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## TOPOLOGICAL DESCRIPTION OF RIEMANNIAN FOLIATIONS WITH DENSE LEAVES

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**A foliation is called Riemannian if its holonomy pseudogroup consists of local isometries for some Riemannian metric. By combining the work on Hilbert’s fifth problem for local groups with our work on equicontinuous foliated spaces, we prove that, if a foliated space is strongly equicontinuous, locally connected and of finite dimension, has a dense leaf, and has holonomy pseudogroup whose closure is quasianalytic, then it is a Riemannian foliation.**

### Introduction

Riemannian foliations occupy an important place in geometry. An excellent survey is A. Haefliger’s Bourbaki seminar [1989], and the book of P. Molino [1988] is the standard reference for Riemannian foliations. In one of the appendices to this book, E. Ghys proposes the problem of developing a theory of equicontinuous foliated spaces paralleling that of Riemannian foliations; he uses the suggestive term “qualitative Riemannian foliations” for such foliated spaces.

In our previous paper [AC 2009], we discussed the structure of equicontinuous foliated spaces and, more generally, of equicontinuous pseudogroups of local homeomorphisms of topological spaces. This concept was difficult to develop because of the local nature of pseudogroups and the lack of an infinitesimal characterization of local isometries, as one has in the Riemannian case. These difficulties give rise to two versions of equicontinuity: A weaker one seems to be more natural, but a stronger one is more useful for generalizing topological properties of Riemannian foliations. Another relevant property for this purpose is that of quasieffectiveness, which is a generalization to pseudogroups of effectiveness for group actions. For locally connected foliated spaces, quasieffectiveness is equivalent to the quasianalyticity introduced by Haefliger [1985]. For instance, the following well-known topological properties of Riemannian foliations were generalized to

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strongly equicontinuous quasieffective compact foliated spaces [AC 2009] (we also assume that all foliated spaces are locally compact and Polish):

- Leaves without holonomy are quasiisometric to one another (this was our original motivation for that study).
- Leaf closures define a partition of the space. So the foliated space is transitive (there is a dense leaf) if and only if it is minimal (all leaves are dense).
- The holonomy pseudogroup has a closure defined by using the compact-open topology on small enough open subsets.

In this paper we show, in fact, that there are few ways of constructing nice equicontinuous foliated spaces beyond Riemannian foliations. The definition of Riemannian foliation used here is slightly more general than usual: A foliation is called Riemannian when its holonomy pseudogroup is given by local isometries of some Riemannian manifold (a Riemannian pseudogroup); thus leafwise smoothness is not required. Our main result is the following purely topological characterization of Riemannian foliations with dense leaves on compact manifolds.

**Theorem.** *Suppose  $(X, \mathcal{F})$  is a transitive compact foliated space. Then  $\mathcal{F}$  is a Riemannian foliation if and only if  $X$  is locally connected and finite-dimensional,  $\mathcal{F}$  is strongly equicontinuous, and the closure of its holonomy pseudogroup is quasi-analytic.*

This theorem follows directly from the corresponding result for pseudogroups, whose proof uses the material developed in [AC 2009], as well as the local version of R. Jacoby's solution [1957] of Hilbert's fifth problem. For some time, Jacoby's work has been known to contain some dubious passages. Fortunately, I. Goldbring [2010] has established a complete new proof of the local Hilbert fifth problem. We learned of these developments while revising this paper, and determined that none of the shortcomings of [Jacoby 1957] directly affect our work below.

M. Kellum [1993; 1994] made some progress toward a topological characterization of Riemannian foliations, by proving a related result for certain pseudogroups of uniformly Lipschitz diffeomorphisms of Riemannian manifolds. R. Sacksteder [1965] gave a result that can be interpreted as giving a characterization of Riemannian pseudogroups of one-dimensional manifolds. More recently, C. Tarquini [2004] proved that equicontinuous transversely conformal foliations are Riemannian; in the case of dense leaves, this result follows easily from our main theorem.

## 1. Local groups and local actions

For ease of reference, we recall some of the terminology pertaining to local groups as developed in [Jacoby 1957].

**Definition 1.1.** A local group is a quintuple  $(G, e, \cdot, ', \mathfrak{D})$  satisfying the following conditions:

- (1)  $(G, \mathfrak{D})$  is a topological space;
- (2)  $\cdot$  is a function from a subset of  $G \times G$  to  $G$ ;
- (3)  $'$  is a function from a subset of  $G$  to  $G$ ;
- (4) there is a subset  $O$  of  $G$  such that
  - (a)  $O$  is an open neighborhood of  $e$  in  $G$ ,
  - (b)  $O \times O$  is a subset of the domain of  $\cdot$ ,
  - (c)  $O$  is a subset of the domain of  $'$ ,
  - (d) for all  $a, b, c \in O$ , if  $a \cdot b$  and  $b \cdot c \in O$ , then  $(a \cdot b) \cdot c = (a \cdot b) \cdot c$ ,
  - (e)  $a \cdot e = e \cdot a = a$  and  $a' \cdot a = a \cdot a' = e$  for all  $a \in O$  and  $a' \in O$ ,
  - (f) the map  $\cdot : O \times O \rightarrow G$  is continuous,
  - (g) the map  $' : O \rightarrow G$  is continuous;
- (5) the set  $\{e\}$  is closed in  $G$ .

Jacoby employs the notation  $\mathfrak{G}$  for the quintuple  $(G, e, \cdot, ', \mathfrak{D})$ , but here it will be simply denoted by  $G$ .

The collection of all sets  $O$  satisfying condition (4) will be denoted by  $\Psi G$ . This is a neighborhood base of  $e \in G$ ; all of these neighborhoods are symmetric with respect to the inverse operation (3). To say that the local group  $G$  enjoys a certain topological property (for example, local compactness, metrizability, finite-dimensionality) means that some element  $O \in \Psi G$ , with the induced topology, has that property. Let  $\Phi(G, n)$  denote the collection of subsets  $A$  of  $G$  such that the product of any collection of no more than  $n$  elements of  $A$  is defined, and the set  $A^n$  of such products is contained in some  $O \in \Psi G$ .

If  $G$  is a local group, then  $H$  is a subgroup of  $G$  if  $H \in \Phi(G, 2)$ ,  $e \in H$ ,  $H' = H$  and  $H \cdot H = H$ .

If  $G$  is a local group, then  $H \subset G$  is a sublocal group of  $G$  in case  $H$  is itself a local group with respect to the induced operations and topology.

If  $G$  is a local group, then  $\Upsilon G$  denotes the set of all pairs  $(H, U)$  of subsets of  $G$  such that  $e \in H$ ,  $U \in \Psi G$ ,  $a \cdot b \in H$  for all  $a, b \in U \cap H$ , and  $c' \in H$  for all  $c \in U \cap H$ .

Jacoby [1957, Theorem 26] proved that  $H \subset G$  is a sublocal group if and only if there exists  $U$  such that  $(H, U) \in \Upsilon G$ .

Let  $G$  be a local group and let  $\Pi G$  denote the pairs  $(H, U)$  such that  $e \in H$ ,  $U \in \Psi G \cap \Phi(G, 6)$ ,  $a \cdot b \in H$  for all  $a, b \in U^6 \cap H$ ,  $c' \in H$  for all  $c \in U^6 \cap H$ , and  $U^2 \setminus H$  is open. Given such a pair  $(H, U) \in \Pi G$ , there is a (completely regular, Hausdorff) topological space  $G/(U, H)$  and a continuous open surjection

$$T : U^2 \rightarrow G/(U, H)$$

such that  $T(a) = T(b)$  if and only if  $a' \cdot b \in H$ ; see [Jacoby 1957, Theorem 29].

If  $(H, V)$  is another pair in  $\Pi G$ , then the spaces  $G/(H, U)$  and  $G/(H, V)$  are locally homeomorphic in an obvious way. Thus the concept of coset space of  $H$  is well-defined in this sense, as a germ of a topological space. The notation  $G/H$  will be used in this sense; and to say that  $G/H$  has a certain topological property will mean that some  $G/(H, U)$  has such a property.

Let  $\Delta G$  be the set of pairs  $(H, U)$  such that  $(H, U) \in \Pi G$  and  $b' \cdot (a \cdot b) \in H$  for all  $a \in H \cap U^4$  and  $b \in U^2$ . A subset  $H \subset G$  is called a normal sublocal group of  $G$  if there exists  $U$  such that  $(H, U) \in \Delta G$ . If  $(H, U) \in \Delta G$ , the quotient space  $G/(H, U)$  admits the structure of a local group (see [Jacoby 1957, Theorem 35] for the pertinent details) and the natural projection  $T : U^2 \rightarrow G/(H, U)$  is a local homomorphism. As before, another such pair  $(H, V)$  produces a locally isomorphic quotient local group.

