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A link in the 3-sphere is homotopically trivial, according to Milnor, if its components bound disjoint maps of disks in the 4-ball. This paper is concerned with the question of what spaces, when used in place of disks in an analogous definition, give rise to the same class of homotopically trivial links. We show that there are 4-manifolds for which this property depends on their embedding in the 4-ball. This work is motivated by the A-B slice problem, a reformulation of the 4-dimensional topological surgery conjecture. As a corollary, this provides a new, secondary obstruction in the A-B slice problem for a certain class of decompositions of D^4.

1. Introduction

The classification of knots and links up to concordance, in both the smooth and topological categories, is an important and difficult problem in 4-dimensional topology. Recall that two links in the 3-sphere are concordant if they bound disjoint embeddings of (smooth or locally flat, depending on the category) annuli in $S^3 \times [0, 1]$. Milnor [1954] introduced the notion of *link homotopy*, often referred to as the "theory of links modulo knots", which turned out to be much more tractable. In particular, there is an elegant characterization of homotopically trivial links using the *Milnor group*, a certain rather natural nilpotent quotient of the fundamental group of the link. Two links are link homotopic if they bound disjoint maps of annuli in $S^3 \times [0, 1]$, so the annuli are disjoint from each other, but (unlike the definition of concordance) they are allowed to have self-intersections. (Strictly speaking, this defines the notion of a singular concordance of links; however, it is known [Giffen 1979; Goldsmith 1979] to be equivalent to Milnor's original notion [Milnor 1954] of link homotopy.)

The subject of link homotopy brings together 4-dimensional geometric topology and the classical techniques of nilpotent group theory. An area where both approaches are important, and which is a motivation for the results in this paper, is the

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A-B slice problem, a reformulation of the topological 4-dimensional surgery conjecture [Freedman and Lin 1989; Freedman and Quinn 1990]. Roughly speaking, this paper is concerned with the problem of characterizing spaces (in interesting cases, 4-manifolds with a specified curve in the boundary) which, when used in place of disks in a definition analogous to Milnor's, give rise to the same class of homotopically trivial links.

A specific question about the A-B slice problem, and related to link-homotopy theory, is the following: Suppose M is a codimension-zero submanifold of the 4-ball, $i : M \hookrightarrow D^4$, with a specified curve $\gamma \subset \partial M$ forming a knot in the 3-sphere: $i(\gamma) \subset S^3 = \partial D^4$. Given such a pair (M, γ), does there exist a homotopically essential link $\big(i_1(\gamma), \dots, i_n(\gamma)\big)$ in the 3-sphere, formed by disjoint embeddings $i_1, \dots, i_n$ of (M, γ) into (D^4, S^3)? If the answer is negative, the pair (M, γ) is called *robust*. The analysis of this problem is substantially more involved than the classical link homotopy case where one considers disks with self-intersections: in general the 1- and 2-handles of 4-manifolds embedded in D^4 may link, and the relations in the fundamental group of the complement do not have the "standard" form implied by the Clifford tori in Milnor's theory.

The main result of this paper is the existence of 4-manifolds for which this property depends on their embedding in the 4-ball:

Theorem 1. *There exist submanifolds* $i : (M, \gamma) \hookrightarrow (D^4, S^3)$ *such that*

(i) *there are disjoint embeddings of copies* (M_i, γ_i) *of* (M, γ) *into* (D^4, S^3) *forming a homotopically essential link* $(\gamma_1, \dots, \gamma_n)$ *in the* 3*-sphere, and*

(ii) *given any disjoint embeddings of* (M, γ) *into* (D^4, S^3), *each one isotopic to the original embedding* i, *the link* $(\gamma_1, \dots, \gamma_n)$ *formed by their attaching curves in the* 3*-sphere is homotopically trivial.*

In the A-B slice problem, one considers *decompositions* of the 4-ball, $D^4 = A \cup B$, where the specified curves α, β of the two parts form the Hopf link in $S^3 = \partial D^4$ (see [Freedman and Lin 1989; Krushkal 2008], and Section 2 below). It is shown in [Krushkal 2008] that there exist decompositions where neither of the two sides is robust. This result left open the question of whether, in fact, these decompositions may be used to solve the general 4-dimensional topological surgery conjecture.

