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OSAMU SAEKI

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Using the technique of singular fibers of C^∞ stable maps, we give a new proof to the theorem, originally due to Rohlin, that the oriented cobordism group of 4-dimensional manifolds is infinite cyclic and is generated by the cobordism class of the complex projective plane. A byproduct is a new and transparent proof of the signature formula, originally due to T. Yamamoto and the author, for 4-dimensional manifolds in terms of singular fibers.

1. Introduction

Saeki [2004] developed the theory of singular fibers of generic differentiable maps $f : M \to N$ between manifolds with $\dim M > \dim N$. In particular, C^∞ stable maps with $(\dim M, \dim N) = (2, 1)$, $(3, 2)$ and $(4, 3)$ were thoroughly studied and their singular fibers were completely classified up to a natural equivalence. (For precise definitions, see [Golubitsky and Guillemin 1973] and Section 2. For the case of maps between nonorientable manifolds, see [Yamamoto 2006] as well.)

In this paper, we use these classifications of singular fibers to determine the structures of $\mathfrak{N}_2$, Ω_2, Ω_3 and Ω_4, where $\mathfrak{N}_n$ and Ω_n are the cobordism group and oriented cobordism group, respectively, of manifolds of dimension n. These groups were central objects of study in differential topology in the middle of 20th century, and their structures have been completely clarified. Our main objective here is to use classifications of singular fibers of C^∞ stable maps to show that Ω_4 is an infinite cyclic group generated by the cobordism class of the complex projective plane, and that the signature function $\Omega_4 \to \mathbb{Z}$ gives an isomorphism. This theorem is originally due to Rohlin [1952]; see also [Guillou and Marin 1986].

The idea of our proof, which is constructive, is as follows. If we have a smooth map f of a closed oriented 4-manifold M into a 3-manifold N, then a regular fiber is a finite disjoint union of circles. In particular, if f is nonsingular, then M is a circle bundle over a 3-manifold and therefore bounds the associated 2-disk bundle.

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If f has singularities and is generic enough, then the 4-manifold M is decomposed into several pieces according to the classification of singular fibers. For example, regular fibers correspond to a circle bundle over a 3-manifold. Furthermore, we can find a "canonical" 5-manifold for each such piece, except for that corresponding to specific singular fibers, called singular fibers of type III^8 in [Saeki 2004]. By gluing these 5-manifold pieces to $M \times [0, 1]$, we get a cobordism between M and a finite disjoint union of copies of a certain 4-manifold, each corresponding to a singular fiber of type III^8. In this paper, we will show that this 4-manifold is in fact diffeomorphic to the complex projective plane $\mathbb{C}P^2$ up to orientation. This observation shows that $\mathbb{C}P^2$ is a natural representative of the generator of the 4-dimensional oriented cobordism group Ω_4, since our proof is natural and the appearance of $\mathbb{C}P^2$ is not artificial. We may also say that we give a modern proof of the classical Rohlin's theorem from a singularity theoretical viewpoint.

As a corollary to our argument, we get a new and constructive proof of the signature formula proved in [Saeki and Yamamoto 2006]: the signature of a 4-manifold M coincides with the number of singular fibers of type III^8 counted with signs for any C^∞ stable map $f : M \to N$ into a 3-manifold. The proof depended on the classification of singular fibers of C^∞ stable maps of n-dimensional manifolds into $(n-1)$-dimensional manifolds for $n \le 5$, whereas our proof here needs only the classification of such singular fibers for $n \le 4$.

The paper is organized as follows. In Section 2, we review some prerequisite notions about cobordisms of manifolds and singular fibers of differentiable maps. In Section 3, we show that $\mathfrak{N}_2 \cong \mathbb{Z}_2$ and $\Omega_2 = 0$ using the classification of singular fibers of Morse functions on surfaces. We will see that the real projective plane $\mathbb{R}P^2$ is a natural representative of the generator of $\mathfrak{N}_2$ and that the Euler characteristic modulo two gives an isomorphism $\mathfrak{N}_2 \to \mathbb{Z}_2$. Although the argument is quite elementary, the proof will turn out to be a good guideline for the 4-dimensional case. In Section 4, we will show that $\Omega_3 = 0$ using the classification of singular fibers of C^∞ stable maps of 3-manifolds into surfaces. A similar idea has been used by Costantino and D. Thurston [2008] in a proof that every 3-manifold efficiently bounds a 4-manifold. In fact, the idea of this paper is based on their work. In Section 5, we will show that $\Omega_4 \cong \mathbb{Z}$ by using the classification of singular fibers of C^∞ stable maps of 4-manifolds into 3-manifolds. As a corollary, we will also show that if $f : M \to N$ is a generic differentiable map between manifolds with $\dim M - \dim N = 1$ that has only singular fibers of codimension ≤ 3 and no singular fiber of type III^8, then M is null cobordant. We note that the results in this paper depend on a bundle structure theorem for singular fibers of stable maps due to Kalmár [2008, Section 5; 2009, Section 6].

Throughout the paper, all manifolds and maps are differentiable of class C^∞. The symbol $\cong$ denotes a diffeomorphism between manifolds or an appropriate isomorphism between algebraic objects. For a closed surface Σ and a positive integer m, we denote by $\Sigma_{(m)}$ the surface Σ with m open disks removed. We denote by $\mathrm{cl}(A)$ the closure of a subset A of a topological space.

2. Preliminaries

Let n be a nonnegative integer. Two closed oriented (possibly disconnected) n-dimensional manifolds M_0 and M_1 are *oriented cobordant* if there is a compact oriented $(n+1)$-dimensional manifold V such that $\partial V = (-M_0) \cup M_1$ as oriented manifolds, where $-M_0$ denotes the manifold obtained by reversing the orientation of M_0. This defines an equivalence relation on the set of all closed oriented manifolds of dimension n, and the oriented cobordism class of a closed oriented manifold M is denoted by $[M]$.

We denote by Ω_n the set of all oriented cobordism classes of closed oriented n-dimensional manifolds. This clearly forms an additive group under the operation given by $[M]+[M'] = [M \cup M']$. The abelian group Ω_n is called the *n-dimensional oriented cobordism group*.

If we ignore the orientations of the manifolds in these definitions above, then we get the usual notion of a *cobordism*, and the set of all cobordism classes of closed (possibly nonorientable) n-dimensional manifolds is denoted by $\mathfrak{N}_n$, which is called the (*unoriented*) *n-dimensional cobordism group*.

These groups were formulated and studied in the middle of 20th century, and their structures have been completely determined. For example, see [Milnor and Stasheff 1974; Pontryagin 1955; Thom 1954; Wall 1959]. In particular, Ω_n and $\mathfrak{N}_n$ are a finitely generated $\mathbb{Z}$-module and $\mathbb{Z}_2$-module, respectively. (Historically, Pontrjagin [1955] first introduced such groups to compute certain homotopy groups of spheres. Thom [1954] reduced the computation of the cobordism groups to the study of homotopy groups of certain spaces, and then the structures of the cobordism groups have been determined by several authors.)

We now recall some definitions about singular fibers. See [Saeki 2004].

Definition 2.1. Let $f_i : M_i \to N_i$ be maps between manifolds and take points $y_i \in N_i$ for $i = 0, 1$. We say that the fibers over y_0 and y_1 are *C^∞ equivalent* if for some open neighborhoods U_i of y_i in N_i, there exist diffeomorphisms $\tilde{\varphi} : (f_0)^{-1}(U_0) \to (f_1)^{-1}(U_1)$ and $\varphi : U_0 \to U_1$ with $\varphi(y_0) = y_1$ such that the following diagram is commutative:

$$\begin{array}{ccc} ((f_0)^{-1}(U_0), (f_0)^{-1}(y_0)) & \xrightarrow{\tilde{\varphi}} & ((f_1)^{-1}(U_1), (f_1)^{-1}(y_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, y_0) & \xrightarrow{\varphi} & (U_1, y_1). \end{array}$$

When $y \in N$ is a regular value of a map $f : M \to N$ between manifolds, we call the C^∞ equivalence class of the fiber over y (or the space $f^{-1}(y)$) a *regular fiber*; otherwise, we call it *singular*.

