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We adapt Topping's $\mathscr{L}$-optimal transportation theory for Ricci flow to a more general situation, in which a complete manifold $(M, g_{ij}(t))$ evolves by $\partial_t g_{ij} = -2S_{ij}$, where S_{ij} is a symmetric 2-tensor field on M. We extend some recent results of Topping, Lott and Brendle, generalize the monotonicity of the $\mathscr{W}$-entropy of List (and hence also of Perelman), and recover the monotonicity of the reduced volume of Müller (and hence also of Perelman).

1. Introduction

Since Monge introduced the optimal transportation problem, much beautiful work has been done, especially in the last several decades. For an extensive discussion, see [Villani 2009]. Recently, Topping, Lott, Brendle and others considered this problem on a manifold evolving by Hamilton's Ricci flow. Topping [2009] introduced $\mathscr{L}$-optimal transportation for Ricci flow. He studied the behavior of Boltzmann–Shannon entropy along $\mathscr{L}$-Wasserstein geodesics, and obtained a natural monotonic quantity from which the monotonicity of Perelman's $\mathscr{W}$-entropy was recovered, among other things. Using Topping's work, both Lott [2009] and Brendle [2009] were able to prove again the monotonicity of Perelman's reduced volume. Lott did so by showing the convexity of a certain entropy-like function, while Brendle proved a Prékopa–Leindler-type inequality for Ricci flow.

List [2008] considered an extended Ricci flow in his thesis, and generalized the monotonicity of Perelman's $\mathscr{W}$-entropy to his flow. Müller [2010] studied more general evolving closed manifolds $(M, g_{ij}(t))$ with metrics $g_{ij}(t)$ satisfying the equation

$$\frac{\partial g_{ij}}{\partial t} = -2S_{ij}, \tag{1-1}$$

where $\mathscr{S} = (S_{ij})$ is a symmetric 2-tensor field on M. He generalized the monotonicity of Perelman's reduced volume to this flow satisfying a certain constraint condition, which will be stated later; see [Müller 2010, Theorem 1.4].

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Here we adapt Topping's $\mathcal{L}$-optimal transportation theory for Ricci flow to the general flow (1-1). We obtain some analogs of results of Topping, Lott and Brendle mentioned above, and using this we can generalize the monotonicity of List's (and hence also of Perelman's) $\mathcal{W}$-entropy, and recover the monotonicity of Müller's (and hence also of Perelman's) reduced volume.

Now we consider the flow (1-1) backwards in time on a complete manifold. Let τ be some parameter increasing backward in time, that is, $\tau = C - t$ for some constant $C \in \mathbb{R}$. Consider the reverse flow

$$\frac{\partial g_{ij}}{\partial \tau} = 2S_{ij}(\tau), \tag{1-2}$$

defined on a time interval including $[\tau_1, \tau_2]$, with $0 \le \tau_1 < \tau_2$. Following Perelman [2002] and Müller [2010], we define the $\mathcal{L}$-length of a curve $\gamma : [\tau_1, \tau_2] \to M$ by

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau}(S(\gamma(\tau), \tau) + |\gamma'(\tau)|^2_{g(\tau)}) d\tau,$$

where S is the trace of $\mathcal{S}$ (with respect to $g(\tau)$). Then we define the $\mathcal{L}$-distance by

$$\begin{aligned} Q(x, \tau_1; y, \tau_2) \\ := \inf\{\mathcal{L}(\gamma) \mid \gamma : [\tau_1, \tau_2] \to M \text{ is smooth and } \gamma(\tau_1) = x,\ \gamma(\tau_2) = y\}. \end{aligned} \tag{1-3}$$

Given two Borel probability measures ν_1 and ν_2 viewed at times τ_1 and τ_2, respectively, following [Topping 2009] we define the $\mathcal{L}$-Wasserstein distance by

$$V(\nu_1, \tau_1; \nu_2, \tau_2) := \inf\left\{ \int_{M\times M} Q(x, \tau_1; y, \tau_2) d\pi(x, y) \;\middle|\; \pi \in \Gamma(\nu_1, \nu_2) \right\}, \tag{1-4}$$

where $\Gamma(\nu_1, \nu_2)$ is the space of Borel probability measures π on $M \times M$ with marginals ν_1 and ν_2, that is, $\pi(\Omega \times M) = \nu_1(\Omega)$ and $\pi(M \times \Omega) = \nu_2(\Omega)$ for Borel subsets Ω in M.

To state our theorems we need to introduce a quantity from [Müller 2010]. Let $g(\tau)$ evolve by (1-2), and let $X \in \Gamma(TM)$ be a vector field on M. Following Müller, we set

$$\begin{aligned} \mathcal{D}(\mathcal{S}, X) := -\partial_\tau S - \Delta S - 2|\mathcal{S}|^2 \\ - 4\delta\mathcal{S}(X) - 2X(S) + 2\operatorname{Ric}(X, X) - 2\mathcal{S}(X, X), \end{aligned} \tag{1-5}$$

where $\delta\mathcal{S} := -\operatorname{tr}_{12} \nabla\mathcal{S}$. (Here, tr_{12} means to trace over the first and second entries.)