We now recall the main results of [Jacoby 1957] on the structure of locally compact local groups.

**Definition 1.2.** A local group  $G$  is a *local Lie group* if there is an  $O \in \Psi G$  and a homeomorphism  $\varphi$  of  $O \cdot O$  onto an open subset of an Euclidean space such that the function  $(x, y) \mapsto \varphi(\varphi^{-1}x \cdot \varphi^{-1}y)$  is real analytic at  $\varphi e$ .

**Theorem 1.3** [Jacoby 1957, Theorem 96]. *Any locally compact local group without small subgroups is a local Lie group.*

In this result, a local group without small subgroups is a local group where some neighborhood of the identity element contains no nontrivial subgroup.

**Theorem 1.4** [Jacoby 1957, Theorems 97–103]. *Any locally compact and second countable local group  $G$  can be approximated by local Lie groups. More precisely, given  $V \in \Psi G \cap \Phi(G, 2)$ , there exists  $U \in \Psi G$  with  $U \subset V$  and there exists a sequence of compact normal subgroups  $F_n \subset U$  such that  $F_{n+1} \subset F_n$ ,  $\bigcap_n F_n = \{e\}$ ,  $(F_n, U) \in \Delta G$ , and  $G/(F_n, U)$  is a local Lie group.*

**Theorem 1.5** [Jacoby 1957, Theorem 107]. *Any finite-dimensional and metrizable locally compact local group is locally isomorphic to the direct product of a Lie group and a compact zero-dimensional topological group.*

An immediate consequence of **Theorem 1.5** is that any locally Euclidean local group is a local Lie group, which is known as the local Hilbert's fifth problem. A solution was originally proposed by Jacoby, but has only recently been established correctly by Goldbring [2010].

All local groups appearing in this paper will be assumed, or proved, to be locally compact and second countable.

**Definition 1.6.** A local group  $G$  is a *local transformation group* on a subspace  $X \subset Y$  if there is given a continuous map  $G \times X \rightarrow Y$ , written  $(g, x) \mapsto gx$ , such

that  $ex = x$  for all  $x \in X$ , and  $g_1(g_2x) = (g_1g_2)x$ , provided both sides are defined. This map  $G \times X \rightarrow Y$  is called a *local action* of  $G$  on  $X \subset Y$ .

The standard example of local action is the following. Let  $H$  be a sublocal group of  $G$ . If  $(H, U) \in \Pi G$  and  $T : U^2 \rightarrow G/(H, U)$  is the natural projection, then  $U$  is a sublocal group of  $G$  and the map  $(u, T(g)) \mapsto T(u \cdot g)$  defines a local action of  $U$  on the open subspace  $T(U)$  of  $G/(H, U)$ .

If  $G$  is a local group acting on  $X \subset Y$  and the action is locally transitive at  $x \in X$  in that there is a neighborhood  $V \in \Psi G$  such that  $Vx$  includes a neighborhood of  $x$  in  $X$ , then there is a sublocal group  $H$  of  $G$  and an open subset  $U \subset G$  such that  $(H, U) \in \Pi G$  and the orbit map  $g \in G \mapsto gx \in X$  induces a local homeomorphism  $G/(H, U) \rightarrow X$  at  $x$ , which is equivariant with respect to the action of  $U$ .

**Theorem 1.7.** *Let  $G$  be a locally compact, separable and metrizable local group. Suppose that there is a local action of  $G$  on a finite-dimensional subspace  $X \subset Y$  and that the action is locally transitive at some  $x \in X$ . Fix some  $(H, U) \in \Pi G$  so that the orbit map  $g \mapsto gx$  induces a local homeomorphism  $G/(H, U) \rightarrow X$  at  $x$ . Then there exists a connected normal subgroup  $K$  of  $G$  such that  $K \subset H$ ,  $(K, U) \in \Pi G$  and  $G/(K, U)$  has finite dimension.*

This is a local version of [Montgomery and Zippin 1955, Theorem 6.2.2], whose proof also establishes the following fact.

**Claim 1.** *Let  $A$  be a locally compact, separable and metrizable topological group, and let  $B$  be a closed subgroup of  $A$  such that  $A/B$  is of finite dimension and connected. Let  $N_n$  be a sequence of compact normal subgroups such that  $\bigcap_n N_n = \{e\}$  and every  $A/N_n$  is a Lie group. Then there is some index  $n_0$  such that the connected component of the identity of  $N_{n_0}$  is contained in  $B$ .*

**Claim 2.** *Let  $A$  be a local group, let  $(B, V) \in \Pi A$ , let  $T : A \rightarrow A/(B, V)$  denote the natural projection, and let  $C$  be a compact subgroup of  $A$  contained in  $V^2 \cap V^6$ . Then  $B \cap C$  is a compact subgroup of  $C$ , a map  $C/(B \cap C) \rightarrow A/(B, V)$  is well-defined by the assignment  $a(B \cap C) \mapsto T(a)$ , and this map is an embedding.*

*Proof of Claim 2.* On the one hand,  $B \cap C$  is compact because  $B$  is closed and  $C$  is compact. On the other hand,  $B \cap C$  is a subgroup of  $C$  because  $C$  is a subgroup,  $C \subset V^6$ , and  $a \cdot b \in B$  and  $a' \in B$  for all  $a, b \in V^6$  since  $(B, V) \in \Pi A$ . The map  $C/(B \cap C) \rightarrow A/(B, V)$  is well-defined and injective because  $C \subset V^2$  and  $T(a) = T(b)$  if and only if  $a \cdot b' \in B$  for  $a, b \in V^2$ . This injection is continuous because it is induced by the inclusion  $C \hookrightarrow V^2$ . Thus this map is an embedding since  $C/(B \cap C)$  is compact and  $A/(B, V)$  is Hausdorff.  $\square$

*Proof of Theorem 1.7.* Let  $F_n$  be a sequence of compact normal subgroups of  $G$  as provided by [Jacoby 1957, Theorems 97–103] (stated here as Theorem 1.4). It may be assumed that  $(F_n, U) \in \Delta G$  and  $F_n \subset U^2 \cap U^6$  for all  $n$ . Let  $K_n$

be the identity component of  $F_n$ . Then the natural quotient map  $G/(K_n, U) \rightarrow G/(F_n, U)$  has zero-dimensional fibers because they are locally homeomorphic to the zero-dimensional group  $F_n/K_n$ . Since  $G/(F_n, U)$  is a local Lie group, it has finite dimension, and so it follows that  $G/(K_n, U)$  has also finite dimension; see [Hurewicz and Wallman 1941, Chapter VII, Section 4].

By Claim 2,  $K_1 \cap H$  is a compact subgroup of  $K_1$ , and there is a canonical embedding  $K_1/(K_1 \cap H) \rightarrow G/(H, U)$ . Also  $K_1/(K_1 \cap H)$  is connected because  $K_1$  is connected. Therefore the dimension of  $K_1/(K_1 \cap H)$  is less than or equal to the dimension of  $G/(H, U)$  by [Hurewicz and Wallman 1941, Theorem III.1], and thus  $K_1/(K_1 \cap H)$  has finite dimension. Furthermore, each canonical embedding  $K_1/(K_1 \cap F_n) \rightarrow G/(F_n, U)$ , given by Claim 2, realizes  $K_1/(K_1 \cap F_n)$  as a compact subgroup of the local Lie group  $G/(F_n, U)$  because  $K_1 \cap F_n$  is a normal subgroup of  $K_1$ . So every  $K_1/(K_1 \cap F_n)$  is a Lie group. Then, by Claim 1 with  $A = K_1$ ,  $B = K_1 \cap H$ , and  $N_n = K_1 \cap F_n$ , there is some index  $n_0$  such that the identity component  $K$  of  $F = K_1 \cap F_{n_0}$  is contained in  $K_1 \cap H$ . This  $F$  is a normal subgroup of  $G$ , and thus  $K$  is a connected normal subgroup of  $G$ . Furthermore  $(K, U), (F, U) \in \Delta G$  and

$$\dim G/(K, U) = \dim G/(F, U) \leq \dim G/(K_1, U) + \dim K_1/(K_1 \cap F_{n_0})$$

by [Hurewicz and Wallman 1941, Theorem III.4], confirming that  $G/(K, U)$  has finite dimension.  $\square$

## 2. Equicontinuous pseudogroups

A pseudogroup of local transformations of a topological space  $Z$  is a collection  $\mathcal{H}$  of homeomorphisms between open subsets of  $Z$  that contains the identity on  $Z$  and is closed under composition (wherever defined), inversion, restriction and combination of maps. A pseudogroup  $\mathcal{H}$  is generated by a set  $E \subset \mathcal{H}$  if every element of  $\mathcal{H}$  can be obtained from  $E$  by using the pseudogroup operations; sets of generators will always be assumed to be symmetric:  $h^{-1} \in E$  if  $h \in E$ .