Given $f : M \to N$ and a point $y \in N$, consider the map $f \times \mathrm{id}_{\mathbb{R}} : M \times \mathbb{R} \to N \times \mathbb{R}$, where $\mathrm{id}_{\mathbb{R}}$ is the identity map of the real line $\mathbb{R}$. Then the fiber of $f \times \mathrm{id}_{\mathbb{R}}$ over the point $(y, 0) \in N \times \mathbb{R}$ is called the *suspension* of the fiber of f over y.

For certain dimension pairs $(\dim M, \dim N)$, singular fibers of C^∞ stable maps (defined below) of M into N have been classified up to C^∞ equivalence. For details, see [Saeki 2004; Yamamoto 2006; Yamamoto 2007].

Definition 2.2. For manifolds M and N, we denote by $C^\infty(M, N)$ the space of all smooth maps of M into N, endowed with the Whitney C^∞ topology. We say that a smooth map $f : M \to N$ is a C^∞ *stable map* if there exists a neighborhood U_f of f in $C^\infty(M, N)$ such that for each $g \in U_f$, the diagram

$$\begin{array}{ccc} M & \xrightarrow{\tilde{\psi}} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ N & \xrightarrow{\psi} & N \end{array}$$

commutes for some diffeomorphisms $\tilde{\psi}$ and ψ; for details, see [Golubitsky and Guillemin 1973].

It is known that a smooth function $M \to \mathbb{R}$ on a closed manifold M is C^∞ stable if and only if it is a *Morse function*, that is, if and only if its critical points are all nondegenerate and have distinct critical values. Furthermore, if $\dim N \leq 5$, then the set of C^∞ stable maps is open and dense in $C^\infty(M, N)$; see [Mather 1971].

Let us recall the following notion of a Stein factorization, which will play an important role in this paper.

Definition 2.3. Let $f : M \to N$ be a smooth map between smooth manifolds. For two points $x, x' \in M$, we define $x \sim_f x'$ if $f(x) = f(x')$, and the points x and x' belong to the same connected component of a fiber of f. We define $W_f = M/\sim_f$ to be the quotient space with respect to this equivalence relation, and $q_f : M \to W_f$ denotes the quotient map. Then it is easy to see that there exists a unique continuous map $\bar{f} : W_f \to N$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ & {\scriptstyle q_f}\searrow \quad \nearrow{\scriptstyle \bar{f}} & \\ & W_f & \end{array}$$

This diagram is called the *Stein factorization* of f; see [Levine 1985].

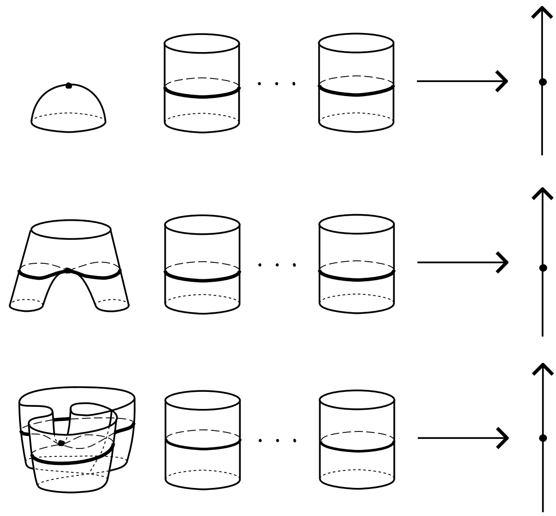

Figure 1. List of C^∞ equivalence classes of singular fibers for Morse functions on surfaces.

If f is a proper C^∞ stable map, then W_f is a polyhedron and all the maps appearing in this diagram are triangulable; for details, see [Hiratuka 2001].

The Stein factorization is a very useful tool for studying topological properties of C^∞ stable maps.

3. Two-dimensional cobordism group

In this section, we show that $\mathfrak{N}_2 \cong \mathbb{Z}_2$ and $\Omega_2 = 0$ using the classification of singular fibers of Morse functions on surfaces.

Let M be an arbitrary closed 2-dimensional manifold, possibly disconnected or nonorientable. It is known that there always exists a Morse function $f : M \to \mathbb{R}$. The singular fibers of such Morse functions are classified as in Figure 1, which may be folklore; for details, see [Saeki 2004].

Construct a compact 3-dimensional manifold V whose boundary includes M by attaching certain pieces to $M \times [0, 1]$, as follows.

Let W_f be the quotient space in the Stein factorization of $f = \bar{f} \circ q_f$. It is a graph whose vertices correspond to connected components of singular fibers: the degree of a vertex is equal to 1, 3 or 2 if it corresponds to the connected component of the singular fiber as (1), (2) or (3) of Figure 1, respectively, containing the critical point; see Figure 2 for an example.

For an edge e of W_f, set $e' = \mathrm{cl}(e \setminus N_0)$, where N_0 is a small regular neighborhood of the set of vertices in W_f and cl denotes the closure in W_f. Since the map

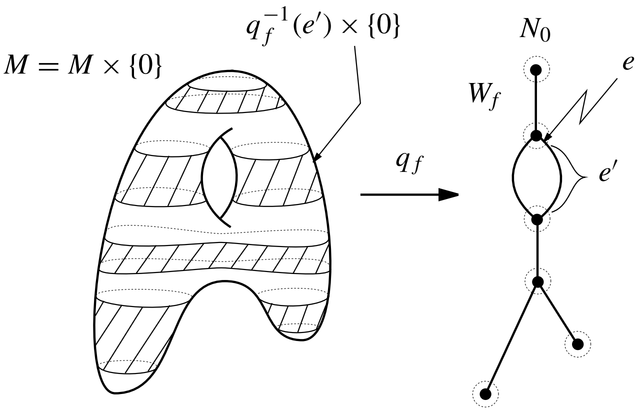

Figure 2. Constructing the 3-manifold V.

q_f restricted to $q_f^{-1}(e')$ is a locally trivial fiber bundle over e' with fiber S^1, it is diffeomorphic to $S^1 \times e'$. Let us glue a 2-handle $D^2 \times e'$ to $M \times [0, 1]$ by identifying $\partial D^2 \times e'$ and $q_f^{-1}(e') \times \{0\}$ by using the diffeomorphism above. Let us perform this operation for each edge of W_f and denote by V the resulting compact 3-dimensional manifold. For an example of the union of $M \times \{0\}$ and the 2-handles, see Figure 2.

We see that the boundary ∂V is a disjoint union of $M \times \{1\}$ and some closed surfaces F_j, where each F_j corresponds to a singular fiber of f. More precisely, let v be a vertex of W_f and $N_0(v)$ its small regular neighborhood in W_f, which is a component of N_0. Then, the corresponding surface F_j is diffeomorphic to the union of $q_f^{-1}(N_0(v))$ and some 2-disks attached to the regular fibers corresponding to $N_0(v) \cap \mathrm{cl}(W_f \setminus N_0)$.

Thus, according to the classification of singular fibers as in Figure 1, we see that each surface F_j is connected and is diffeomorphic to S^2 for the singular fibers as in (1) and (2) of Figure 1, and to $\mathbb{R}P^2$ for that in Figure 1(3). See Figure 3.

Since $S^2 = \partial D^3$ is null cobordant, we have proved the following.

Lemma 3.1. *Every closed surface is cobordant to the disjoint union of a finite number of copies of* $\mathbb{R}P^2$.

The cobordism class $[\mathbb{R}P^2 \cup \mathbb{R}P^2]$ is zero since $\mathbb{R}P^2 \cup \mathbb{R}P^2$ is the boundary of the compact 3-manifold $\mathbb{R}P^2 \times [0, 1]$. Let us consider the homomorphism

$$\varphi : \mathbb{Z}_2 \to \mathfrak{N}_2, \quad 1 \mapsto [\mathbb{R}P^2],$$

where $1 \in \mathbb{Z}_2$ is the generator. This is a well-defined homomorphism, and is surjective by Lemma 3.1.