Our first result generalizes [Topping 2009, Theorem 1.1] and a result of von Renesse and Sturm [2005]. Following Topping, we refer to a family of smooth probability measures $\nu(\tau)$ on M as a diffusion if the density $u(\tau)$ relative to the Riemannian volume measure $\mu(\tau)$ of $g(\tau)$ (that is, the density with $d\nu(\tau) = u(\tau)d\mu(\tau)$)

satisfies the equation

$$\frac{\partial u}{\partial \tau} = \triangle u - Su. \tag{1-6}$$

Theorem 1.1. *Given* $0 < \bar{\tau}_1 < \bar{\tau}_2$, *suppose* $(M, g(\tau))$ *is a closed*, n-*dimensional manifold evolving by* (1-2) *for* τ *in some open interval containing* $[\bar{\tau}_1, \bar{\tau}_2]$, *such that the quantity* $\mathfrak{D}(\mathcal{S}, X)$ *is nonnegative for all vector fields* $X \in \Gamma(TM)$ *and all times for which the flow exists. Let* $\nu_1(\tau)$ *and* $\nu_2(\tau)$ *be two diffusions* (*as defined above*) *for* τ *in some neighborhoods of* $\bar{\tau}_1$ *and* $\bar{\tau}_2$, *respectively. Set* $\tau_1 = \tau_1(s) := \bar{\tau}_1 e^s$ *and* $\tau_2 = \tau_2(s) := \bar{\tau}_2 e^s$, *and define the renormalized* $\mathcal{L}$-*Wasserstein distance by*

$$\Theta(s) := 2(\sqrt{\tau_2} - \sqrt{\tau_1}) V(\nu_1(\tau_1), \tau_1; \nu_2(\tau_2), \tau_2) - 2n(\sqrt{\tau_2} - \sqrt{\tau_1})^2$$

for s *in a neighborhood of* 0 *such that* $\nu_i(\tau_i(s))$ *are defined for* $i = 1, 2$.

Then $\Theta(s)$ *is a weakly decreasing function of* s.

The constraint condition on $\mathfrak{D}(\mathcal{S}, X)$ in Theorem 1.1 is the same as that appeared in [Müller 2010, Theorem 1.4]. Müller pointed out that it is satisfied, for example, by the static manifolds with nonnegative Ricci curvature, by Hamilton's Ricci flow, by List's flow [2008], by the Ricci flow coupled with harmonic map heat flow introduced in Müller's thesis [2010], and by mean curvature flow in an ambient Lorentzian manifold with nonnegative sectional curvature.

Our second result generalizes [Lott 2009, Theorem 1].

Theorem 1.2. *Given* $0 < \tau_1 < \tau_2$, *suppose that* $(M, g(\tau))$ *is a connected closed*, n-*dimensional manifold evolving by* (1-2) *for* τ *in some open interval including* $[\tau_1, \tau_2]$, *such that the quantity* $\mathfrak{D}(\mathcal{S}, X)$ *is nonnegative for all vector fields* X *in* $\Gamma(TM)$ *and all times for which the flow exists. Let* $\mathcal{V}_\tau (\tau \in [\tau_1, \tau_2])$ *be an* $\mathcal{L}$-*Wasserstein geodesic induced by a reflexive function* $\varphi : M \to \mathbb{R}$, *with* $\mathcal{V}_{\tau_1}$ *and* $\mathcal{V}_{\tau_2}$ *both absolutely continuous probability measures. Set*

$$\phi(y, \tau) := \frac{1}{2\sqrt{\tau}} \inf_{x \in M} (Q(x, \tau_1; y, \tau) - \varphi(x)) \tag{1-7}$$

for $y \in M$ *and* $\tau \in [\tau_1, \tau_2]$. *Then* $E(\mathcal{V}_\tau) + \int_M \phi(\,\cdot\,, \tau) d\mathcal{V}_\tau + (n/2) \ln \tau$ *is convex in the variable* $\tau^{-1/2}$.

For the definition of $\mathcal{L}$-Wasserstein geodesic, see the paragraph after [Topping 2009, Theorem 2.14] and also the paragraph following our Theorem 2.1. Here $E(\mathcal{V}_\tau)$ is the Boltzmann–Shannon entropy of $\mathcal{V}_\tau$ (recalled in Section 2).

Our third theorem generalizes [Brendle 2009, Theorem 2] and also a result in [Cordero-Erausquin et al. 2001]. Note that we do not assume that M is compact in this theorem.

