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PARASURFACE GROUPS

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A residually nilpotent group is *k-parafree* if all of its lower central series quotients match those of a free group of rank k. Magnus proved that k-parafree groups of rank k are themselves free. We mimic this theory with surface groups playing the role of free groups. Our main result shows that the analog of Magnus' theorem is false in this setting.

Introduction

This article is motivated by three stories. The first story concerns a theorem of Wilhelm Magnus. Recall that the *lower central series* of a group G is defined to be

$$\gamma_1(G) := G \text{ and } \gamma_k(G) := [G, \gamma_{k-1}(G)] \quad \text{for } k \geq 2,$$

where $[A, B]$ denotes the group generated by commutators of elements of A with elements of B. The *rank* of G is the size of a minimal generating set of G. In [1939], Magnus gave a beautiful characterization of free groups in terms of their lower central series.

Theorem (Magnus' theorem on parafree groups). *Let F_k be a nonabelian free group of rank k and G a group of rank k. If $G/\gamma_i(G) \cong F_k/\gamma_i(F_k)$ for all i, then $G \cong F_k$.*

Following this result, Hanna Neumann inquired whether it was possible for two residually nilpotent groups G and G' to have $G/\gamma_i(G) \cong G'/\gamma_i(G')$ for all i without having $G \cong G'$; see [Liriano 2007]. Gilbert Baumslag [1967] gave a positive answer to this question by constructing what are now known as parafree groups that are not themselves free. A group G is *parafree* if

(1) G is residually nilpotent, and

(2) there exists a finitely generated free group F such that $G/\gamma_i(G) \cong F/\gamma_i(F)$ for all i.

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By Magnus' theorem, Baumslag's examples necessarily have rank different from the corresponding free group. In this paper we give new examples addressing Neumann's question. Specifically, we construct residually nilpotent groups G that share the same lower central series quotients with a surface group but are not themselves surface groups. These examples are analogous to Baumslag's parafree groups, where the role of free groups is replaced by surface groups. Consequently, we call such groups *parasurface groups*. Since the parasurface examples constructed in this paper have the same rank as their corresponding surface groups, the analog of Magnus' theorem for parasurface groups is false.

Theorem 1. *Let Γ_g be the genus g surface group. There exists a rank $2g$ residually nilpotent group G such that $G/\gamma_i(G) \cong \Gamma_g/\gamma_i(\Gamma_g)$ for all i.*

We now turn towards the second story, which concerns residual properties in free groups. Our second story requires some notation. We say two elements $g, h \in G$ are *nilpotent-conjugacy equivalent* if the images of g and h in all nilpotent quotients of G are conjugate. We say a group G is *conjugacy-nilpotent separable* if any pair of nilpotent-conjugacy equivalent elements must be conjugate. Free groups are known to be conjugacy-nilpotent separable [Lyndon and Schupp 2001; Paris 2009]. A natural question to ask is whether the role of inner automorphisms may be played by automorphisms. In this vein, we say two elements $g, h \in G$ are *automorphism equivalent* if there exists some $\phi \in \operatorname{Aut} G$ with $\phi(g) = h$. Further, we say g is *nilpotent-automorphism equivalent* to h if the images of g and h in all nilpotent quotients of G are automorphism equivalent. A group G is *automorphism-nilpotent separable* if all pairs of nilpotent-automorphism equivalent elements must be automorphism equivalent. Free groups are conjugacy-nilpotent separable; however:

Theorem 2. *Nonabelian free groups of even rank are not automorphism-nilpotent separable.*

In the same flavor, Orin Chein [1969] showed that there exist automorphisms of nilpotent quotients of F_3 that do not lift to automorphisms of F_3.

Our third and final story concerns Magnus' conjugacy theorem for groups with one defining relator [Magnus et al. 1976, Theorem 4.11, page 261], which we state below. Let $\langle\langle w \rangle\rangle_H$ be the group generated by the H-conjugates of w for a subgroup H of G.

Theorem (Magnus' conjugacy theorem for groups with one defining relator). *Let s and t be elements in F_k such that $\langle\langle s \rangle\rangle_{F_k} = \langle\langle t \rangle\rangle_{F_k}$. Then s is conjugate to t^ϵ for $\epsilon = 1$ or -1.*

Our next result demonstrates that this theorem does *not* generalize to one-relator nilpotent groups. Let $F_{k,i} = F_k/\gamma_i(F_k)$ be the *free rank k, i-step nilpotent quotient*. Let $\phi_{k,i}$ be the projection $F_k \to F_{k,i}$.