This paper provides a detailed analysis of the construction in [Krushkal 2008]. The proof of Theorem 1 shows that, in this case, one of the two parts of the decomposition is robust provided that the re-embeddings forming the link $(\gamma_1, \dots, \gamma_n)$ are topologically equivalent to the original embedding into the 4-ball. This provides a new obstruction in the A-B slice problem for this class of decompositions of the 4-ball, exhibiting a new phenomenon where the obstruction depends not just on the submanifold but also on its specific embedding into D^4. In particular, this shows that the construction in [Krushkal 2008] does not satisfy the equivariance condition which is necessary for solving the canonical 4-dimensional surgery problems. An important open question is whether, given *any* decomposition $D^4 = A \cup B$, the conclusion (ii) of Theorem 1 holds for either A or B.

The main tool in the proof of part (ii) is the Milnor group, in the context of the *relative-slice problem*. This context is substantially different from Milnor's original work, since we use it to analyze embeddings of more general submanifolds in the 4-ball. The topology of these spaces is richer than the setting of disks with self-intersections in the 4-ball, considered in classical link homotopy. To find an obstruction, one has to consider in detail the structure of the graded Lie algebra associated to the lower central series of the link group. The strategy of the proof should be useful for further study of the A-B slice problem.

Section 2 gives a detailed definition of robust 4-manifolds and robust embeddings, and discusses its relation with the A-B slice problem. The construction [Krushkal 2008] of the submanifolds, used in the proof of Theorem 1, is recalled in Section 3. In Section 4, we review the Milnor group in the 4-dimensional setting and complete the proof of Theorem 1.

2. Robust 4-manifolds and the A-B slice problem

This section states the definition of robust 4-manifolds and robust embeddings, notions which provide a convenient setting for the results of this paper and are important for the A-B slice problem. Let M be a 4-manifold with a specified curve γ in its boundary. Let $i : M \hookrightarrow D^4$ be an embedding into the 4-ball with $i(\gamma) \subset S^3 = \partial D^4$.

Definition 2.1. The pair (M, γ) is called *robust* if, given any $n \geq 2$ and disjoint embeddings $i_1, \ldots, i_n$ of (M, γ) into (D^4, S^3), the link formed by the curves $i_1(\gamma), \ldots, i_n(\gamma)$ in the 3-sphere is homotopically trivial.

An *embedding* $i : (M, \gamma) \hookrightarrow (D^4, S^3)$ is *robust* if, given any $n \geq 2$ and disjoint embeddings $i_1, \ldots, i_n$ of (M, γ) into (D^4, S^3), each isotopic to the original embedding i, the link formed by the curves $i_1(\gamma), \ldots, i_n(\gamma)$ in the 3-sphere is homotopically trivial. (In this case, we say that the re-embeddings are *standard*.)

In these terms, Theorem 1 states that there exist pairs (M, γ) that are not robust but admit robust embeddings; these are the first examples of this phenomenon.

It follows easily from the definition that the 2-handle $(D^2 \times D^2, \{0\} \times \partial D^2)$, and more generally any kinky handle (a regular neighborhood in the 4-ball of a disk with self-intersections) is robust.

It is not difficult to give further examples: it follows from the link composition lemma [Freedman and Lin 1989; Krushkal and Teichner 1997] that the 4-manifold (B_0, β) in Figure 1, obtained from the collar $\beta \times D^2 \times [0, 1]$ by attaching 2-handles to the Bing double of the core of the solid torus, is robust. This example illustrates the important point that the disjoint copies $i_j(M)$ in the definition above are *embedded*: it is easy to see that, if the 2-handles H_1, H_2 in Figure 1 were allowed to intersect, this 4-manifold may be mapped to the collar on its attaching curve, and therefore there exist disjoint *singular* maps of copies of this manifold such that their attaching curves $\{\gamma_i\}$ form a homotopically essential link in the 3-sphere.