A pseudogroup of local transformations  $\mathcal{H}$  on  $Z$  induces an equivalence relation on  $Z$  whose equivalence classes are the orbits: The orbit of a point  $x$  in  $Z$  under  $\mathcal{H}$  is the set  $\mathcal{H}(x)$  of all points  $h(x)$ , for all  $h \in \mathcal{H}$  whose domain contains  $x$ .

Pseudogroups of local transformations naturally generalize group actions on topological spaces (the restrictions to open subsets of the space of the homeomorphism of the action generate a pseudogroup of local transformations of that space), and include as an important example the holonomy pseudogroup of a foliated space generated by the holonomy transformations between transversals of a regular covering by foliation charts of the foliated space [Candel and Conlon 2000; Haefliger 1985; 1988; Hector and Hirsch 1981].

The study of the geometry and dynamics of pseudogroups can be simplified by using certain equivalence relation introduced by Haefliger [1985; 1988]. This equivalence relation is generated by the following basic example. If  $\mathcal{H}$  is a pseudogroup of local transformations of  $Z$  and  $U \subset Z$  is an open subset of  $Z$  that has nonempty intersections with every orbit of  $\mathcal{H}$ , then the pseudogroup  $\mathcal{H}_U$  generated by the restrictions of elements of  $\mathcal{H}$  to  $U$  is a pseudogroup of local transformations of  $U$  that is equivalent to  $\mathcal{H}$ . This concept of pseudogroup equivalence is very important in the study of foliated spaces because the equivalence class of the holonomy pseudogroup depends only on each foliated space; it is independent of the choice of a regular covering by flow boxes.

Haefliger also introduced the concept of compact generation for pseudogroups, a property that is preserved under equivalence of pseudogroups. A pseudogroup of local transformations  $\mathcal{H}$  of a locally compact space  $Z$  is *compactly generated* if there is a relatively compact open subset  $U$  of  $Z$  that meets each orbit of  $\mathcal{H}$ , and is such that the restriction  $\mathcal{G}$  of  $\mathcal{H}$  to  $U$  is generated by a finite symmetric collection  $E \subset \mathcal{G}$  such that each  $g \in E$  is the restriction of an element  $\bar{g}$  of  $\mathcal{H}$  defined on some neighborhood of the closure of the source of  $g$ . Any such  $E$  is called a *system of compact generation* of  $\mathcal{H}$  on  $U$ .

In [AC 2009], we introduced the concepts of strong and weak equicontinuity for pseudogroups of local transformations of spaces whose topology is induced by the following type of structure. Let  $\{(Z_i, d_i)\}_{i \in I}$  be a family of metric spaces such that  $\{Z_i\}_{i \in I}$  is a covering of a set  $Z$ , each intersection  $Z_i \cap Z_j$  is open in  $(Z_i, d_i)$  and  $(Z_j, d_j)$ , and for all  $\varepsilon > 0$  there is some  $\delta(\varepsilon) > 0$  such that for all  $i, j \in I$  and  $z \in Z_i \cap Z_j$ , there is some open neighborhood  $U_{i,j,z}$  of  $z$  in  $Z_i \cap Z_j$  (with respect to the topology induced by  $d_i$  and  $d_j$ ) such that

$$d_i(x, y) < \delta(\varepsilon) \quad \text{implies} \quad d_j(x, y) < \varepsilon$$

for all  $\varepsilon > 0$  and all  $x, y \in U_{i,j,z}$ . Such a family is called a *cover of  $Z$  by quasilocally equal metric spaces*. This term refers to a covering of  $Z$  by sets endowed with metrics whose restrictions to the overlaps are almost equal locally. Compare it with the notion of covering by locally equal metric spaces, which is given below. Two such families are called *quasilocally equal* when their union also is a cover of  $Z$  by quasilocally equal metric spaces. This is an equivalence relation whose equivalence classes are called *quasilocal metrics* on  $Z$ . For each quasilocal metric  $\Omega$  on  $Z$ , the pair  $(Z, \Omega)$  is called a *quasilocal metric space*. Such a  $\Omega$  induces a topology on  $Z$  such that, for each  $\{(Z_i, d_i)\}_{i \in I} \in \Omega$ , the family of open balls of all metric spaces  $(Z_i, d_i)$  form a base of open sets. Any topological concept or property of  $(Z, \Omega)$  refers to this underlying topology. We also observed that  $(Z, \Omega)$  is a locally compact Polish space if and only if it is Hausdorff, paracompact, separable and locally compact.



In [AC 2009], we defined the strongest version of equicontinuity as follows.

**Definition 2.1.** A pseudogroup  $\mathcal{H}$  of local homeomorphisms of a quasilocal metric space  $(Z, \Omega)$  is *strongly equicontinuous* if there exists some  $\{(Z_i, d_i)\}_{i \in I} \in \Omega$  and some symmetric set  $S$  of generators of  $\mathcal{H}$  that is closed under compositions such that, for every  $\varepsilon > 0$ , there is some  $\delta(\varepsilon) > 0$  such that

$$d_i(x, y) < \delta(\varepsilon) \quad \text{implies} \quad d_j(h(x), h(y)) < \varepsilon$$

for all  $h \in S$ ,  $i, j \in I$  and  $x, y \in Z_i \cap h^{-1}(Z_j \cap \text{im } h)$ .

A pseudogroup  $\mathcal{H}$  of local homeomorphisms of a topological space  $Z$  is *strongly equicontinuous* if it is strongly equicontinuous with respect to some quasilocal metric inducing the topology of  $Z$ .

Strong equicontinuity is invariant under equivalences of pseudogroups acting on locally compact Polish spaces [AC 2009, Lemma 8.8].

The condition that the symmetric generating set  $S$  in Definition 2.1 be closed under compositions is precisely what distinguishes strong and weak equicontinuity [AC 2009, Lemma 8.3]. A typical choice of  $S$  is the set of all possible composites of some symmetric set of generators. In fact, given any  $S$  satisfying the condition of strong equicontinuity, it is obviously possible to find a symmetric set of generators  $E$  consisting of restrictions of elements of  $S$ , and so the set of all composites of elements of  $E$  also satisfies the condition of strong equicontinuity.

A key property of strong equicontinuity is the following.

**Proposition 2.2** [AC 2009, Proposition 8.9]. *Let  $\mathcal{H}$  be a compactly generated and strongly equicontinuous pseudogroup acting on a locally compact Polish quasilocal metric space  $(Z, \Omega)$ , and let  $U$  be any relatively compact open subset of  $(Z, \Omega)$  that meets every  $\mathcal{H}$ -orbit. Suppose that  $\{(Z_i, d_i)\}_{i \in I} \in \Omega$  satisfies the condition of strong equicontinuity. Let  $E$  be any system of compact generation of  $\mathcal{H}$  on  $U$ , and let  $\bar{g}$  be an extension of each  $g \in E$  with  $\overline{\text{dom } g} \subset \text{dom } \bar{g}$ . Also, let  $\{Z'_i\}_{i \in I}$  be any shrinking of  $\{Z_i\}_{i \in I}$ . Then there is a finite family  $\mathcal{V}$  of open subsets of  $(Z, \Omega)$  whose union contains  $U$  such that, for any  $V \in \mathcal{V}$ ,  $x \in U \cap V$ , and  $h \in \mathcal{H}$  with  $x \in \text{dom } h$  and  $h(x) \in U$ , the domain of  $\tilde{h} = \bar{g}_n \circ \cdots \circ \bar{g}_1$  contains  $V$  for any composite  $h = g_n \circ \cdots \circ g_1$  defined around  $x$  with  $g_1, \dots, g_n \in E$ , and moreover  $V \subset Z'_{i_0}$  and  $\tilde{h}(V) \subset Z'_{i_1}$  for some  $i_0, i_1 \in I$ .*

In this statement,  $\{Z'_i\}_{i \in I}$  is a *shrinking* of  $\{Z_i\}_{i \in I}$  in that it is also an open cover of  $Z$  and satisfies  $\bar{Z}'_i \subset Z_i$  for all  $i \in I$ .