Theorem 1.3. *Given* $0 < \tau_1 < \tau_2$, *suppose that* $(M, g(\tau))$ *is a complete manifold evolving by* (1-2) *for* τ *in some open interval including* $[\tau_1, \tau_2]$, *with* S_{ij} *uniformly* *bounded in compact time intervals, and such that the quantity* $\mathfrak{D}(\mathcal{S}, X)$ *is non-negative for all vector fields* $X \in \Gamma(TM)$ *and all times for which the flow exists. Fix* $\bar{\tau} \in (\tau_1, \tau_2)$, *and write*

$$\frac{1}{\sqrt{\bar{\tau}}} = \frac{1-\lambda}{\sqrt{\tau_1}} + \frac{\lambda}{\sqrt{\tau_2}},$$

for some $0 < \lambda < 1$. *Let* $u_1, u_2, v : M \to \mathbb{R}$ *be nonnegative measurable functions such that*

$$\Big(\frac{\bar{\tau}}{\tau_1^{1-\lambda}\tau_2^{\lambda}}\Big)^{n/2} v(\gamma(\bar{\tau})) \geq \exp\Big(-\frac{1-\lambda}{2\sqrt{\tau_1}} Q(\gamma(\tau_1), \tau_1; \gamma(\bar{\tau}), \bar{\tau})\Big) u_1(\gamma(\tau_1))^{1-\lambda} \\ \times \exp\Big(\frac{\lambda}{2\sqrt{\tau_2}} Q(\gamma(\bar{\tau}), \bar{\tau}; \gamma(\tau_2), \tau_2)\Big) u_2(\gamma(\tau_2))^{\lambda}$$

for each minimizing $\mathcal{L}$*-geodesic* $\gamma : [\tau_1, \tau_2] \to M$. *Then*

$$\int_M v \, d\mu(\bar{\tau}) \geq \Big(\int_M u_1 \, d\mu(\tau_1)\Big)^{1-\lambda} \Big(\int_M u_2 \, d\mu(\tau_2)\Big)^{\lambda}.$$

The proofs of our theorems, given in Section 2, rely heavily on [Topping 2009]. In Section 3 we give some applications, (following Topping and Brendle).

2. Proofs of theorems

Part of Topping's $\mathcal{L}$-optimal transportation theory for Ricci flow extends to the general flow (1-1) without any change. In particular, virtually all theorems in [Topping 2009, Section 2] hold in our more general situation.

Consider $(M, g(\tau))$ satisfying (1-2) on an open time interval including some interval $[\tau_1, \tau_2]$ with $0 < \tau_1 < \tau_2$. Recall [Perelman 2002; Müller 2010] that a path $\gamma : [\tau_1, \tau_2] \to M$ is an $\mathcal{L}$-geodesic if the first variation of the $\mathcal{L}$-length at γ is zero. For $x \in M$ and $Z \in T_x M$, let $\gamma : [\tau_1, \tau_2] \to M$ be the (unique) $\mathcal{L}$-geodesic with $\gamma(\tau_1) = x$ and $\sqrt{\tau_1}\gamma'(\tau_1) = Z$. We define

$$\mathcal{L}_{\tau_1,\tau_2} \exp_x(Z) = \gamma(\tau_2).$$

Theorem 2.1 (see [Topping 2009, Section 2, in particular Theorem 2.14]). *Given* $0 < \tau_1 < \tau_2$, *suppose that* $(M, g(\tau))$ *is a closed manifold evolving by* (1-2) *for* τ *in some open interval including* $[\tau_1, \tau_2]$. *Suppose that* ν_1 *and* ν_2 *are absolutely continuous probability measures, with respect to (any) volume measure. Then there exists an optimal transference plan* π *in* (1-4) *that is given by the push-forward of* ν_1 *under the map* $x \mapsto (x, F(x))$, *where* $F : M \to M$ *is a Borel map defined by*

$$F(x) := \mathcal{L}_{\tau_1,\tau_2} \exp_x(-\tfrac{1}{2}\nabla\varphi(x)), \tag{2-1}$$

at points of differentiability of some reflexive function $\varphi : M \to R$, *where the gradient is with respect to* $g(\tau_1)$.

There exists a Borel set $K \subset M$ with $\nu_1(K) = 1$ such that for each $x \in K$, φ admits a Hessian at x and

$$f_{\tau_1}(x) = f_{\tau_2}(F(x)) \det(dF)_x \neq 0, \tag{2-2}$$

where f_{τ_i} is the density defined by $d\nu_i = f_{\tau_i} d\mu(\tau_i)$ for $i = 1, 2$.

For definition of reflexive function, see [Topping 2009, Definition 2.1], and for push-forward measure, see for example [Villani 2009, Conventions, page 11].

As in [Topping 2009], we refer to $\mathcal{V}_\tau := (F_\tau)_\sharp(\nu_1)$ as an $\mathcal{L}$-Wasserstein geodesic, where $F_\tau : M \to M$ is a Borel map defined by

$$F_\tau(x) := \mathcal{L}_{\tau_1,\tau} \exp_x(-\tfrac{1}{2}\nabla\varphi(x))$$

at points of differentiability of φ (as in the theorem above) for $\tau \in [\tau_1, \tau_2]$.