The complement in D^4 of the standard embedding of the 4-manifold in Figure 1 is the 4-manifold $A =$ (genus-one surface with one boundary component α) $\times D^2$. It is easy to see that (A, α) is not robust: for example, the Borromean rings form a homotopically essential link bounding disjoint standard genus-one surfaces in the 4-ball.

To review the relation of these results to the 4-dimensional topological surgery conjecture, recall the definition of an A-B slice link (see [Freedman and Lin 1989; Krushkal 2008] for a more detailed discussion.)

Definition 2.2. A *decomposition* of D^4 is a pair of compact codimension-zero submanifolds with boundary $A, B \subset D^4$, satisfying conditions (1)–(3) below, where

$$\partial^+ A = \partial A \cap \partial D^4, \quad \partial^+ B = \partial B \cap \partial D^4, \quad \partial^- A = \overline{\partial A \setminus \partial^+ A}, \quad \partial^- B = \overline{\partial B \setminus \partial^+ B}.$$

(1) $A \cup B = D^4$.

(2) $A \cap B = \partial^- A = \partial^- B$.

(3) $S^3 = \partial^+ A \cup \partial^+ B$ is the standard genus-one Heegaard decomposition of S^3.

Given an n-component link $L = (l_1, \ldots, l_n) \subset S^3$, let $D(L) = (l_1, l_1', \ldots, l_n, l_n')$ denote the $2n$-component link obtained by adding an untwisted parallel copy L' to L. The link L is *A-B slice* if there exist decompositions (A_i, B_i), $i = 1, \ldots, n$, of D^4, and self-homeomorphisms φ_i, ψ_i of D^4, for $i = 1, \ldots, n$, such that all sets in the collection $\varphi_1 A_1, \ldots, \varphi_n A_n, \psi_1 B_1, \ldots, \psi_n B_n$ are disjoint, and the following boundary data is satisfied: $\varphi_i(\partial^+ A_i)$ is a tubular neighborhood of l_i and $\psi_i(\partial^+ B_i)$ is a tubular neighborhood of l_i', for each i.

The surgery conjecture is equivalent to the statement that the Borromean rings, and a certain family of their generalizations, are A-B slice. In [Krushkal 2008], we constructed a decomposition $D^4 = A \cup B$ and disjoint embeddings A_i, B_i into D^4 so that the attaching curves $\{\alpha_i\}$ of the A_i formed the Borromean rings (or, more generally, any given link with trivial linking numbers) and the curves $\{\beta_i\}$ formed an untwisted parallel copy. The validity of one of the conditions necessary for solving the canonical surgery problems was unknown at the time of that construction, namely the equivariance (the existence of the homeomorphisms φ_i, ψ_i); phrased differently, it was not known whether there exist disjoint re-embeddings of the submanifolds A, B which are *standard*. It follows from Theorem 1 that standard disjoint embeddings for these decompositions do not exist. Therefore, an open question (important in the search for an obstruction to surgery in the context of the A-B slice problem) is: *Given any decomposition $D^4 = A \cup B$, is one of the two embeddings $A \hookrightarrow D^4$, $B \hookrightarrow D^4$ necessarily robust?*

3. Construction of the submanifolds

This section reviews the construction [Krushkal 2008] of the submanifolds of D^4, which will be used in the proof of Theorem 1 in Section 4. The construction consists of a series of modifications of the handle structures, starting with a standard surface and its complement in the 4-ball. Consider the genus-one surface S with a single boundary component α, and set $A_0 = S \times D^2$. Consider the standard embedding $(S, \alpha) \subset (D^4, S^3)$ (take an embedding of the surface in S^3, push it into the 4-ball and take a regular neighborhood). Then, A_0 is identified with a regular neighborhood of S in D^4. The complement B_0 of A_0 in the 4-ball is obtained from the collar on its attaching curve, $S^1 \times D^2 \times I$, by attaching a pair of zero-framed 2-handles to the Bing double of the core of the solid torus $S^1 \times D^2 \times \{1\}$, as in Figure 1. (See for example [Freedman and Lin 1989] for a proof of this statement.)