In [AC 2009], we introduced the following terminology for the study of strongly equicontinuous pseudogroups. A pseudogroup  $\mathcal{H}$  of local transformations of a space  $Z$  is said to be *quasieffective* if it is generated by some symmetric set  $S$  that is closed under compositions, and if any transformation in  $S$  is the identity on its domain if it is the identity on some nonempty open subset of its domain. The

family  $S$  may also be assumed to be closed under restrictions to open sets, and in that case every map in  $\mathcal{H}$  is a combination of maps in  $S$ . Moreover, if  $\mathcal{H}$  is strongly equicontinuous and quasieffective, then  $S$  can be chosen to satisfy the conditions of both strong equicontinuity and quasieffectiveness.

We proved in [AC 2009, Lemma 9.5] that quasieffectiveness is preserved by equivalences of pseudogroups acting on locally compact Polish spaces, and for pseudogroups of local homeomorphisms of locally connected and locally compact Polish spaces, it is equivalent to quasianalyticity [AC 2009, Lemma 9.6], where recall that a pseudogroup  $\mathcal{H}$  is called *quasianalytic* if every  $h \in \mathcal{H}$  is the identity around some  $x \in \text{dom } h$  whenever  $h$  is the identity on some open set whose closure contains  $x$  [Haefliger 1985].

**Proposition 2.3** [AC 2009, Proposition 9.9]. *Let  $\mathcal{H}$  be a compactly generated, strongly equicontinuous and quasieffective pseudogroup of local homeomorphisms of a locally compact Polish space  $Z$ . Suppose that the conditions of strong equicontinuity and quasieffectiveness are satisfied with a symmetric set  $S$  of generators of  $\mathcal{H}$  that is closed under compositions. Let  $A, B$  be open subsets of  $Z$  such that  $\bar{A}$  is compact and contained in  $B$ . If  $x$  and  $y$  are close enough points in  $Z$ , then  $f(x) \in A$  implies  $f(y) \in B$  for all  $f \in S$  whose domain contains  $x$  and  $y$ .*

Recall that a pseudogroup is said to be *transitive* if it has a dense orbit, and is said to be *minimal* if all of its orbits are dense.

**Theorem 2.4** [AC 2009, Theorem 11.1]. *Let  $\mathcal{H}$  be a compactly generated and strongly equicontinuous pseudogroup of local transformations of a locally compact Polish space  $Z$ . If  $\mathcal{H}$  is transitive, then  $\mathcal{H}$  is minimal.*

For spaces  $Y, Z$ , let  $C(Y, Z)$  denote the set of continuous maps  $Y \rightarrow Z$ , and let  $C_{c-o}(Y, Z)$  denote this set when it is endowed with the compact-open topology. For open subspaces  $O$  and  $P$  of a space  $Z$ , the space  $C_{c-o}(O, P)$  will be considered as an open subspace of  $C_{c-o}(O, Z)$  in the canonical way.

**Theorem 2.5** [AC 2009, Theorem 12.1]. *Let  $\mathcal{H}$  be a quasieffective, compactly generated and strongly equicontinuous pseudogroup of local transformations of a locally compact Polish space  $Z$ . Let  $S$  be a symmetric set of generators of  $\mathcal{H}$  that is closed under compositions and restrictions to open subsets, and satisfies the conditions of strong equicontinuity and quasieffectiveness. Let  $\tilde{\mathcal{H}}$  be the set of maps  $h$  between open subsets of  $Z$  that satisfy the following property: For every  $x \in \text{dom } h$ , there exists a neighborhood  $O_x$  of  $x$  in  $\text{dom } h$  such that the restriction  $h|_{O_x}$  is in the closure of  $C(O_x, Z) \cap S$  in  $C_{c-o}(O_x, Z)$ . Then*

- $\tilde{\mathcal{H}}$  is closed under composition, combination and restriction to open sets;
- every map in  $\tilde{\mathcal{H}}$  is a homeomorphism around every point of its domain;
- the maps of  $\tilde{\mathcal{H}}$  that are homeomorphisms form a pseudogroup  $\bar{\tilde{\mathcal{H}}}$  containing  $\mathcal{H}$ ;

- $\overline{\mathcal{H}}$  is strongly equicontinuous;
- the orbits of  $\overline{\mathcal{H}}$  are equal to the closures of the orbits of  $\mathcal{H}$ ; and
- $\widetilde{\mathcal{H}}$  and  $\overline{\mathcal{H}}$  are independent of the choice of  $S$ .

If a pseudogroup  $\mathcal{H}$  satisfies the conditions of [Theorem 2.5](#), then the pseudogroup  $\overline{\mathcal{H}}$  is called the *closure* of  $\mathcal{H}$ .

**Remark.** In [Theorem 2.5](#), the closure of  $C(O_x, Z) \cap S$  in  $C_{c-o}(O_x, Z)$  is compact by the equicontinuity condition. So the restriction of the Tikhonov product topology of  $Z^{O_x}$  to  $C(O_x, Z)$  would produce the same closure. In the study of distal flows — R. Ellis [[1958](#); [1969](#); [1978](#)] and H. Furstenberg [[1963](#)] — the Tikhonov topology is used to define the Ellis (semi)group because the more general condition of distality does not guarantee the mentioned precompactness, and also because it applies to compact Hausdorff spaces that need not be metrizable [[Ellis 1978](#)].

**Lemma 2.6.** *Let  $\mathcal{H}$  be a compactly generated, strongly equicontinuous and quasi-effective pseudogroup of local transformations of a locally compact Polish space  $Z$ . Then  $\overline{\mathcal{H}}$  is quasieffective if and only if there is a symmetric set  $S$  of generators of  $\mathcal{H}$  that is closed under compositions and restrictions to open subsets, and such that the restriction map  $\rho_W^V : S \cap C(V, Z) \rightarrow S \cap C(W, Z)$  is a homeomorphism with respect to the compact-open topologies for small enough open subsets  $V$  and  $W$  of  $Z$  with  $W \subset V$ .*

*Proof.* The result follows directly by observing that, according to [Theorem 2.5](#),  $\overline{\mathcal{H}}$  is quasieffective just when there is some symmetric set  $S$  of generators of  $\mathcal{H}$  that is closed under compositions and satisfies the condition that for any sequence  $h_n$  in  $S$  and open nonempty subsets  $V$  and  $W$  of  $Z$ , with  $W \subset V \subset \text{dom } h_n$  for all  $n$ , if  $h_n|_W \rightarrow \text{id}_W$  in  $C_{c-o}(W, Z)$ , then  $h_n|_V \rightarrow \text{id}_V$  in  $C_{c-o}(V, Z)$ .  $\square$

**Corollary 2.7.** *Suppose  $\mathcal{H}$  is a compactly generated, strongly equicontinuous and quasianalytic pseudogroup of local transformations of a locally connected and locally compact Polish space  $Z$ . Then  $\overline{\mathcal{H}}$  is quasianalytic if and only if there is a symmetric set  $S$  of generators of  $\mathcal{H}$  that is closed under compositions and restrictions to open subsets and for which  $\rho_W^V : S \cap C(V, Z) \rightarrow S \cap C(W, Z)$  is a homeomorphism with respect to the compact-open topologies for small enough open subsets  $V$  and  $W$  of  $Z$  with  $W \subset V$ .*

A pseudogroup  $\mathcal{H}$  of local transformations of a locally compact space  $Z$  is quasi-effective precisely when there is a symmetric set  $S$  of generators of  $\mathcal{H}$  that is closed under compositions and restrictions to open subsets, and for which the restriction map  $\rho_W^V : S \cap C(V, Z) \rightarrow S \cap C(W, Z)$  is injective for all open subsets  $V$  and  $W$  of  $Z$  with  $W \subset V$ . If moreover  $Z$  is a locally compact Polish space, and  $\mathcal{H}$  is compactly generated and strongly equicontinuous, then any such  $\rho_W^V$  is bijective for  $V$  and  $W$  small enough by [Proposition 2.2](#). Moreover  $\rho_W^V$  is continuous with

respect to the compact-open topology [Munkres 1975, page 289], but it may not be a homeomorphism as shown by the following example.

**Example 2.8.** Let  $Z$  be the union of two tangent spheres in  $\mathbb{R}^3$ , and let  $h : Z \rightarrow Z$  be the combination of two rotations, one on each sphere, around the common axis and with rationally independent angles. Then  $h$  generates a compactly generated, strongly equicontinuous and quasieffective pseudogroup  $\mathcal{H}$  of local transformations of  $Z$ ; indeed,  $h$  is an isometry for the path metric space structure on  $Z$  induced from that of  $\mathbb{R}^3$ . Nevertheless, it is easy to see that the closure  $\overline{\mathcal{H}}$  is not quasieffective.

