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We find new relations in the mapping class group of a genus 2 surface with n boundary components for $n = 1, \ldots, 8$ that induce a genus 2 Lefschetz fibration $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2 \to S^2$ with n disjoint sections. As a consequence, we show any holomorphic genus 2 Lefschetz fibration without separating singular fibers admits a section.

1. Introduction

The study of Lefschetz fibrations is important in low-dimensional topology because of a close relationship between symplectic 4-manifolds and Lefschetz fibrations, [Donaldson 1999; Gompf and Stipsicz 1999]. Sections of Lefschetz fibrations play an important role in the theory. For example, in the presence of a section, the fundamental group and the signature of a Lefschetz fibration can be easily computed.

Here, we provide sections for genus 2 Lefschetz fibrations $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2 \to S^2$, with global monodromy given by the relation $(t_{c_1}t_{c_2}t_{c_3}t_{c_4}t_{c_5}^2t_{c_4}t_{c_3}t_{c_2}t_{c_1})^2 = 1$ in the mapping class group Γ_2 of a closed genus 2 surface, where each c_i is a simple closed curve as in Figure 3, and t_{c_i} is a right-handed Dehn twist about c_i for $i = 1, \ldots, 5$. In [Korkmaz and Ozbagci 2008], similar relations were found in the mapping class group $\Gamma_{1,n}$ of a genus 1 surface with n boundary components for $n = 4, \ldots, 9$, giving an elliptic Lefschetz fibration $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2 \to S^2$ with n disjoint sections.

In Section 2, we recall definitions and relations in the mapping class group to be used in our computations, and we fix notation. In Section 3, we give brief background information on Lefschetz fibrations. In Section 4, we provide the necessary relations in the mapping class group $\Gamma_{2,n}$ for a genus 2 Lefschetz fibration $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2 \to S^2$ with n disjoint sections for $n = 1, \ldots, 6$. In Section 5 we list several observations and open problems related to sections of Lefschetz fibrations. We show that a genus 2 Lefschetz fibration $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2 \to S^2$ may admit at most 12 disjoint sections. We provide relations in the corresponding mapping class

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group that give $n = 7$ and $n = 8$ disjoint sections for genus 2 Lefschetz fibrations $\mathbb{CP}^2 \# 13 \overline{\mathbb{CP}}^2 \to S^2$. We conclude that any holomorphic genus 2 Lefschetz fibration without separating singular fibers admits a section.

2. Mapping class groups

Let $\Sigma^k_{g,n}$ denote an oriented, connected, genus g surface, with n boundary components and k marked points. *The mapping class group* of $\Sigma^k_{g,n}$ is defined as the isotopy classes of orientation-preserving self-diffeomorphisms of $\Sigma^k_{g,n}$ that fix the marked points and the points on the boundary. Denote the mapping class group of $\Sigma^k_{g,n}$ by $\Gamma^k_{g,n}$. When $k = 0$, denote the mapping class group of $\Sigma_{g,n}$ by $\Gamma_{g,n}$.

Let a be a simple closed curve on $\Sigma^k_{g,n}$. A *right-handed Dehn twist* t_a about a is the isotopy class of a self-diffeomorphism of $\Sigma^k_{g,n}$ obtained by cutting $\Sigma^k_{g,n}$ along a and gluing it back after twisting one side by 2π to the right. The inverse of a right-handed Dehn twist is a *left-handed Dehn twist*, denoted by t_a^{-1}.

We now briefly mention the facts and relations to be used in our computations; for the proofs, see [Farb and Margalit 2005; Ivanov 2002]. If $f : \Sigma^k_{g,n} \to \Sigma^k_{g,n}$ is an orientation-preserving diffeomorphism, then $f t_a f^{-1} = t_{f(a)}$ for a a simple closed curve on $\Sigma^k_{g,n}$.

For simplicity, we will denote a right-handed Dehn twist t_a along a by a, and a left-handed Dehn twist t_a^{-1} by $\bar{a}$. The product ab means that we first apply the Dehn twist b, then the Dehn twist a. A simple closed curve parallel to a boundary component of a given surface will be called a *boundary curve* of the surface.

The following relations will be useful:

The *commutativity relation.* If a and b are two disjoint simple closed curves on $\Sigma^k_{g,n}$, then the Dehn twists along a and b commute: $ab = ba$.

The *braid relation.* If a and b are two simple closed curves on $\Sigma^k_{g,n}$ intersecting transversely at a single point, then their Dehn twists satisfy $aba = bab$.

The *lantern relation.* Consider a sphere with four holes, the boundary curves $\delta_1, \delta_2, \delta_3, \delta_4$, and the simple closed curves α, γ, σ, as shown in Figure 1. We have $\delta_1\delta_2\delta_3\delta_4 = \gamma\sigma\alpha$. Dehn discovered this relation; it was then rediscovered and named by D. Johnson.

The *star relation* [Gervais 2001]. Let $\Sigma_{1,3}$ be a torus with three boundary curves $\delta_1, \delta_2, \delta_3$. In $\Gamma_{1,3}$ we have $\delta_1\delta_2\delta_3 = (a_1a_2a_3b)^3$ for the simple closed curves a_1, a_2, a_3, b from Figure 1.

The *chain relation* for a two-holed torus. Consider a torus $\Sigma_{1,2}$ with two boundary curves δ_1, δ_2, and the simple closed curves c_1, c_2, b, as shown in Figure 1. We have $\delta_1\delta_2 = (c_1bc_2)^4$.

The *chain relations* for the genus 2 case: If c_1, c_2, c_3, c_4, c_5 is the chain of curves

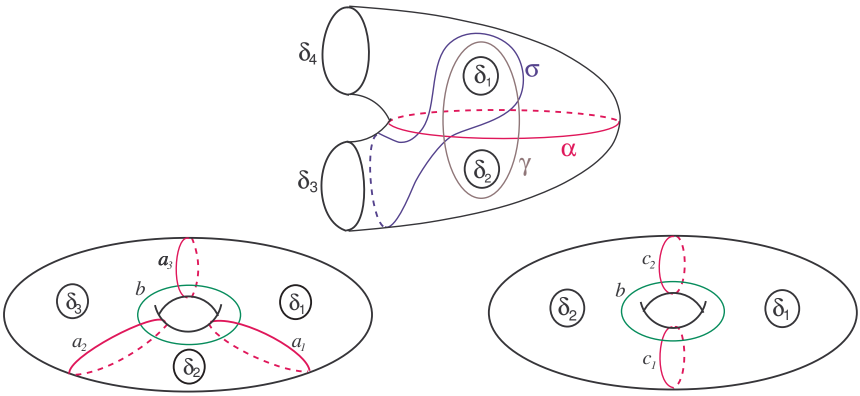

Figure 1. Counterclockwise from the top: The lantern relation, $\delta_1\delta_2\delta_3\delta_4 = \gamma\sigma\alpha$, the star relation, $\delta_1\delta_2\delta_3 = (a_1a_2a_3b)^3$, and the two-holed torus relation, $\delta_1\delta_2 = (c_1bc_2)^4$.