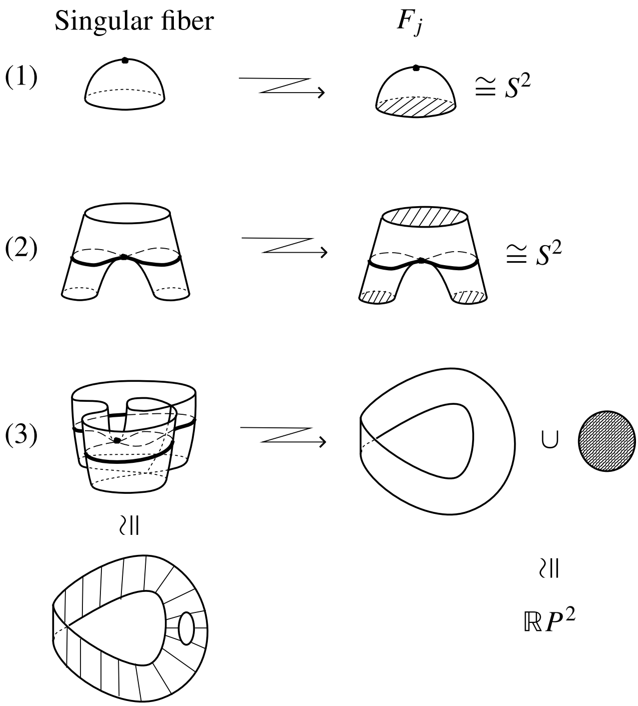

Figure 3. Surface F_j appearing around each singular fiber.

Let

$$\chi_2 : \mathfrak{N}_2 \to \mathbb{Z}_2 \tag{3-1}$$

be the homomorphism defined by associating to each cobordism class the Euler characteristic modulo two of its representative. Using standard techniques in algebraic topology, we can show that this defines a well-defined homomorphism; for example, see [Thom 1952].

Since the Euler characteristic of $\mathbb{R}P^2$ is equal to 1, we see that the composition $\chi_2 \circ \varphi : \mathbb{Z}_2 \to \mathbb{Z}_2$ is the identity. Therefore, φ must be injective. Thus, we have proved the following.

Theorem 3.2. *The* 2*-dimensional cobordism group* $\mathfrak{N}_2$ *is cyclic of order two and is generated by the cobordism class of* $\mathbb{R}P^2$. *In fact*, *the homomorphism* (3-1) *is an isomorphism.*

Our proof does not depend on the classification of closed surfaces.

As a corollary to the proof, we also get the following, which was originally obtained in [Saeki 2004].

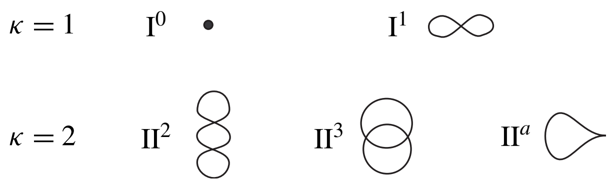

Figure 4. List of C^∞ equivalence classes of singular fibers of C^∞ stable maps of orientable 3-manifolds into surfaces.

Corollary 3.3. *Let $f : M \to \mathbb{R}$ be a Morse function on a closed surface M. Then, the number of singular fibers of f as in Figure 1(3) has the same parity as the Euler characteristic of M.*

If a closed surface M is oriented, then a singular fiber as in Figure 1(3) never appears, since its neighborhood is nonorientable. Furthermore, the 3-manifold V constructed above is orientable, since $M \times [0, 1]$ is orientable and attaching a 2-handle does not alter the orientability of a 3-manifold. Moreover, V can be oriented so that its oriented boundary consists of M and some 2-spheres. Thus:

Corollary 3.4. *The 2-dimensional oriented cobordism group Ω_2 vanishes.*

4. Three-dimensional oriented cobordism group

In this section, we show that $\Omega_3 = 0$ by using the classification of singular fibers of C^∞ stable maps of closed orientable 3-manifolds into surfaces.

For M a closed oriented 3-manifold, there always exists a C^∞ stable map f from M into any surface N; for example, see [Kushner et al. 1984; Levine 1985]. Singular fibers of such maps have been classified in [Saeki 2004] up to C^∞ equivalence; see also [Kushner et al. 1984; Levine 1985]. The connected components of singular fibers containing singular points are as depicted in Figure 4, where each figure represents the connected component of the inverse image of a point in the target. (In fact, it is known that two such singular fibers are C^∞ equivalent if and only if the corresponding inverse images are diffeomorphic to each other. For details, see [Saeki 2004].) Furthermore, in Figure 4, κ denotes the *codimension* of a singular fiber, that is, it is the codimension of the set of those points in the target over which lies a relevant singular fiber. Following the convention introduced in [Saeki 2004], we use the notation I^0, I^1 and so on for the C^∞ equivalence classes of singular fibers. The I^0-type (or I^1-type) singular fiber is the suspension of the singular fiber for Morse functions as in Figure 1(1) (respectively Figure 1(2)).

A C^∞ stable map of a 3-manifold into a surface has only fold and cusp singularities [Kushner et al. 1984; Levine 1985]. A singular fiber (or the corresponding inverse image) can be regarded as a graph: a fold point corresponds to an isolated vertex or to a vertex of degree four, and a cusp point corresponds to a cuspidal vertex of degree two (see Figure 4).

Let W_f be the quotient space in the Stein factorization of a C^∞ stable map f from a closed oriented 3-manifold M into a surface N. The space W_f is a compact 2-dimensional polyhedron and its local structure is completely determined [Kushner et al. 1984; Levine 1985]. Let $W^{(0)}$ denote the q_f-image of the singular fibers of $\kappa = 2$, and let $W^{(1)}$ denote the q_f-image of the singular fibers of $\kappa \geq 1$ (more precisely, they are the q_f-images of the components of the relevant singular fibers containing singular points). Note that $W^{(0)}$ is a finite set of points and $W^{(1)}$ is a 1-dimensional subcomplex of W_f whose complement is a nonsingular surface. For $i = 0, 1$, we denote by $N^{(i)}$ a small regular neighborhood of $W^{(i)}$ in W_f. We set $N_0 = N^{(0)}$, $N_1 = \mathrm{cl}(N^{(1)} \setminus N^{(0)})$ and $N_2 = \mathrm{cl}(W_f \setminus N^{(1)})$, where N_1 is regarded as a regular neighborhood of $\mathrm{cl}(W^{(1)} \setminus N^{(0)})$ in $\mathrm{cl}(W_f \setminus N^{(0)})$. Note that W_f is decomposed as

$$W_f = N_0 \cup N_1 \cup N_2.$$

Let us construct a compact 4-dimensional manifold V by attaching certain pieces to $M \times [0, 1]$ as follows. First note that q_f restricted to $q_f^{-1}(N_2)$ is a locally trivial fibration with fiber S^1 over the surface N_2. Thus, we can attach the total space of the associated D^2-bundle over N_2 to $M \times [0, 1]$ by identifying the associated (∂D^2)-bundle with $q_f^{-1}(N_2) \times \{0\}$. (Here, we use the well-known fact that the structure group of every smooth S^1-bundle can be reduced to the orthogonal group $O(2)$.) The resulting 4-manifold is denoted by V_1. Note that V_1 is orientable, since M and $q_f^{-1}(N_2)$ are orientable as 3-manifolds.

Let $V_1'(\subset V_1)$ denote the union of $M \times \{0\}$ and the D^2-bundle over N_2. There is a natural map $q_1 : V_1' \to W_f$, which on $M \times \{0\}$ is defined by $q_f : M \times \{0\} = M \to W_f$, and on the D^2-bundle is defined by the projection to N_2.