Remark 2.2. Theorem 2.1 extends to the case of noncompact M with suitable modifications: One imposes in addition that S_{ij} is uniformly bounded (in compact time intervals) and that $V(\nu_1, \tau_1; \nu_2, \tau_2)$ is finite. Then the results in Theorem 2.1 still hold, with the gradient in (2-1), the differential in (2-2), and the Hessian replaced by approximate versions. (Of course, φ need not be reflexive any more.) For more details, see [Fathi and Figalli 2010; Figalli 2007; Villani 2009]. Moreover, in the noncompact case, one can still say something even if one does not impose the finiteness condition on $V(\nu_1, \tau_1; \nu_2, \tau_2)$; see [Figalli 2007; Villani 2009].

Note that Müller [2010] has established some properties of $\mathcal{L}$-geodesics and $\mathcal{L}$-distance in our situation.

As in [Müller 2010], (following R. Hamilton) we introduce

$$\mathcal{H}(\mathcal{S}, X) := -\partial_\tau S - (1/\tau)S - 2X(S) + 2\mathcal{S}(X, X) \quad \text{for } X \in \Gamma(TM). \tag{2-3}$$

The following key lemma generalizes [Topping 2009, Lemma 3.1].

Lemma 2.3. *Let $\gamma : [\tau_1, \tau_2] \to M$ be an $\mathcal{L}$-geodesic, and $\{Y_i(\tau)\}_{i=1,\dots,n}$ be a set of $\mathcal{L}$-Jacobi fields along γ that form a basis of $T_{\gamma(\tau)}M$ for each $\tau \in [\tau_1, \tau_2]$, with $\{Y_i(\tau_1)\}$ orthonormal and $\langle D_\tau Y_i, Y_j\rangle$ symmetric in i and j at $\tau = \tau_1$. Define $\alpha : [\tau_1, \tau_2] \to \mathbb{R}$ by $\alpha(\tau) = -\frac{1}{2} \ln\det\langle Y_i(\tau), Y_j(\tau)\rangle_{g(\tau)}$, and write $\sigma = \sqrt{\tau}$. Then*

$$\frac{d^2\alpha}{d\sigma^2} = 4\sqrt{\tau}\frac{d}{d\tau}\Big(\sqrt{\tau}\frac{d\alpha}{d\tau}\Big) \geq 2\tau(\mathcal{H}(\mathcal{S}, X) + \mathcal{D}(\mathcal{S}, X)),$$
$$\frac{d^2(\sigma\alpha)}{d\sigma^2} = 4\frac{d}{d\tau}\Big(\tau^{3/2}\frac{d\alpha}{d\tau}\Big) \geq 2\tau^{3/2}(\mathcal{H}(\mathcal{S}, X) + \mathcal{D}(\mathcal{S}, X)) - n\tau^{-1/2},$$

where $X = \gamma'(\tau)$.

Proof. The proof follows closely that of [Topping 2009, Lemma 3.1] with some necessary modifications. Recall the $\mathscr{L}$-geodesic equation [Müller 2010]

$$D_\tau X = \tfrac{1}{2}\nabla S - 2\widetilde{\mathcal{S}}(X) - \tfrac{1}{2}\tau^{-1}X,$$

where $\widetilde{\mathcal{S}}$ is $\mathcal{S}$ viewed as an endomorphism (using $g(\tau)$), which is defined by $\langle\widetilde{\mathcal{S}}(Z), W\rangle = \mathcal{S}(Z, W)$, and $D_\tau = D_\tau^{g(\tau)}$ represents the pull-back of the Levi-Civita connection on $(M, g(\tau))$ under γ, acting in the direction $\partial/\partial\tau$. Then at $\tau = \hat{\tau}$,

$$\begin{aligned} D_\tau^2 Y &:= D_\tau^{g(\tau)}(D_\tau^{g(\tau)} Y) \\ &= D_\tau^{g(\hat{\tau})}(D_\tau^{g(\hat{\tau})} Y) + (\nabla_Y \widetilde{\mathcal{S}})(X) + (\nabla_X \widetilde{\mathcal{S}})(Y) - ((\nabla\mathcal{S})(\,\cdot\,, X, Y))^\sharp, \end{aligned}$$

where $\sharp$ represents the musical isomorphism $\Gamma(\gamma^*(T^*M)) \to \Gamma(\gamma^*(TM))$ via the Riemannian metric and is given by

$$\langle\omega^\sharp, Z\rangle = \omega(Z)$$

for a 1-form $\omega \in \Gamma(\gamma^*(T^*M))$.

Then we can derive the $\mathscr{L}$-Jacobi equation for $Y(\tau)$,

$$\begin{aligned} D_\tau^2 Y = -R(X, Y)X + \tfrac{1}{2}\nabla_Y(\nabla S) - \nabla_Y \widetilde{\mathcal{S}}(X) - 2\widetilde{\mathcal{S}}(D_\tau Y) \\ - \tfrac{1}{2}\tau^{-1} D_\tau Y + \nabla_X \widetilde{\mathcal{S}}(Y) - (\nabla\mathcal{S}(\,\cdot\,, X, Y))^\sharp \end{aligned}$$

where $R(X, Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$.

