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**METRIC PROPERTIES OF HIGHER-DIMENSIONAL
THOMPSON'S GROUPS**

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Higher-dimensional Thompson's groups nV are finitely presented groups that generalize dyadic self-maps of the unit interval to dyadic self-maps of n -dimensional unit cubes. We describe some of the metric properties of these groups. We describe their elements based upon tree-pair diagrams and give upper and lower bounds for word length in terms of the size of the diagrams. Though these bounds are somewhat separated, we show that there are elements realizing the lower bounds and that the fraction of elements that are close to the upper bound converges to 1, showing that the bounds are optimal and that the upper bound is generically achieved.

Introduction

Thompson's groups provide a wide range of interesting examples of unusual group-theoretic behavior. The family of Thompson's groups includes the original groups described by Thompson, commonly denoted F , T and V , as well as generalizations in many different directions. The bulk of these generalizations includes groups that can be regarded as self-maps of the unit interval. Brin [2004; 2005] generalized V to higher-dimensional groups nV , which are described naturally in terms of dyadic self-maps of n -dimensional cubes. Little is known about these groups aside from their simplicity. Brin describes their elements geometrically in terms of dyadic interpolations of collections of dyadic blocks, and gives presentations for $2V$.

We describe elements in higher-dimensional Thompson's groups as being given by tree-pair diagrams. Though these diagrams usually take advantage of the natural left-to-right ordering of subintervals of the unit interval, by using several types of carets, Brin describes elements of nV via tree-pair diagrams. A natural question is how the size of these tree-pair diagrams corresponds to the word length of elements with respect to finite generating sets.

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We give upper and lower bounds for the word lengths of elements of higher-dimensional Thompson's groups nV with respect to finite generating sets, in terms of the size of tree-pair descriptions of elements. An element with a minimal tree-pair description of size N has word length between $\log N$ and $N \log N$ (up to the standard affine equivalences) with respect to the standard finite generating set. This of course also thus holds for any finite generating set.

1. Background on higher-dimensional Thompson's groups

Brin [2004] describes higher-dimensional Thompson's groups nV , giving presentations and showing that the group $2V$ is simple. Brin [2004; 2005] defines nV as a subgroup of the homeomorphism group of the n -fold product of the Cantor set with itself, and then shows there are a number of equivalent characterizations. The one we use here is Brin's characterization [2005] in terms of equivalence classes of labeled tree pair diagrams. The properties of the groups nV relevant here are as follows.

We denote the unit interval $[0, 1]$ by I , and the n -dimensional unit cube by I^n .

An element of V is given by two finite-rooted binary trees with the same number of leaves and a permutation, which represents a bijection between the leaves of the two trees. Hence an element of V can be seen as a triple (T_+, π, T_-) , where T_+ and T_- are trees with k leaves, and $\pi \in \mathcal{S}_k$.

Each binary tree can be seen as a way of subdividing the interval I into dyadic subintervals of the type $[i/2^r, (i+1)/2^r]$, where $r > 0$ and $0 \leq i < 2^r$. A rooted binary tree gives instructions for successive halvings of subintervals to obtain a particular dyadic subdivision. Given two binary trees with n leaves and a permutation in \mathcal{S}_n , an element in V is represented as a left-continuous map of the interval into itself, sending each interval in the first subdivision to a corresponding interval in the second subdivision, as specified by the given permutation. Though not every dyadic subdivision of I can be obtained by a successive halving process described by trees, every dyadic subdivision has a refinement that can be obtained by a successive halving process.

For more details on the group V , including presentations and a proof of its simplicity, see Cannon, Floyd and Parry [1996] as well as Brin [2004].

To obtain an element in $2V$, we will define a partition of I^2 into dyadic rectangles of the type

$$\left[\frac{i}{2^r}, \frac{i+1}{2^r} \right] \times \left[\frac{j}{2^s}, \frac{j+1}{2^s} \right].$$

Though again, not every such partition can be realized by successive halving processes, every such partition has a refinement that can be realized in that way. For the two-dimensional successive halving process, there are two possible halvings