Let e be a connected component of $\mathrm{cl}(W^{(1)} \setminus N_0)$. Note that e is an arc or a circle. Let $N_1(e)$ denote the connected component of N_1 containing e. If the singular fiber lying over a point in e is of type I^0, then $q_f^{-1}(N_1(e))$ is diffeomorphic to the total space of a D^2-bundle over e [Kushner et al. 1984; Levine 1985]. In fact, $N_1(e)$ is diffeomorphic to $e \times [0, 1]$ by a diffeomorphism that induces the identity $e(\subset N_1(e)) \to e \times \{0\}$, and if J $(\cong [0, 1])$ is a fiber of the natural fibration $N_1(e) \to e$, then q_f restricted to $q_f^{-1}(J) \cong D^2$ is equivalent to the function

$$(x, y) \mapsto x^2 + y^2. \tag{4-1}$$

(In fact, the commutative diagram

$$\begin{array}{ccc} q_f^{-1}(N_1(e)) & \xrightarrow{\;q_f\;} & N_1(e) \\ & \searrow \quad \swarrow & \\ & e & \end{array}$$

can be regarded as a fiber bundle with the map $D^2 \to [0, 1]$ defined by (4-1) as fiber in an appropriate sense. For details, see [Kalmár 2008, Section 5; Kalmár 2009, Section 6].)

Therefore, q_1 restricted to $q_1^{-1}(N_1(e))$ followed by the natural projection from $N_1(e)$ to e is an S^2-bundle, where the fiber of this fibration can be identified with the 2-sphere as in Figure 3(1) in Section 3. Then, we can attach the associated D^3-bundle over e to V_1, where we identify the associated (∂D^3)-bundle with $q_1^{-1}(N_1(e))$. Here, we use the fact that the structure group of every smooth S^2-bundle can be reduced to $O(3)$; see [Smale 1959]. Note that the resulting 4-manifold is orientable, since so is V_1 and the orientability of $q_1^{-1}(N_1(e))$ coincides with that of $q_f^{-1}(N_1(e))$ $(\subset M)$.

If the singular fiber lying over a point in e is of type I^1, then the natural projection $N_1(e) \to e$ defines a Y-bundle, where

$$Y = \{r \exp(2\pi\sqrt{-1}k/3) \in \mathbb{C} \mid 0 \leq r \leq 1,\ k = 0, 1, 2\}.$$

Moreover, $q_f^{-1}(N_1(e))$ is diffeomorphic to the total space of an $S^2_{(3)}$-bundle over e, where $S^2_{(3)}$ denotes the 2-sphere with three open disks removed. Then, q_1 restricted to $q_1^{-1}(N_1(e))$ followed by the natural projection $N_1(e) \to e$ is again an S^2-bundle, but with fiber as in Figure 3(2), and we can attach the associated D^3-bundle.

We perform the operation described above for each e. The resulting 4-manifold, denoted by V, is a compact 4-dimensional manifold that is orientable. Furthermore, it can be oriented so that

$$\partial V = (M \times \{1\}) \cup \bigl(-\textstyle\bigcup_j F_j\bigr),$$

where each F_j is a closed oriented 3-manifold corresponding to a singular fiber of f of $\kappa = 2$.

Lemma 4.1. *The closed 3-manifold F_j is diffeomorphic to the 3-sphere S^3 for every singular fiber of $\kappa = 2$.*

In fact, for the singular fibers of types II^2 and II^3, this lemma has been essentially obtained in [Costantino and Thurston 2008]. Here we give a proof from a different viewpoint in a way that is useful in Section 5.

Proof. Let v be a point in W_f that is the q_f-image of a singular fiber of type II^2. Then, its regular neighborhood $N_0(v)$ in W_f is of the form depicted in Figure 5, where $N_0(v)$ is the component of N_0 containing v. Note that $\bar{f}(N_0(v)) \cong J_1 \times J_2$ with $J_1 = J_2 = [-1, 1]$.

Then, the map f restricted to $q_f^{-1}(N_0(v))$ can be regarded as a 1-parameter family of functions on $S^2_{(4)}$ with only nondegenerate critical points as depicted in Figure 6. This family is parametrized by J_1, where for each parameter value the relevant function is regarded as a height function $S^2_{(4)} \to J_2$.

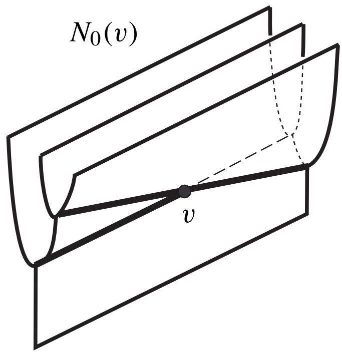

Figure 5. Neighborhood of the q_f-image of a II^2-type singular fiber.

In constructing V_1, we attached to each $S^2_{(4)}$ 2-disks along the four boundary circles so that we get a 2-sphere. Along the 2-spheres for $t = \pm 1$, we attached 3-disks to construct V. Thus, the relevant 3-manifold F_j is diffeomorphic to a manifold obtained by attaching two 3-disks to $S^2 \times [-1, 1]$ along the boundaries, and is therefore diffeomorphic to the 3-sphere S^3.

The same argument can be applied for the singular fiber of type II^3. The regular neighborhood of a corresponding point in W_f is shown in Figure 7.

Finally, for the singular fiber of type II^a, a similar argument can be applied as follows. The regular neighborhood of a corresponding point in W_f is shown in the top of Figure 8. The map f restricted to the inverse image of the neighborhood

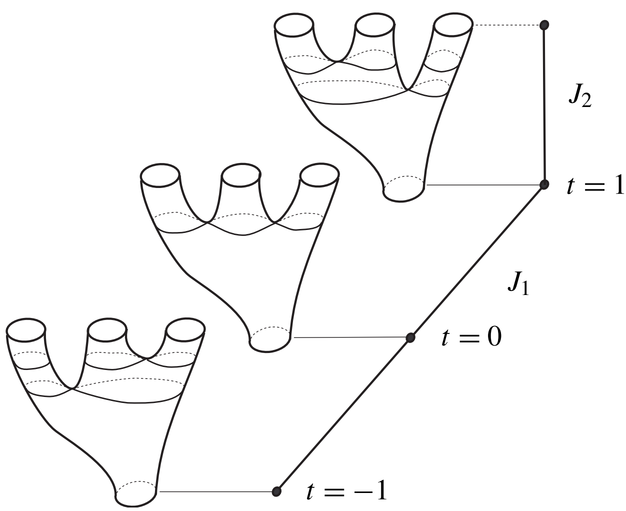

Figure 6. 1-parameter family of functions on $S^2_{(4)}$ corresponding to a II^2-type singular fiber.

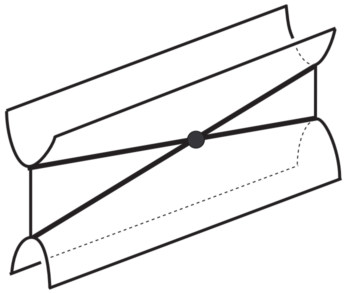

Figure 7. Neighborhood of the q_f-image of a II^3-type singular fiber.

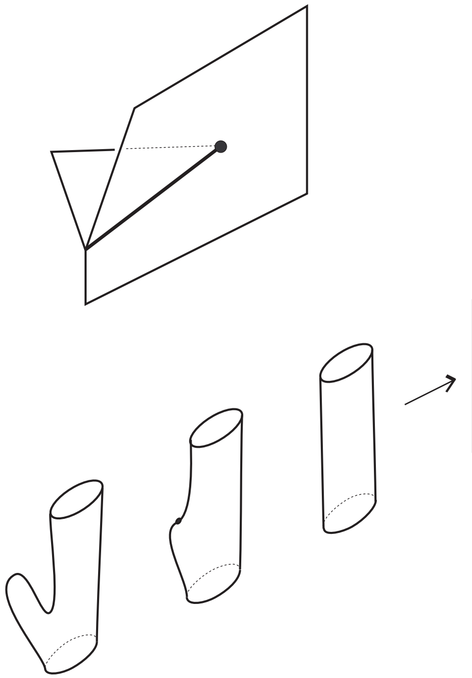

Figure 8. The case of a singular fiber of type II^a.

with respect to q_f can be regarded as a 1-parameter family of smooth functions on the annulus corresponding to a birth-death of a pair of nondegenerate critical points as shown in the bottom of Figure 8. Thus, the resulting 3-manifold F_j is again diffeomorphic to S^3.