shown in Figure 2, then for a genus 2 surface $\Sigma_{2,1}$ with one boundary curve δ_1 we have $\delta_1 = (c_1c_2c_3c_4)^{10}$, while for a genus 2 surface $\Sigma_{2,2}$ with two boundary curves δ_1, δ_2 we have $\delta_1\delta_2 = (c_1c_2c_3c_4c_5)^6$.

3. Lefschetz fibrations

A *Lefschetz fibration* on a closed, connected, oriented smooth 4-manifold X is a map $f : X \to \Sigma$, where Σ is a closed, connected, oriented smooth surface, such that f is surjective, has isolated critical points, and for each critical point p there is an orientation-preserving local complex coordinate chart on which f takes the form $f(z_1, z_2) = z_1^2 + z_2^2$.

The Lefschetz fibration f is a smooth fiber bundle away from critical points. A regular fiber of f is diffeomorphic to a closed, oriented smooth genus g surface. We define the *genus* of the Lefschetz fibration to be the genus of a regular fiber.

A *singular fiber* is a fiber containing a critical point. We assume that each singular fiber contains only one critical point. The singular fiber can be obtained by taking a simple closed curve on a regular fiber and shrinking it to a point. This

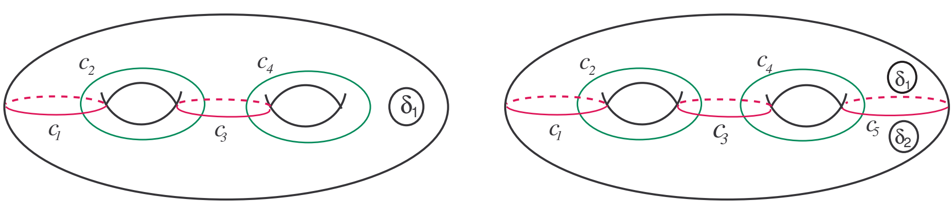

Figure 2. The chain relations: $\delta_1 = (c_1c_2c_3c_4)^{10}$ and $\delta_1\delta_2 = (c_1c_2c_3c_4c_5)^6$.

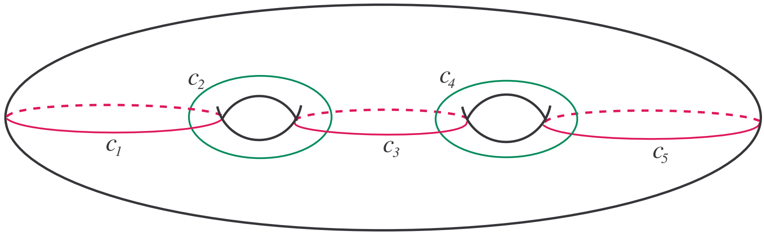

Figure 3. Σ_2.

simple closed curve describing the singular fiber is called a *vanishing cycle*. If this curve is a nonseparating curve, then the singular fiber is called *nonseparating*; otherwise it is called *separating*. For a genus g Lefschetz fibration over S^2, the product of Dehn twists along the vanishing cycles gives us the *global monodromy* of the Lefschetz fibration.

On a closed surface Σ_g, the right-handed Dehn twists c_i along the simple closed curves c_i for $i = 1, \dots, s$, with the relation $c_1 c_2 \cdots c_s = 1$, define a genus g Lefschetz fibration over S^2 with vanishing cycles $c_1, \dots, c_s$. In particular, in Γ_2 we have

$$(c_1c_2c_3c_4c_5^2c_4c_3c_2c_1)^2 = 1,$$
$$(c_1c_2c_3c_4c_5)^6 = 1,$$
$$(c_1c_2c_3c_4)^{10} = 1,$$

where $c_1, \dots, c_5$ are simple closed curves as in Figure 3. For each relation above, we have genus 2 Lefschetz fibrations over S^2 with total spaces $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$, $K3 \# 2\overline{\mathbb{CP}}^2$, and the Horikawa surface H, respectively. For more on Lefschetz fibrations, see [Auroux 2003; Gompf and Stipsicz 1999].

A section of a Lefschetz fibration is a map $\sigma : \Sigma \to X$ such that $f\sigma = \mathrm{id}_\Sigma$. Consider a collection of simple closed curves $c_1, \dots, c_s$ on a genus g surface $\Sigma_{g,n}$ with the relation

$$c_1 \cdots c_s = \delta_1^{k_1} \cdots \delta_n^{k_n}$$

in $\Gamma_{g,n}$, where $\delta_1, \dots, \delta_n$ are boundary curves, and $k_1, \dots, k_n$ are positive integers. This relation defines a genus g Lefschetz fibration over S^2 admitting n disjoint sections, with global monodromy $c_1 \cdots c_s = 1$. Moreover, the self-intersection of the i-th section is $-k_i$. To see this, note that after gluing a disk along each boundary curve one gets the relation $c_1 \cdots c_s = 1$ in Γ_g. Thus, this relation will give us a genus g Lefschetz fibration over S^2 as before. One can then use the centers of the capping disks to construct sections. For details, see [Gompf and Stipsicz 1999]; for more on self-intersection of sections, see [Smith 1998].

In the following sections, we will find relations of the above type, $c_1 \cdots c_s = \delta_1 \cdots \delta_n$, in the mapping class group $\Gamma_{2,n}$ for $n = 1, \dots, 8$.