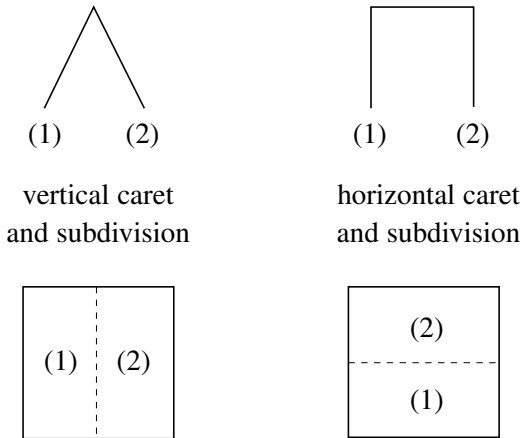
that can occur at each stage to a specified dyadic rectangle—a horizontal or a vertical subdivision. We can obtain a refinement of any dyadic partition of I^2 via iterated horizontal and vertical subdivisions. This gives a natural means of describing elements of $2V$ as pairs of dyadic subdivisions, each with m rectangles, and a permutation in S_m giving the bijection between the rectangles.

We can describe elements of $2V$ also with pairs of binary trees, with the nodes representing steps in successive halving processes to represent the refined subdivisions. Since there are two types of subdivisions, vertical and horizontal, we will consider binary trees that contain two types of caret, *vertical* and *horizontal* carets, represented by *triangular* and *square* carets, respectively.

As with other tree-based definitions of Thompson's groups, there are numerous representatives of a given group element, with natural equivalence relations coming from expansions and contractions of leaves of trees. To multiply two elements, we typically find representatives for each of them where the trees are compatible for a natural multiplication via composition.

We define the group $2V$ as the set of equivalence classes of triples (T_+, π, T_-) in which the trees are labeled to indicate the successive halving process and π is a permutation between the leaves of the trees (where each leaf represents a dyadic rectangle).

To fix a labeling convention, we will represent a vertical subdivision with a traditional triangular caret, where the left and right leaves naturally represent the left and right rectangles. A horizontal subdivision will be represented by a square caret; in it, the left leaf represents the bottom rectangle, and the right leaf represents the top rectangle, as shown below.



We say a representative of an element is *reduced* if for each caret c in T_- of with two leaves, the permutation π does not map both leaves of c to leaves of the same caret of the same type in T_+ in the same order. We say a representative

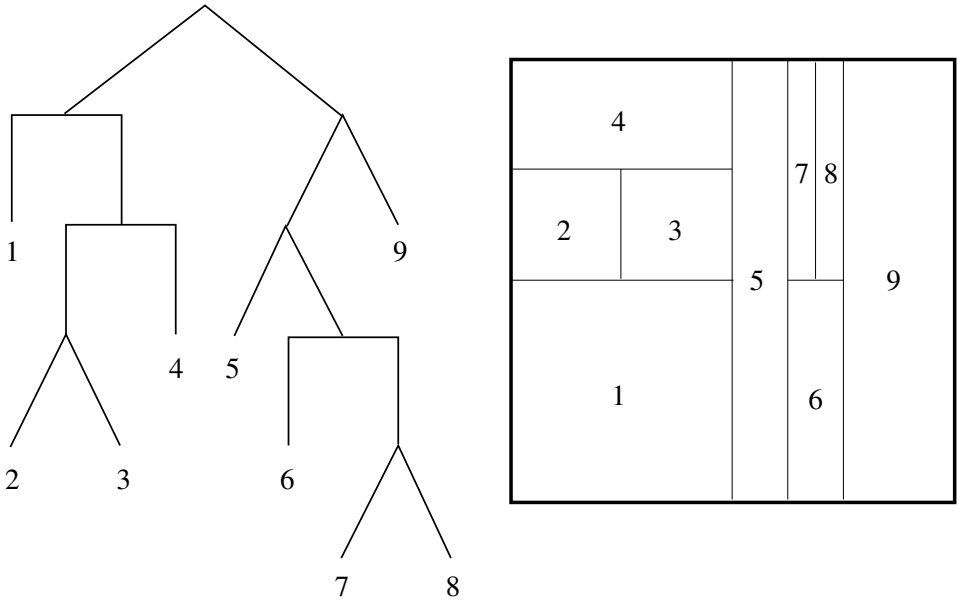


Figure 1. An example of a binary tree with the two types of carets, and its corresponding partition of I^2 .

of an element is of *minimal size* if it has the minimal number of carets among all representatives in its equivalence class, and we denote by $N(x)$ the number of carets in a minimal size representative of an element x . Unlike the case of V , but similar to that of $F(n, m)$ (see Wladis [2007]), each element of minimal size may have multiple distinct representatives.

