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PAINLEVÉ ANALYSIS OF GENERALIZED ZAKHAROV EQUATIONS

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We consider the invariance and integrability properties of the generalized Zakharov equations and obtain an exact invariant solution. We derive the Painlevé property of these equations and obtain some exact solutions using the truncated Painlevé expansion.

1. Introduction

We consider the generalized Zakharov equations (GZEs) for the complex envelope $E(x,t)$ of the high-frequency wave and the real low-frequency field $\eta(x,t)$ in the form

$$iE_t + E_{xx} - 2\beta|E|^2E + 2E\eta = 0, \tag{1}$$

$$\eta_{tt} - \eta_{xx} = -(|E|^2)_{xx}, \tag{2}$$

where the cubic term in Equation (1) describes nonlinear self-interaction in the high-frequency subsystem; such a term corresponds to a self-focusing effect in plasma physics. The coefficient β is a real constant that can be positive or negative. The sound velocity and the coupling constant in Equation (2) have been normalized to unity for simplicity. The GZEs are a universal model of interaction between high- and low-frequency waves in one dimension. The collisions between solitary waves of GZEs have been simulated in detail in [Hadouaj et al. 1991]. When $\beta = 0$, this system is reduced to the classical Zakharov equations for plasma physics.

Several aspects of the GZEs have been studied. Malomed, Anderson, Lisak, Quiroga-Teixeiro and Stenflo [1997] analyzed them using a variational approach. Wang and Li [2005] introduced periodic wave solutions using the extended F-expansion method. By considering the modified Adomian decomposition method, Wang, Dai, Wu, Lei and Zhang [2007] calculated exact and numerical solutions. Zhang [2007] constructed exact traveling wave solutions by a direct algebraic method. Li, Li and Lin [2008] used the exp-function method to obtain exact solutions. Javidi and Golbabi [2008] obtained exact and numerical solutions by the

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variational iteration method. Abbasbandy, Babolian and Ashtiani [2009] applied homotopy analysis method to obtain a solution.

The objective of this work is to investigate the symmetry and conduct Painlevé analysis of GZEs. It will be organized as follows. In Section 2, we obtain the Lie point symmetry group of GZEs and its similarity reductions. Moreover, we also construct explicit analytic invariant solutions. In Section 3, we inspect the singularity structure of GZEs by means of the Weiss–Tabor–Carnevale procedure. In Section 4, we obtain some exact solutions using the truncated Painlevé expansion.

2. Symmetry analysis

Symmetry is one of the most important concepts in the area of partial differential equations [Bluman and Kumei 1989]. To find the Lie symmetries of the GZEs, we express the complex envelope as $E(x,t) = u(x,t) + iv(x,t)$, with real high-frequency waves $u(x,t)$ and $v(x,t)$. Substituting it into Equations (1) and (2) and separating the imaginary and real parts we obtain the equations

$$\begin{aligned} u_{xx} - v_t - 2\beta(u^2+v^2)u + 2\eta u &= 0,\\ u_t + v_{xx} - 2\beta(u^2+v^2)v + 2\eta v &= 0,\\ \eta_{tt} - \eta_{xx} &= -(u^2+v^2)_{xx}. \end{aligned} \tag{3}$$

Consider a one-parameter Lie group of infinitesimal transformations of the form

$$\begin{aligned} u &\to U = u + \varepsilon\phi_1(x,t,u,v,\eta),\\ v &\to V = v + \varepsilon\phi_2(x,t,u,v,\eta),\\ \eta &\to \Lambda = \eta + \varepsilon\phi_3(x,t,u,v,\eta),\\ x &\to X = x + \varepsilon\xi_1(x,t,u,v,\eta),\\ t &\to T = t + \varepsilon\xi_2(x,t,u,v,\eta) \quad \text{for } \varepsilon \ll 1, \end{aligned} \tag{4}$$

with infinitesimal generator

$$\widetilde{X} = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial t} + \phi_1\frac{\partial}{\partial u} + \phi_2\frac{\partial}{\partial v} + \phi_3\frac{\partial}{\partial \eta}. \tag{5}$$

Require that (3) are invariant under (4) by direct substitution [Bluman and Kumei 1989]. Eliminate u_t, v_t, η_{tt} using (3), and set to zero all coefficients of the independent terms of the polynomials of u, v and η and their partial derivatives. We then obtain the determining equations for the infinitesimals. By solving these equations we have

$$\begin{aligned}
\phi_1 &= -(k_3 + k_4 t + k_5 t^2)v,\\
\phi_2 &= (k_3 + k_4 t + k_5 t^2)u,\\
\phi_3 &= \tfrac{1}{2}k_4 + k_5 t,\\
\xi_1 &= k_1, \quad \xi_2 = k_2,
\end{aligned}$$

where k_1, k_2, k_3, k_4 and k_5 are arbitrary parameters. The infinitesimal generator of the Lie algebra associated with each parameter k_i is obtained from the generator (5):

$$\begin{aligned}
L_1 &= \frac{\partial}{\partial x}, \quad L_2 = \frac{\partial}{\partial t}, \quad L_4 = tu\frac{\partial}{\partial v} - tv\frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial}{\partial \eta},\\
L_3 &= u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}, \qquad L_5 = t^2 u\frac{\partial}{\partial v} - t^2 v\frac{\partial}{\partial u} + t\frac{\partial}{\partial \eta}.
\end{aligned}$$

The similarity solution can be obtained from the invariant surface equation

$$\frac{dt}{k_2} = \frac{dx}{k_1} = \frac{du}{-(k_3+k_4t+k_5t^2)v} = \frac{dv}{(k_3+k_4t+k_5t^2)u} = \frac{d\eta}{k_4/2 + k_5 t}.$$