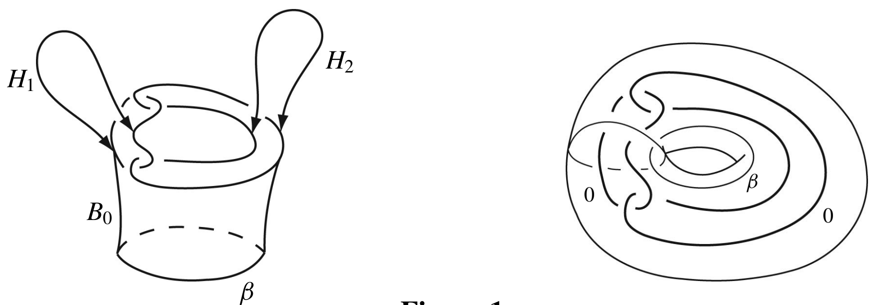

Figure 1

Note that a distinguished pair of curves α_1, α_2, forming a symplectic basis in the surface S, is determined as the meridians (linking circles) to the cores of the 2-handles H_1, H_2 of B_0 in D^4. In other words, α_1, α_2 are fibers of the circle normal bundles over the cores of H_1, H_2 in D^4.

An important observation [Freedman and Lin 1989] is that this construction may be iterated: Consider the 2-handle H_1 in place of the original 4-ball. The pair of curves (α_1 and the attaching circle β_1 of H_1) form the Hopf link in the boundary of H_1. In H_1, consider the standard genus-one surface T bounded by β_1. As discussed above, its complement is given by two zero-framed 2-handles attached to the Bing double of α_1. Assembling this data, consider the new decomposition $D^4 = A_1 \cup B_1$ (in this paper we need only the B-side of the decomposition, shown in Figure 2.) As above, the diagrams are drawn in solid tori (complements in S^3 of the unknotted circles drawn dashed in the figures.)

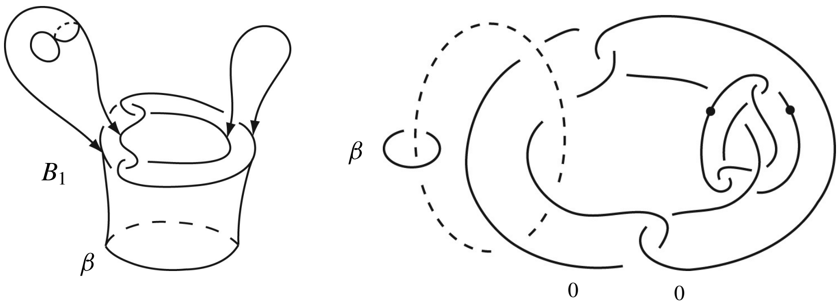

Figure 2

The handlebodies A_1, B_1 are examples of *model decompositions* [Freedman and Lin 1989] obtained by iterated applications of the construction above. It is known that such model handlebodies are robust and, in particular, the Borromean rings are not weakly A-B slice when restricted to the class of model decompositions. (A link L is *weakly* A-B slice if the submanifolds $\{A_i, B_i\}$ in Definition 2.2 may be embedded into D^4 disjointly, but the equivariance condition encoded by the existence of the homeomorphisms φ_i, ψ_i is omitted; see [Krushkal 2008].)

We are now in a position to define the decomposition $D^4 = A \cup B$ used in the proof of Theorem 1.

Definition 3.1. Consider $B = (B_1 \cup \text{zero-framed 2-handle})$, attached as shown in the Kirby diagram in Figure 3.