See Figure 1 for an example of a partition of I^2 and its corresponding binary tree. Every dyadic partition of the unit square has a refinement that can be obtained with a binary tree with these two types of carets.

For the general groups nV , the partitions are divisions of the unit n -cube in n -rectangles of dyadic lengths, and the corresponding binary trees have n types of carets. For simplicity, we state our results for $n = 2$, but all of the results below extend naturally to $n > 2$.

The fact that we have vertical and horizontal subdivisions brings new relations to the group. These relations arise when both types of subdivisions are combined in different orders to obtain different descriptions of the same dyadic partition of I^2 . The obvious relation (and the one from which all other relations are deducible) is the combination of one subdivision of each type in the two possible orders, as illustrated in Figure 2.

Subdividing in both the vertical and horizontal directions once, but in the two possible orders, gives the same dyadic partitions, but according to our convention

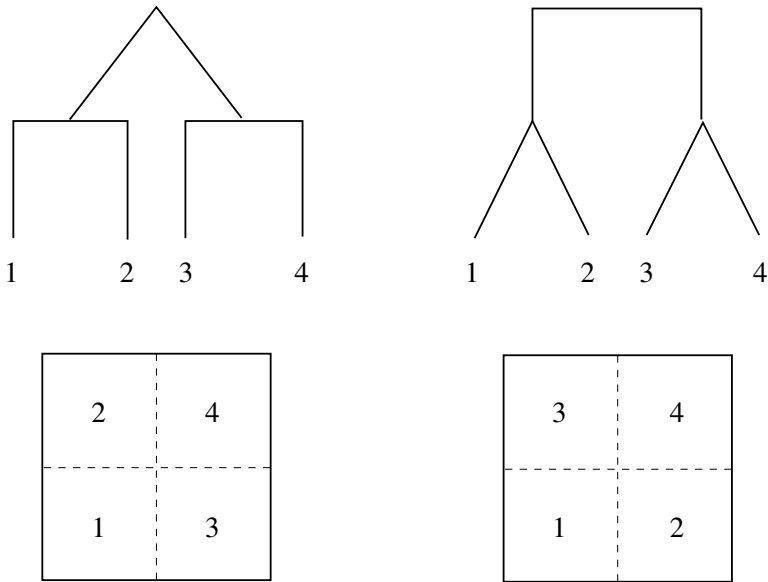


Figure 2. The relation obtained when performing vertical and horizontal subdivisions.

the resulting rectangles are numbered in different way. This makes using tree diagrams to express the multiplication of elements tricky and quite unwieldy for large elements. Even though the leaves of the two-caret-type tree diagrams are ordered in a natural way, this order is not apparent in the square, and it is not preserved when different diagrams represent the same partition. Thus, there are (nonminimal) diagrams representing the identity whose leaves are ordered in different ways — an example is illustrated in [Figure 3](#).

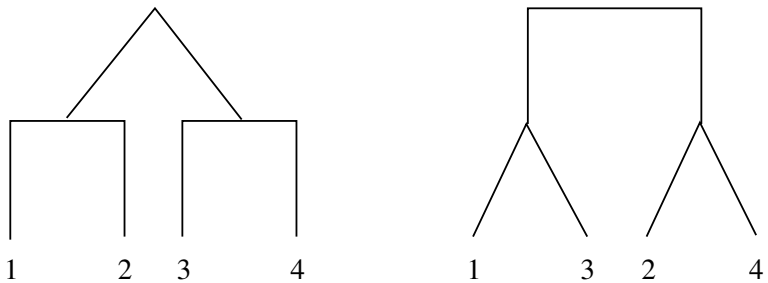


Figure 3. A tree-pair diagram for the identity element that has a nonidentity permutation.

2. Families of generators

Brin [2004; 2005] showed that the groups nV are finitely generated and gave several generating sets for $2V$. As is common with groups of the Thompson family, there is an infinite presentation, which is useful for its symmetry and regularity and which has a finite subpresentation. For the purposes of word length, we work with a standard finite generating set as described by Brin. We next define the families of generators for $2V$, following the usual procedure in Thompson's groups; see Cannon, Floyd and Parry [1996]. We build the generators on a backbone of an all-right tree with only vertical caret.