We next recall an isometrization theorem from [AC 2009] that establishes that in a certain sense equicontinuous quasieffective pseudogroups are indeed pseudogroups of local isometries. Two metrics on the same set are said to be *locally equal* when they induce the same topology and each point has a neighborhood where both metrics are equal. Let  $\{(Z_i, d_i)\}_{i \in I}$  be a family of metric spaces such that  $\{Z_i\}_{i \in I}$  is a covering of a set  $Z$ , each intersection  $Z_i \cap Z_j$  is open in  $(Z_i, d_i)$  and  $(Z_j, d_j)$ , and the metrics  $d_i, d_j$  are locally equal on  $Z_i \cap Z_j$  whenever this is a nonempty set. Such a family will be called a *cover of  $Z$  by locally equal metric spaces*. Two such families are called *locally equal* when their union also is a cover of  $Z$  by locally equal metric spaces. This is an equivalence relation whose equivalence classes are called *local metrics* on  $Z$ . For each local metric  $\mathfrak{D}$  on  $Z$ , the pair  $(Z, \mathfrak{D})$  is called a *local metric space*. Observe that every metric induces a unique local metric in a canonical way. In turn, every local metric canonically determines a unique quasilocal metric. Note also that local metrics induced by metrics can be considered as germs of metrics around the diagonal. Moreover, a local or quasilocal metric is induced by some metric if and only if it is Hausdorff and paracompact [AC 2009, Theorems 13.5 and 15.1].

We call a local homeomorphism  $h$  of a local metric space  $(Z, \mathfrak{D})$  a *local isometry* if there is some  $\{(Z_i, d_i)\}_{i \in I} \in \mathfrak{D}$  such that, for  $i, j \in I$  and  $z \in Z_i \cap h^{-1}(Z_j \cap \text{im } h)$ , there is some neighborhood  $U_{h,i,j,z}$  of  $z$  in  $Z_i \cap h^{-1}(Z_j \cap \text{im } h)$  such that  $d_i(x, y) = d_j(h(x), h(y))$  for all  $x, y \in U_{h,i,j,z}$ . This definition is independent of the choice of the family  $\{(Z_i, d_i)\}_{i \in I} \in \mathfrak{D}$ .

**Theorem 2.9** [AC 2009, Theorem 15.1]. *Let  $\mathcal{H}$  be a compactly generated, quasieffective and strongly equicontinuous pseudogroup of local transformations of a locally compact Polish space  $Z$ . Then  $\mathcal{H}$  is a pseudogroup of local isometries with respect to some local metric inducing the topology of  $Z$ .*

### 3. Riemannian pseudogroups

**Definition 3.1.** A pseudogroup  $\mathcal{H}$  of local transformations of a space  $Z$  is called a *Riemannian pseudogroup* if  $Z$  is a Hausdorff paracompact  $C^\infty$  manifold and all maps in  $\mathcal{H}$  are local isometries with respect to some Riemannian metric on  $Z$ .

**Example 3.2.** Let  $G$  be a local Lie group, and  $G_0 \subset G$  a compact subgroup. Then the canonical local action of some neighborhood of the identity in  $G$  on some neighborhood of the identity class in  $G/G_0$  generates a transitive Riemannian pseudogroup. In fact, since  $G_0$  is compact, there is a  $G$ -left invariant and  $G_0$ -right invariant Riemannian metric on some neighborhood of the identity in  $G$ ; this metric induces a  $G$ -invariant Riemannian metric on some neighborhood of the identity class in  $G/G_0$ . More generally, if  $\Gamma \subset G$  is a dense sublocal group, then the canonical local action of some neighborhood of the identity in  $\Gamma$  on some neighborhood of the identity class in  $G/G_0$  generates a transitive Riemannian pseudogroup that is complete in the sense of [Haefliger 1985]. The pseudogroup version of the Molino description of Riemannian foliations establishes that any transitive complete Riemannian pseudogroup is equivalent to a pseudogroup of this type.

The pseudogroup version of our main result here is the following topological characterization of transitive compactly generated Riemannian pseudogroups.

**Theorem 3.3.** *Let  $\mathcal{H}$  be a transitive, compactly generated pseudogroup of local transformations of a locally compact Polish space  $Z$ . Then  $\mathcal{H}$  is a Riemannian pseudogroup if and only if  $Z$  is locally connected and finite-dimensional,  $\mathcal{H}$  is strongly equicontinuous, and  $\overline{\mathcal{H}}$  is quasianalytic.*

**Remark.** The closure  $\overline{\mathcal{H}}$  of  $\mathcal{H}$  exists by virtue of Theorem 2.5, because the space  $Z$  is locally connected (hence the pseudogroup  $\mathcal{H}$  is quasieffective because it is quasianalytic [AC 2009, Lemma 9.6]).

**Corollary 3.4.** *Let  $\mathcal{H}$  be a compactly generated, strongly equicontinuous pseudogroup of local transformations of a locally compact Polish space  $Z$ . Then the  $\mathcal{H}$ -orbit closures are  $C^\infty$  manifolds if and only if they are locally connected and finite-dimensional, and the induced pseudogroup  $\overline{\mathcal{H}}$  is quasianalytic on them.*

*Proof.* This follows from Theorem 3.3 because the closure of  $\mathcal{H}$  acting on the closure of an orbit is equivalent to a pseudogroup like the one in Example 3.2.  $\square$

The proof of Theorem 3.3 will be given in the next section; in the interim, we describe some examples illustrating the necessity of several hypotheses.

**Example 3.5.** Let  $Z$  be the product of countably infinitely many circles. This is a compact, locally connected Polish group that acts on itself by translations in an equicontinuous way. Let  $\mathbb{Z} \rightarrow Z$  be an injective homomorphism with dense image. Then the action of  $\mathbb{Z}$  on  $Z$  induced by this homomorphism is minimal and equicontinuous, and so it generates a minimal, quasianalytic and equicontinuous pseudogroup, which is not Riemannian because  $Z$  is of infinite dimension.

**Example 3.6.** Suppose  $Z$  is the set of  $p$ -adic numbers  $x \in \mathbb{Q}_p$  with  $p$ -adic norm  $|x|_p \leq 1$ . Then the operation  $x \mapsto x + 1$  defines an action of  $\mathbb{Z}$  on  $Z$  that is minimal and equicontinuous (it preserves the  $p$ -adic metric on  $Z$ ). Thus it generates a minimal, quasianalytic and equicontinuous pseudogroup, which is not Riemannian because  $Z$  is zero-dimensional.

**Example 3.7.** Let  $C$  be the standard Cantor set in  $[0, 1] \subset \mathbb{R}$  and  $Z = \bigcup_{n=-\infty}^{\infty} C + n$  be the union of  $C$  and all of its integer translates. Then there is a pseudogroup  $\mathcal{H}$  acting on  $Z$  generated by translations of the line that locally preserve  $Z$ . In fact,  $\mathcal{H}$  is a pseudogroup of local isometries for two geometrically distinct metrics, the Euclidean and the dyadic.

**Example 3.8.** The previous example can be generalized, replacing  $Z$  by the universal Menger curve [Blumenthal and Menger 1970, Chapter 15]. This space  $Z$  (to be precise, a modification of it) is constructed as an invariant set of the pseudogroup of local homeomorphisms of  $\mathbb{R}^3$  generated by the map  $f(x) = 3x$  and the three unit translations parallel to the coordinate axes. There is a pseudogroup acting on  $Z$  generated by Euclidean isometries that locally preserve  $Z$ . It is fairly easy to see that such a pseudogroup is minimal, quasianalytic and equicontinuous. Moreover  $Z$  is locally connected and of dimension one. However, this pseudogroup is not compactly generated.

The most elusive hypotheses of [Theorem 3.3](#) is the one concerning the quasianalyticity of the closure of a quasianalytic pseudogroup. It is used explicitly in the proof of [Lemma 4.2](#). We do not have an example of a pseudogroup  $\mathcal{H}$  as in the main theorem whose closure fails to be quasianalytic; the remaining of this section offers some examples and observations relevant to this problem.

For metric spaces, being a length space is a local property; thus the two theorems of [[AC 2009](#), Section 15] are valid for length spaces.

**Definition 3.9.** A length space  $X$  is analytic at a point  $x \in X$  if the following holds: If  $\gamma$  and  $\gamma'$  are geodesic arcs (parametrized by arc-length) defined on an interval about  $0 \in \mathbb{R}$ , such that  $\gamma(0) = \gamma'(0) = x$  and that  $\gamma = \gamma'$  on some interval  $(-a, 0)$ , then they have the same germ at 0. The space  $X$  is analytic if it is analytic at every point.