Now consider the solution $e_i \in \Gamma(\gamma^*(TM))$ for $i = 1, \dots, n$ of the ODE

$$D_\tau e_i + \widetilde{\mathcal{S}}(e_i) = 0,$$

with initial condition $e_i(\tau_1) = Y_i(\tau_1)$. Then $\{e_i(\tau)\}$ is an orthonormal frame for all $\tau \in [\tau_1, \tau_2]$. Write $Y_j(\tau) = A_{kj} e_k(\tau)$ for a τ-dependent $n \times n$ matrix A. Then

$$\begin{aligned} A'_{ij} &= \langle D_\tau Y_j, e_i\rangle + A_{kj}\mathcal{S}(e_k, e_i), \\ A''_{ij} &= \langle D_\tau^2 Y_j, e_i\rangle + 2A'_{kj}\mathcal{S}(e_k, e_i) + A_{kj}\langle D_\tau(\widetilde{\mathcal{S}}(e_k)), e_i\rangle. \end{aligned}$$

Using the $\mathscr{L}$-Jacobi equation we get

$$\begin{aligned} &\langle D_\tau^2 Y_j, e_i\rangle \\ &= A_{kj}\big(-\operatorname{Rm}(X, e_k, X, e_i) + \tfrac{1}{2}\operatorname{Hess}(S)(e_i, e_k) + \nabla_X\mathcal{S}(e_i, e_k) - \langle\nabla_{e_k}\widetilde{\mathcal{S}}(X), e_i\rangle \\ &\quad - \langle\nabla_{e_i}\widetilde{\mathcal{S}}(X), e_k\rangle + 2\langle\widetilde{\mathcal{S}}^2(e_k), e_i\rangle + \tfrac{1}{2}\tau^{-1}\mathcal{S}(e_i, e_k)\big) - 2A'_{kj}\mathcal{S}(e_k, e_i) - \tfrac{1}{2}\tau^{-1}A'_{ij}. \end{aligned}$$

We also have

$$\langle D_\tau(\widetilde{\mathcal{S}}(e_k)), e_i\rangle = \frac{\partial\mathcal{S}}{\partial\tau}(e_i, e_k) + \nabla_X\mathcal{S}(e_i, e_k) - 3\langle\widetilde{\mathcal{S}}^2(e_k), e_i\rangle.$$

Then we get

$$A'' + \tfrac{1}{2}\tau^{-1}A' = MA,$$

where M is the τ-dependent $n \times n$ symmetric matrix given by

$$M_{ik} = -\operatorname{Rm}(X, e_k, X, e_i) + \tfrac{1}{2}\operatorname{Hess}(S)(e_i, e_k) + 2\nabla_X \mathcal{S}(e_i, e_k) - \langle \nabla_{e_k} \widetilde{\mathcal{S}}(X), e_i\rangle \\ - \langle \nabla_{e_i} \widetilde{\mathcal{S}}(X), e_k\rangle - \langle \widetilde{\mathcal{S}}^2(e_k), e_i\rangle + \tfrac{1}{2}\tau^{-1}\mathcal{S}(e_i, e_k) + \frac{\partial \mathcal{S}}{\partial \tau}(e_i, e_k).$$

Using $\operatorname{tr} \nabla_X \mathcal{S} = \nabla_X \operatorname{tr} \mathcal{S}$ and the definition of the operator δ (recalled in Section 1), we compute

$$\operatorname{tr} M = -\operatorname{Ric}(X, X) + \tfrac{1}{2}\triangle S + 2X(S) + 2\delta\mathcal{S}(X) - |\mathcal{S}|^2 + \frac{S}{2\tau} + \operatorname{tr}\frac{\partial \mathcal{S}}{\partial \tau}.$$

Using (1-5), (2-3) and the equation $\operatorname{tr} \partial\mathcal{S}/\partial\tau = \partial S/\partial\tau + 2|\mathcal{S}|^2$, we see that

$$\operatorname{tr} M = -\tfrac{1}{2}(\mathcal{H}(\mathcal{S}, X) + \mathcal{D}(\mathcal{S}, X)).$$

Now define $B := (dA/d\tau)A^{-1}$. Then as in [Topping 2009], using $\alpha = -\ln\det A$ and $d\alpha/d\tau = -\operatorname{tr}((dA/d\tau)A^{-1})$, we have

$$\tau^{-1/2}\frac{d}{d\tau}\Big(\sqrt{\tau}\frac{d\alpha}{d\tau}\Big) = \operatorname{tr} B^2 - \operatorname{tr} M = \operatorname{tr} B^2 + \tfrac{1}{2}(\mathcal{H}(\mathcal{S}, X) + \mathcal{D}(\mathcal{S}, X)),$$

$$\tau^{-3/2} d/d\tau\Big(\tau^{3/2}\frac{d\alpha}{d\tau}\Big) = \operatorname{tr}(B - \tfrac{1}{2}\tau^{-1}I)^2 + \tfrac{1}{2}(\mathcal{H}(\mathcal{S}, X) + \mathcal{D}(\mathcal{S}, X)) - \frac{n}{4\tau^2}.$$

Similarly to [Topping 2009], one can show that B is symmetric, and our result follows. □

Now we begin to study the behavior along an $\mathcal{L}$-Wasserstein geodesic of the Boltzmann–Shannon entropy, the entropy defined for a probability measure $f\,d\mu$ by

$$E(f\,d\mu) = \int_M f \ln f\,d\mu,$$

where μ is a Riemannian volume measure and f is a reasonably regular weakly positive function on M. As before we set $\sigma = \sqrt{\tau}$. Then we have the following lemma, which generalizes [Topping 2009, Lemma 3.2].