- (1) The generators A_n involve only vertical subdivisions, and they are the traditional generators of Thompson's group F .
- (2) The generators B_n have the one nonright caret replaced by a horizontal one.
- (3) The generators C_n have the last caret of the all-right tree of horizontal type, and they are used to build horizontal carets on the right-hand-side of the tree, as will be seen later.
- (4) The generators π_n and $\bar{\pi}_n$ are permutations built on an all-right tree with only vertical carets. These generators are exactly equal to those for the subgroup V appearing as the purely vertical elements.

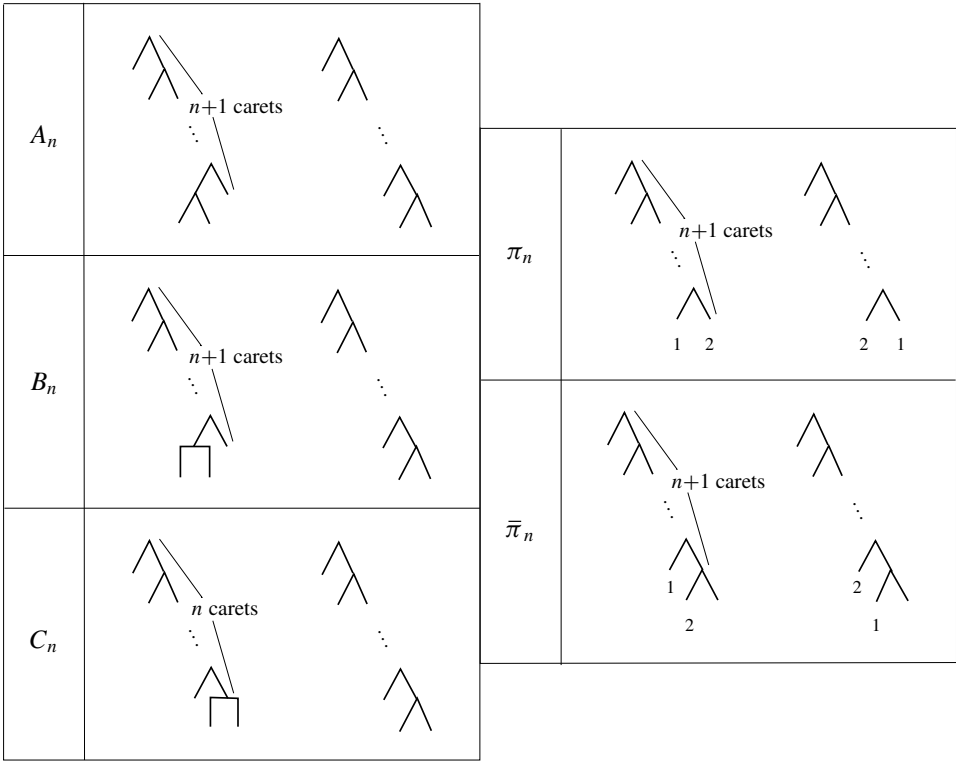
Theorem 2.1 [Brin 2004; 2005]. *The families A_n , B_n , C_n , π_n and $\bar{\pi}_n$ generate the group $2V$.*

Proof. Since our proof is quite different from Brin's, we will include it here and use aspects of it later for metric considerations.

An element of $2V$ is given by a triple (T_+, π, T_-) , where the two trees are composed of the two types of carets. First, we subdivide a given element into three elements: (T_+, id, R_k) , (R_k, π, R_k) , and (R_k, id, T_-) , where R_k is the all-right tree with k leaves and only vertical carets. Clearly (R_k, π, R_k) is product of the permutation generators, as is the case in V already.

To obtain the element (T_+, id, R_k) , we will concentrate first on the backbone of the tree T_+ , that is, the sequence of carets in the right hand side. If this sequence of carets has horizontal carets in the positions m_1, m_2, \dots, m_p , then the product of the generators $C_{m_1} C_{m_2} \cdots C_{m_p}$ produces exactly a backbone with horizontal carets in the desired positions. We denote this backbone by K ; that is, K is the subtree of T_+ that consists only of right carets. Thus, at this stage, we have constructed the element (K, id, R_{m_p}) .