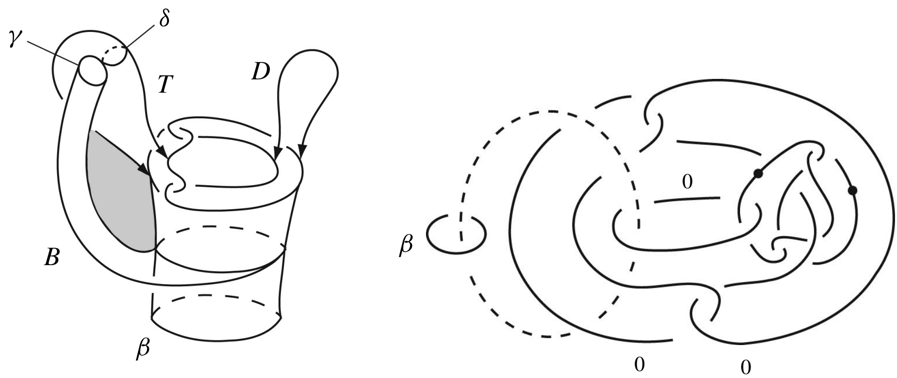

Figure 3

Imprecisely (up to homotopy, on the level of spines) B may be viewed as the union of B_1 with a 2-cell, attached along the composition of the attaching circle β of B_1 and a curve representing a generator of H_1(the second-stage surface of B_1). This 2-cell is schematically shown in the spine picture of B in Figure 3, left, as a cylinder connecting the two curves. The shading indicates that the new generator of π_1 created by adding the cylinder is filled-in with a disk. The figure showing

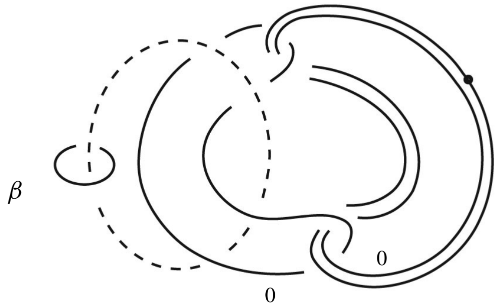

Figure 4

the spine is provided only as motivation for the construction; a precise description is given by the handle diagram.

Note that, by canceling a (1-handle, 2-handle) pair, one gets the diagram for B shown in Figure 4; this fact will be used in the proof of Theorem 1 in the next section. (Observe that the handle diagram in Figure 4 may also be obtained from the handle diagram of its complement, [Krushkal 2008, Figure 12].)

4. The Milnor group and the proof of Theorem 1

We start this section by summarizing the relevant information about the Milnor group, which will be used in the proof of Theorem 1. The reader is referred to the original [Milnor 1954] for a more complete introduction to the Milnor group of links in the 3-sphere; see also [Freedman and Teichner 1995, Section 2] for a discussion of the Milnor group in the more general 4-dimensional context.

Definition 4.1. Given a group G, normally generated by elements $g_1, \dots, g_n$, the *Milnor group* of G relative to the given normal generating set $\{g_i\}$ is defined as

$$MG = G \,/\, \langle\!\langle\, [g_i^x, g_i^y] \mid x, y \in G, i = 1, \dots, n \,\rangle\!\rangle. \tag{4-1}$$

The Milnor group is a finitely presented nilpotent group of class $\leq n$, where n is the number of normal generators in the previous definition. In this paper, an example of interest is $G = \pi_1(D^4 \setminus \Sigma)$, where Σ is a collection of surfaces with boundary, properly and disjointly embedded in (D^4, S^3). In this case, a choice of normal generators is provided by the meridians m_i to the components Σ_i of Σ. Here, a meridian m_i is an element of G which is obtained by following a path α_i in $D^4 \setminus \Sigma$ from the basepoint to the boundary of a regular neighborhood of Σ_i, followed by a small circle (a fiber of the circle normal bundle) linking Σ_i, then followed by α_i^{-1}.