For the most general generator X, we obtain the similarity solution

$$\begin{aligned}
u(x,t) &= c\cos\left(\frac{k_5}{3k_2}t^3 + \frac{k_4}{2k_2}t^2 + \frac{k_3}{k_2}t + F_1(z)\right),\\
v(x,t) &= c\sin\left(\frac{k_5}{3k_2}t^3 + \frac{k_4}{2k_2}t^2 + \frac{k_3}{k_2}t + F_1(z)\right),\\
\eta(x,t) &= \frac{k_5}{2k_2}t^2 + \frac{k_4}{2k_2}t + F_2(z),
\end{aligned}$$

where F_1 and F_2 are similarity functions of the similarity variable $z = k_1 t - k_2 x$, satisfying the similarity equations

$$\begin{aligned}
0 &= -k_2^3(F_1'' - (F_1')^2) + k_1k_2F_1' - 2k_2F_2 + k_3 + 2c^2\beta k_2,\\
0 &= k_2^3(F_1'' + (F_1')^2) + k_1k_2F_1' - 2k_2F_2 + k_3 + 2c^2\beta k_2,\\
0 &= F_2'' + k_5/(k_2(k_1^2 - k_2^2)),
\end{aligned} \tag{6}$$

where prime denotes differentiation with respect to z. Solving equations (6), we obtain the solution

$$\begin{aligned}
F_1 &= \frac{1}{2k_2^3}\left(-k_1k_2 + \sqrt{(k_1k_2)^2 - 4k_2^3(k_3 - 2k_2(a_2 - c^2\beta))}\right)z + a_0,\\
F_2 &= a_1,
\end{aligned}$$

where a_0 and a_1 are arbitrary constants. Hence the exact invariant solution to

GZEs (1) and (2) can be written as

$$E(x,t) = c\exp\Bigl(\frac{i}{2k_2^3}\bigl(k_2^2\bigl(\tfrac{2}{3}k_5t^3+k_4t^2+2k_3t\bigr) \\ +(k_1t-k_2x)\sqrt{(k_1k_2)^2-4k_2^3(k_3-2k_2(a_1-c^2\beta))}-k_1k_2+2k_2^3a_0\bigr)\Bigr),$$

$$\eta(x,t) = \frac{k_5}{2k_2}t^2+\frac{k_4}{2k_2}t+a_1.$$

3. Painlevé analysis

The Painlevé test is one of the impressive ways to test whether partial differential equations are integrable or nonintegrable. Myrzakulov [1999] has confirmed the Painlevé nature of the (2+1)-dimensional Zakharov equation. In this paper we will show that GZEs possess the Painlevé property. Various approaches can be applied to investigate the Painlevé integrability. Here we will use WTC method [Weiss et al. 1983]. Consider Laurent expansion of the solutions of (3) in a local neighborhood of a movable singular manifold $\phi(x,t)=0$:

$$u(x,t)=\sum_{j=0}^{\infty}u_j\phi^{j+\alpha_1},\quad v(x,t)=\sum_{j=0}^{\infty}v_j\phi^{j+\alpha_2},\quad \eta(x,t)=\sum_{j=0}^{\infty}\eta_j\phi^{j+\alpha_3},$$

where $u_j(x,t)$, $v_j(x,t)$ and $\eta_j(x,t)$ are analytic functions and the three α_i are integers to be determined. It is sufficient to substitute

$$u(x,t)=u_0\phi^{\alpha_1},\quad v(x,t)=v_0\phi^{\alpha_2},\quad \eta(x,t)=\eta_0\phi^{\alpha_3},$$

into equations (3) to find the dominant behavior and leading exponents α_i. Balancing the dominant terms, we get $\alpha_1=\alpha_2=-1$ and $\alpha_3=-2$, and the equations

$$u_0^2+v_0^2=\phi_x^2-\frac{\phi_x^4}{\beta^2\phi_t^2+\beta(1-\beta)\phi_x^2}\quad\text{and}\quad \eta_0=-\frac{\phi_x^4}{\beta^2\phi_t^2+(1-\beta)\phi_x^2}.$$

The first shows that either u_0 or v_0 is arbitrary. Substituting the series expansion

$$u=u_0\phi^{-1}+u_r\phi^{r-1},\quad v=v_0\phi^{-1}+v_r\phi^{r-1},\quad \eta=\eta_0\phi^{-2}+\eta_r\phi^{r-2},$$

into equations (3) and collecting the coefficients of u_r, v_r and η_r for leading terms, the resonance are found to have the values $r=-1,0,2,3,3,4$. As usual, the resonance $r=-1$ represents the arbitrariness of the singularity manifold $\phi(x,t)=0$, and the remaining resonance values indicate the arbitrariness of five functions (one of $\{u_0, v_0, \eta_0\}$, one of $\{u_2, v_2, \eta_2\}$, two of $\{u_3, v_3, \eta_3\}$ and one of $\{u_4, v_4, \eta_4\}$).