Once the backbone is constructed, each new caret is obtained by a generator of the type A_i or B_i . To attach a vertical caret from the leaf labeled i , we only need to multiply by A_i on the right. Similarly, to attach a horizontal caret to leaf i , we

**Figure 4.** Generators of $2V$.

multiply by B_i on the right. This multiplication process is shown in [Figure 5](#), with an example illustrated in [Figure 6](#).

This proves that the element (T_+, id, R_k) is product of the generators A_i , B_i and C_i , and using inverses, we see that the entire group is generated by the full family of generators. \square

An element of the type (T_+, id, R_k) is called a *positive* element of $2V$. Positive elements can always be written as products of the generators A_i , B_i and C_i without using their inverses.

In the process of proving Brin's theorem, we have obtained this:

Theorem 2.2. *Elements of $2V$, with respect to the standard infinite generating set $\{A_i, B_i, C_i, \pi_i, \bar{\pi}_i\}$, have these properties:*

- Any positive element always admits an expression of the type

$$C_{m_1} \cdots C_{m_p} W_1(A_{i_1}, B_{i_1}) \cdots W_r(A_{i_r}, B_{i_r}).$$

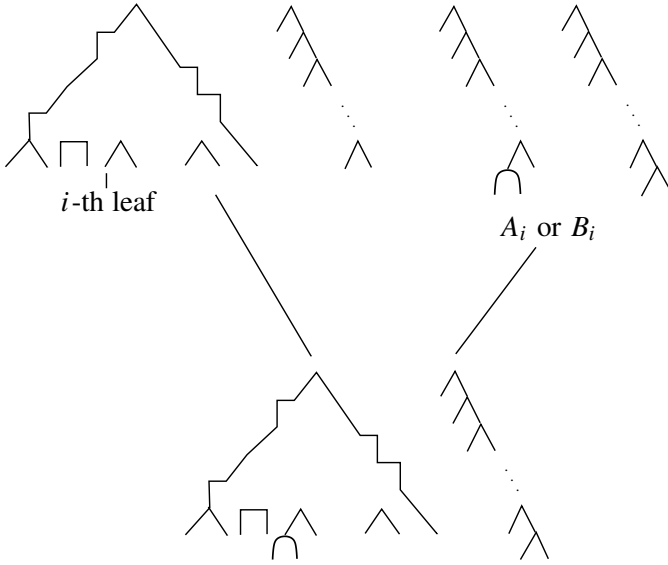


Figure 5. The building process: attaching a caret to the i -th leaf by multiplying by A_i or B_i .

where the W_i are words on the positive generators A_i and B_i and never their inverses, and where $m_1 < m_2 < \dots < m_p$ and $i_1 < i_2 < \dots < i_r$.

- Each element admits an expression $P\Pi Q^{-1}$, where P and Q are positive elements and Π is a permutation on an all-right vertical tree, and thus a word in the π_i and $\bar{\pi}_i$.

This expression will be used as a seminormal form for elements of $2V$.

Brin also shows that several finite generating sets suffice to generate each nV . We will use the finite generating set $\{A_0, A_1, B_0, B_1, C_0, C_1, \pi_0, \pi_1, \bar{\pi}_0, \bar{\pi}_1\}$. This set is larger than the smallest ones used by Brin, but is more convenient for our methods below.

3. Metric properties

We will be interested in metric properties up to the standard affine equivalence, defined here.

Definition 3.1. Given two functions $f, g : G \rightarrow \mathbb{R}$, we say that $f < g$ if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for all $x \in G$. We say that f and g are equivalent, written $f \sim g$, if $f < g$ and $g < f$.

Elements of Thompson's groups admit representations as diagrams with binary trees in one form or another, perhaps with associated permutations or braids between the leaves. For several of these groups, the distance $|x|$ from an element x