Denote by $F_{g_1,\dots,g_n}$ the free group generated by the g_i, and consider the Magnus expansion

$$M : F_{g_1,\dots,g_n} \to \mathbb{Z}[x_1, \dots, x_n] \tag{4-2}$$

into the ring of formal power series in noncommuting variables $\{x_i\}$, defined by

$$M(g_i) = 1 + x_i, \qquad M(g_i^{-1}) = 1 - x_i + x_i^2 \mp \dots$$

We will keep the same notation for the homomorphism

$$M : MF_{g_1,\dots,g_n} \to R_{x_1,\dots,x_n}, \tag{4-3}$$

induced by the Magnus expansion, into the quotient $R_{x_1,\dots,x_n}$ of $\mathbb{Z}[x_1, \dots, x_n]$ by the ideal generated by all monomials $x_{i_1} \cdots x_{i_k}$ with some index occurring at least twice. It is established in [Milnor 1954] that the homomorphism (4-3) is well defined and injective.

We now turn to the proof of Theorem 1. Consider the submanifold $i : (B, \beta) \subset (D^4, S^3)$ that was constructed in Definition 3.1. Part (i) of the theorem follows from [Krushkal 2008, Theorem 1], which showed that there exist disjoint embeddings of three copies (B_i, β_i), such that the link formed by the curves β_1, β_2, β_3 in the 3-sphere is the Borromean rings. It is convenient to introduce the next definition.

Definition 4.2. An embedding $j : (B, \beta) \hookrightarrow (D^4, S^3)$ is *standard* if there exists an ambient isotopy between j and the original embedding i from Definition 3.1.

Examining the proof of [Krushkal 2008, Theorem 1], one may check that the embeddings of the B_i, constructed there and giving rise to the Borromean rings on the boundary, are not standard. (In terms of the spine picture of B in Figure 3, for the standard embedding, the curve δ bounds a disk in D^4 which is disjoint from the 2-sphere formed by the core of the 2-handle D capped off with a null-homotopy for its attaching curve; on the other hand, for the embedding constructed in [Krushkal 2008], δ has linking number 1 with this 2-sphere.)

We will now show that given disjoint *standard* embeddings of several copies (B_i, β_i) into the 4-ball, the link formed by the curves $\beta_1, \dots, \beta_n$ in the 3-sphere is necessarily homotopically trivial. We will show that the Borromean rings do not bound disjoint standard embeddings of three copies of (B, β). The Borromean rings case is the most interesting example from the perspective of the A-B slice problem, while the case of other homotopically essential links is proved analogously.

Suppose to the contrary that the Borromean rings bound disjoint standard embeddings B_1, B_2, B_3. We will consider the *relative-slice* reformulation of the problem; see [Freedman and Lin 1989] and also [Krushkal 2008] for a more detailed introduction. Using the handle diagram in Figure 4, one then observes that there is a solution to the relative-slice problem shown in Figure 5. This means that the six components $l_1, \ldots, l_6$ (drawn solid in the figure) bound disjoint embedded disks in the handlebody $D^4 \cup_{a,b,c}$ 2-handles, where the 2-handles are attached to the 4-ball with zero framings along the curves a, b, c (drawn dashed) from Figure 5. The fact that the embeddings $B_i \hookrightarrow D^4$ are standard is reflected by the fact that the slices bounded by the "solid" curves of each B_i do not go over the 2-handles (dashed curves) corresponding to the same B_i. This means that the slices for l_1, l_2 do not go over a and, similarly, l_3, l_4 do not go over b, nor l_5, l_6 over c. (Note that, without this restriction, there is a rather straightforward solution to this relative-slice problem.)