4. Some solutions to the generalized Zakharov equation

When an equation has the Painlevé property, one can find some kinds of exact solutions by truncating the Painlevé expansion of the solution about the movable singularity manifold at the constant level term, leading to the so-called Backlund transformation of the equation. For GZEs, the transformation has the form

$$u = u_0\phi^{-1} + u_1, \quad v = v_0\phi^{-1} + v_1, \quad \eta = \eta_0\phi^{-2} + \eta_1\phi^{-1} + \eta_2. \tag{7}$$

Our approach to finding an exact solution for u, v and η is similar to that of [Choudhury 2006]. Substituting equations (7) into equations (3) and equating the coefficients of ϕ^{-i} for $i = 0, 1, 2, 3, 4$ to zero, we obtain this system of PDEs:

$$0 = -2\beta u_1^2 v_1 - 2\beta v_1^3 + 2v_1\eta_2 + u_{1,t} + v_{1,xx}, \tag{8}$$

$$0 = -2\beta u_1^2 v_0 - 4\beta u_0 u_1 v_1 - 6\beta v_0 v_1^2 + 2v_1\eta_1 + 2v_0\eta_2 + u_{0,t} + v_{0,xx}, \tag{9}$$

$$0 = -4\beta u_0 u_1 v_0 - 2\beta u_0^2 v_1 - 6\beta v_0^2 v_1 + 2v_1\eta_0 + 2v_0\eta_1 - u_0\phi_t - 2v_{0,x}\phi_x - v_0\phi_{xx}, \tag{10}$$

$$0 = -2\beta u_0^2 v_0 - 2\beta v_0^3 + 2v_0\eta_0 + 2v_0\phi_x^2, \tag{11}$$

$$0 = -2\beta u_1^3 - 2\beta u_1 v_1^2 + 2u_1\eta_2 - v_{1,t} + u_{1,xx}, \tag{12}$$

$$0 = -6\beta u_0 u_1^2 - 4\beta u_1 v_0 v_1 - 2\beta u_0 v_1^2 + 2u_1\eta_1 + 2u_0\eta_2 - v_{0,t} + u_{0,xx}, \tag{13}$$

$$0 = -6\beta u_0^2 u_1 - 2\beta u_1 v_0^2 - 4\beta u_0 v_0 v_1 + 2u_1\eta_0 + 2u_0\eta_1 + v_0\phi_t - 2u_{0,x}\phi_x - u_0\phi_{xx}, \tag{14}$$

$$0 = -2\beta u_0^3 - 2\beta u_0 v_0^2 + 2u_0\eta_0 + 2u_0\phi_x^2, \tag{15}$$

$$0 = \eta_{2,tt} + 2u_{1,x}^2 + 2v_{1,x}^2 + 2u_1 u_{1,xx} + 2v_1 v_{1,xx} - \eta_{2,xx}, \tag{16}$$

$$\begin{aligned}0 = \eta_{1,tt} &+ 4u_{0,x}u_{1,x} + 4v_{0,x}v_{1,x} + 2u_1 u_{0,xx} + 2u_0 u_{1,xx}\\ &+ 2v_1 v_{0,xx} + 2v_0 v_{1,xx} - \eta_{1,xx},\end{aligned} \tag{17}$$

$$\begin{aligned}0 = -2\eta_{1,t}\phi_t &- \eta_1\phi_{tt} + \eta_{0,tt} - 4u_1\phi_x u_{0,x} + 2u_{0,x}^2 - 4u_0\phi_x u_{1,x}\\ &- 4v_1\phi_x v_{0,x} + 2v_{0,x}^2 - 4v_0\phi_x v_{1,x} + 2\phi_x\eta_{1,x} - 2u_0 u_1\phi_{xx}\\ &- 2v_0 v_1\phi_{xx} + \eta_1\phi_{xx} + 2u_0 u_{0,xx} + 2v_0 v_{0,xx} - \eta_{0,xx},\end{aligned} \tag{18}$$

$$\begin{aligned}0 = 2\eta_1\phi_t^2 &- 4\phi_t\eta_{0,t} - 2\eta_0\phi_{tt} + 4u_0 u_1\phi_x^2 + 4v_0 v_1\phi_x^2 - 2\eta_1\phi_x^2\\ &- 8u_0\phi_x u_{0,x} - 8v_0\phi_x v_{0,x} + 4\phi_x\eta_{0,x} - 2u_0^2\phi_{xx} - 2v_0^2\phi_{xx} + 2\eta_0\phi_{xx},\end{aligned} \tag{19}$$

$$0 = 6\eta_0\phi_t^2 + 6u_0^2\phi_x^2 + 6v_0^2\phi_x^2 - 6\eta_0\phi_x^2. \tag{20}$$

Substituting a trial solution $\phi(x,t) = 1 + \exp(iQ(x,t))$ into system (8)–(20), we consider two cases:

Case 1. Let $v_0(x,t) = \exp(iQ(x,t))$ be the arbitrary function; then equation (11), (15) and (10) become

$$0 = -2e^{iQ}(e^{2iQ}(\beta + Q_x^2) + \beta u_0^2 - \eta_0), \tag{21}$$

$$0 = -2u_0(e^{2iQ}(\beta + Q_x^2) + \beta u_0^2 - \eta_0), \tag{22}$$

$$\begin{aligned}0 = -2\beta u_0^2 v_1 &+ v_1(-6\beta e^{2iQ} + 2\eta_0) + 2e^{iQ}\eta_1\\ &- e^{iQ}u_0(4\beta u_1 + iQ_t) + 3e^{2iQ}Q_x^2 - ie^{2iQ}Q_{xx}.\end{aligned} \tag{23}$$

It is convenient to separate (23) into two equations:

$$0 = -e^{iQ}u_0(iQ_t) - ie^{2iQ}Q_{xx},$$
$$0 = -2\beta u_0^2 v_1 + v_1(-6\beta e^{2iQ} + 2\eta_0) + 2e^{iQ}\eta_1 - 4\beta e^{iQ}u_0 u_1 + 3e^{2iQ}Q_x^2.$$

Solving the first for u_0 we get $u_0(x,t) = -Q_{xx}e^{iQ}/Q_t$. Substituting this into (21) we obtain

$$\eta_0(x,t) = (\beta Q_t^2 + Q_t^2 Q_x^2 + \beta Q_{xx}^2)e^{2iQ}/Q_t^2.$$

Letting $Q(x,t) = f_1(t)x + f_2(t)$ and substituting this into system (8)–(20), we find that (20) has the form

$$0 = 6e^{4i(f_1 x + f_2)}(f_1^4 - \beta(xf_1' + f_2')^2 + f_1^2(-1 + \beta - x^2 f_1'^2 - 2xf_1'f_2' - f_2'^2)),$$
$$0 = f_1^4 - \beta(xf_1' + f_2')^2 + f_1^2(-1 + \beta - x^2 f_1'^2 - 2xf_1'f_2' - f_2'^2).$$