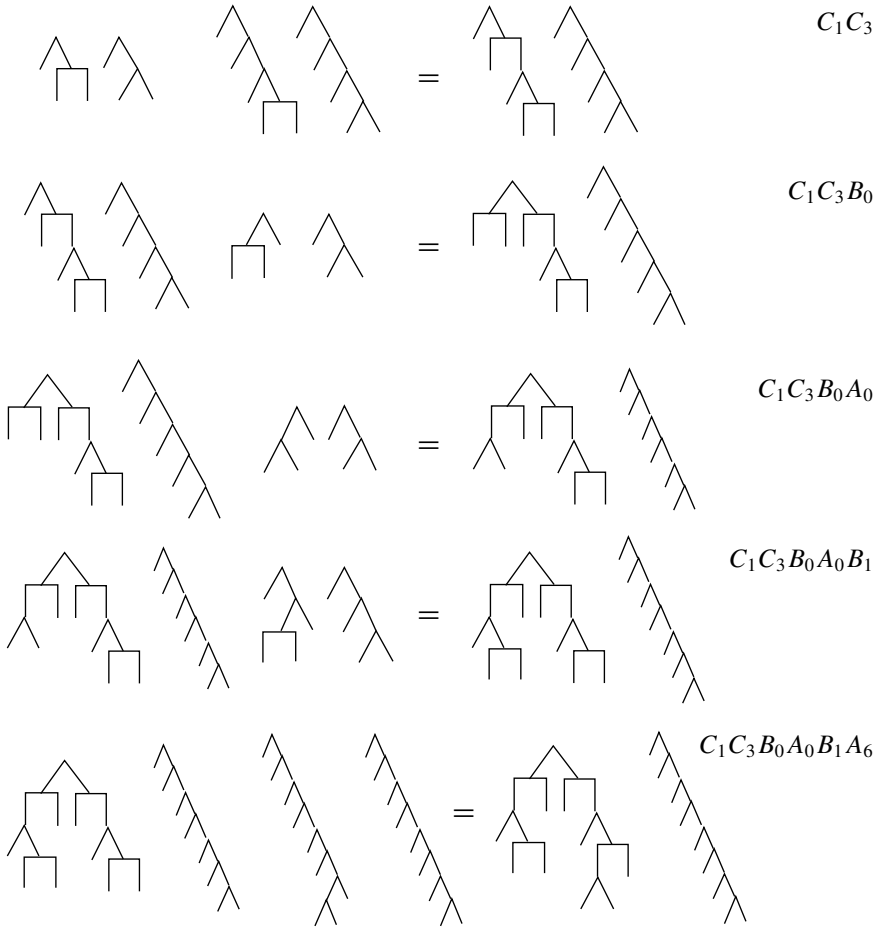


Figure 6. Constructing an element generator-by-generator.

to the identity is closely related with the number of carets $N(x)$ of the minimal diagram, indicating that the complexity of the minimal (reduced) diagram is a good indication of the size of the element.

In Thompson's groups F and T , the two functions are equivalent: for both groups we have $|x| \sim N(x)$; see Burillo, Cleary and Stein [2001] and Burillo, Cleary, Stein and Taback [2009]. This result cannot hold in V , for counting reasons. The number of elements of V whose trees have N carets is at least the order of $N!$ (due to the permutations), while the number of distinct elements of length N can be at most exponential in N in any group.

The best possible bound for the number of carets in terms of word length in the group V is an inequality proved by Birget as [2004, Theorem 3.8], which says that

$N(x) \prec |x| \prec N(x) \log N(x)$ and that these bounds are optimal, in a sense similar to that which will be made precise in [Section 4](#).

In the case of V , the lower bounds on word length are linear in the number of carets. Therefore, the standard inclusions $F \subset T \subset V$ are all quasiisometric embeddings; that is, F is undistorted in T , and T is undistorted in V . This property will be no longer true for inclusions in $2V$, as we will show in [Section 5](#).

In this section we will prove the analog for $2V$ of the inequality above. In the next section, we address optimality of the bounds.

Theorem 3.2. *In the group $2V$, the word length of an element $|x|$ and number of carets $N(x)$ in a minimal size tree-pair representative satisfy*

$$\log N(x) \prec |x| \prec N(x) \log N(x),$$

and the bounds cannot be improved.

Proof. We defer the proof of its optimality to the following section.

The upper bound is proved the standard way. We take a positive word P , and according to [Theorem 2.2](#), write it as

$$C_{m_1} \cdots C_{m_p} W_1(A_{i_1}, B_{i_1}) \cdots W_r(A_{i_r}, B_{i_r}).$$

Then, we rewrite each generator using the relations

$$A_{i+1} = A_0^{-i+1} A_1 A_0^{i-1}, \quad B_{i+1} = A_0^{-i+1} B_1 A_0^{i-1}, \quad C_{i+1} = A_0^{-i+1} C_1 A_0^{i-1}$$

for all $i > 1$. Since the conjugating element is always A_0 , cancellations ensure that the length stays approximately the same. The bound $N \log N$ appears because of the permutations, since $N \log N$ is the diameter of \mathcal{S}_N with respect to the relevant transpositions.

