpret this action in terms of classical modular forms.

Proposition 24. *Let U_1 and T_1 be the operators in the local Hecke algebra $H(\gamma)$, where $\gamma(-1) = -i^r$. Recall that $\zeta = (1 - i^r)/\sqrt{2}$. For $f(z) \in S_{r/2}(4)$, we have*

(i) $(\boldsymbol{q}(U_1)f)(z) = \bar{\zeta}(2z)^{-r/2} f(-1/(4z))$ *and*

(ii) $(\boldsymbol{q}(T_1)f)(z) = 2^{-r/2}\sum_{u=0}^{3} f((z+u)/4)$.

Proof. (i) Suppose $\varphi = Q^{-1}(f) \in A_{r/2}(4)$. For every place v, let $w_v = w(2^{-1})$ be the element in $G(\mathbb{Q}_v)$ defined in Section 2. By Proposition 3(iv),

$$(U_1\varphi)(g_\infty) = \int_{K_0 w_2 K_0} U_1(k)\varphi(g_\infty k)dk = U_1(w_2)\varphi(g_\infty w_2) = \bar{\zeta}\varphi(g_\infty w_2).$$

Next, consider $\underline{w}(2^{-1})$ in $\mathrm{SL}_2(\mathbb{Q})$. By [Gelbart 1976, (2.30)], $s_{\mathbb{Q}}(\underline{w}(2^{-1})) = \prod w_v$. Since φ is left $\mathrm{SL}_2(\mathbb{Q})$-invariant, and right K_p-invariant for $p \neq 2$,

$$\bar{\zeta}\varphi(g_\infty w_2) = \bar{\zeta}\varphi(s_{\mathbb{Q}}(\underline{w}(2^{-1}))^{-1} g_\infty w_2) = \bar{\zeta}\varphi\Big(\Big(\prod_{v\neq 2} w_v^{-1}\Big) g_\infty\Big) = \bar{\zeta}\varphi(w_\infty^{-1} g_\infty).$$

Applying Q to this equation gives (i). Part (ii) is proved analogously. □

8. Kohnen's plus space

Hecke eigenforms in $S_{r/2}(4)$ correspond to cuspidal automorphic representations π such that π_∞ is a discrete series representation of lowest weight $r/2$, π_p is unramified for all $p \neq 2$, and π_2 has $K_1(4)$-fixed vectors. In particular, $\pi_2^{K_0,\gamma} \neq 0$ for the central character $\gamma(-1) = -i^r$. If π_2 is a principal series representation, then $\pi_2^{K_0,\gamma}$ is 2-dimensional and therefore the corresponding Hecke eigenspace in $S_{r/2}(4)$ is also 2-dimensional. Kohnen's plus space is introduced to resolve this ambiguity. In terms of the space of automorphic functions $A_{r/2}(4)$, it is clear what to do. Decompose $A_{r/2}(4) = A^+_{r/2}(4) \oplus A^-_{r/2}(4)$, where $A^+_{r/2}(4)$ is the eigenspace of the local Hecke operator T_w with eigenvalue 2, while $A^-_{r/2}(4)$ is the eigenspace with eigenvalue -1. Since the presence of the eigenvalue 2 for T_w acting on π_2 eliminates a possibility that π_2 is a Steinberg representation, we see that there is a one-to-one correspondence between Hecke eigenforms in $A^+_{r/2}(4)$ and cuspidal automorphic representations π (as above) such that π_2 is a γ-unramified representation. The classical Kohnen plus space is (essentially) $Q(A^+_{r/2}(4))$, as will be explained in a moment. Niwa [1977] defines two operators $\boldsymbol{T}_4$ and $\boldsymbol{W}_4$ on $S_{r/2}(4)$ by

$$(\boldsymbol{T}_4 f)(z) = \frac{1}{4}\sum_{u=0}^{3} f((z+u)/4) \quad \text{and} \quad (\boldsymbol{W}_4 f)(z) = (-2iz)^{-r/2} f(-1/(4z)).$$

Note that $\boldsymbol{W}_4^2 = 1$. Let $\kappa = (r-1)/2$. Niwa shows that the operator

$$\boldsymbol{W} = (-1)^{(r^2-1)/8} 2^{1-\kappa} \boldsymbol{W}_4 \boldsymbol{T}_4$$

on $S_{r/2}(4)$ satisfies[1] the quadratic relation $(\boldsymbol{W}+1)(\boldsymbol{W}-2) = 0$. Kohnen defines $S^+_{r/2}(4)$ and $S^-_{r/2}(4)$ to be the eigenspaces of $\boldsymbol{W}$ on $S_{r/2}(4)$ of eigenvalues 2 and -1, respectively [Kohnen 1980]. Proposition 24 says that

$$\boldsymbol{q}(U_1) = (-1)^{r^2-1/8} \boldsymbol{W}_4 \quad \text{and} \quad \boldsymbol{q}(T_1) = 2^{3/2-\kappa} \boldsymbol{T}_4,$$

where the sign $(-1)^{r^2-1/8}$ is the quotient of

$$\bar{\zeta} = (1+i^r)/\sqrt{2} \quad \text{and} \quad i^{r/2} = ((1+i)/\sqrt{2})^r.$$