For example, a Riemannian manifold is an analytic length space. In relation to [Theorem 3.3](#), if a local length space is known to be analytic at one point and admits a transitive action of a pseudogroup of local isometries, then it is analytic.

Real trees give rise to many examples, like the one below, of metric spaces that are also length space and admit actions of pseudogroups of isometries that are not quasianalytic; see [[Shalen 1987](#)].

**Example 3.10.** Let  $X = \mathbb{R}^2$  be endowed with the metric given by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2, \\ |y_1 - y_2| & \text{if } x_1 = x_2. \end{cases}$$

Given a subset  $F$  of the real axis there is an isometry  $f_F$  of  $X$  given by

$$f_F(x, y) = \begin{cases} (x, y) & \text{if } x \in F, \\ (x, -y) & \text{if } x \notin F. \end{cases}$$

This family of isometries  $\{f_F \mid F \subset X\}$  forms a normal subgroup of the group of isometries of  $X$ . Thus this group is not quasianalytic.

**Proposition 3.11.** *Let  $\mathcal{H}$  be a pseudogroup of local isometries of an analytic local length space  $X$ . Then  $\mathcal{H}$  is quasianalytic.*

*Proof.* If  $\mathcal{H}$  is not quasianalytic, then, by definition, there exists an element  $f$  of  $\mathcal{H}$ , an open set  $U$  in  $\text{dom}(f)$  such that  $f|_U = \text{id}$ , and a point  $x_0$  in the closure of  $U$  such that  $f$  is not the identity in any neighborhood of  $x_0$ . Therefore, there is a sequence of points  $x_n$  converging to  $x_0$  such that  $f(x_n) \neq x_n$ . If  $n$  is sufficiently large, then there is a geodesic arc contained in the domain of  $f$  and joining  $x_n$  and a point  $y \in U$ . This geodesic arc is mapped by  $f$  onto a distinct geodesic arc that has the same germ at one of its endpoints as the first one.  $\square$

The following example shows that the converse is false.

**Example 3.12.** Let  $X$  be the Euclidean plane  $\mathbb{R}^2$  endowed with the metric induced by the supremum norm  $\|(x, y)\| = \max\{|x|, |y|\}$ . Then  $X$  is a length space that is not locally analytic. Indeed, if  $f: I \rightarrow \mathbb{R}$  is a function such that  $|f(s) - f(t)| \leq |s - t|$  for all  $s, t \in I$ , then  $t \in I \mapsto (t, f(t)) \in X$  is a geodesic. However, every local isometry is locally equal to a linear isometry; hence the pseudogroup of local isometries is quasianalytic.

#### 4. Equicontinuous pseudogroups and Hilbert's fifth problem

This section is devoted to the proof of [Theorem 3.3](#). The “only if” part is obvious, so it is enough to show the “if” part, which has essentially two steps. In the first one, a local group action on  $Z$  is obtained as the closure of the set of elements of  $\mathcal{H}$  that are sufficiently close to the identity map on an appropriate subset of  $Z$ . This construction follows Kellum [[1993](#)]. The second step invokes the theory behind the solution to the local version of Hilbert's fifth problem in order to show that the local group is a local Lie group, and thus this local action is isometric for some Riemannian metric if its isotropy subgroups are compact. So  $\mathcal{H}$  will be shown to be Riemannian by proving that it is of the type described in [Example 3.2](#).

By [Theorem 2.9](#), there is a local metric structure  $\mathcal{D}$  on  $Z$  with respect to which the elements of  $\mathcal{H}$  are local isometries. Let  $\{(Z_i, d_i)\}_{i \in I} \in \mathcal{D}$  be any local cover of

$Z$  satisfying the condition of strong equicontinuity. Let  $U$  be a relatively compact nontrivial open subset of  $Z$ , and let  $\mathcal{V}$  be a family of open subsets that cover  $U$  as in Proposition 2.2. Let  $V$  be an element of  $\mathcal{V}$  having nonempty intersection with  $U$  and contained in  $Z_{i_0}$  for some  $i_0 \in I$ , and let  $D \subset V$  be an open nonempty connected subset whose closure is compact and also contained in  $V$ . According to Proposition 2.2, if  $h \in \mathcal{H}$  is such that  $\text{dom } h \subset D$  and  $\text{im } h \cap U \neq \emptyset$ , then there exists an element  $\tilde{h} \in \mathcal{H}$  that extends  $h$  and whose domain contains  $V$ . Moreover, since  $\mathcal{H}$  is quasianalytic and  $D$  is connected, such extension  $\tilde{h}$  is unique on  $D$ . In particular, such  $h$  admits a unique extension to a homeomorphism of  $\bar{D}$  onto its image.

Under the current hypothesis, the completion  $\bar{\mathcal{H}}$  of  $\mathcal{H}$  is a quasianalytic pseudo-group of transformations of  $Z$  whose action on  $Z$  has a single orbit. Let  $\bar{\mathcal{H}}_D$  be the collection of all homeomorphisms  $h|_D$ , where  $h \in \bar{\mathcal{H}}$  is any element whose domain contains  $D$ . Let  $D' \subset D$  be a connected, compact set with nonempty interior, and let  $\bar{\mathcal{H}}_{DD'} = \{h \in \bar{\mathcal{H}}_D \mid h(D') \cap D' \neq \emptyset\}$ . By the strong equicontinuity of  $\bar{\mathcal{H}}$  and Proposition 2.3, the set  $D'$  can be chosen so that all the translates  $h(D')$  for  $h \in \bar{\mathcal{H}}_{DD'}$  are contained in a fixed compact subset  $K$  of  $D$ . Once this choice of  $D'$  is made, let  $G = \bar{\mathcal{H}}_{DD'}$  be the resulting space.

The space  $G$  is endowed with the compact open topology as a subset of  $C(D, Z)$ . Every element of  $G$  is actually defined on  $V$ , and hence on  $\bar{D}$ , and so the compact open topology can be described by the supremum metric given by  $d(g_1, g_2) = \sup_{x \in D} d_{i_0}(g_1(x), g_2(x))$ , where  $d_{i_0}$  is the distance function on  $Z_{i_0} \subset Z$  as above.

**Lemma 4.1.** *Endowed with this topology,  $G$  is a compact space.*

*Proof.* It has to be shown that any sequence  $g_n$  of elements of  $G$  has a convergent subsequence. By equicontinuity, the sequence  $g_n$  may be assumed to be consist of elements of  $\mathcal{H}$ . By Proposition 2.2 and the definition of  $G$ , each  $g_n$  can be extended to a homeomorphism whose domain contains  $V$ . According to Theorem 2.5, the sequence  $g_n$  converges uniformly on  $D$  to a map  $g \in \bar{\mathcal{H}}$ . It needs to be shown that  $g : D \rightarrow g(D)$  is a homeomorphism and that it satisfies  $g(D') \cap D' \neq \emptyset$ .

To verify this last condition, note that, for each  $n$ , there exists  $x_n \in D'$  such that  $g_n(x_n) \in D'$ , by the definition of  $G$ . Since  $D'$  is compact, it may be assumed (after passing to a subsequence if needed) that  $x_n \rightarrow x \in D'$ , which implies that  $g_n(x_n) \rightarrow g(x) \in D'$  since  $g_n \rightarrow g$  uniformly on  $D$ . Thus,  $g(D') \cap D' \neq \emptyset$ .

To verify that  $g : D \rightarrow g(D)$  is a homeomorphism, we argue by contradiction and thus assume that there are points  $x, y \in D$  with  $d_{i_0}(x, y) > 0$  and  $g(x) = g(y) = z$ . The map  $g$  is a homeomorphism around each point of  $D$ , as Theorem 2.5 shows. Thus there are disjoint neighborhoods  $O_x$  and  $O_y$  of  $x$  and  $y$ , respectively, such that  $g$  maps each of them homeomorphically onto a neighborhood  $W$  of  $z$ .