For the lower bound, we note that if an element has k carets, when it is multiplied by a generator it is possible that the multiplication could have up to $4k$ carets. If an element has, for instance, only horizontal subdivisions, when multiplied by A_1 the number of subdivisions (and hence the number of carets) becomes $4k$, since each one of the previous subdivision is divided in four, as illustrated in [Figure 7](#).

If each multiplication by a generator can multiply the number of carets by 4, an element of length ℓ could have up to 4^ℓ carets. Other generators may increase word length by additive factors, but the worst case is exactly the increase by a factor of 4. Hence $N(x) \leq 4^\ell$, so $\log N(x) \leq |x|$. \square

4. Optimality of bounds

To show the optimality of the bounds in [Theorem 3.2](#), we first describe some properties of the lower bound for word length in terms of the number of carets.

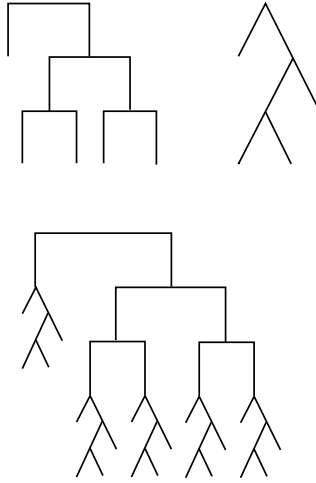


Figure 7. An illustration of why when two types of carets are involved, the number of carets gets multiplied. The tree on the top left has only horizontal subdivisions, so when joined with a tree with only vertical ones, these have to be put at every leaf.

Lemma 4.1. *The word length of an element of $2V$ that is represented by a tree-pair diagram of depth D is at least $D/3$.*

Proof. Right multiplying by a generator in the finite generating set can add levels to the tree-pair diagram. When we right multiply by a generator, we may need to add carets to T_- (and thus to T_+) if the carets in the generator are not already present in T_- . This can add at most 3 levels, which happens only in the case that we right-multiply an element (T_+, π, T_-) with T_- having no right subtree by any of A_1 , B_1 or C_1 from the finite generating set. In that case, the right child of the root of T_- is a leaf labeled n and we will need to add carets to it and the corresponding leaf from T_+ to obtain a representative compatible with multiplication by the generator. If leaf $\pi^{-1}(n)$ in T_+ is at maximal level, then the level could increase but only by the number of carets added to leaf $\pi^{-1}(n)$ in T_- . Thus, any product of less than $D/3$ generators cannot have depth D . \square

From this it follows that the cyclic subgroups generated by any single generator from the A_n or B_n families are undistorted in $2V$, for example.

As seen in the proof of [Theorem 3.2](#), multiplying by a generator can increase the number of carets by a multiplicative factor. The powers of the element C_0 have exponentially many carets. In fact, the minimal number of carets needed to represent C_0^n is 2^n . We give further related specific examples later in [Section 5](#). Any of these examples show that the lower bound of [Theorem 3.2](#) is optimal.

To analyze the genericity of the upper bound in [Theorem 3.2](#), we use arguments analogous to those for V . Birget [2004] showed not only that $n \log n$ is an upper bound on the growth of elements in V with respect to the number of carets n in minimal length representatives, but also that the fraction of elements close to this bound converges exponentially fast to 1. In $2V$, we note analogous behavior:

Theorem 4.2. *Let H_n be the set of elements of $2V$ representable with n carets and having no representatives with fewer than n carets. The fraction of elements of H_n that have word length greater than $n \log n$ converges exponentially fast to 1.*

Proof. Here we use a counting argument, analogous to that used by Birget [2004] to count elements in V . In $2V$, we consider the subset of representatives of elements that have only horizontal subdivisions in the domain and only vertical subdivisions in the range. Their tree-pair diagrams are guaranteed to be reduced and of minimal size by a straightforward argument. So to count the set of elements of this type of diagram size n , we have C_n choices (where C_n is the n -th Catalan number) for the first all-square tree of size n and C_n choices for the all-triangular tree of size n , and $n!$ choices for the permutation. This set of elements has size $C_n^2 n!$, which is, by the Stirling formula,

$$(C_n)^2 n! = \left(\frac{(2n)!}{(n+1)!n!} \right)^2 n! = \sqrt{\frac{2}{\pi}} \frac{16^n}{e^n} n^{(n-2)} (1 + o(1)),$$

where the $o(1)$ term goes to zero as n increases.