Since $T_w = \sqrt{2}^{\,-1} T_1 U_1$, it follows that $\boldsymbol{q}(T_w)$ and $\boldsymbol{W}$ are conjugates of each other by $\boldsymbol{W}_4$. Thus Kohnen's plus space is simply a conjugate of our space:

$$Q(A^+_{r/2}(4)) = \boldsymbol{W}_4(S^+_{r/2}(4)).$$

Because $\boldsymbol{W}_4$ commutes with the classical Hecke operators $\boldsymbol{T}_{p^2}$ whenever $p \neq 2$, $Q(A^+_{r/2}(4))$ and $S^+_{r/2}(4)$ are isomorphic as $\mathbb{C}[\boldsymbol{T}_{3^2}, \boldsymbol{T}_{5^2}, \dots]$-modules.

There is another description of $S^+_{r/2}(4)$ in terms of Fourier coefficients. It consists of the cusp forms whose n-th Fourier coefficient vanishes whenever $(-1)^\kappa n \equiv 2, 3 \pmod 4$. Kohnen defines a Hecke operator $\boldsymbol{T}^+_4$ that preserves $S^+_{r/2}(4)$ in the following way: For $f(z) = \sum_n a_n q^n \in S^+_{r/2}(4)$, set $(\boldsymbol{T}^+_4 f)(z) = \sum_n b_n q^n$ where the sum is taken over integers $n > 0$ and $(-1)^\kappa n \equiv 0, 1 \pmod 4$, and

$$b_n = a_{4n} + \left(\frac{(-1)^\kappa n}{2}\right) 2^{\kappa-1} a_n + 2^{r-2} a_{n/4}.$$

[1] In [Kohnen 1980], the operator is $\boldsymbol{T}_4 \boldsymbol{W}_4$ acting on the right, that is, $\boldsymbol{T}_4$ acts first and $\boldsymbol{W}_4$ follows.

Here $a_{n/4} = 0$ if n is not a multiple of 4. The large parentheses denote the Legendre symbol.

We can now formulate and prove our main global results.

Theorem 25. *There is a one-to-one correspondence between Hecke eigenforms f in $S^+_{r/2}(4)$ and irreducible cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ in $L^2(\mathrm{SL}_2(\mathbb{Q})\backslash G(\mathbb{A}))$ such that*

(i) *π_∞ is the discrete series representation of $G(\mathbb{R})$ with lowest weight $r/2$;*

(ii) *π_p is unramified for all odd primes p;*

(iii) *π_2 is γ-unramified, where $\gamma(-1) = -i^r$; and*

(iv) *if $\boldsymbol{T}^+_4 f = \lambda_2 f$, then a γ-spherical vector in π_2 is an eigenvector for $Z = T_1/2 + (T_1/2)^{-1}$ with eigenvalue $2^{1-r/2}\lambda_2$.*

Note that λ_2 determines the eigenvalue of Z on a γ-spherical vector, which in turn determines π_2 uniquely by Corollary 20.

Proof. The first three statements are clear, since $Q^{-1}(\boldsymbol{W}_4 f)$ is a Hecke eigenform in $A^+_{r/2}(4)$ that is contained in a cuspidal automorphic representation π with these properties. It remains to show (iv).

Lemma 26. *Let f be in $S^+_{r/2}(4)$. Then $\boldsymbol{T}^+_4 f = 2^{r/2-1}\boldsymbol{q}(Z) f$.*

Proof. Recall that T_1 is invertible by Proposition 8. Hence, it suffices to show that $2^{2-r/2}\boldsymbol{q}(T_1)\boldsymbol{T}^+_4 = \boldsymbol{q}(T_1^2 + 4)$. If $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{r/2}(4)$, then $(\boldsymbol{q}(T_1) f)(z) = 2^{2-r/2}\sum_{n=0}^{\infty} a_{4n} q^n$ by Proposition 24. Thus, if $f(z) \in S^+_{r/2}(4)$, then one computes

$$2^{2-r/2}(\boldsymbol{q}(T_1)\boldsymbol{T}^+_4 f)(z) = (\boldsymbol{q}(T_1^2+4) f)(z) = \sum_n (2^{4-r} a_{16n} + 4a_n) q^n. \qquad \square$$