4. Relations in $\Gamma_{2,n}$

For each $n = 1, \ldots, 6$, we write the product of right-handed Dehn twists along the boundary curves $\delta_1, \ldots, \delta_n$ as a product of twenty right-handed Dehn twists along nonboundary parallel simple closed curves on a genus 2 surface $\Sigma_{2,n}$. Namely, we provide relations of the form $\delta_1 \cdots \delta_n = \beta_1 \cdots \beta_{20}$, where $\beta_1, \ldots, \beta_{20}$ are nonboundary parallel simple closed curves on $\Sigma_{2,n}$. After gluing disks along the boundary curves $\delta_1, \ldots, \delta_n$, we get the relation $1 = \beta_1 \cdots \beta_{20}$ in the mapping class group Γ_2. By using the commutativity relation and the braid relation, one can simplify the right-hand side of the equation $1 = \beta_1 \cdots \beta_{20}$ so that it gives us the global monodromy $(c_1c_2c_3c_4c_5^2c_4c_3c_2c_1)^2 = 1$ of a Lefschetz fibration $\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2 \to S^2$. For the simple closed curves $c_1, \ldots, c_5$, see Figure 3. Here, twenty right-handed Dehn twists along nonboundary parallel, nonseparating simple closed curves c_i correspond to twenty nonseparating singular fibers. The consecutive simple closed curves c_i are the vanishing cycles.

In the following subsections, the relations we find in $\Gamma_{2,n}$ will give us a genus 2 Lefschetz fibration $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2 \to S^2$ admitting n disjoint sections of self-intersection -1, for $n = 1, \ldots, 6$.

4.1. ***Genus two surface with one hole.*** Consider the genus 2 surface $\Sigma_{2,1}$ with one boundary curve δ_1, as in Figure 4. We have

$$\delta_1 = (a_1b_1a_2b_2)^{10} = (a_1b_1a_2b_2)^5(a_1b_1a_2b_2)^5.$$

Using the commutativity and braid relations, one can show

$$(a_1b_1a_2b_2)^5 = (a_1b_1a_2)^4(b_2a_2b_1a_1^2b_1a_2b_2).$$

(For the proof, see the appendix.) Notice the two-holed torus embedded in $\Sigma_{2,1}$ with two boundary curves a_3, a_4; then, by the chain relation for this torus, we have $(a_1b_1a_2)^4 = a_3a_4$.

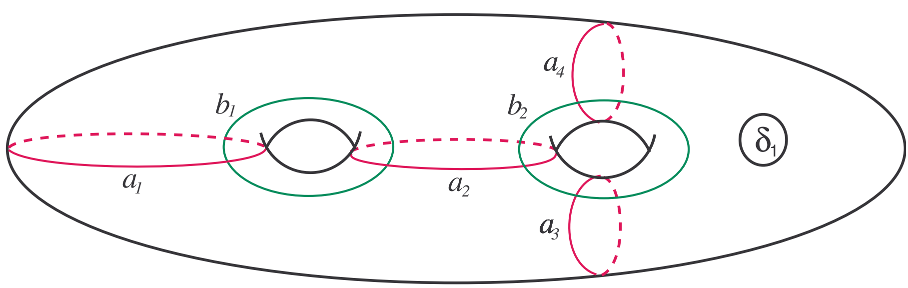

Figure 4. $\Sigma_{2,1}$.

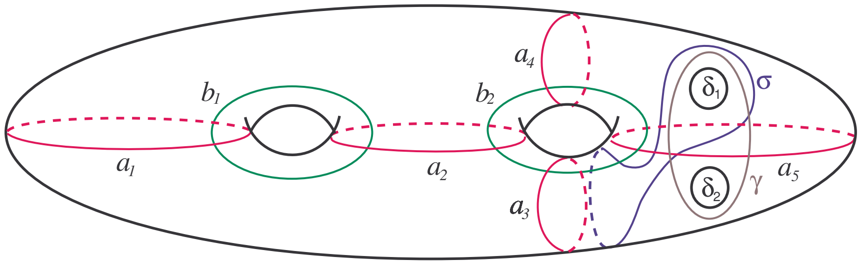

Figure 5. $\Sigma_{2,2}$.

By combining the relations above, we get

$$\begin{aligned}\delta_1 &= (a_1b_1a_2)^4(b_2a_2b_1a_1^2b_1a_2b_2)(a_1b_1a_2)^4(b_2a_2b_1a_1^2b_1a_2b_2)\\ &= a_3a_4(b_2a_2b_1a_1^2b_1a_2b_2)a_3a_4(b_2a_2b_1a_1^2b_1a_2b_2)\\ &= (a_3a_4b_2a_2b_1a_1^2b_1a_2b_2)^2.\end{aligned}$$

4.2. ***Genus two surface with two holes.*** Consider the genus 2 surface $\Sigma_{2,2}$ with two boundary curves δ_1, δ_2 from Figure 5, and notice the embedded sphere embedded in $\Sigma_{2,2}$ with four boundary curves $\delta_1, \delta_2, a_3, a_4$. Then, using the lantern relation, we have

$$a_3a_4\delta_1\delta_2 = \gamma\sigma a_5.$$

Notice in Figure 5 the genus 2 surface with one boundary curve γ; we thus have the chain relation $\gamma = (a_1b_1a_2b_2)^{10}$. Substituting γ in the lantern relation and then using the two-holed torus relation $(a_1b_1a_2)^4 = a_3a_4$ in the equation, we get

$$\begin{aligned}a_3a_4\delta_1\delta_2 &= \gamma\sigma a_5\\ &= (a_1b_1a_2b_2)^{10}\sigma a_5\\ &= (a_1b_1a_2b_2)^5(a_1b_1a_2b_2)^5\sigma a_5\\ &= (a_1b_1a_2)^4(b_2a_2b_1a_1^2b_1a_2b_2)(a_1b_1a_2)^4(b_2a_2b_1a_1^2b_1a_2b_2)\sigma a_5\\ &= a_3a_4(b_2a_2b_1a_1^2b_1a_2b_2)a_3a_4(b_2a_2b_1a_1^2b_1a_2b_2)\sigma a_5.\end{aligned}$$

We simplify this equation to

$$\delta_1\delta_2 = b_2a_2b_1a_1^2b_1a_2b_2a_3a_4(b_2a_2b_1a_1^2b_1a_2b_2)\sigma a_5.$$

4.3. ***Genus two surface with three holes.*** First, the lantern relation for the sphere with four boundary curves $\delta_1, \delta_2, a_4, a_5$ in $\Sigma_{2,3}$ from Figure 6 is $a_4a_5\delta_1\delta_2 = \gamma\sigma a_6$. For the three-holed torus with boundary curves γ, a_1, a_2, we have the star relation $\gamma a_1a_2 = (a_4a_5a_3b_2)^3$, while for the three-holed torus with boundary curves δ_3, a_4, a_5, we have $\delta_3a_4a_5 = (a_1a_2a_3b_1)^3$.