The number of elements of word length n in any finitely generated group with d generators is no more than $(2d)^n$. Thus we see that the ratio of elements that have word length less than $n \log_{2d} n$ out of elements that have tree-pair diagrams of size n is less than

$$\frac{n^n}{\sqrt{\frac{2}{\pi}} \frac{16^n}{e^n} n^{(n-2)} (1 + o(1))} \sim c \frac{n^2 e^n}{16^n},$$

which converges to 0 exponentially fast, as desired. So by complementing we have the result. \square

5. Distortion

Elements of $2V$ can be represented with pairs of binary trees, where the carets have been subdivided in two types. This fact makes it more difficult to multiply elements whose caret types disagree, as illustrated already in [Theorem 3.2](#), and the number of carets can grow faster because of this situation. This phenomenon has been described already by Wladis [2007] for the group $F(2, 3)$, which has also carets of two types (binary and ternary). This feature of $2V$ implies now that

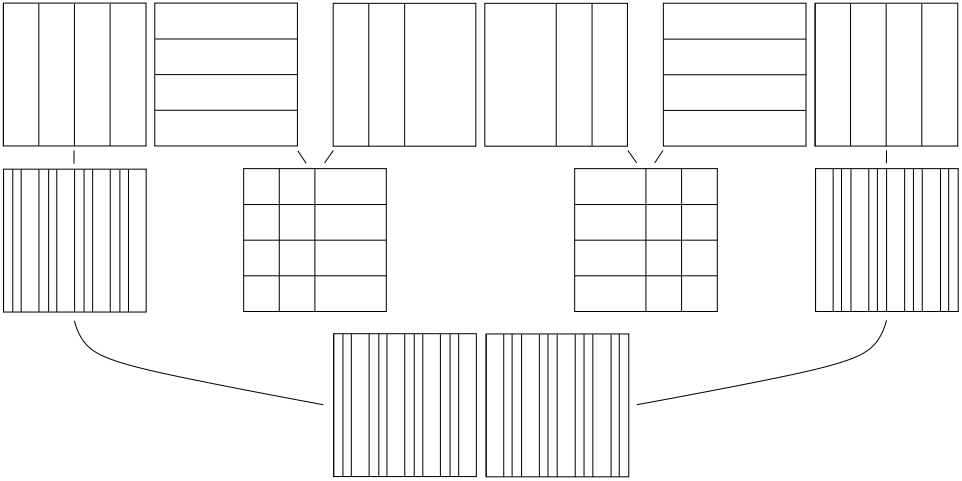


Figure 8. The process of multiplication for $C_0^{-2} A_0 C_0^2$, illustrating the exponential number of subdivisions.

the groups F , T and V , when seen as subgroups of $2V$ (by doing only vertical subdivisions, for instance), are exponentially distorted.

Theorem 5.1. *The groups F , T and V are at least exponentially distorted in $2V$.*

Proof. We consider the specific element $C_0^{-n} A_0 C_0^n$, illustrated in Figure 8. This element lies in a copy of F in $2V$ obtained by putting F into $2V$ using only vertical subdivisions.

The element C_0^n has 2^n carets, as seen with an easy induction. Its two trees are balanced trees of depth n , one with only vertical subdivisions, and one with only horizontal subdivisions. By matching the horizontal subdivisions with the vertical ones in A_0 , we see the element $C_0^{-n} A_0 C_0^n$ has a number of carets of order 2^n , and all the carets are of vertical type, so the element is in F . This element has length in V no more than $2n + 1$, but the number of carets is exponential in n and thus its word length as an element of the vertical copy of F in $2V$ is also exponential.

Thus F is at least exponentially distorted in $2V$. Since F is undistorted in T and V , we see that T and V (as subgroups using only vertical subdivisions) are also at least exponentially distorted in $2V$. \square

Acknowledgments

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