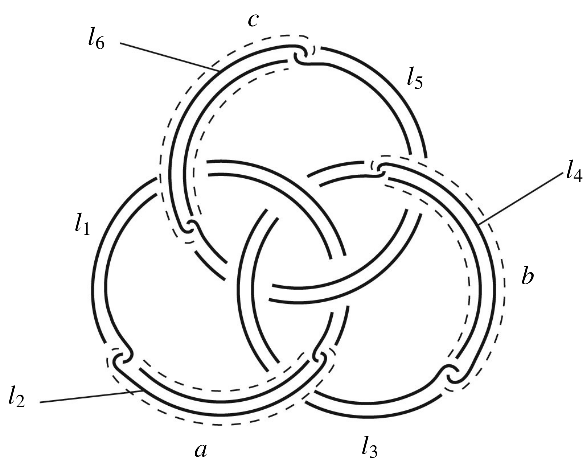

Figure 5

Therefore, assume the link in Figure 5 is relatively slice, subject to the "standard" condition discussed previously. Denote by D_i the slice bounded by l_i, $i = 1, \ldots, 6$, and let $D = \bigcup_{i=2,\ldots,6} D_i$. Consider

$$X := (D^4 \cup_{a,b,c} \text{2-handles}) \setminus D.$$

Denote by m_i the meridians to the components l_i, and by m_a, m_b, m_c the meridians to the curves a, b, c, respectively. The first homology $H_1(X)$ is generated by $m_2, \ldots, m_6$. In fact, we view $\{m_i\}$ as based loops in X normally generating $\pi_1(X)$.

If we omit the first component l_1, the remaining link $(l_2, \ldots, l_6, a, b, c)$ in Figure 5 is the unlink. This implies that the second homology $H_2(X)$ is spherical. Indeed, its generators may be represented by parallel copies of the cores of the 2-handles attached to a, b, c, capped off by disks in the complement of a neighborhood of the link $(l_2, \ldots, l_6, a, b, c)$ in the 3-sphere. Therefore, the Milnor group $M\pi_1(X)$ with respect to the normal generators m_i is isomorphic to the free

Milnor group:

$$M\pi_1(X) \cong MF_{m_2,\dots,m_6}.$$

Indeed, since the Milnor group is nilpotent, it is obtained from the quotient by a term of the lower central series, $\pi_1(X)/(\pi_1(X))^n$, by adding the Milnor relations (4-1). The relations in the nilpotent group $\pi_1(X)/(\pi_1(X))^n$ may be read off from surfaces representing generators of $H_2(X)$; see [Krushkal 1998, Lemma 13]. In particular, the relations corresponding to spherical classes are trivial. Therefore, all relations in $M\pi_1(X)$ are the standard relations (4-1) or, in other words, $M\pi_1(X)$ is the free Milnor group.

It follows that the Magnus expansion (4-3)

$$M : M\pi_1(X) \cong MF_{m_2,\dots,m_6} \to R_{x_2,\dots,x_6} \tag{4-4}$$

is well defined. Connecting the first component l_1 to the basepoint, consider it as an element of $M\pi_1(X)$. From the assumption that the link in Figure 5 is relatively slice, it follows that l_1 bounds a disk in X and, in particular, that it is trivial in $M\pi_1(X)$. We will find a nontrivial term in the Magnus expansion (4-4) $M(l_1) \in R_{x_2,\dots,x_6}$, giving a contradiction with the relative-slice assumption.

Consider the meridians m_i to the components l_i in S^3, for $i = 2, \dots, 6$, and also the meridians m_a, m_b, m_c to a, b, c. The meridians m_i will also serve as meridians to the slices D_i bounded by l_i, for $i = 2, \dots, 6$, that were discussed above. Consider

$$l_1 = \big[m_a m_2, \big[[m_3, m_b m_4], [m_5, m_6 m_c]\big]\big] \in M\pi_1(X). \tag{4-5}$$

In this expression, m_a, m_b, m_c are elements of $M\pi_1(X)$ that depend on how the hypothetical slices D_i go over the 2-handles attached to a, b, c. The expression (4-5) may be read from the capped grope (see Figure 6) bounded by l_1 in the complement of the other components in the 3-sphere. (Note that the components $l_2, \dots, l_6, a, b, c$ intersect only the caps and not the body of the grope.)