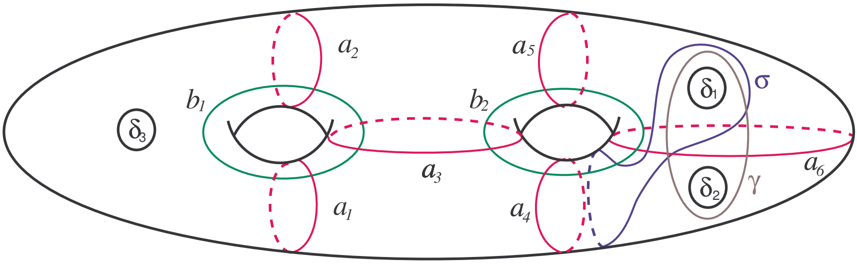

Figure 6. $\Sigma_{2,3}$.

Now, combine these relations, substitute δ_3 and γ, then simplify by using the commutativity and braid relations. Note that all the a_i commute for $i = 1, \ldots, 6$. The simple closed curves a_1, a_2, a_3 intersect b_1 transversely at a single point, and a_4, a_5, a_3 intersect b_2 transversely at a single point. Thus, with $\beta = \bar{a}_5\bar{a}_4 b_2 a_4 a_5$,

$$\begin{aligned}
\delta_1\delta_2 &= \bar{a}_4\bar{a}_5\gamma\sigma a_6,\\
\delta_1\delta_2\delta_3 &= \delta_3\bar{a}_4\bar{a}_5\gamma\sigma a_6\\
&= \delta_3\bar{a}_4\bar{a}_5\bar{a}_1\bar{a}_2(a_4a_5a_3b_2)^3\sigma a_6\\
&= \bar{a}_1\bar{a}_2(\delta_3)\bar{a}_4\bar{a}_5(a_4a_5a_3b_2)(a_4a_5a_3b_2)^2\sigma a_6\\
&= \bar{a}_1\bar{a}_2((a_1a_2a_3b_1)^3\bar{a}_4\bar{a}_5)a_3b_2(a_4a_5a_3b_2)^2\sigma a_6\\
&= a_3b_1(a_1a_2a_3b_1)^2a_3(\bar{a}_4\bar{a}_5b_2a_4a_5)a_3b_2a_4a_5a_3b_2\sigma a_6\\
&= a_3b_1(a_1a_2a_3b_1)^2a_3\beta a_3b_2a_4a_5a_3b_2\sigma a_6.
\end{aligned}$$

4.4. ***Genus two surface with four holes.*** We will use the three-holed genus two relation we found in Section 4.3. Notice in Figure 7 the genus 2 surface with three boundary curves δ_3, δ_4, γ. Then

$$\begin{aligned}
\delta_3\delta_4\gamma &= a_3b_2(a_5a_4a_3b_2)^2a_3\beta_1a_3b_1a_2a_1a_3b_1\sigma_1a_7\\
&= a_3\beta_1a_3b_1a_2a_1a_3b_1\sigma_1a_7a_3b_2(a_5a_4a_3b_2)^2.
\end{aligned}$$

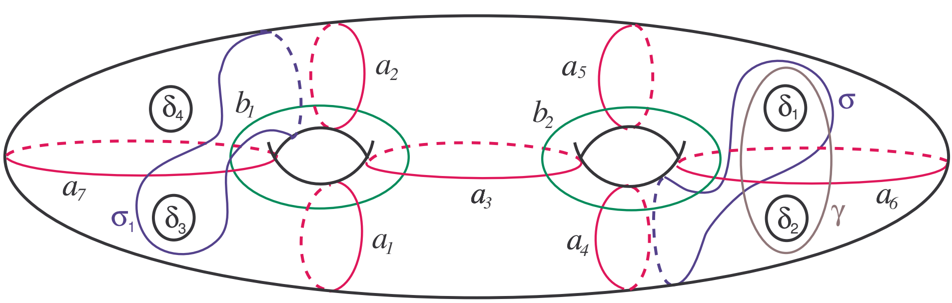

Figure 7. $\Sigma_{2,4}$.

where $\beta_1 = \bar{a}_1\bar{a}_2 b_1 a_2 a_1$. Note that we identify the curves $(a_1, a_2, a_3, a_4, a_5, a_6)$ in $\Sigma_{2,3}$ from Figure 6 with the curves $(a_5, a_4, a_3, a_2, a_1, a_7)$ in $\Sigma_{2,4}$ from Figure 7.

By the lantern relation for the sphere with four boundary curves $\delta_1, \delta_2, a_4, a_5$, we have

$$a_4 a_5 \delta_1 \delta_2 = \gamma\sigma a_6.$$

Now, combine the above relations to get

$$\begin{aligned}\delta_1\delta_2\delta_3\delta_4 &= a_3\beta_1 a_3 b_1 a_2 a_1 a_3 b_1 \sigma_1 a_7 a_3 b_2 (a_5 a_4 a_3 b_2)^2 \bar{\gamma}\bar{a}_4\bar{a}_5\gamma\sigma a_6\\ &= a_3\beta_1 a_3 b_1 a_2 a_1 a_3 b_1 \sigma_1 a_7 a_3 b_2 (a_5 a_4 a_3 b_2) a_3 a_5 a_4 b_2 (\bar{a}_4\bar{a}_5\sigma a_6)\\ &= a_3\beta_1 a_3 b_1 a_2 a_1 a_3 b_1 \sigma_1 a_7 a_3 b_2 (a_5 a_4 a_3 b_2) a_3 \beta_2 \sigma a_6,\end{aligned}$$

where $\beta_1 = \bar{a}_1\bar{a}_2 b_1 a_2 a_1$ and $\beta_2 = a_5 a_4 b_2 \bar{a}_4\bar{a}_5$.

4.5. ***Genus two surface with five holes.*** The lantern relation for the sphere with four boundary curves $\delta_1, \delta_2, a_4, a_5$ in $\Sigma_{2,5}$ from Figure 8 is $a_4 a_5 \delta_1\delta_2 = \gamma\sigma a_6$. In Figure 8, notice the genus 2 surface with four boundary curves $\delta_3, \delta_4, \delta_5, \gamma$. Identify the curves $(\delta_1, \delta_2, a_5, a_6, \sigma)$ in $\Sigma_{2,4}$ from Figure 7 with $(\delta_5, \gamma, a_8, a_5, \sigma_2)$ in $\Sigma_{2,5}$ from Figure 8. Then, by the relation given in Section 4.4, we have

$$\delta_3\delta_4\delta_5\gamma = a_3\beta_1 a_3 b_1 a_2 a_1 a_3 b_1 \sigma_1 a_7 a_3 b_2 (a_8 a_4 a_3 b_2) a_3 \beta_2 \sigma_2 a_5,$$

where $\beta_1 = \bar{a}_1\bar{a}_2 b_1 a_2 a_1$ and $\beta_2 = a_8 a_4 b_2 \bar{a}_4 \bar{a}_8$.