Because each  $g_n : D \rightarrow g_n(D) \subset V$  is a homeomorphism, [Proposition 2.2](#) implies that there are maps  $h_n \in \mathcal{H}$  defined on  $V$  such that  $h_n \circ g_n = \text{id}$  on  $g_n(D)$ . Since the sequences  $g_n(x)$  and  $g_n(y)$  both converge to  $z$ , it may be assumed, after passing to a subsequence if needed, that they are contained in  $W$ . Furthermore, perhaps after further shrinking  $W$ , the restrictions  $h_n|_W$  form an equicontinuous family, and so  $h_n$  converges uniformly on  $W$  to a map  $h$  that inverts  $g$  on  $W$ , and whose image contains both  $x$  and  $y$ , a contradiction.  $\square$

**Lemma 4.2** (compare [[Kellum 1993](#)]). *The space  $G$ , endowed with the compact-open topology and the operations just described, is a locally compact local group.*

*Proof.* Let  $g_1$  and  $g_2$  be two elements of  $G$ . Then the composition  $g_1 \circ g_2$  is defined on  $D'$  because  $g_1(D') \subset D$ . Therefore there exists  $h \in \overline{\mathcal{H}}_D$  that extends  $g_1 \circ g_2$ . By quasianalyticity of  $\overline{\mathcal{H}}$ , this extension is unique and thus it defines a map  $(g_1, g_2) \mapsto g_1 \cdot g_2$  from  $G \times G$  into  $\mathcal{H}$ . If  $g_1$  and  $g_2$  are sufficiently close to the identity of  $D$  in the compact open topology of  $C(D, Z)$ , then also  $g_1 \cdot g_2 \in G$ .

The existence of a unique identity element  $e$  for  $G$  and the existence of an inverse operation on  $G$  are proved in a similar fashion.

Finally, it follows easily from [Corollary 2.7](#) and the quasianalyticity of  $\overline{\mathcal{H}}$  that the local group multiplication and inverse map are continuous operations with respect to the compact open topology on  $G$ .  $\square$

**Remark.** By [Theorem 2.9](#), we can assume that all elements of  $G$  are isometries with respect to  $d_{i_0}$ . Then it easily follows that the above distance  $d$  on  $G$  is left invariant.

The proof of the following lemma is straightforward; see [[Kellum 1993](#)].

**Lemma 4.3.** *The map  $G \times D' \rightarrow D$  defined by  $(g, x) \mapsto g(x)$  makes  $G$  into a local group of transformations on  $D' \subset D$ .*

Let  $\Gamma = \mathcal{H} \cap G$ , which is a finitely generated dense sublocal group of  $G$ . The following is a direct consequence of the minimality of  $\mathcal{H}$ .

**Lemma 4.4.**  *$\mathcal{H}$  is equivalent to the pseudogroup generated by the local action of  $\Gamma$  on any nonempty open subset of  $D' \subset Z$ .*

Let  $x_0$  be a point in the interior of  $D'$ , which will remain fixed from now on. Note that, by construction, all elements of  $G$  are defined at  $x_0$ . Let  $\phi : G \rightarrow D$  be the orbit map given by  $\phi(g) = g(x_0)$ . This map is continuous because the action is continuous.

**Lemma 4.5.** *The image of the orbit map  $\phi$  contains a neighborhood of  $x_0$ .*

*Proof.*  $\mathcal{H}$  is minimal by [Theorem 2.4](#), and therefore the space  $Z$  is locally homogeneous with respect to the pseudogroup  $\overline{\mathcal{H}}$  by [Theorem 2.5](#). More precisely, [Proposition 2.2](#) and [Theorem 2.5](#) show that, given  $x \in D'$ , there exists  $h \in \overline{\mathcal{H}}$

with domain  $\text{dom } h = D$  such that  $h(x_0) = x$ . Since both  $x, x_0 \in D'$ , it follows that  $h \in G$ . The statement follows immediately from this.  $\square$

Let  $G_0$  denote the collection of elements  $g \in G$  such that  $g(x_0) = x_0$ .

**Lemma 4.6.** *The set  $G_0$  is a compact subgroup of  $G$ .*

*Proof.* First,  $G_0$  is compact because, being the stabilizer of a point, it is a closed subset of  $G$  and  $G$  is a compact Hausdorff space.

Second, it follows from the definitions of  $G$  and of its group multiplication that the product of two elements of  $G_0$  is defined and belongs to  $G_0$ , and likewise the inverse of every element. More precisely, if  $g_1, g_2 \in G_0$ , then  $g_1 \circ g_2$  is an element of  $\mathcal{H}$  that fixes  $x_0$ ; hence  $g_1 \circ g_2(D') \cap D' \neq \emptyset$ .  $\square$

In the special case just considered of the group  $G_0$  that stabilizes  $x_0$ , the equivalence relation  $\sim$  on  $G$  used to define a representative coset space of  $G_0$  can also be defined as  $h \sim g$  if and only if  $h(x_0) = g(x_0)$ . Therefore:

**Lemma 4.7.** *The orbit map  $\phi : G \rightarrow Z$  induces a map  $\psi : G/G_0 \rightarrow Z$  that is a homeomorphism of a neighborhood of the identity class in  $G/G_0$  onto a neighborhood of  $x_0$  in  $Z$ .*

**Corollary 4.8.**  *$\mathcal{H}$  is equivalent to the pseudogroup induced by the canonical local action of some neighborhood of the identity in  $\Gamma$  on some neighborhood of the identity class in  $G/G_0$ .*

*Proof.* This follows from Lemmas 4.4 and 4.7.  $\square$

**Corollary 4.9.**  *$G/G_0$  is finite-dimensional.*

*Proof.* This follows directly from Lemma 4.7 and the finite dimensionality of  $Z$ .  $\square$

**Lemma 4.10.** *The group  $G_0$  contains no nontrivial normal sublocal group of  $G$ .*

*Proof.* If  $N \subset G$  is a normal sublocal group contained in  $G_0$  and  $n \in N$ , then for each  $g$  in a suitable neighborhood of  $e$  in  $G$  there is some  $n' \in N$  such that  $n\phi(g) = n g(x_0) = g n'(x_0) = g(x_0) = \phi(g)$ . Thus  $n$  acts trivially on a neighborhood of  $x_0$  in  $D'$ . This is possible only if  $n = e$ , because  $\overline{\mathcal{H}}$  is quasianalytic.  $\square$

**Lemma 4.11.** *The local group  $G$  is finite-dimensional.*

*Proof.* By Corollary 4.9,  $G/G_0$  is finite-dimensional. By Theorem 1.7, there exists a compact normal subgroup  $(K, U)$  in  $\Delta G$  such that  $K \subset G_0$  and  $G/(K, U)$  is finite-dimensional. Lemma 4.10 implies that  $K$  is trivial; thus  $G$  is finite-dimensional because it is locally isomorphic to  $G/K$ .  $\square$

Finally, since  $G_0$  is a compact subgroup of  $G$ , the following finishes the proof of Theorem 3.3 according to Example 3.2 and Corollary 4.8.

**Lemma 4.12.** *The group  $G$  is a local Lie group.*

*Proof.* This is a local version of [Pontriaguin 1978, Theorem 73]. By Theorem 1.3, it is enough to show that  $G$  has no small subgroups. The local group  $G$  is finite-dimensional and metrizable, so Theorem 1.5 implies that there is a neighborhood  $U$  of  $e$  in  $G$  that decomposes as the direct product of a local Lie group  $L$  and a compact zero-dimensional normal subgroup  $N$ . Then  $P = N \cap G_0$  is a normal subgroup of  $G_0$ , and  $G_0/P$  is a Lie group because it is a group that is locally isomorphic to the local Lie group  $G/(N, U)$ ; see [Jacoby 1957, Theorem 36].

Furthermore, since  $N$  is zero-dimensional,  $P$  is also zero-dimensional and so there exists a neighborhood  $V$  of  $e$  in  $G_0$  that is the direct product of a connected local Lie group  $M$  and the normal subgroup  $P$ . It may be assumed that  $V \subset U$ . Since  $M$  is connected and  $N$  is zero-dimensional, it follows that  $M \subset L$ .

In summary, there is a local isomorphism between  $G$  and the direct product  $L \times N$ , which restricts to a local isomorphism of  $G_0$  to  $M \times P$ . Therefore, there exists a neighborhood of the class  $G_0$  in  $G/G_0$  that is homeomorphic to a neighborhood of the class of the identity in the product  $L/M \times N/P$ . It follows that a neighborhood of  $x_0$  in  $Z$  is homeomorphic to the product of a Euclidean ball and an open subspace  $T \subset N/P$ . Since  $Z$  is by assumption locally connected and  $N/P$  is zero-dimensional, it follows that  $T$  is finite, and hence that  $N/P$  is a discrete space. So  $P$  is an open subset of  $N$  and thus there exists a neighborhood  $W$  of  $e$  in  $G$  such that  $W \cap P = W \cap N$ . By the local approximation of Jacoby (Theorem 1.4), there exists a compact normal subgroup  $K \subset W$  such that  $G/K$  is a local Lie group. Then  $G_0$  contains  $P \cap K$ , which is equal to the normal subgroup  $N \cap K$  of  $G$  because  $K \subset W$ . Thus, by Lemma 4.10,  $N \cap K$  is trivial. On the other hand,  $N/(N \cap K)$  is a zero-dimensional Lie group; hence  $N \cap K$  is open in  $N$ . It follows that  $N$  is finite, and thus that  $G$  is a local Lie group.  $\square$

## 5. A description of transitive, compactly generated, strongly equicontinuous and quasieffective pseudogroups

The following example is slightly more general than Example 3.2.