Lemma 2.4. *Let $(M, g(\tau))$ be as in Theorem 2.1. Let $\mathcal{V}_\tau$ $(\tau \in [\tau_1, \tau_2])$ be an $\mathcal{L}$-Wasserstein geodesic induced by a potential $\varphi : M \to \mathbb{R}$, with $\mathcal{V}_{\tau_1}$ and $\mathcal{V}_{\tau_2}$ both absolutely continuous probability measures, and write $d\mathcal{V}_\tau = f_\tau\,d\mu(\tau)$, where $\mu(\tau)$ is the volume measure of $g(\tau)$. Then for all $\tau \in [\tau_1, \tau_2]$, we have $f_\tau \in L\ln L(\mu(\tau))$, and the function $E(\mathcal{V}_\tau)$ is semiconvex in τ and satisfies, for almost all $\tau \in [\tau_1, \tau_2]$,*

$$\frac{d^2}{d\sigma^2}E(\mathcal{V}_\tau) = 4\sqrt{\tau}\frac{d}{d\tau}\Big(\sqrt{\tau}\frac{dE(\mathcal{V}_\tau)}{d\tau}\Big) \geq 2\tau\int_M (\mathcal{H}(\mathcal{S}, X(\tau)) + \mathcal{D}(\mathcal{S}, X(\tau)))d\mathcal{V}_{\tau_1}$$

and

$$\frac{d^2}{d\sigma^2}(\sigma E(\mathcal{V}_\tau)) = 4\frac{d}{d\tau}\Big(\tau^{3/2}\frac{dE(\mathcal{V}_\tau)}{d\tau}\Big) \geq 2\tau^{3/2}\int_M (\mathcal{H}(\mathcal{S}, X(\tau)) + \mathfrak{D}(\mathcal{S}, X(\tau)))d\mathcal{V}_{\tau_1} - n\tau^{-1/2},$$

where $\sigma \mapsto E(\mathcal{V}_\tau)$ admits a second derivative in the sense of Alexandrov, and where $X(\tau) = \gamma'(\tau)$ at a point $x \in M$ where φ admits a Hessian, for $\gamma : [\tau_1, \tau_2] \to M$ the minimizing $\mathcal{L}$-geodesic from x to $F(x)$. Moreover, the one-sided derivatives of $E(\mathcal{V}_\tau)$ at τ_1 and τ_2 exist, with

$$\frac{d}{d\tau}\Big|_{\tau_1^+} E(\mathcal{V}_\tau) \geq -\int_M \Big(S(\,\cdot\,, \tau_1) + \Big\langle \frac{\nabla\varphi}{2\sqrt{\tau_1}}, \nabla \ln f_{\tau_1}\Big\rangle_{g(\tau_1)}\Big)d\mathcal{V}_{\tau_1}.$$

Proof. Using Theorem 2.1 and Lemma 2.3, one can follow exactly the steps of [Topping 2009]. □

Let $g(\tau)$ be defined on $(\hat\tau_1, \hat\tau_2) \supset [\tau_1, \tau_2]$, where $\hat\tau_1 > 0$. As in [Topping 2009], let $\Upsilon := \{(x, \tau_a; y, \tau_b) \mid x, y \in M, \hat\tau_1 < \tau_a < \tau_b < \hat\tau_2\}$. Suppose $(x, \tau_1; y, \tau_2) \in \Upsilon \setminus \mathcal{L}\mathrm{Cut}$, (for the definition of $\mathcal{L}$Cut, see [Topping 2009, Appendix A]), let $\gamma : [\tau_1, \tau_2] \to M$ be the minimizing $\mathcal{L}$-geodesic from x to y, and write $X(\tau) = \gamma'(\tau)$ as before. Following [Perelman 2002; Topping 2009; Müller 2010], define

$$\mathcal{K} = \mathcal{K}(x, \tau_1, y, \tau_2) := \int_{\tau_1}^{\tau_2} \tau^{\frac{3}{2}}\mathcal{H}(\mathcal{S}, X(\tau))d\tau.$$

Then we have the following result that generalizes [Topping 2009, Corollary 3.3].