This completes the proof of Lemma 4.1. □

Since S^3 is the oriented boundary of an oriented 4-disk, we see that M bounds a compact oriented 4-manifold. Therefore, we have proved the following.

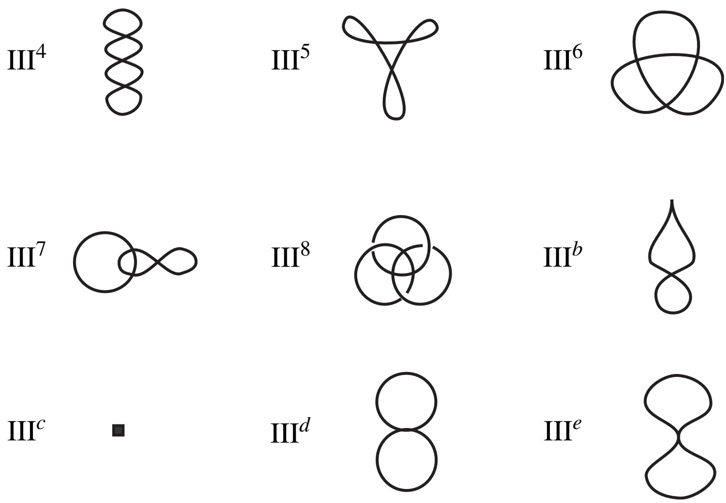

Figure 9. Singular fibers of C^∞ stable maps of orientable 4-manifolds into 3-manifolds of $\kappa = 3$.

Theorem 4.2. *The* 3*-dimensional oriented cobordism group* Ω_3 *vanishes.*

This result is originally due to Rohlin [1951] and Thom [1952]. In fact, our argument resembles that in [Costantino and Thurston 2008].

Remark 4.3. Kalmár [2009] showed that for every C^∞ stable map f of a closed orientable 3-manifold M into the plane, singular fibers of types II^2, II^3 and II^a can be eliminated by cobordism. Using this result, he showed that $\Omega_3 = 0$. For details, see [Kalmár 2009, Remark 2.8].

5. Four-dimensional oriented cobordism group

In this section, we show that the oriented cobordism group Ω_4 is infinite cyclic and is generated by the cobordism class of $\mathbb{C}P^2$, using the classification of singular fibers of C^∞ stable maps of orientable 4-manifolds into 3-manifolds.

For M a closed oriented 4-manifold, there always exists a C^∞ stable map f from M into any 3-manifold N; for example, see [Saeki 2004]. Singular fibers of such maps were classified up to C^∞ equivalence in [Saeki 2004]. The connected components of singular fibers containing singular points are shown in Figure 9 in addition to the singular fibers of codimension $\kappa = 1$ and 2 that are suspensions of the singular fibers as in Figure 4. (For the singular fibers of $\kappa \leq 2$, we continue to use the same notation I^0, I^1, and so on as in Figure 4 for the suspensions as well.)

A C^∞ stable map of a 4-manifold into a 3-manifold has only fold, cusp, and swallowtail singularities. A fold point corresponds to an isolated point or to a transverse crossing point of two line segments, a cusp point corresponds to a cuspidal vertex of degree two in a singular fiber, and a swallowtail point corresponds to an isolated point (depicted by a black square in Figure 9, III^c) or to a tangency point of two touching parabolas.

Let W_f be the quotient space in the Stein factorization of a C^∞ stable map f of a closed oriented 4-manifold M into a 3-manifold N. The space W_f is a compact 3-dimensional polyhedron, and its local structure has been completely determined. A complete list of local structures can be found in [Hiratuka 2001], although we do not need it here. Let $W^{(j)}$ denote the q_f-image of the components of singular fibers of $\kappa \geq 3 - j$ containing singular points for $j = 0, 1, 2$, and let $N^{(j)}$ denote a small regular neighborhood of $W^{(j)}$ in W_f. Then, set

$$N_0 = N^{(0)}, \quad N_1 = \mathrm{cl}(N^{(1)} \setminus N^{(0)}), \quad N_2 = \mathrm{cl}(N^{(2)} \setminus N^{(1)}), \quad N_3 = \mathrm{cl}(W_f \setminus N^{(2)}),$$

where N_1 is seen as a regular neighborhood of $\mathrm{cl}(W^{(1)} \setminus N^{(0)})$ in $\mathrm{cl}(W_f \setminus N^{(0)})$ and N_2 is seen as a regular neighborhood of $\mathrm{cl}(W^{(2)} \setminus N^{(1)})$ in $\mathrm{cl}(W_f \setminus N^{(1)})$.

Let us now construct a compact 5-dimensional manifold V by attaching certain pieces to $M \times [0, 1]$ as follows. First note that q_f restricted to $q_f^{-1}(N_3)$ is a locally trivial fibration with fiber S^1 over the 3-manifold possibly with boundary N_3. Thus, we can attach the total space of the associated D^2-bundle over N_3 to $M \times [0, 1]$ by identifying the associated (∂D^2)-bundle with $q_f^{-1}(N_3) \times \{0\}$. The resulting 5-manifold, denoted by V_1, is orientable.

Let V_1' (a subset of V_1) be the union of $M \times \{0\}$ and the D^2-bundle over N_3. There is a natural map $q_1 : V_1' \to W_f$ that on $M \times \{0\}$ is defined by $q_f : M \times \{0\} = M \to W_f$, and on the D^2-bundle is defined by the projection to N_3.

Let S be a connected component of $\mathrm{cl}(W^{(2)} \setminus N^{(1)})$. Note that S is a compact surface possibly with boundary. Let $N_2(S)$ denote the connected component of N_2 containing S. If the singular fiber lying over a point in S is of type I^0, then $q_f^{-1}(N_2(S))$ is diffeomorphic to the total space of a D^2-bundle over S. In fact, $N_2(S)$ is diffeomorphic to $S \times [0, 1]$ by a diffeomorphism that induces the identity S $(\subset N_2(S)) \to S \times \{0\}$, and if J $(\cong [0, 1])$ is a fiber of the natural fibration $N_2(S) \to S$, then q_f restricted to $q_f^{-1}(J) \cong D^2$ is equivalent to the function (4-1). (For this, we need the bundle structure theorem mentioned in [Kalmár 2008, Section 5; Kalmár 2009, Section 6].) Therefore, the map q_1 restricted to $q_1^{-1}(N_2(S))$ followed by the natural projection $N_2(S) \to S$ is an S^2-bundle whose fiber can be identified with the 2-sphere as in Figure 3(1). Then, we can attach the associated D^3-bundle over S to V_1, where we identify the associated (∂D^3)-bundle with $q_1^{-1}(N_2(S))$ $(\subset V_1')$. The resulting 5-manifold is orientable, since so is V_1 and the orientability of $q_1^{-1}(N_2(S))$ coincides with that of $q_f^{-1}(N_2(S))$ $(\subset M)$.

If the singular fiber lying over a point in S is of type I^1, then $q_f^{-1}(N_2(S))$ is diffeomorphic to the total space of an $S^2_{(3)}$-bundle over S. Then, q_1 restricted to $q_1^{-1}(N_2(S))$ followed by the natural projection $N_2(S) \to S$ is again an S^2-bundle,

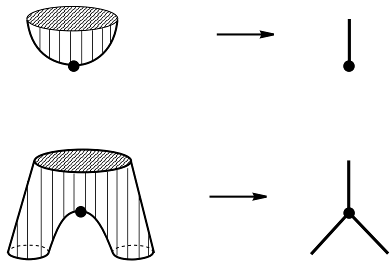

Figure 10. Function on each D^3-fiber.

but with fiber as in Figure 3(2), and we can attach the associated D^3-bundle. As before, the resulting 5-manifold is orientable.