Equating the coefficients of x^2, x^1 and x^0 in the second equation to zero, we obtain

$$0 = -f_1^2 + \beta f_1^2 + f_1^4 - \beta f_2'^2 - f_1^2 f_2'^2,$$
$$0 = -2\beta f_1' f_2' - 2f_1^2 f_1' f_2',$$
$$0 = -f_1'^2(\beta + f_1^2).$$

The first two are satisfied if $f_1(t) = c_1$. Substituting this into the last and solving for f_2, we get

$$f_2(t) = \sqrt{\frac{-c_1^2(c_1^2 + \beta - 1)}{-\beta - c_1^2}}\,t + c_2.$$

Substituting all the above results into equations (10), (13) and (14) and solving for v_1, u_1 and η_1 we find $v_1(x,t) = -1/2$,

$$u_1(x,t) = \frac{i\sqrt{(\beta + c_1^2)(\beta + c_1^2 - 1)}}{2c_1(\beta + c_1^2)} \quad \text{and} \quad \eta_1(x,t) = -(\beta + c_1^2)\exp(ih(x,t)),$$

where $h(x,t) = c_1 x + c_2 + \sqrt{c_1^2(1 - \beta - c_1^2)}\,t/\sqrt{-\beta - c_1^2}$. Solving (8) or (9) or (12) for η_2, we get

$$\eta_2(x,t) = \frac{\beta(1 - c_1^2 + c_1^4) - \beta^2(1 - c_1^2)}{4c_1^2(\beta + c_1^2)},$$

Substituting all the results into the truncated expansion (7), we obtain the corresponding exact solution of GZEs (3):

$$u(x,t) = -\frac{i\sqrt{1 - \beta - c_1^2}}{2c_1\sqrt{-\beta - c_1^2}}, \quad v(x,t) = -\frac{1}{2} + \frac{\exp(ih(x,t))}{1 + \exp(ih(x,t))},$$

$$\eta(x,t) = \frac{\beta(1 - c_1^2 + c_1^4) - \beta^2(1 - c_1^2)}{4c_1^2(\beta + c_1^2)} + \frac{(\beta + c_1^2)\exp(ih(x,t))}{1 + \exp(ih(x,t))}\left(\frac{\exp(ih(x,t))}{1 + \exp(ih(x,t))} - 1\right).$$

Figures 1, 2, 3, 4 and 5 illustrate the solution for certain β, c_1 and c_2.

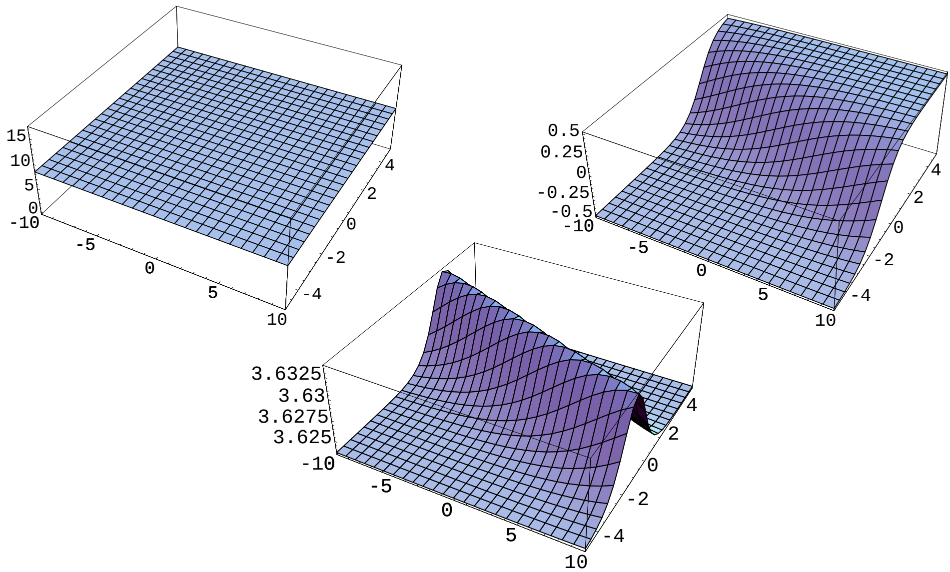

Figure 1. Clockwise from top left: Graphs of $u(x,t)$, $v(x,t)$ and $\eta(x,t)$ for $\beta = 0.05$, $c_1 = -0.3i$ and $c_2 = i$.