Now, combine the above relations and simplify the equation as

$$\begin{aligned}\delta_1\delta_2\delta_3\delta_4\delta_5 &= a_3\beta_1 a_3 b_1 a_2 a_1 a_3 b_1 \sigma_1 a_7 a_3 b_2 (a_8 a_4 a_3 b_2) a_3 \beta_2 \sigma_2 a_5 \bar{\gamma}\bar{a}_4\bar{a}_5\gamma\sigma a_6\\ &= a_3\beta_1 a_3 b_1 a_2 a_1 a_3 b_1 \sigma_1 a_7 a_3 b_2 (a_8 a_4 a_3 b_2) a_3 \beta_2 \sigma_2 \bar{a}_4 \sigma a_6\\ &= a_3\beta_1 a_3 b_1 a_2 a_1 a_3 b_1 \sigma_1 a_7 a_3 (\bar{a}_4) b_2 (a_4 a_8 a_3 b_2) a_3 \beta_2 \sigma_2 \sigma a_6\\ &= a_3\beta_1 a_3 b_1 a_2 a_1 a_3 b_1 \sigma_1 a_7 a_3 \beta_3 (a_8 a_3 b_2) a_3 \beta_2 \sigma_2 \sigma a_6,\end{aligned}$$

where $\beta_1 = \bar{a}_1\bar{a}_2 b_1 a_1 a_2$, $\beta_2 = a_8 a_4 b_2 \bar{a}_4 \bar{a}_8$, and $\beta_3 = \bar{a}_4 b_2 a_4$.

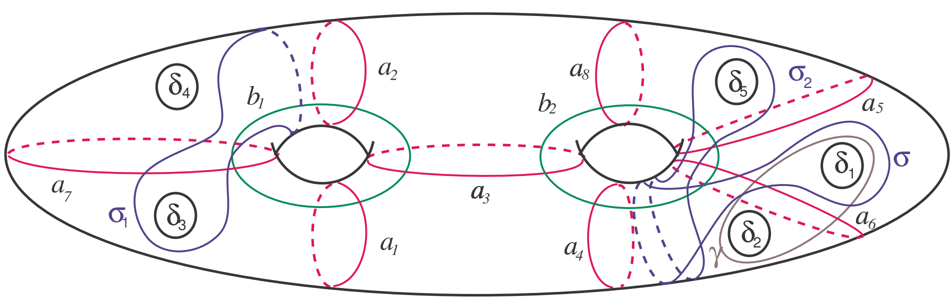

Figure 8. $\Sigma_{2,5}$.

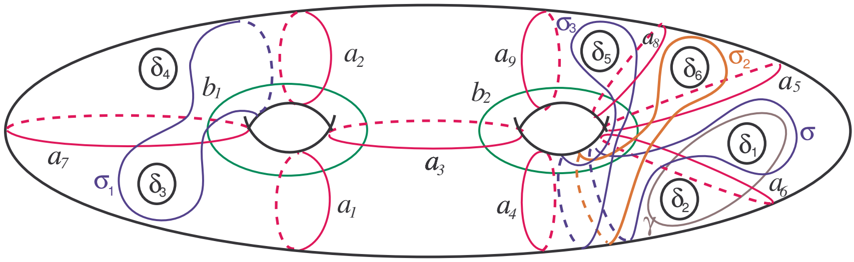

Figure 9. $\Sigma_{2,6}$.

4.6. ***Genus two surface with six holes.*** The lantern relation for the sphere with four boundary curves δ_1, δ_2, a_4, a_5 in $\Sigma_{2,6}$ from Figure 9 is $a_4a_5\delta_1\delta_2 = \gamma\sigma a_6$. Now, identify the curves $(\delta_1, \delta_2, a_6, a_8, \sigma, \sigma_2)$ in $\Sigma_{2,5}$ from Figure 8 with the curves $(\delta_6, \gamma, a_5, a_9, \sigma_2, \sigma_3)$ in $\Sigma_{2,6}$ from Figure 9. By the relation given in Section 4.5 for the genus 2 surface with five boundary curves δ_3, δ_4, δ_5, δ_6, γ, we have

$$\delta_6\gamma\delta_3\delta_4\delta_5 = a_3\beta_1a_3b_1a_2a_1a_3b_1\sigma_1a_7a_3\beta_3a_9a_3b_2a_3\beta_2\sigma_3\sigma_2a_5.$$

where $\beta_1 = \bar{a}_1\bar{a}_2b_1a_2a_1$, $\beta_2 = a_9a_4b_2\bar{a}_4\bar{a}_9$, and $\beta_3 = \bar{a}_4b_2a_4$.

Now, combine the above relations to get

$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 &= a_3\beta_1a_3b_1a_2a_1a_3b_1\sigma_1a_7a_3\beta_3a_9a_3b_2a_3\beta_2\sigma_3\sigma_2a_5\bar{\gamma}\bar{a}_4\bar{a}_5\gamma\sigma a_6\\
&= a_3\beta_1a_3b_1a_2a_1a_3b_1\sigma_1a_7a_3\beta_3a_9a_3b_2a_3\beta_2\sigma_3\sigma_2(\bar{a}_4)\sigma a_6\\
&= a_3\beta_1a_3b_1a_2a_1a_3b_1\sigma_1a_7a_3(\bar{a}_4)(\beta_3)a_9a_3b_2a_3\beta_2\sigma_3\sigma_2\sigma a_6\\
&= a_3\beta_1a_3b_1a_2a_1a_3b_1\sigma_1a_7a_3(\bar{a}_4)(\bar{a}_4b_2a_4)a_9(a_3b_2a_3)\beta_2\sigma_3\sigma_2\sigma a_6\\
&= a_3\beta_1a_3b_1a_2a_1a_3b_1\sigma_1a_7a_3(\bar{a}_4)(b_2a_4\bar{b}_2)a_9(b_2a_3b_2)\beta_2\sigma_3\sigma_2\sigma a_6\\
&= a_3\beta_1a_3b_1a_2a_1a_3b_1\sigma_1a_7a_3\beta_3\beta_4a_3b_2\beta_2\sigma_3\sigma_2\sigma a_6
\end{aligned}$$

where $\beta_1 = \bar{a}_1\bar{a}_2b_1a_2a_1$, $\beta_2 = a_5a_4b_2\bar{a}_4\bar{a}_5$, $\beta_3 = \bar{a}_4b_2a_4 = b_2a_4\bar{b}_2$, and $\beta_4 = \bar{b}_2a_9b_2$.