Theorem 3. *Let $F_4 = \langle a, b, c, d\rangle$ be the free group of rank 4. Let $w = [a, b][c, d]$. Then*

$$\langle\!\langle \phi_{4,i}(w[w, bwb^{-1}])\rangle\!\rangle_{F_{4,i}} = \langle\!\langle \phi_{4,i}\,(w)\rangle\!\rangle_{F_{4,i}} \quad \textit{for all } i.$$

However, for large i, the element w is not conjugate to $w[w, bwb^{-1}]$ in $F_{4,i}$.

1. Almost surface groups

Let Γ_g be the fundamental group of a closed hyperbolic surface of genus g. A group G is a *weakly g-parasurface group* if $G/\gamma_k(G) \cong \Gamma_g/\gamma_k(\Gamma_g)$ for all $k \geq 1$. If G is weakly g-parasurface and residually nilpotent, we say that G is *g-parasurface*. Let $G = F/N$ be a weakly g-parasurface group where F is a free group of rank $2g$ on generators $a_1, a_2, \ldots, a_{2g-1}, a_{2g}$. Set $w = [a_1, a_2]\cdots[a_{2g-1}, a_{2g}]$. Recall that $\langle\!\langle w\rangle\!\rangle_F$ is the normal closure of w in F. Then we have the following trichotomy (see Theorem 4 below) for such groups G:

(Type I) There exists an isomorphism $\phi : F \to F$ such that $\phi(N) \geq \langle\!\langle w\rangle\!\rangle_F$.

(Type II) There exists an isomorphism $\phi : F \to F$ such that $\phi(N) < \langle\!\langle w\rangle\!\rangle_F$.

(Type III) G is not of Type I or II.

The following theorem demonstrates that only examples of Type I or III may be residually nilpotent.

Theorem 4. *Groups of Type I must be surface groups. Further, groups of Type II are never parasurface.*

Our next two theorems show that although surface groups are residually nilpotent, there exist examples of Type II and III. That is, weakly parasurface groups that are not parasurface groups exist. Further, parasurface groups that are not surface groups exist.

Theorem 5. *Let $k > 2$ be even, and let $F_k = \langle a_1, a_2, \ldots, a_k\rangle$. Suppose $w = [a_1, a_2]\cdots[a_{k-1}, a_k]$, and let γ be an element in F_k. If $w[w, \gamma w\gamma^{-1}]$ is cyclically reduced and of different word length than w in F_k, then the group*

$$G = \langle a_1, a_2, \ldots, a_k : w[w, \gamma w\gamma^{-1}]\rangle$$

is weakly $k/2$-parasurface of Type II.

In Theorem 5, one can take $\gamma = a_2$, for example.

Theorem 6. *Let $k > 2$ be even, let $F_k = \langle a_1, a_2, \ldots, a_k\rangle$, and let δ be in the commutator subgroup of $F(a_1, a_2)$. If $[a_1\delta, a_2]$ is cyclically reduced and is of different word length than $[a_1, a_2]$ in $F(a_1, a_2)$, then the group*

$$G = \langle a_1, a_2, \ldots, a_k : [a_1\delta, a_2][a_3, a_4]\cdots[a_{k-1}, a_k]\rangle$$

is $k/2$-parasurface of Type III.

In Theorem 6 one can take $\delta = [[a_1, a_2], a_1]$, for example.

2. Proofs of the main results

2.1. *Preliminaries.* We first list a couple of results needed in the proofs of our main theorems. The first is from Magnus, Korass and Solitar [1976, Lemma 5.9, page 350].

Lemma 7. *The Frattini subgroup of a nilpotent group contains the derived subgroup.*

For the following theorem, see [Azarov 1998, Theorem 1].

Theorem 8 (Azarov's theorem). *Let A and B be free groups, and let α and β be nonidentity elements of the groups A and B, respectively. Let $G = (A * B; \alpha = \beta)$. Let n be the largest positive integer such that $y^n = \beta$ has a solution in B. If $n = 1$, then G is a residually finite p-group.*

2.2. *The proofs.* Before proving Theorems 1, 2, and 3 from the introduction, we prove Theorems 4, 5, and 6.

Proof of Theorem 4. We first show that groups of Type I must be surface groups. For the sake of a contradiction, suppose that G is weakly g-parasurface of Type I and is not isomorphic to Γ_g. Let F and $K = \langle\langle w \rangle\rangle_F$ be as in the definition of Type I groups. By assumption, there exists an isomorphism $\phi : F \to F$ such that $\phi(N) \geq K$. The isomorphism ϕ^{-1} induces a homomorphism $\rho_i : \Gamma_g/\gamma_i(\Gamma_g) \to G/\gamma_i(G)$ that is surjective for all i. Since finitely generated nilpotent groups are Hopfian (see for example [de la Harpe 2000, Section III.A.19]), the maps ρ_i must be isomorphisms for all i. On the other hand, since G is not isomorphic to Γ_g, we must have some $\gamma \in \phi(N) - K$. Further, $F/K = \Gamma_g$ is residually nilpotent, so there exists some i such that $\gamma \neq 1$ in $\Gamma_g/\gamma_i(\Gamma_g)$. Since $\gamma \in \ker \rho_i$, we have a contradiction.