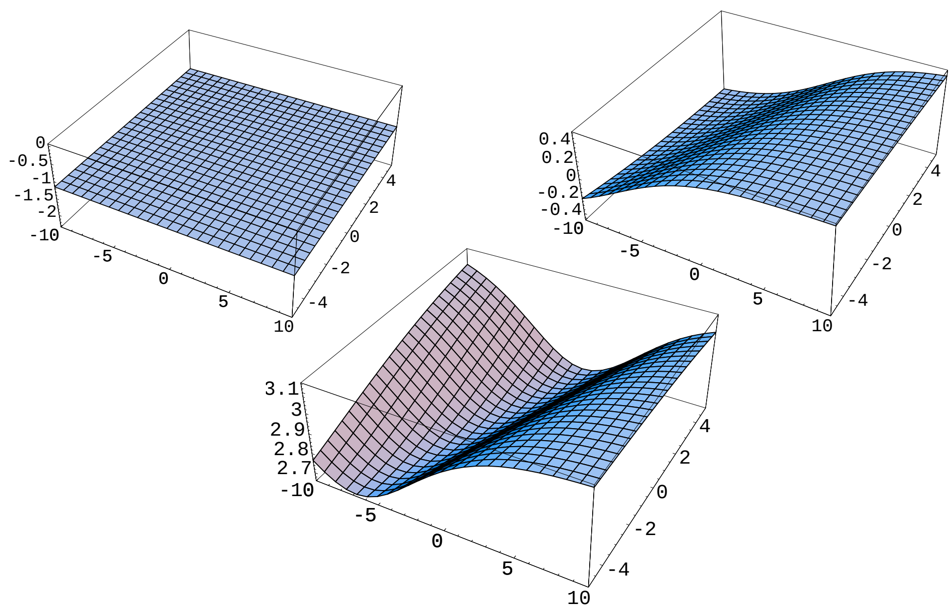

Figure 2. Clockwise from top left: Graphs of $u(x,t)$, $v(x,t)$ and $\eta(x,t)$ for $\beta = 2$, $c_1 = -0.3i$ and $c_2 = -i$.

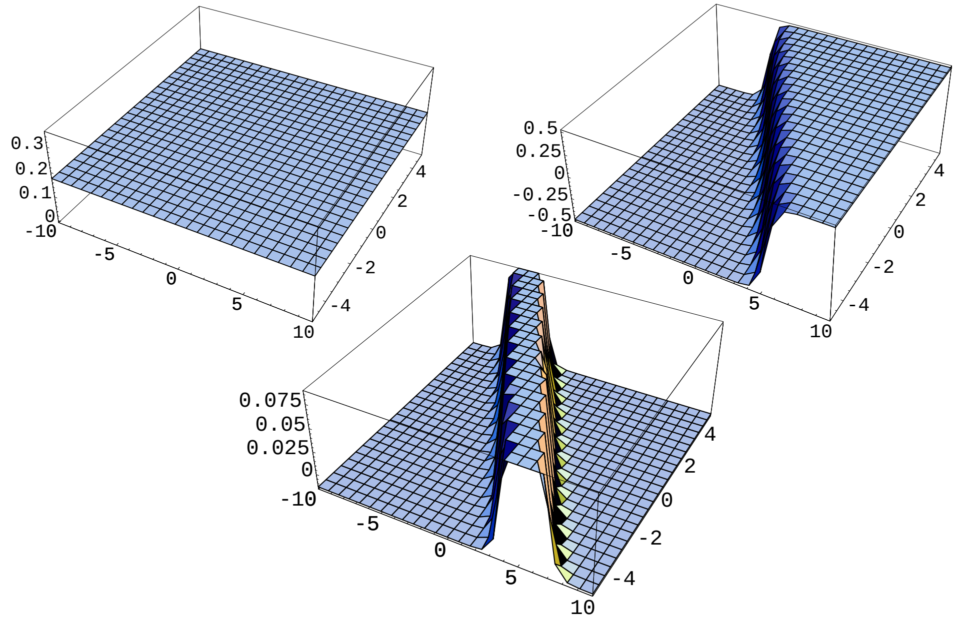

Figure 3. Clockwise from top left: Graphs of $u(x,t)$, $v(x,t)$ and $\eta(x,t)$ for $\beta=-0.05$, $c_1=-3i$ and $c_2=0.4i$.

Case 2: Let

$$u_0(x,t)=\exp(iQ(x,t))$$

be the arbitrary function. Then separate (14) into two equations:

$$0=iv_0Q_t-ie^{iQ}Q_{xx},$$
$$0=-2u_1(3\beta e^{2iQ}+\beta v_0^2-\eta_0)+e^{iQ}(2\eta_1+v_0(-4\beta v_1+iQ_t)+3e^{iQ}Q_x^2).$$

From the first we obtain $v_0(x,t)=Q_{xx}e^{iQ}/Q_t$. Solving (11) for η_0, we get

$$\eta_0(x,t)=(\beta Q_t^2+Q_t^2Q_x^2+\beta Q_{xx}^2)e^{2iQ}/Q_t^2.$$

Letting $Q(x,t)=f_1(t)x+f_2(t)$ and substituting this into system (8)–(20), we find that (20) assumes the form

$$0=6e^{4i(f_1x+f_2)}(f_1^4-\beta(xf_1'+f_2')^2+f_1^2(-1+\beta-x^2f_1'^2-2xf_1'f_2'-f_2'^2)),$$
$$0=f_1^4-\beta(xf_1'+f_2')^2+f_1^2(-1+\beta-x^2f_1'^2-2xf_1'f_2'-f_2'^2).$$