We perform the operation described above for each connected component S of $\mathrm{cl}(W^{(2)} \setminus N^{(1)})$. The resulting 5-manifold, denoted V_2, is orientable. We denote by V_2' the union of V_1' and the D^3-bundle over $\mathrm{cl}(W^{(2)} \setminus N^{(1)})$. There is a natural map $q_2 : V_2' \to W_f$ that on V_1' is defined by $q_1 : V_1' \to W_f$, and on the D^3-bundle X over $\mathrm{cl}(W^{(2)} \setminus N^{(1)})$ is defined by the natural map $X \to N_2$ that makes the diagram

$$\begin{array}{ccc} X & \longrightarrow & N_2 \\ & \searrow \quad \swarrow & \\ & \mathrm{cl}(W^{(2)} \setminus N^{(1)}) & \end{array}$$

commutative and is given by the function on each fiber as shown in Figure 10. (For this, we again need Kalmár's bundle structure theorem.)

Let e be a connected component of $\mathrm{cl}(W^{(1)} \setminus N^{(0)})$ and we denote by $N_1(e)$ the connected component of N_1 containing e. If the singular fiber lying over a point in e is of type II^2, then $q_f^{-1}(N_1(e))$ is diffeomorphic to the total space of an $(S^2_{(4)} \times [0, 1])$-bundle over e; see Figure 6.

Therefore, q_2 restricted to $q_2^{-1}(N_1(e))$ followed by the natural projection from $N_1(e)$ to e is an S^3-bundle by Lemma 4.1. From [Hatcher 1983], the structure group of every smooth S^3-bundle can be reduced to the orthogonal group $O(4)$. Then, we can attach the associated D^4-bundle over e to V_2, where we identify the associated (∂D^4)-bundle with $q_2^{-1}(N_1(e))$ $(\subset V_2')$. The resulting 5-manifold is orientable, since so is V_2 and the orientability of $q_2^{-1}(N_1(e))$ coincides with that of $q_f^{-1}(N_1(e))$ $(\subset M)$.

If the singular fiber lying over a point in e is of another type (II^3 or II^a), then we can still perform the same operation by virtue of Lemma 4.1.

We perform such an operation for each e. The resulting 5-manifold V is compact and orientable. Note that V can be oriented so that

$$\partial V = (M \times \{1\}) \cup \big(-\textstyle\bigcup_j F_j\big),$$

where each F_j is a closed oriented 4-manifold corresponding to a singular fiber of f of $\kappa = 3$.

Lemma 5.1. *F_j is diffeomorphic to the 4-sphere S^4 for the singular fibers of types* III^4, III^5, III^6, III^7, III^b, III^c, III^d *and* III^e.

Proof. Let us first consider the case of the singular fiber of type III^4. Let $g : L \to D^3$ be a representative of the singular fiber; we assume that it has a singular fiber of type III^4 over the center of D^3. Then, we can regard g as a family of functions $\{h_s\}_{s\in\Delta}$ on $S^2_{(5)}$ with only nondegenerate critical points parametrized by $\Delta \cong D^2$ as depicted in Figure 11, where the target D^3 is identified with the product $[-1, 1]\times\Delta$, $\pi : [-1, 1] \times \Delta \to \Delta$ is the projection to the second factor, and the critical points of h_s are denoted by p_1, p_2 and p_3. More precisely, g can be identified with the map $L \cong S^2_{(5)} \times \Delta \to [-1, 1] \times \Delta \cong D^3$, $(x, s) \mapsto (h_s(x), s)$. The singular point set $S(g)$ of g consists of three 2-disks, and their images by g in D^3 intersect at the origin in general position.

Then, from the construction of V it follows that the 4-manifold F_j corresponding to $g^{-1}(0)$ is diffeomorphic to the boundary of a $\widetilde{D}$-bundle over Δ, where $\widetilde{D} \cong D^3$ is the 3-disk as in Figure 12, which is obtained by filling $S^2_{(5)} \times [0, 1]$ by 2- and 3-handles as in Section 3. Hence F_j is diffeomorphic to S^4.

For the singular fibers of types III^5, III^b, III^c and III^e, similar arguments show that $F_j \cong S^4$.

For the singular fiber of type III^6, we can again regard its representative $g : L \to D^3$ as a family of functions parametrized by a 2-disk Δ. Note that L is diffeomorphic to $S^2_{(5)} \times \Delta$. Set $F'_j = L \cup q_1^{-1}((\bar{g})^{-1}(\{\pm 1\}\times\Delta))$ and $F''_j = \mathrm{cl}(F_j \setminus F'_j)$, where $g = \bar{g} \circ q_g$ is the Stein factorization of $g : L \to [-1, 1] \times \Delta$ and we regard W_g as a subset of W_f.

Then, we see that F'_j is diffeomorphic to $S^2 \times D^2$. On the other hand, from the construction of V it follows that F''_j is the union of six copies of $D^3 \times [-1, 1]$ attached to each other along $D^3 \times \{\pm 1\}$ consecutively so that if forms the total space of a D^3-bundle over S^1. Hence, F''_j is diffeomorphic to $D^3 \times S^1$, since it is orientable. Therefore, F_j is diffeomorphic to the union of $S^2 \times D^2$ and $D^3 \times S^1$ attached along their boundaries, where $S^2 \times \{*\} \subset S^2 \times D^2$ and $\partial D^3 \times \{*\} \subset D^3 \times S^1$ are identified. Then a standard argument shows that F_j is diffeomorphic to S^4.

For the singular fibers of types III^7 and III^d, similar arguments give $F_j \cong S^4$.

This completes the proof of Lemma 5.1. □

Lemma 5.2. *For the singular fiber of type* III^8, *F_j is orientation-preservingly diffeomorphic to the complex projective plane $\mathbb{C}P^2$ or its orientation reversal $\overline{\mathbb{C}P^2}$.*

Proof. As in the proof of Lemma 5.1, a representative $g : L \to D^3$ of the singular fiber of type III^8 can be regarded as a family of functions $\{h_s\}_{s\in\Delta}$ on $T^2_{(3)}$ with only nondegenerate critical points parametrized by a small 2-disk Δ. (Here, the torus

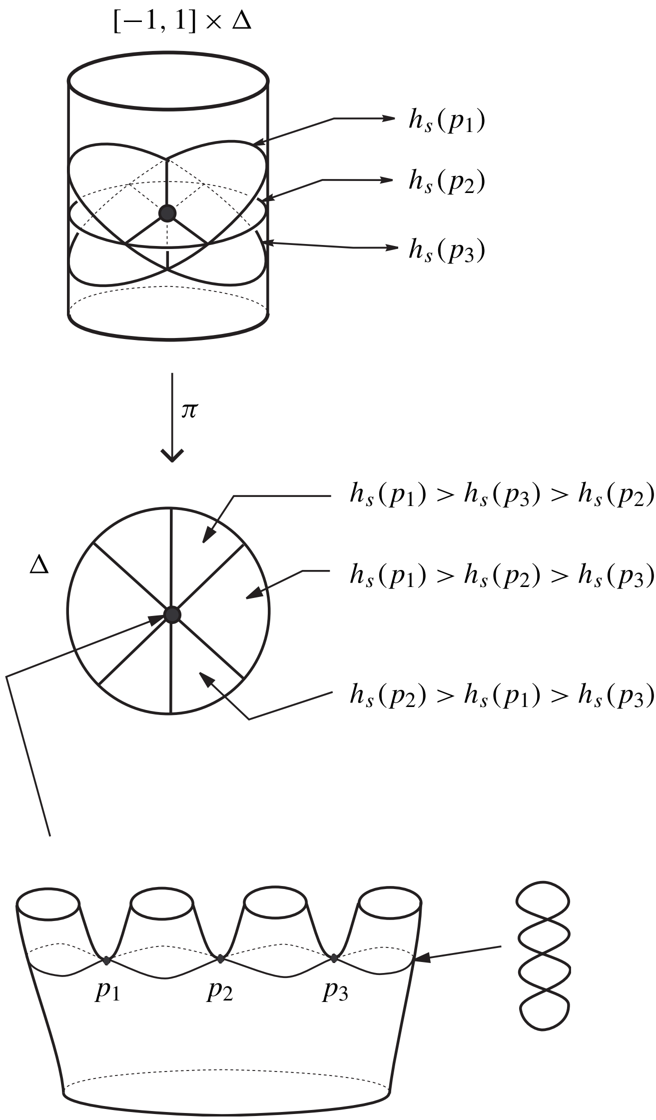

Figure 11. Family of functions corresponding to a singular fiber of type III^4.

with three holes appears, since the natural "thickening" of the III^8-type singular fiber is a compact orientable surface with three boundary circles and has Euler characteristic −3. See Figure 13.) See Figure 14, where $\pi : D^3 \cong [-1, 1] \times \Delta \to \Delta$ is the projection onto the second factor; see also [Saeki 2004, Figure 6.3].