5. Final remarks

Lemma 5.1. *A genus* 2 *Lefschetz fibration* $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2 \to S^2$ *admits at most* 12 *disjoint sections.*

Proof. Suppose that $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2 \to S^2$ admits 13 disjoint sections. Each section is a sphere with self-intersection -1. Furthermore, each section intersects a regular fiber, a genus 2 surface Σ_2 with self-intersection 0, at one point. Now, by blowing down all -1 spheres, we get a genus 2 surface $\widetilde{\Sigma}_2$ with self-intersection 13, which cannot exist in a manifold with second homology $\mathbb{Z}$. □

In Section 4, we found relations giving n disjoint sections for genus 2 Lefschetz fibration $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2 \to S^2$ for $n = 1, \ldots, 6$. The technique applied in Section 4 stops at $n = 6$. However, by using results from [Korkmaz and Ozbagci 2008], we can find relations in the corresponding mapping class group that give $n = 7$ and $n = 8$ disjoint sections for genus 2 Lefschetz fibration $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2 \to S^2$. We will next show how to derive these relations. This method does not go further, and it remains unknown whether there are more than eight sections.

The seven-holed torus relation from [Korkmaz and Ozbagci 2008] sits in $\Sigma_{2,7}$:

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7 = \alpha_3\alpha_4\alpha_1 b_1\sigma_5\alpha_2\beta_5\sigma_3\sigma_6\alpha_6\beta_3\sigma_4,$$

where $\beta_3 = \alpha_3 b_1\overline{\alpha}_3$ and $\beta_5 = \alpha_5 b_1\overline{\alpha}_5$ in $\Sigma_{2,7}$; see Figure 10. We identify the boundary curves $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7)$ in $\Sigma_{1,7}$ from Figure 10 with the curves $(\delta_6, \delta_5, a_2, a_1, \delta_2, \delta_1, \delta_7)$ in $\Sigma_{2,7}$. The seven-holed torus relation gives

$$a_1a_2\delta_1\delta_2\delta_5\delta_6\delta_7 = a_3a_4a_9b_2\sigma_5a_{10}\beta_5\sigma_3\sigma_6a_5\beta_3\sigma_4$$

where $\beta_3 = a_3b_2\overline{a}_3$ and $\beta_5 = a_6b_2\overline{a}_6$.

Next, we combine this with the lantern relation $a_1a_2\delta_3\delta_4 = \gamma\sigma a_7$ for the sphere with four boundary curves $\delta_3, \delta_4, a_1, a_2$ in $\Sigma_{2,7}$. The star relation for the torus with three boundary curves γ, a_4, a_{10} is $a_4a_{10}\gamma = (a_1a_2a_3b_1)^3$. We now substitute

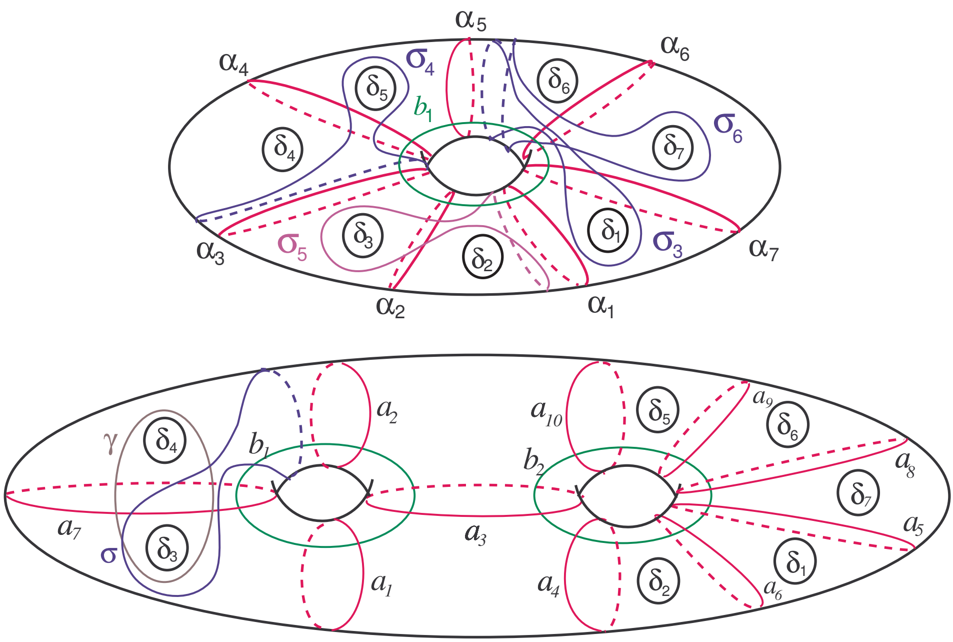

Figure 10. Seven-holed torus relation.

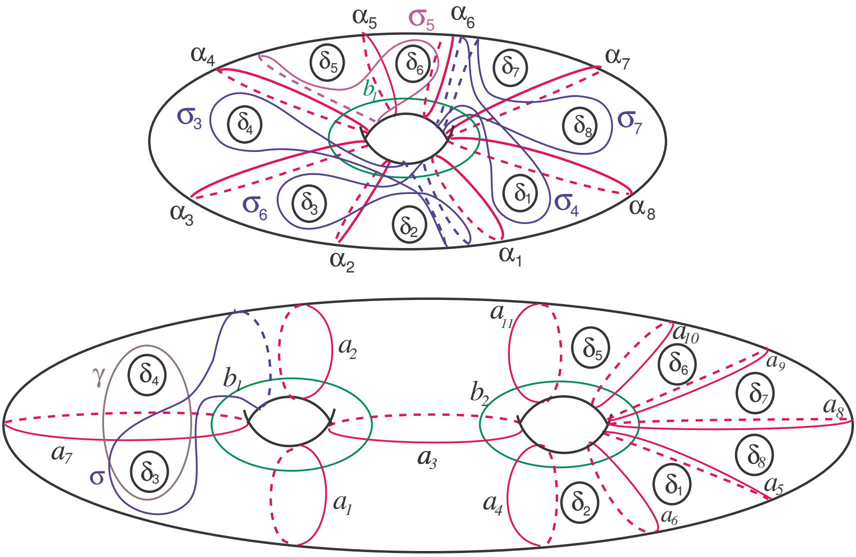

Figure 11. Eight-holed torus relation.