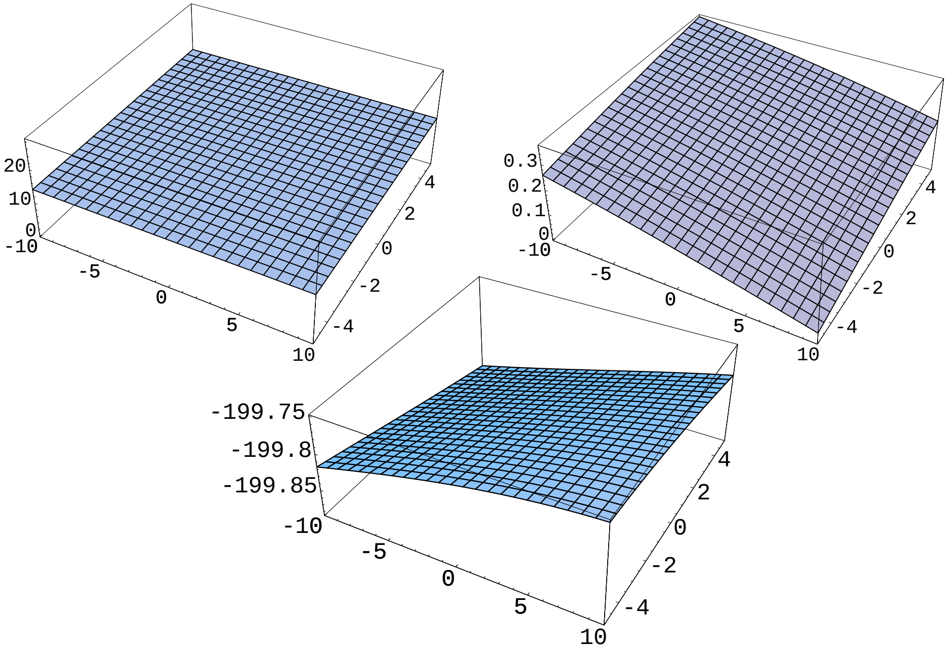

Figure 4. Clockwise from top left: Graphs of $u(x,t)$, $v(x,t)$ and $\eta(x,t)$ for $\beta = -1$, $c_1 = 0.05i$ and $c_2 = -i$.

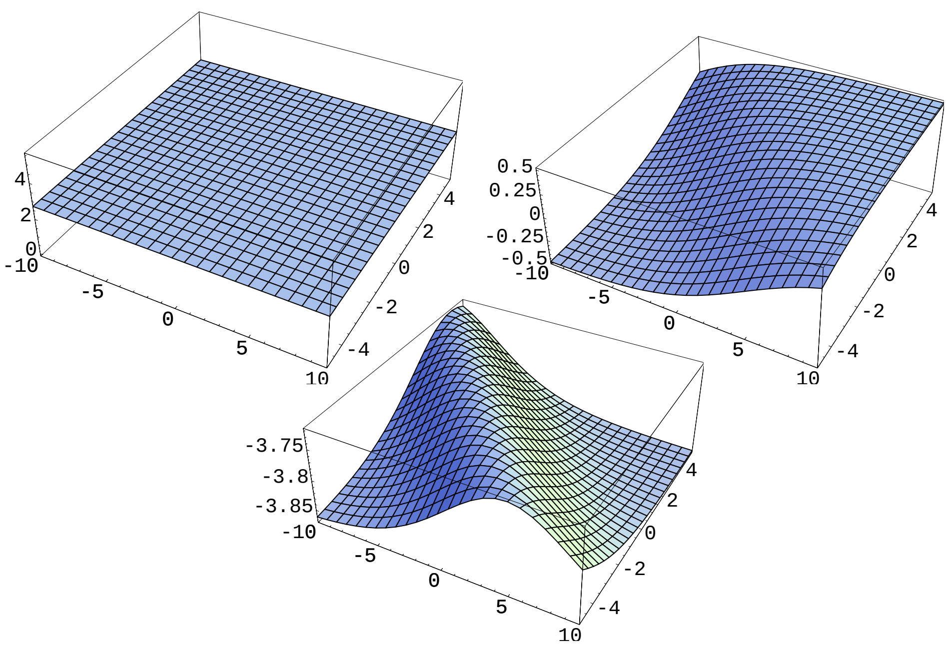

Figure 5. Clockwise from top left: Graphs of $u(x,t)$, $v(x,t)$ and $\eta(x,t)$ for $\beta = -0.5$, $c_1 = -0.3$ and $c_2 = -i$.

Equating the coefficients of x^2, x^1 and x^0 in the second of these to zero, we obtain

$$\begin{aligned} 0 &= -f_1^2 + \beta f_1^2 + f_1^4 - \beta f_2'^2 - f_1^2 f_2'^2, \\ 0 &= -2\beta f_1' f_2' - 2 f_1^2 f_1' f_2', \\ 0 &= -f_1'^2(\beta + f_1^2). \end{aligned}$$

The latter two are satisfied if $f_1(t) = c_1$. Substituting this into the first equation above and solving for f_2, we get

$$f_2(t) = t\sqrt{\frac{-c_1^2(c_1^2+\beta-1)}{-\beta-c_1^2}} + c_2.$$

Substituting all the above results into Equations (10), (14) and (19) and solving for υ_1, u_1 and η_1 we find

$$\begin{aligned} u_1(x,t) &= -1/2, \\ \upsilon_1(x,t) &= \frac{i\sqrt{(-\beta-c_1^2+1)}}{2c_1(\beta+c_1^2)}, \\ \eta_1(x,t) &= -(\beta+c_1^2)\exp(ih(x,t)), \end{aligned}$$

where $h(x,t) = c_1 x + c_2 + t\sqrt{c_1^2(1-\beta-c_1^2)}/\sqrt{-\beta-c_1^2}$. Solving (8) or (12) or (13) for η_2, we get

$$\eta_2(x,t) = \frac{\beta(1-c_1^2+c_1^4) - \beta^2(1-c_1^2)}{4c_1^2(\beta+c_1^2)},$$

Substituting all the results into the truncated expansion (7), we obtain the corresponding exact solution of GZEs (3):

$$\begin{aligned} \upsilon(x,t) &= \frac{i\sqrt{1-\beta-c_1^2}}{2c_1\sqrt{-\beta-c_1^2}} \\ u(x,t) &= -\frac{1}{2} + \frac{\exp(ih(x,t))}{1+\exp(ih(x,t))} \\ \eta(x,t) &= \frac{\beta(1-c_1^2+c_1^4) - \beta^2(1-c_1^2)}{4c_1^2(\beta+c_1^2)} \\ &\qquad + \frac{(\beta+c_1^2)\exp(ih(x,t))}{1+\exp(ih(x,t))}\left(\frac{\exp(ih(x,t))}{1+\exp(ih(x,t))} - 1\right). \end{aligned}$$

Figures 6, 7, 8, 9 and 10 illustrate the solution for certain β, c_1 and c_2.