Set $K = g^{-1}(\partial D^3) = \partial L$, which is a closed orientable 3-manifold. Then, $g|_K : K \to \partial D^3$ can be regarded as a stable map as in Section 4. Note that F_j is $L \cup W$

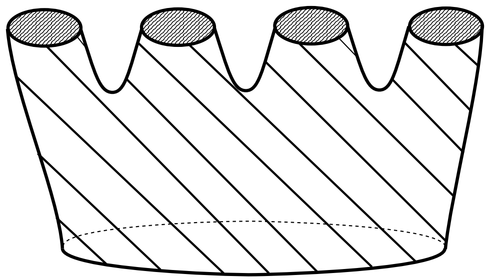

Figure 12. $\widetilde{D} \cong D^3$.

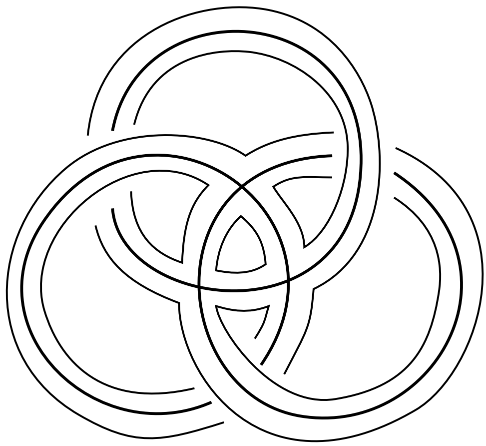

Figure 13. Natural "thickening" of the III^8-type singular fiber.

attached along K, where W is the compact orientable 4-manifold bounded by K constructed as in Section 4 from the stable map $g|_K$.

Set

$$F'_j = L \cup q_1^{-1}((\bar{g})^{-1}(\{\pm 1\} \times \Delta)) \quad \text{and} \quad F''_j = \text{cl}(F_j \setminus F'_j)$$

as in the proof of the previous lemma. Note that F'_j is diffeomorphic to a T^2-bundle over Δ. (More precisely, the map $\pi \circ g : L \to \Delta$ is a smooth fiber bundle with $T^2_{(3)}$ as fibers, and F'_j is obtained from L by attaching three 2-disks to each of the fibers.)

Let us consider the piece P_i in F''_j corresponding to the arc α_i for $i = 1, 2, 3$, on $\partial\Delta$ as depicted in Figure 14. More precisely, P_i is the compact 4-manifold described as follows. First, note that $g|_{(\pi \circ g)^{-1}(\alpha_i)} : (\pi \circ g)^{-1}(\alpha_i) \to [-1, 1] \times \alpha_i$ can be regarded as a 1-parameter family of smooth functions on $T^2_{(3)}$ with exactly three nondegenerate critical points corresponding to interchanging the heights of the top two critical points. Furthermore, the singular fiber (of codimension two)

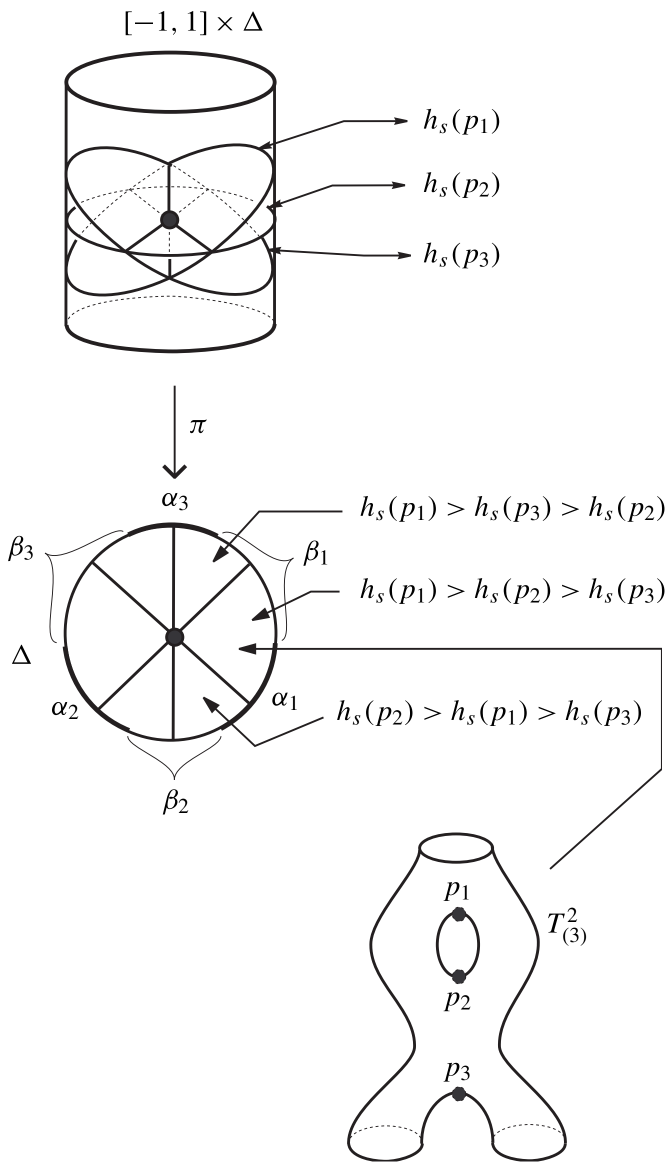

Figure 14. Family of functions corresponding to a singular fiber of type III^8.

over the middle point of α_i corresponds to the singular fiber of type II^3. The compact 4-manifold P_i is obtained from $(\pi \circ g)^{-1}(\alpha_i) \times [0,1]$ by attaching D^2-bundles, D^3-bundles and a 4-disk as in Section 4, where the last 4-disk corresponds to the II^3-type singular fiber.

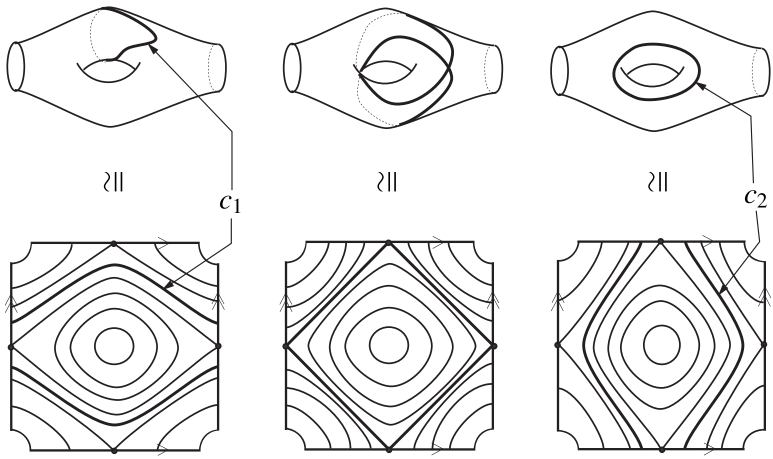

Figure 15. Family of functions corresponding to II^3.