Now we can finish the proof of Theorem 25. If $\boldsymbol{T}^+_4 f = \lambda_p f$, then Lemma 26 implies that $Q^{-1}(f)$ is an eigenform for Z with eigenvalue $2^{1-r/2}\lambda_2$. Since $\boldsymbol{W}_4 = (-1)^{(r^2-1)/8}\boldsymbol{q}(U_1)$ and Z commutes with U_1, we see that $Q^{-1}(\boldsymbol{W}_4 f)$ is also an eigenform for Z with the same eigenvalue. $\square$

If f is a Hecke eigenform in $S^+_{r/2}(4)$, then by [Kohnen 1980, Theorem 1(ii)] the corresponding Shimura lift $f' = \mathrm{Sh}(f)$ is a Hecke eigenform in $S_{r-1}(\mathrm{SL}_2(\mathbb{Z}))$. Recall that $G' = \mathrm{PGL}_2$. There is a bijection between Hecke eigenforms f' in $S_{r-1}(\mathrm{SL}_2(\mathbb{Z}))$ and irreducible cuspidal automorphic representations $\pi' = \bigotimes_v \pi'_v$ in $L^2(G'(\mathbb{Q})\backslash G'(\mathbb{A}))$ such that π'_∞ is a discrete series representation with lowest weight $r-1$ and π'_p is unramified for all primes p; see [Gelbart 1975, Proposition 3.1]. Recall the local Shimura lift $\mathrm{Sh}(\pi_2)$ in Proposition 11 of a γ-unramified representation π_2 of $G(\mathbb{Q}_2)$. The following corollary gives a precise representation-theoretic description of the Shimura correspondence at the place $p = 2$.

Corollary 27. *Let f be a Hecke eigenform in $S^+_{r/2}(4)$. Let $\pi = \bigotimes_v \pi_v$ be the cuspidal automorphic representation corresponding to f in Theorem 25. Let $\pi' = \bigotimes_v \pi'_v$ be the cuspidal automorphic representations of $L^2(G'(\mathbb{Q})\backslash G'(\mathbb{A}))$ corresponding to the Hecke eigenform $f' = \mathrm{Sh}(f)$ in $S_{r-1}(\mathrm{SL}_2(\mathbb{Z}))$. Then $\mathrm{Sh}(\pi_2) = \pi'_2$.*

Proof. If $\boldsymbol{T}^+_4 f = \lambda_2 f$, then $\boldsymbol{T}_2 f' = \lambda_2 f'$ by [Kohnen 1980, Theorem 1(ii)], where $\boldsymbol{T}_2$ is the classical Hecke operator action on $S_{r-1}(\mathrm{SL}_2(\mathbb{Z}))$. By [Gelbart 1975, Proposition 5.2.1], one checks that π'_2 is indeed isomorphic to $\mathrm{Sh}(\pi_2)$. □

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ as in Theorem 25 and π' be the corresponding cuspidal automorphic representation of $G'(\mathbb{A})$ as in Corollary 27. By the Ramanujan conjecture, proved by Deligne, $\pi'_2 = \mathrm{Sh}(\pi_2)$ is a tempered irreducible unramified representation, so $\pi'_2 = I'(s)$ for some $s \in i\mathbb{R}$. This implies that $\pi_2 = I(\gamma, s)$ by Theorem 22 and Corollary 23. Thus $\pi_2^{K_0,\gamma}$ is an irreducible $H(\gamma)$-module of dimension 2. It corresponds under Q to a two-dimensional subspace of $S_{r/2}(4)$ spanned by a line in $S^+_{r/2}(4)$ and a line in $S^-_{r/2}(4)$.

On the other hand, if $\pi_2 = St(\epsilon)$ is a Steinberg representation of $G(\mathbb{Q}_2)$ (see the definition before Corollary 18), then π corresponds under Q to an Hecke eigenform in $S^-_{r/2}(4)$. More precisely:

Theorem 28. *There is a one-to-one correspondence between Hecke eigenforms f in $S^-_{r/2}(4)$ such that $\mathbf{W}_4 f = -\epsilon(-1)^{(r^2-1)/8} f$ for some $\epsilon = \pm 1$ and irreducible cuspidal automorphic representations $\pi = \bigotimes_v \pi_v$ in $L^2(\mathrm{SL}_2(\mathbb{Q})\backslash G(\mathbb{A}))$ such that*

(i) *π_∞ is the discrete series representation of $G(\mathbb{R})$ with lowest weight $r/2$,*

(ii) *π_p is unramified for all odd primes p, and*

(iii) *π_2 is the Steinberg representation $St(\epsilon)$.*

Proof. Recall by Corollary 21 that T_w and U_1 act on the one-dimensional space $St(\epsilon)^{K_0,\gamma}$ by -1 and $-\epsilon$. The theorem now follows from Proposition 24 and the definition of $S^-_{r/2}(4)$. □

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