Corollary 2.5. *Assume again the hypothesis of Lemma 2.4, and assume further that the quantity $\mathfrak{D}(\mathcal{S}, X)$ is nonnegative for all vector fields $X \in \Gamma(TM)$ and all times for which the flow exists. Then*

$$\int_{M\times M} \big(\mathcal{K} - 2\tau_1^{3/2}S(x, \tau_1) - \tau_1\langle\nabla_1 Q, \nabla \ln f_{\tau_1}(x)\rangle_{g(\tau_1)} + 2\tau_2^{3/2}S(y, \tau_2) - \tau_2\langle\nabla_2 Q, \nabla \ln f_{\tau_2}(y)\rangle_{g(\tau_2)}\big)d\pi(x, y) \leq n(\sqrt{\tau_2} - \sqrt{\tau_1}),$$

where we denote by $\nabla_1 Q$ the gradient of Q with respect to its x argument and with respect to $g(\tau_1)$ and by $\nabla_2 Q$ the gradient of Q with respect to its y argument and with respect to $g(\tau_2)$; also π is the optimal transference plan from $\mathcal{V}_{\tau_1}$ to $\mathcal{V}_{\tau_2}$ (for $\mathcal{L}$-optimal transportation).

The following result generalizes [Topping 2009, Lemma A.6].

Lemma 2.6. *Under the flow (1-2), we have*

$$\tau_2\frac{\partial Q}{\partial \tau_2} + \tau_1\frac{\partial Q}{\partial \tau_1} = 2\tau_2^{3/2}S(y, \tau_2) - 2\tau_1^{3/2}S(x, \tau_1) + \mathcal{K} - \tfrac{1}{2}Q.$$

Proof. Similarly to [Topping 2009, (A.4) and (A.5)], we have (compare with [Müller 2010, Section 5])

$$\frac{\partial Q}{\partial \tau_1}(x,\tau_1;y,\tau_2) = \sqrt{\tau_1}(|X(\tau_1)|^2 - S(x,\tau_1));\ \nabla_1 Q(x,\tau_1;y,\tau_2) = -2\sqrt{\tau_1}X(\tau_1), \tag{2-4}$$

$$\frac{\partial Q}{\partial \tau_2}(x,\tau_1;y,\tau_2) = \sqrt{\tau_2}(S(y,\tau_2)) - |X(\tau_2)|^2);\ \nabla_2 Q(x,\tau_1;y,\tau_2) = 2\sqrt{\tau_2}X(\tau_2). \tag{2-5}$$

Similarly to [Topping 2009, (A.9)], we have (see [Müller 2010, Section 5])

$$\tau_2^{3/2}(S(y,\tau_2)+|X(\tau_2)|^2) - \tau_1^{3/2}(S(x,\tau_1)+|X(\tau_1)|^2) = -\mathscr{K}(x,\tau_1,y,\tau_2) + \tfrac{1}{2}Q(x,\tau_1;y,\tau_2). \tag{2-6}$$ □

Finally, Theorem 1.1 follows from Corollary 2.5 and Lemma 2.6; compare with [Topping 2009, Section 4].

Proof of Theorem 1.2. We follow [Lott 2009, Proposition 16].

Since the manifold M is compact, for $y \in M$ and $\tau \in [\tau_1, \tau_2]$ the infimum in (1-7) is attained at some point $x \in M$. Assume that the functions φ and $Q(\cdot,\tau_1;y,\tau)$ are both differentiable at x (which is almost always the case). Then

$$\nabla_1 Q(x,\tau_1;y,\tau) - \nabla\varphi(x) = 0. \tag{2-7}$$

We may assume that there is a unique minimizing $\mathscr{L}$-geodesic $\gamma : [\tau_1,\tau] \to M$ from x to y. Set $X(\cdot) = \gamma'(\cdot)$ as before. Then using (2-4) and (2-7) we get $\sqrt{\tau_1}X(\tau_1) = -\nabla\varphi(x)/2$. So $y = \gamma(\tau) = \mathscr{L}_{\tau_1,\tau}\exp_x(-\nabla\varphi(x)/2) = F_\tau(x)$. Now we have

$$2\sqrt{\tau}\phi(\gamma(\tau),\tau) = Q(x,\tau_1;\gamma(\tau),\tau) - \varphi(x).$$

Then we get

$$\frac{d}{d\tau}(2\sqrt{\tau}\phi(\gamma(\tau),\tau)) = \frac{d}{d\tau}Q(x,\tau_1;\gamma(\tau),\tau) = \sqrt{\tau}(S(\gamma(\tau),\tau)+|X(\tau)|^2)$$

by definition of the $\mathscr{L}$-distance, and

$$\tau^{3/2}\frac{d}{d\tau}\phi(\gamma(\tau),\tau) = -\tfrac{1}{2}\sqrt{\tau}\phi(\gamma(\tau),\tau) + \tfrac{1}{2}\tau^{3/2}(S(\gamma(\tau),\tau)+|X(\tau)|^2).$$

From the proof of [Müller 2010, Lemma 5.1], we have

$$\frac{d}{d\tau}(S(\gamma(\tau),\tau)+|X(\tau)|^2) = -\mathscr{H}(\mathscr{S},X) - \frac{1}{\tau}(S(\gamma(\tau),\tau)+|X(\tau)|^2).$$

It follows that

$$\Big(\tau^{3/2}\frac{d}{d\tau}\Big)^2\phi(\gamma(\tau),\tau) = -\tfrac{1}{2}\tau^3\mathscr{H}(\mathscr{S},X).$$