By attaching D^2-bundles over arcs and three copies of D^3 to each component of $(\pi \circ g)^{-1}(\partial \alpha_i)$, we get two copies of the solid torus. Hence ∂P_i is diffeomorphic to the union of two solid tori attached along their boundaries. The attaching map sends the boundary c_1 of a meridian disk[1] to a simple closed curve on the boundary of the other solid torus that intersects with the boundary c_2 of its meridian disk transversely at one point. For details, see Figure 15, which shows how the fibers change around a singular fiber of type II^3. (Note that each component of a regular fiber bounds a disk by virtue of the construction of V.) Hence, ∂P_i is diffeomorphic to S^3. Since P_i is obtained from $\partial P_i \times [0, 1]$ by attaching a 4-disk, we see that P_i is diffeomorphic to D^4.

On the other hand, the piece Q_i corresponding to the arc β_i for $i = 1, 2, 3$ on $\partial \Delta$ as shown in Figure 14 is diffeomorphic to $(S^1 \times D^2) \times [-1, 1]$. This is because the singular fiber of type II^3 over the middle point of β_i corresponds to interchanging the heights of the bottom two critical points of the function $h_s : T^2_{(3)} \to [-1, 1]$, where $s \in \beta_i$. In terms of the quotient space, its behavior is similar to what is depicted in Figure 7. Therefore, Q_i is obtained from a 4-disk (corresponding to the II^3-type singular fiber) by attaching $D^3 \times \beta_i$ (corresponding to the top critical point of h_s)[2] along $(D^2_1 \cup D^2_2) \times \beta_i$, where D^2_1 and D^2_2 are disjoint 2-disks in ∂D^3. Therefore, Q_i is diffeomorphic to $(S^1 \times D^2) \times \beta_i \cong (S^1 \times D^2) \times [-1, 1]$, since over each end point of β_i we have a solid torus, which is orientable.

Consequently, F''_j is diffeomorphic to the compact 4-manifold obtained from three 4-disks P_1, P_2 and P_3 by attaching them appropriately along solid tori. Thus

[1] A properly embedded 2-disk in a solid torus is a *meridian disk* if its boundary is not null homotopic in the boundary torus.

[2] Each 3-disk $D^3 \times \{*\}$ corresponds to that in the lower-left figure of Figure 10.

P_1 can be regarded as a 0-handle and then P_2 is regarded as a 2-handle attached to P_1 along an unknotted circle on ∂P_1, since the exterior of the attaching circle in ∂P_1 is again a solid torus. Furthermore, F'_j is diffeomorphic to $T^2 \times D^2$, and $P_3 \cup F'_j$ is diffeomorphic to D^4 since $P_3 \cap F'_j$ is diffeomorphic to $T^2 \times \alpha_3$.

Therefore, F_j is diffeomorphic to the closed 4-manifold consisting of the 0-handle P_1, the 2-handle P_2 attached to P_1 along an unknotted circle on ∂P_1, and a 4-handle. In particular, the boundary of $P_1 \cup P_2$ must be diffeomorphic to S^3 so that the framing of the 2-handle P_2 must be equal to ± 1; see [Kirby 1989], for example. Therefore, F_j must be diffeomorphic to the complex projective plane $\mathbb{C}P^2$ up to orientation. □

Remark 5.3. It is easy to see that the singular fibers appearing in Figure 9 can be embedded in the 2-sphere, except for the III^8-type singular fiber. This fact implies that the corresponding singular fiber is associated with a 2-parameter family of smooth functions on a punctured 2-sphere, as pointed out in the proof of Lemma 5.1. The III^8-type singular fiber cannot be embedded in the 2-sphere, but can be embedded in the 2-dimensional torus (see the proof of Lemma 5.2). The proofs above show that this fact is essential in distinguishing the III^8-type singular fiber from the others.

We have proved that every closed oriented 4-manifold is oriented cobordant to the disjoint union of a finite number of copies of $\pm\mathbb{C}P^2$.

Let us consider the homomorphism $\varphi : \mathbb{Z} \to \Omega_4$ defined by $\varphi(1) = [\mathbb{C}P^2]$. This is a well-defined homomorphism, and is surjective by the argument above.

Let

$$\sigma : \Omega_4 \to \mathbb{Z} \tag{5-1}$$

be the homomorphism defined by associating to each oriented cobordism class the signature of its representative. Classical techniques in algebraic topology show that this defines a well-defined homomorphism; for example, see [Thom 1952].

Since $\sigma([\mathbb{C}P^2]) = 1$, we see that the composition $\sigma \circ \varphi : \mathbb{Z} \to \mathbb{Z}$ is the identity. Therefore, φ is injective. Thus, we get the following, which was originally proved by Rohlin [1952]; see also [Guillou and Marin 1986].

Theorem 5.4. *The 4-dimensional oriented cobordism group Ω_4 is infinite cyclic and is generated by the oriented cobordism class of $\mathbb{C}P^2$. In fact, the homomorphism (5-1) is an isomorphism.*

Our proof shows that the complex projective plane naturally appears around each singular fiber of type III^8, and therefore $\mathbb{C}P^2$ can be regarded as the genuinely natural representative of the generator of $\Omega_4 \cong \mathbb{Z}$.

As a corollary to our proof, we get the following, which was originally obtained in [Saeki and Yamamoto 2006]. The proof given there is somewhat complicated since it depends on the classification of singular fibers of C^∞ stable maps of n-dimensional manifolds into $(n-1)$-dimensional manifolds for $n \le 5$, whereas our proof needs only the classification of such singular fibers for $n \le 4$.

Corollary 5.5. *Let f be a C^∞ stable map of a closed oriented 4-manifold M into a 3-manifold N. Then the number of singular fibers of f of type III^8 counted with signs coincides with the signature of M.*

The sign of a singular fiber of type III^8 is $+1$ (or -1) if the corresponding manifold F_j is oriented diffeomorphic to $\mathbb{C}P^2$ (respectively $\overline{\mathbb{C}P^2}$). This sign convention must coincide with the one in [Saeki and Yamamoto 2006] since Corollary 5.5 determines the sign uniquely.

Gromov [2009] studied estimates for the number of self-intersections of the critical value set of a generic map from one manifold to another in terms of the topology of the source manifold. Corollary 5.5 gives a model case for such a study, as pointed out by Gromov.

Corollary 5.6. *Let f be a smooth map of a closed oriented n-dimensional manifold M into an $(n-1)$-dimensional manifold N for $n \ge 4$, and suppose its singular fibers are (iterated suspensions of) those of C^∞ stable maps of codimension ≤ 3 not of type III^8 (that is, the singular fibers of f are as in Figures 4 and 9 but without the III^8-type). Then the manifold M is oriented null cobordant.*

For the proof, we again need Kalmár's bundle structure theorem concerning the structure group since we need to deal with smooth fiber bundles with fiber S^4. Note that this corollary generalizes [Kalmár 2009, Corollary 2.7] about simple fold maps.

In particular, if M is not oriented null cobordant, then every generic map $M \to N$ has a singular fiber of codimension ≥ 4 or (an iterated suspension of) a singular fiber of type III^8.

Remark 5.7. Unfortunately, our technique in this paper does not directly apply for computing the 3-dimensional unoriented cobordism group $\mathfrak{N}_3$. This is because the 2-dimensional one is not trivial, and we cannot fill $\mathbb{R}P^2$-bundles over arcs and circles. For similar reasons, our method cannot directly be used for computating $\mathfrak{N}_m$ for $m \ge 4$ and Ω_n for $n \ge 5$.

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OSAMU SAEKI
FACULTY OF MATHEMATICS
KYUSHU UNIVERSITY
MOTOOKA 744, NISHI-KU
FUKUOKA 819-0395
JAPAN

saeki@math.kyushu-u.ac.jp
http://www2.math.kyushu-u.ac.jp/∼saeki/index.html

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