**Example 5.1.** Let  $G$  be a locally compact, metrizable and separable local group,  $G_0 \subset G$  a compact subgroup, and  $\Gamma \subset G$  a dense sublocal group. Suppose that there is a left invariant metric on  $G$  inducing its topology. This metric can be assumed to be also  $G_0$ -right invariant by the compactness of  $G_0$ . Then the canonical local action of  $\Gamma$  on some neighborhood of the identity class in  $G/G_0$  induces a transitive strongly equicontinuous and quasieffective pseudogroup of local transformations of a locally compact Polish space. In fact, this is a pseudogroup of local isometries in the sense of [AC 2009].

The proof of the following is a straightforward adaptation of the first part of the proof of Theorem 3.3, using quasieffectiveness instead of quasianalyticity.

**Theorem 5.2.** *Suppose  $\mathcal{H}$  is a transitive, compactly generated, strongly equicontinuous pseudogroup of local transformations of a locally compact Polish space, and suppose that  $\overline{\mathcal{H}}$  is quasieffective. Then  $\mathcal{H}$  is equivalent to a pseudogroup of the type described in [Example 5.1](#).*

The study of compact generation for the pseudogroups of [Example 5.1](#) is very delicate [[Meigniez 1992](#)]. Since those pseudogroups are obviously complete, one could try to replace compact generation by completeness in [Theorem 5.2](#). This would seem to require the generalization of our work [[AC 2009](#)] to complete strongly equicontinuous pseudogroups.

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### References

- [AC 2009] J. A. Álvarez López and A. Candel, “Equicontinuous foliated spaces”, *Math. Z.* **263**:4 (2009), 725–774. [MR 2010i:53040](#) [Zbl 1177.53026](#)
- [Blumenthal and Menger 1970] L. M. Blumenthal and K. Menger, *Studies in geometry*, W. H. Freeman, San Francisco, CA, 1970. [MR 42 #8370](#) [Zbl 0204.53401](#)
- [Candel and Conlon 2000] A. Candel and L. Conlon, *Foliations, I*, Graduate Studies in Mathematics **23**, American Mathematical Society, Providence, RI, 2000. [MR 2002f:57058](#) [Zbl 0936.57001](#)
- [Ellis 1958] R. Ellis, “Distal transformation groups”, *Pacific J. Math.* **8** (1958), 401–405. [MR 21 #96](#)
- [Ellis 1969] R. Ellis, *Lectures on topological dynamics*, W. A. Benjamin, New York, 1969. [MR 42 #2463](#) [Zbl 0193.51502](#)
- [Ellis 1978] R. Ellis, “The Furstenberg structure theorem”, *Pacific J. Math.* **76**:2 (1978), 345–349. [MR 58 #7574](#) [Zbl 0359.54035](#)
- [Furstenberg 1963] H. Furstenberg, “The structure of distal flows”, *Amer. J. Math.* **85** (1963), 477–515. [MR 28 #602](#) [Zbl 0199.27202](#)
- [Goldbring 2010] I. Goldbring, “Hilbert’s fifth problem for local groups”, *Annals of Math.* **172** (2010), 1269–1314.
- [Haefliger 1985] A. Haefliger, “Pseudogroups of local isometries”, pp. 174–197 in *Differential geometry* (Santiago de Compostela, 1984), edited by L. A. Cordero, Res. Notes in Math. **131**, Pitman, Boston, MA, 1985. [MR 88i:58174](#) [Zbl 0656.58042](#)
- [Haefliger 1988] A. Haefliger, “Leaf closures in Riemannian foliations”, pp. 3–32 in *A fête of topology*, edited by T. Mizutani and S. Morita, Academic Press, Boston, MA, 1988. [MR 89c:57033](#) [Zbl 0667.57012](#)
- [Haefliger 1989] A. Haefliger, “Feuilletages riemanniens”, pp. 183–197 in *Séminaire Bourbaki*, 1988/89, Astérisque **177-178**, Soc. Math. de France, Paris, 1989. [MR 91e:57047](#) [Zbl 0703.57012](#)
- [Hector and Hirsch 1981] G. Hector and U. Hirsch, *Introduction to the geometry of foliations, A: Foliations on compact surfaces, fundamentals for arbitrary codimension, and holonomy*, Aspects of Mathematics **1**, Friedr. Vieweg, Braunschweig, 1981. [MR 83d:57019](#) [Zbl 0486.57002](#)

- [Hurewicz and Wallman 1941] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Mathematical Series **4**, Princeton University Press, 1941. [MR 3,312b](#) [Zbl 0060.39808](#)
- [Jacoby 1957] R. Jacoby, “Some theorems on the structure of locally compact local groups”, *Ann. of Math.* (2) **66** (1957), 36–69. [MR 19,751c](#) [Zbl 0084.03202](#)
- [Kellum 1993] M. Kellum, “Uniformly quasi-isometric foliations”, *Ergodic Theory Dynam. Systems* **13**:1 (1993), 101–122. [MR 94c:57048](#) [Zbl 0784.58060](#)
- [Kellum 1994] M. Kellum, “Uniform Lipschitz distortion, invariant measures and foliations”, pp. 313–326 in *Geometric study of foliations* (Tokyo, 1993), edited by T. Mizutani et al., World Sci. Publ., River Edge, NJ, 1994. [MR 96k:58175](#)
- [Meigniez 1992] G. Meigniez, “Sous-groupes de génération compacte des groupes de Lie résolubles”, preprint number 33, Mathématiques de l’Université Paris VII, 1992.
- [Molino 1988] P. Molino, *Riemannian foliations*, Progress in Mathematics **73**, Birkhäuser, Boston, MA, 1988. [MR 89b:53054](#) [Zbl 0633.53001](#)
- [Montgomery and Zippin 1955] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience Publishers, New York, 1955. [MR 17,383b](#) [Zbl 0068.01904](#)
- [Munkres 1975] J. R. Munkres, *Topology: a first course*, Prentice-Hall, Englewood Cliffs, N.J., 1975. [MR 57 #4063](#) [Zbl 0306.54001](#)
- [Pontriaguin 1978] L. S. Pontriaguin, *Continuous groups*, Editorial Mir, Moscow, 1978. [MR 80c:22001](#)
- [Sacksteder 1965] R. Sacksteder, “Foliations and pseudogroups”, *Amer. J. Math.* **87** (1965), 79–102. [MR 30 #4268](#) [Zbl 0136.20903](#)
- [Shalen 1987] P. B. Shalen, “Dendrology of groups: an introduction”, pp. 265–319 in *Essays in group theory*, edited by S. M. Gersten, Math. Sci. Res. Inst. Publ. **8**, Springer, New York, 1987. [MR 89d:57012](#) [Zbl 0649.20033](#)
- [Tarquini 2004] C. Tarquini, “Feuilletages conformes”, *Ann. Inst. Fourier (Grenoble)* **54**:2 (2004), 453–480. [MR 2005f:53040](#) [Zbl 1064.53014](#)

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# PACIFIC JOURNAL OF MATHEMATICS

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Topological description of Riemannian foliations with dense leaves	257
JESÚS A. ÁLVAREZ LÓPEZ and ALBERTO CANDEL	
The nonexistence of quasi-Einstein metrics	277
JEFFREY S. CASE	
Twisted symmetric group actions	285
AKINARI HOSHI and MING-CHANG KANG	
Optimal transportation and monotonic quantities on evolving manifolds	305
HONG HUANG	
Hopf structures on the Hopf quiver $Q((g), g)$	317
HUA-LIN HUANG, YU YE and QING ZHAO	
Minimal surfaces in $S^3$ foliated by circles	335
NIKOLAI KUTEV and VELICHKA MILOUSHEVA	
Prealternative algebras and prealternative bialgebras	355
XIANG NI and CHENGMING BAI	
Some remarks about closed convex curves	393
KE OU and SHENGLIANG PAN	
Orbit correspondences for real reductive dual pairs	403
SHU-YEN PAN	
Graphs of bounded degree and the $p$ -harmonic boundary	429
MICHAEL J. PULS	
Invariance of the BFV complex	453
FLORIAN SCHÄTZ	
Some elliptic PDEs on Riemannian manifolds with boundary	475
YANNICK SIRE and ENRICO VALDINOCI	
Representations of Lie superalgebras in prime characteristic, III	493
LEI ZHAO	