We now show that groups of Type II are never residually nilpotent. For the sake of a contradiction, suppose that G is a residually nilpotent group of Type II. Let F and $K = \langle\langle w \rangle\rangle_F$ be as in the definition of Type II groups. By assumption, the map $\phi : F \to F$ induces a map $\psi : G \to \Gamma_g$ that is onto with nontrivial kernel. Let $\gamma \in \ker \psi$. Since G is residually nilpotent, there exists i such that $g \notin \gamma_i(G)$. Hence, the induced map $\rho_i : G/\gamma_i(G) \to \Gamma_g/\gamma_i(\Gamma_g)$ is onto but not bijective, which is impossible since finitely generated nilpotent groups are Hopfian. □

Proof of Theorem 5. Let G, w, and γ be as in the statement of Theorem 5. We first show that G is weakly parasurface:

Claim 9. *We have $w = 1$ in all quotients $G/\gamma_i(G)$.*

Proof of claim. Let $H_1 = \langle\langle \phi_{k,i}(w[w, \gamma w \gamma^{-1}]) \rangle\rangle$ and $H_2 = \langle\langle \phi_{k,i}(w) \rangle\rangle$ in $F_k/\gamma_i(F_k)$. Clearly $H_1 \le H_2$. Further, the image of H_1 in $H_2/[H_2, H_2]$ generates as

$$\phi_{k,i}(w[w, \gamma w \gamma^{-1}]) = \phi_{k,i}(w) \mod [H_2, H_2].$$

Hence, since H_2 is nilpotent, Lemma 7 implies that $H_1 = H_2$, and so the claim follows. □

If $w \neq 1$ in G, then the claim also shows that G is not residually nilpotent. Suppose, for the sake of a contradiction, that $w = 1$ in G. Then by Magnus' conjugacy theorem for groups with one defining relator, w^ϵ and $w[w, \gamma w \gamma^{-1}]$ are conjugate for $\epsilon = 1$ or -1, but this is impossible since they are both cyclically reduced words of different word lengths [Magnus et al. 1976, Theorem 1.3, page 36]. Hence G is not residually nilpotent and G is not parasurface. Further, as $w = 1$ in all nilpotent quotients, G is weakly parasurface. The proof of Theorem 5 is complete. □

Proof of Theorem 6. Let G and δ be as in the statement of Theorem 6. That G is residually nilpotent follows from Theorem 8 applied to

$$A = F_{k-2} = \langle a_1, a_2, \dots, a_{k-2} \rangle \quad \text{and} \quad B = F_2 = \langle a_{k-1}, a_k \rangle,$$

with $\alpha = [a_1\delta, a_2][a_3, a_4] \cdots [a_{k-3}, a_{k-2}]$ and $\beta = ([a_{k-1}, a_k])^{-1}$ and the following claim:

Claim 10. *If $\beta = y^n$ in B, where $y \in B$, then $n = 1$.*

Proof of claim. If $\beta = y^n$ for some $n > 1$, then since $B/\langle\langle \beta \rangle\rangle$ is torsion-free, $y \in \langle\langle \beta \rangle\rangle$. Hence, $\langle\langle y \rangle\rangle = \langle\langle \beta \rangle\rangle$, and so by Magnus' conjugacy theorem for groups with one defining relator, y is conjugate to β or β^{-1}. But then β would have the same cyclically reduced form word length as β^n, contradicting [Magnus et al. 1976, Theorem 1.3, page 36]. □

We now show that G is not a surface group. Let $H = \langle a_1\delta, a_2, \dots, a_{k-1}, a_k \rangle \le G$. The next two claims imply that H is a nonfree proper subgroup of G of rank k. However, if G were a surface group, it would have to be the surface group $\Gamma_{k/2}$, of genus $k/2$. Any rank k proper subgroup of $\Gamma_{k/2}$ must be free, so H must be free, a contradiction.