$\gamma=\bar{a}_4\bar{a}_{10}(a_1a_2a_3b_1)^3$ into the lantern relation; then we simplify the equation and write the product of right-handed Dehn twists along the boundary curves $\delta_1,\ldots,\delta_7$ as a product of twenty right-handed Dehn twists along nonboundary parallel simple closed curves on $\Sigma_{2,7}$:

$$\begin{aligned}
&\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\\
&\quad=a_3a_4a_9b_2\sigma_5a_{10}\beta_5\sigma_3\sigma_6a_5\beta_3\sigma_4\bar{a}_1\bar{a}_2\bar{a}_1\bar{a}_2\gamma\sigma a_7\\
&\quad=a_3a_9b_2\sigma_5a_{10}\beta_5\sigma_3\sigma_6a_5\beta_3\sigma_4(a_4)\bar{a}_1\bar{a}_2\bar{a}_1\bar{a}_2\bar{a}_4\bar{a}_{10}(a_1a_2a_3b_1)^3\sigma a_7\\
&\quad=a_3a_9b_2\sigma_5a_{10}\beta_5\sigma_3\sigma_6a_5\beta_3\sigma_4\bar{a}_1\bar{a}_2a_3b_1(a_1a_2a_3b_1)^2(\bar{a}_{10})\sigma a_7\\
&\quad=a_3a_9(\bar{a}_{10})b_2\sigma_5a_{10}\beta_5\sigma_3\sigma_6a_5\beta_3\sigma_4a_3(\bar{a}_1\bar{a}_2b_1a_1a_2)a_3b_1(a_1a_2a_3b_1)\sigma a_7\\
&\quad=a_3a_9(\bar{a}_{10}b_2a_{10})(\bar{a}_{10}\sigma_5a_{10})\beta_5\sigma_3\sigma_6a_5\beta_3\sigma_4a_3\tilde{\beta}a_3b_1(a_1a_2a_3b_1)\sigma a_7\\
&\quad=a_3a_9\tilde{\beta}_1\tilde{\beta}_2\beta_5\sigma_3\sigma_6a_5\beta_3\sigma_4a_3\tilde{\beta}a_3b_1(a_1a_2a_3b_1)\sigma a_7,
\end{aligned}$$

where $\tilde{\beta}_1=\bar{a}_{10}b_2a_{10}$, $\tilde{\beta}_2=\bar{a}_{10}\sigma_5a_{10}$, $\beta_5=a_6b_2\bar{a}_6$, $\beta_3=a_3b_2\bar{a}_3$, and $\tilde{\beta}=\bar{a}_1\bar{a}_2b_1a_1a_2$. Note that the simple closed curves σ_5 and a_{10} intersect at 2 points.

Similarly, the eight-holed torus relation from [Korkmaz and Ozbagci 2008] sits in $\Sigma_{2,8}$ (see Figure 11):

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8=\alpha_4\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7\beta_4\sigma_5=\alpha_4b_1\sigma_5\alpha_5\beta_1\sigma_3\sigma_6\alpha_2\beta_6\sigma_4\sigma_7\alpha_7,$$

where $\beta_1=\alpha_1b_1\bar{\alpha}_1$, $\beta_4=\alpha_4b_1\bar{\alpha}_4$, and $\beta_6=\alpha_6b_1\bar{\alpha}_6$ in $\Sigma_{2,8}$. We identify the curves

$(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8)$ in $\Sigma_{1,8}$ with $(\delta_1, \delta_8, \delta_7, \delta_6, \delta_5, a_2, a_1, \delta_2)$ in $\Sigma_{2,8}$.

By applying the same technique, we also get the necessary relation for $n = 8$. The eight-holed torus relation gives

$$a_1 a_2 \delta_1 \delta_2 \delta_5 \delta_6 \delta_7 \delta_8 = a_{10} b_2 \sigma_5 a_{11} \beta_1 \sigma_3 \sigma_6 a_8 \beta_6 \sigma_4 \sigma_7 a_4,$$

where $\beta_1 = a_5 b_2 \bar{a}_5$ and $\beta_6 = a_3 b_2 \bar{a}_3$. We combine this with the lantern relation $a_1 a_2 \delta_3 \delta_4 = \gamma \sigma a_7$ for the sphere with four boundary curves $\delta_3, \delta_4, a_1, a_2$ in $\Sigma_{2,8}$. Using the star relation $a_4 a_{11} \gamma = (a_1 a_2 a_3 b_1)^3$ for the torus with three boundary curves γ, a_4, a_{11}, we substitute $\gamma = \bar{a}_4 \bar{a}_{11} (a_1 a_2 a_3 b_1)^3$ in the lantern relation. We simplify the equation as

$$\begin{aligned}
&\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \\
&\quad = a_{10} b_2 \sigma_5 a_{11} \beta_1 \sigma_3 \sigma_6 a_8 \beta_6 \sigma_4 \sigma_7 a_4 \bar{a}_1 \bar{a}_2 \bar{a}_1 \bar{a}_2 \gamma \sigma a_7 \\
&\quad = a_{10} b_2 \sigma_5 a_{11} \beta_1 \sigma_3 \sigma_6 a_8 \beta_6 \sigma_4 \sigma_7 a_4 \bar{a}_1 \bar{a}_2 \bar{a}_1 \bar{a}_2 \bar{a}_4 \bar{a}_{11} (a_1 a_2 a_3 b_1)^3 \sigma a_7 \\
&\quad = a_{10} b_2 \sigma_5 a_{11} \beta_1 \sigma_3 \sigma_6 a_8 \beta_6 \sigma_4 \sigma_7 \bar{a}_1 \bar{a}_2 a_3 b_1 (a_1 a_2 a_3 b_1)^2 (\bar{a}_{11}) \sigma a_7 \\
&\quad = a_{10} (\bar{a}_{11}) b_2 \sigma_5 a_{11} \beta_1 \sigma_3 \sigma_6 a_8 \beta_6 \sigma_4 \sigma_7 a_3 (\bar{a}_1 \bar{a}_2 b_1 a_1 a_2) a_3 b_1 (a_1 a_2 a_3 b_1) \sigma a_7 \\
&\quad = a_{10} (\bar{a}_{11} b_2 a_{11}) (\bar{a}_{11} \sigma_5 a_{11}) \beta_1 \sigma_3 \sigma_6 a_8 \beta_6 \sigma_4 \sigma_7 a_3 \tilde{\beta} a_3 b_1 (a_1 a_2 a_3 b_1) \sigma a_7 \\
&\quad = a_{10} \tilde{\beta}_1 \tilde{\beta}_2 \beta_1 \sigma_3 \sigma_6 a_8 \beta_6 \sigma_4 \sigma_7 a_3 \tilde{\beta} a_3 b_1 (a_1 a_2 a_3 b_1) \sigma a_7,
\end{aligned}$$