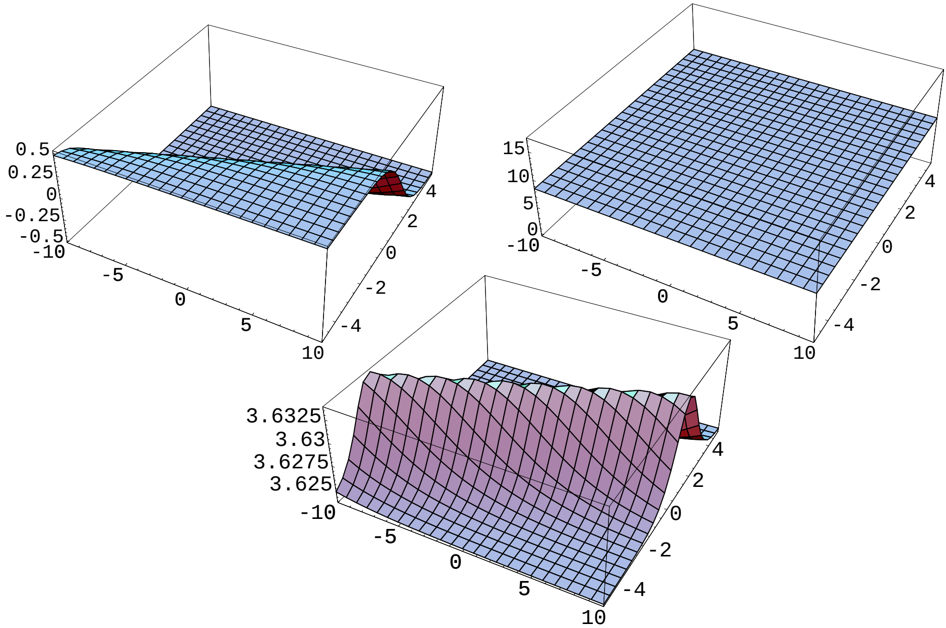

Figure 6. Clockwise from top left: Graphs of $u(x,t)$, $v(x,t)$ and $\eta(x,t)$ for $\beta = 0.05$, $c_1 = -0.3i$ and $c_2 = i$.

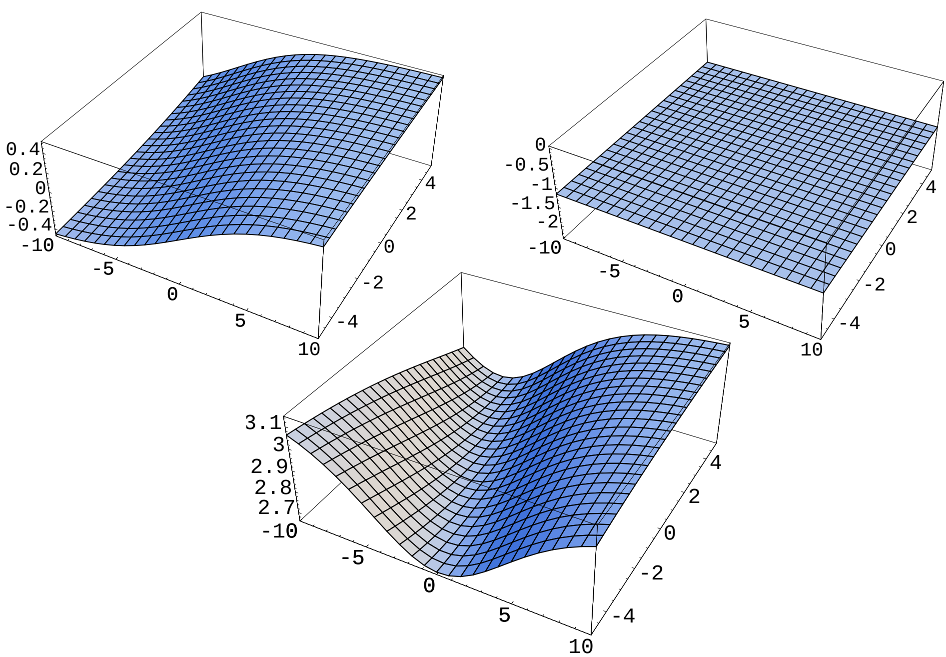

Figure 7. Clockwise from top left: Graphs of $u(x,t)$, $v(x,t)$ and $\eta(x,t)$ for $\beta = 2$, $c_1 = -0.3i$ and $c_2 = -i$.