Claim 11. *H is not a free group and is of rank k.*

Proof of claim. H is rank k, since $\{a_1\delta, a_2, \dots, a_k\}$ generate $G/[G, G]$ and since $G/[G, G] = \mathbb{Z}^k$. If H were free, it would be free of rank k. Let $x_1, x_2, \dots, x_k$ be a free basis for H. The map $H \to H$ given by $x_1 \mapsto a_1\delta$ and $x_k \mapsto a_k$ for $k > 1$ is an isomorphism because H is Hopfian (being a free group). So H is freely generated by $\{a_1\delta, a_2, \dots, a_k\}$, but this is impossible since $[a_1\delta, a_2] \cdots [a_{k-1}, a_k] = 1$. □

Claim 12. *H is a proper subgroup of G.*

Proof of claim. Indeed, the element a_1 cannot be in H. For suppose $a_1 \in H$, and let N be the normal subgroup generated by a_k for $k > 2$. Then we have $G/N = \langle a_1, a_2 : [a_1\delta, a_2]\rangle$. Since $a_1 \in H$, G/N is abelian, and so has presentation $G/N = \langle a_1, a_2 : [a_1, a_2]\rangle$. But then the normal subgroup generated by $[a_1, a_2]$ and the normal subgroup generated by $[a_1\delta, a_2]$ are equal in $F_2 = \langle a_1, a_2\rangle$. By Magnus' conjugacy theorem for groups with one defining relator, $[a_1, a_2]^\epsilon$ must be conjugate to $[a_1\delta, a_2]$ for $\epsilon = 1$ or -1 in $F_2 = \langle a_1, a_2\rangle$, which is impossible since $[a_1, a_2]$ and $[a_1\delta, a_2]$ are cyclically reduced and have different word lengths [Magnus et al. 1976, Theorem 1.3, page 36]. Hence $a_1 \notin H$. □

We finish the proof of Theorem 6 by showing that all of the lower central series quotients of G match those of a surface group of genus $k/2$.

Claim 13. *G is weakly $k/2$-parasurface.*

Proof of claim. Let ψ be the map defined by $a_1 \mapsto a_1\delta$ and $a_k \mapsto a_k$ for $k > 1$. This gives a well-defined map $F_{k,i} \to F_{k,i}$, where $F_{k,i} := F_k/\gamma_i(F_k)$. This is an epimorphism by Lemma 7. Since finitely generated nilpotent groups are Hopfian, $\psi : F_{k,i} \to F_{k,i}$ must be an isomorphism. Therefore, the induced map on $\Gamma_{k/2}/\gamma_i(\Gamma_{k/2}) \to G/\gamma_i(G)$ is an isomorphism, as claimed. □

The proof of Theorem 6 is now complete. □

We are now ready to quickly prove all of our theorems stated in the introduction.

Proof of Theorem 1. Theorem 6 gives the desired parasurface groups. □

Proof of Theorem 2. The proof of Claim 13 with $\delta = [[a_1, a_2], a_1]$ shows that

$$[a_1, a_2]\cdots[a_{k-1}, a_k] \quad \text{and} \quad [a_1\delta, a_2]\cdots[a_{k-1}, a_k]$$

are nilpotent-automorphism equivalent. However, $[a_1, a_2]\cdots[a_{k-1}, a_k]$ is not automorphism equivalent to $[a_1\delta, a_2]\cdots[a_{k-1}, a_k]$ in F_k by Theorem 6. □

Proof of Theorem 3. The element w is not conjugate to $w[w, bwb^{-1}]$ in F_4. Hence, since F_4 is conjugacy-nilpotent separable, there exists some large $N > 0$ such that $\phi_{4,i}(w)$ is not conjugate to $\phi_{4,i}(w[w, bwb^{-1}])$ in $F_{4,i}$ for all $i > N$. Moreover, the equality

$$\langle\langle \phi_{4,i}(w[w, bwb^{-1}])\rangle\rangle_{F_{4,i}} = \langle\langle \phi_{4,i}(w)\rangle\rangle_{F_{4,i}} \quad \text{for all } i$$

is an immediate consequence of Claim 9. □

3. Final remarks

We have shown that there exist groups that are almost surface groups in the sense that they share all their lower central series quotients with a surface group but are not themselves surface groups. In light of our examples, we pose the following question.

Question 14. What properties do parasurface groups share with surface groups?

As a small step in answering this question, we present this:

Theorem 15. *Any finite-index subgroup of a parasurface group is not free.*

Proof. Let G be a parasurface group. Note that G is not free, for if it were, Magnus' theorem would imply that the fundamental group of some compact surface is free. Further, G is torsion-free, for otherwise by residual nilpotence, there would exist torsion elements in $\Gamma_g/\gamma_k(\Gamma_g)$ for some g and k, but this is impossible by [Labute 1970].

Let $\mathrm{cd}(G)$ denote the cohomological dimension of G. If $\Gamma \leq G$ is a free group of finite index, then by [Brown 1994, Theorem 3.1, page 190] and the fact that G is torsion-free, $\mathrm{cd}\, G = \mathrm{cd}\, \Gamma$. Hence, $\mathrm{cd}\, G = 1$, but then by [Stallings 1968] and [Swan 1969], G must itself be free, a contradiction. □

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