Recall a basic commutator identity: any three elements f, g, h in a group satisfy

$$[fg, h] = [f, h]^g \, [g, h]. \tag{4-6}$$

Suppose two elements $s, t \in M\pi_1(X)$ have Magnus expansions

$$M(s) = 1 + \boldsymbol{x} \qquad \text{and} \qquad M(t) = 1 + \boldsymbol{y},$$

where $\boldsymbol{x}, \boldsymbol{y}$ denote the sum of all monomials of nonzero degree in the expansions of s, t. Then, the expansion of the conjugate tst^{-1} is of the form

$$1 + \boldsymbol{x} + \boldsymbol{x}\boldsymbol{y} + \boldsymbol{y}\boldsymbol{x} + \cdots$$

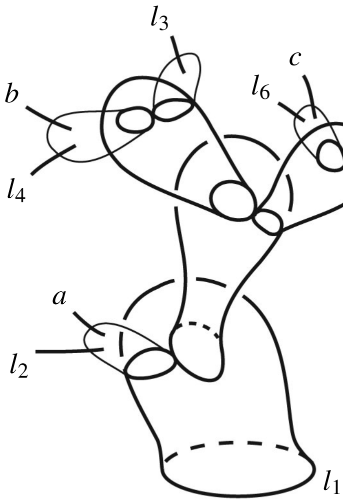

Figure 6

In particular, any first nontrivial term in the expansion $M(s)$ also appears in the expansion of any conjugate of s, $M(s^t)$. The expression (4-5) for l_1 is a 5-fold commutator and, since the ring $R_{x_2,\dots,x_6}$ is defined in terms of *nonrepeating* variables, this implies that any monomial of nonzero degree in the expansion $M(l_1)$ contains all the variables $x_2, \dots x_6$. This means that, to read off the Magnus expansion, any conjugation coming up while using (4-6) to simplify (4-5) may be omitted. These observations imply that the Magnus expansion $M(l_1)$ equals

$$M\big([m_2, [[m_3, m_4], [m_5, m_6]]]\big), \tag{4-7}$$

times the Magnus expansion of seven other terms where some (or all) of m_2, m_4, m_6 are replaced with m_a, m_b, m_c. Moreover, recall that the "standard" embedding assumption implies that the slice D_2 bounded by l_2 does not go over the 2-handle attached to a. Therefore, the Magnus expansion of m_a is of the form

$$M(m_a) = 1 + \sum_{i=3}^{6} \alpha_i x_i + \text{higher terms},$$

for some coefficients α_i. Since the meridians m_3, m_5 are present in each commutator obtained by simplifying (4-5), the only terms in the Magnus expansion of m_a that may contribute to a nontrivial monomial in $M(l_1)$ are x_4 and x_6. Similarly, the only possibly nontrivial contributions to $M(l_1)$ of m_b are x_2, x_6, and of m_c are x_2, x_4.

Using the fact that

$$M([s, t]) = 1 + \boldsymbol{xy} - \boldsymbol{yx} \pm \cdots ,$$

where $M(s) = 1+\boldsymbol{x}$, $M(t) = 1+\boldsymbol{y}$, note that the expansion (4-7) contains the monomial $x_2x_3x_4x_6x_5$. We claim that this monomial does not cancel with any other term in the expansion $M(l_1)$. This claim is proved by a direct inspection: any monomial in the Magnus expansion of a commutator of the form (4-7) with m_2 replaced by m_a has x_4 or x_6 as either the first or last variable. The only other possibility is the expansion of the commutator $\big[m_2, \big[[m_3, m_b], [m_5, m_c]\big]\big]$ with $M(m_b)$ contributing x_6, and $M(m_c)$ contributing x_4. The monomial $x_2x_3x_4x_6x_5$ does not appear in this expansion either. Therefore, we found a nontrivial term in the Magnus expansion $M(l_1) \in M\pi_1(X)$, contradicting the relative-slice assumption. This contradiction completes the proof.

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