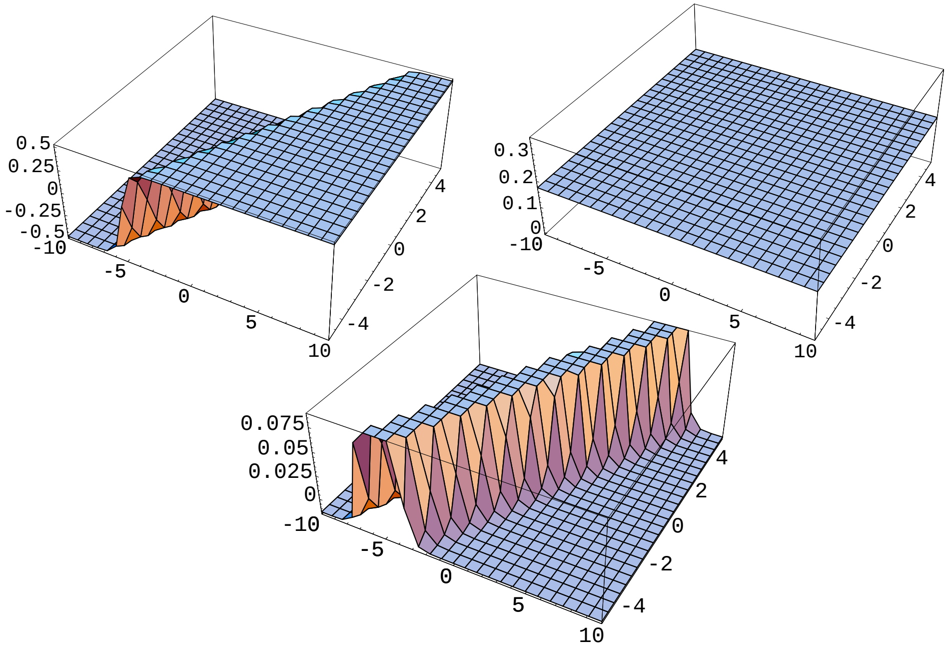

Figure 8. Clockwise from top left: Graphs of $u(x,t)$, $v(x,t)$ and $\eta(x,t)$ for $\beta = -0.05$, $c_1 = -3i$ and $c_2 = 0.4i$.

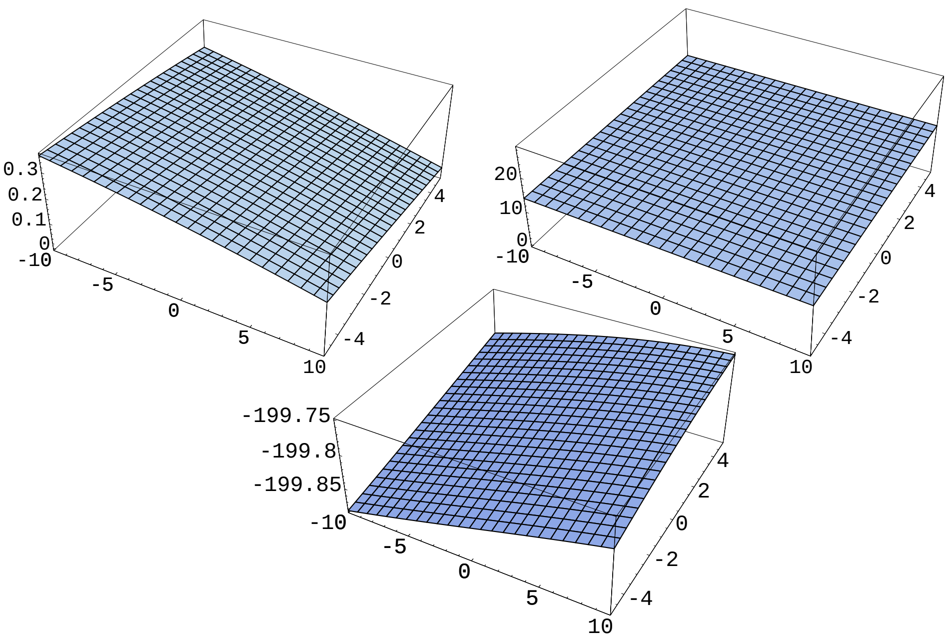

Figure 9. Clockwise from top left: Graphs of $u(x,t)$, $v(x,t)$ and $\eta(x,t)$ for $\beta = -1$, $c_1 = 0.05$ and $c_2 = -i$.

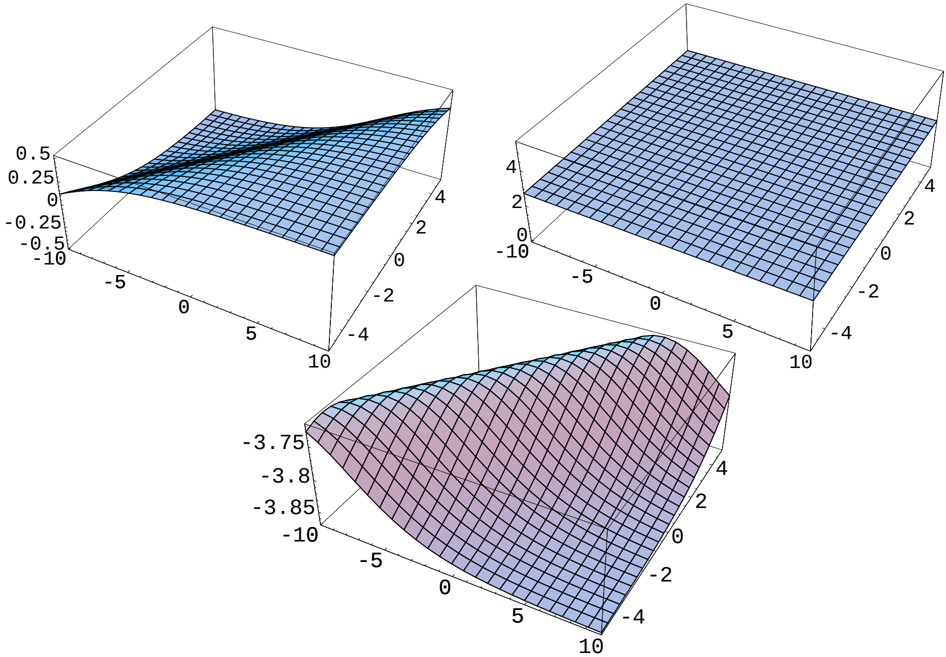

Figure 10. Clockwise from top left: Graphs of $u(x,t)$, $v(x,t)$ and $\eta(x,t)$ for $\beta = -0.5$, $c_1 = -0.3$ and $c_2 = -i$.

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