Combining this with the condition $\mathcal{V}_\tau = (F_\tau)_\sharp \mathcal{V}_{\tau_1}$, we have

$$\Big(\tau^{3/2}\frac{d}{d\tau}\Big)^2 \int_M \phi(\,\cdot\,,\tau)\,d\mathcal{V}_\tau = \Big(\tau^{3/2}\frac{d}{d\tau}\Big)^2 \int_M \phi(F_\tau(\,\cdot\,),\tau)\,d\mathcal{V}_{\tau_1} = -\tfrac{1}{2}\tau^3 \int_M \mathcal{H}(\mathcal{S}, X(\tau))\,d\mathcal{V}_{\tau_1}.$$

Combining this with Lemma 2.4 and the assumption on $\mathcal{D}(\mathcal{S}, X)$, we are done. □

Proof of Theorem 1.3. It suffices to prove the case that u_1 and u_2 have compact support, since then the general case will follow by an approximate technique as in [Cordero-Erausquin et al. 2001]. Now one proceeds as in [Brendle 2009]. A key step is to prove that under our assumption on $\mathcal{D}(\mathcal{S}, X)$, one has

$$\tau^{-3/2}\frac{d}{d\tau}\Big(\tau^{3/2}\frac{d}{d\tau}\Big(\frac{n}{2}\ln\tau + \tfrac{1}{2}\tau^{-1/2}Q(x,\tau_1; F_\tau(x),\tau) - \ln\det A\Big)\Big) \geq 0$$

as in [Brendle 2009], where A is as in the proof of Lemma 2.3. This can be proved by using Lemma 2.3, as we did in the proof of Theorem 1.2. □

3. Some applications

For a closed manifold $(M^n, g(\tau))$ evolving by (1-2) and a solution u of (1-6) we introduce the $\mathcal{W}$-entropy as in [Perelman 2002; List 2008]:

$$\mathcal{W} := \int_M \big(\tau(S + |\nabla f|^2) + f - n\big)(4\pi\tau)^{-n/2}e^{-f}\,d\mu,$$

where f is defined by $u = (4\pi\tau)^{-n/2}e^{-f}$.

Theorem 3.1. *Assume that the quantity $\mathcal{D}(\mathcal{S}, X)$ is nonnegative for all vector fields $X \in \Gamma(TM)$ and all times for which the flow exists. Then $d\mathcal{W}/d\tau \leq 0$.*

Proof. This follows easily from Theorem 1.1 and by generalizing [Topping 2009, Lemma 1.3] by replacing R in [Topping 2009, (1.7)] with S (the proof requires only a minor modification). □

Remark 3.2. Some special cases of Theorem 3.1 appeared in [Perelman 2002; Ni 2004; List 2008]. Of course, one can also prove Theorem 3.1 by a direct computation as in these references.

As in [Topping 2009, Section 1.3], Theorem 1.1 also implies the monotonicity of the enlarged length, generalizing the corresponding result of Perelman [2002]. More precisely, following Perelman, consider $L(y,\tau) := Q(x,0; y,\tau)$ for fixed $x \in M$ and $\bar{L}(y,\tau) := 2\sqrt{\tau}L(y,\tau)$.

Theorem 3.3. *Assume that the quantity $\mathcal{D}(\mathcal{S}, X)$ is nonnegative for all vector fields $X \in \Gamma(TM)$ and all times for which the flow exists. Then the minimum over M of $\bar{L}(\,\cdot\,,\tau) - 2n\tau$ is a weakly decreasing function of τ.*

Following [Perelman 2002; Müller 2010], we define the reduced distance by $l(y,\tau) := L(y,\tau)/(2\sqrt{\tau})$ and the reduced volume by

$$\tilde{V}(\tau) = \int_M \tau^{-n/2} e^{-l(y,\tau)} d\mu(y).$$

The following theorem extends a theorem in [Perelman 2002, Section 7.1], and also extends [Müller 2010, Theorem 1.4] to the noncompact case.

Theorem 3.4. *Suppose that $(M, g(\tau))$ is a complete manifold evolving by (1-2), with the Ricci curvature uniformly bounded below and S_{ij} uniformly bounded in compact time intervals, and such that the quantity $\mathfrak{D}(\mathcal{S}, X)$ is nonnegative for all vector fields $X \in \Gamma(TM)$ and all times for which the flow exists. Then the reduced volume $\tilde{V}(\tau)$ is nonincreasing in τ.*

Proof. This is a corollary of Theorem 1.3; see [Brendle 2009, Section 3]. The assumption on the boundedness of the Ricci curvature and S_{ij} guarantees that the reduced volume is finite. □

Remark 3.5. Our $\mathcal{L}$-length is the same as the $\mathcal{L}_b$-length in [Müller 2010], and corresponds to the $\mathcal{L}_-$-length in [Lott 2009]. One can also develop a parallel theory of $\mathcal{L}_f$- (or $\mathcal{L}_+$-) and $\mathcal{L}_0$-optimal transportation as in [Lott 2009].

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