where $\beta_1 = a_5 b_2 \bar{a}_5$, $\beta_6 = a_3 b_2 \bar{a}_3$, $\tilde{\beta} = \bar{a}_1 \bar{a}_2 b_1 a_1 a_2$, $\tilde{\beta}_1 = \bar{a}_{11} \beta_2 a_{11}$ and $\tilde{\beta}_2 = \bar{a}_{11} \sigma_5 a_{11}$. Note that the simple closed curves σ_5 and a_{11} intersect at 2 points.

By Lemma 5.1, for $n > 12$ there is no relation in the mapping class group $\Gamma_{2,n}$ inducing a genus 2 Lefschetz fibration $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2 \to S^2$ with n disjoint sections. For $n = 1, \dots, 8$, we did find relations giving n disjoint sections for genus 2 Lefschetz fibration $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2 \to S^2$. As a consequence, by using a result of K. Chakiris, we observe:

Corollary 5.2. *Any genus 2 holomorphic Lefschetz fibration without separating singular fibers admits a section.*

Proof. Chakiris [1978] showed that any genus 2 holomorphic Lefschetz fibration without separating singular fibers is obtained by fiber-summing the three genus 2 Lefschetz fibrations given by the relations

$$(c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1)^2 = 1, \quad (c_1 c_2 c_3 c_4 c_5)^6 = 1, \quad \text{and} \quad (c_1 c_2 c_3 c_4)^{10} = 1$$

in Γ_2, where $c_1, \dots, c_5$ are the simple closed curves shown in Figure 3. As noted in Section 2, each relation gives us a genus 2 Lefschetz fibration with total spaces $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$, $\mathrm{K3} \# 2\overline{\mathbb{CP}}^2$, and the Horikawa surface H, respectively.

For the Lefschetz fibrations with total spaces $\mathrm{K3} \# 2\overline{\mathbb{CP}}^2$ and H, it is known that they have sections: The relation $(c_1 c_2 c_3 c_4 c_5)^6 = \delta_1 \delta_2$ in $\Gamma_{2,2}$ gives 2 disjoint sections for the Lefschetz fibration $K3\#2\overline{\mathbb{CP}}^2 \to S^2$, and $(c_1c_2c_3c_4)^{10} = \delta_1$ in $\Gamma_{2,1}$ gives a section for the Lefschetz fibration $H \to S^2$; see Figure 2.

Earlier, we found sections for the genus 2 Lefschetz fibration $\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2 \to S^2$. By sewing the sections during the fiber-sum operation, we get a section for any genus 2 holomorphic Lefschetz fibration without separating singular fibers. □

Remark 5.3. One may continue and try to write similar relations for $9 \le n \le 12$ to see the exact number of disjoint sections that $\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2 \to S^2$ can admit. One can also try to find the exact number of disjoint sections of the genus 2 Lefschetz fibrations with total spaces $K3\#2\overline{\mathbb{CP}}^2$ and H, respectively. It is still not known whether every genus g Lefschetz fibration over S^2 admits a section.

Appendix

In this appendix, we deduce the relation

$$(a_1b_1a_2b_2)^5 = (a_1b_1a_2)^4(b_2a_2b_1a_1^2b_1a_2b_2),$$

used in Section 4.1; for the corresponding curves, see Figure 4.

Note that a_1 intersects b_1 transversely at a single point, and commutes with a_2 and b_2. Also note that a_2 intersects b_1 and b_2 transversely at a single point, and the simple closed curves b_1 and b_2 commute. By the commutativity and braid relations, we have

$$\begin{aligned}
(a_1b_1a_2b_2)^5 &= (a_1b_1a_2(b_2))(a_1b_1a_2b_2)(a_1b_1a_2b_2)^3\\
&= (a_1b_1a_2)a_1b_1(b_2a_2b_2)(a_1b_1a_2b_2)^3\\
&= (a_1b_1a_2)a_1b_1(a_2b_2a_2)(a_1b_1a_2b_2)^3\\
&= (a_1b_1a_2)^2b_2a_2((a_1)b_1a_2b_2)(a_1b_1a_2b_2)^2\\
&= (a_1b_1a_2)^2a_1b_2(a_2b_1a_2)b_2(a_1b_1a_2b_2)^2\\
&= (a_1b_1a_2)^2a_1b_2((b_1)a_2(b_1))b_2(a_1b_1a_2b_2)^2\\
&= (a_1b_1a_2)^2a_1b_1(b_2a_2b_2)(b_1a_1b_1)a_2b_2(a_1b_1a_2b_2)\\
&= (a_1b_1a_2)^2a_1b_1(a_2b_2a_2)(a_1b_1a_1)a_2b_2(a_1b_1a_2b_2)\\
&= (a_1b_1a_2)^3b_2a_2((a_1)b_1a_1)(a_2)b_2(a_1b_1a_2b_2)\\
&= (a_1b_1a_2)^3a_1b_2(a_2b_1a_2)a_1b_2(a_1b_1a_2b_2)\\
&= (a_1b_1a_2)^3a_1b_2((b_1)a_2b_1)a_1(b_2)(a_1b_1a_2b_2)\\
&= (a_1b_1a_2)^3a_1b_1(b_2a_2b_2)b_1a_1(a_1b_1a_2b_2)\\
&= (a_1b_1a_2)^3a_1b_1(a_2b_2a_2)b_1a_1(a_1b_1a_2b_2)\\
&= (a_1b_1a_2)^4(b_2a_2b_1a_1^2b_1a_2b_2).
\end{aligned}$$

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