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We investigate pointed Hopf algebras via quiver methods. We classify all the possible Hopf structures arising from the simplest Hopf quiver $Q(\langle g \rangle, g)$, which serves as a basic ingredient for the general ones. This provides the very local structure information for the general pointed Hopf algebras.

1. Introduction

Quivers are oriented diagrams consisting of vertices and arrows. Due to a well-known theorem of Gabriel [1972], elementary associative algebras over fields can be presented by path algebras of quivers modulo admissible ideals in some unique manner, with their representations given by representations of the corresponding bound quivers. This makes the abstract algebras and their representation theory visible and plays a central role in the modern representation theory of associative algebras.

After Gabriel, quiver theory has been established for some other algebraic structures, in particular for coalgebras and Hopf algebras. A coalgebra over a field is said to be pointed if its simple subcoalgebras are one-dimensional, or equivalently, its simple comodules are one-dimensional. Chin and Montgomery [1997] gave a Gabriel-type theorem for pointed coalgebras. Cibils and Rosso [2002] introduced the notion of Hopf quivers, which are determined by groups with ramification data, and observed that the path coalgebra of a quiver admits a graded Hopf structure if and only if the quiver is a Hopf quiver. A Hopf algebra is called pointed if its underlying coalgebra is so. Van Oystaeyen and Zhang [2004] established a Gabriel-type theorem for graded pointed Hopf algebras.

These results motivate our project of studying general pointed Hopf algebras by taking advantage of quiver methods. The quiver setting gives a visible frame to the classification problem, representation theory, and other respects. The project of quiver approaches to pointed Hopf algebras consists mainly of the following:

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(1) Classify graded Hopf structures on Hopf quivers. This amounts to a complete classification of graded pointed Hopf algebras.

(2) Carry out a proper deformation process for graded Hopf structures to get the general pointed Hopf algebras.

(3) With these Hopf algebras in hand, investigate their (co)representation theory as well as other aspects with the help of their quiver presentation.

This paper is conceived as the first step of the project. We classify Hopf algebra structures on the minimal Hopf quiver $Q(\langle g\rangle, g)$. Here $\langle g\rangle$ denotes a cyclic group generated by g and $Q(\langle g\rangle, g)$ is a cyclic quiver if the order of g is finite, or the infinite linear quiver if the order of g is infinite. Intuitively, it is easy to see that a general Hopf quiver is a compatible combination of such minimal Hopf quivers. Moreover, on a given Hopf quiver there is a clear relation between the sub-Hopf algebras and sub-Hopf quivers, as can be seen from Cibils and Rosso's description [2002] of graded Hopf structures on Hopf quivers. Therefore, our result provides the complete local structures needed for general pointed Hopf algebras.

Most of the Hopf structures arising from the Hopf quiver $Q(\langle g\rangle, g)$ are by no means novel. They appear sporadically in various work on Hopf algebras, for example [Taft 1971; Andruskiewitsch and Schneider 2002; Radford 1999], and on quantum groups, for example [Lusztig 1990; De Concini and Kac 1990]. Quiver methods provide these Hopf algebras in a unified setting. Aside from quiver techniques, our arguments also rely on Bergman's diamond lemma [1978], which helps to present the Hopf algebras by generators with relations. This is useful in carrying out the preferred deformation procedure in the sense of Gerstenhaber and Schack [1990].

Although our result settles the local structure of a general pointed Hopf algebra, it is far from providing a full understanding of it, especially when the group of its group-like elements is nonabelian. In the situation of finite-dimensional pointed Hopf algebras with abelian group-likes, Andruskiewitsch and Schneider [2010] have made substantial progress in the classification problem by a different method, as they surveyed in [2002].

The paper is organized as follows. In Section 2 we review some necessary facts about Hopf quivers and pointed Hopf algebras. In Sections 3 and 4, we give the explicit classifications of Hopf structures on the cyclic quiver and the infinite linear quiver, respectively. Section 5 is devoted to some applications of the classification results.

Throughout the paper, we work over an algebraically closed field of characteristic zero. See [Gabriel and Roiter 1997; Assem et al. 2006] for general knowledge about quivers and representations, and [Sweedler 1969; Montgomery 1993] for that of coalgebras and Hopf algebras.

2. Quiver approaches to pointed Hopf algebras

For the convenience of the reader, we recall some basic notions and facts from [Cibils and Rosso 2002; van Oystaeyen and Zhang 2004]. There is a dual approach to elementary Hopf algebras via quivers; see [Cibils 1993; Green 1995; Cibils and Rosso 1997; Green and Solberg 1998] for related work.

A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows, and $s, t : Q_1 \to Q_0$ are two maps assigning respectively the source and the target for each arrow. A path of length $l \geq 1$ in the quiver Q is a finitely ordered sequence of l arrows $a_l \cdots a_1$ such that $s(a_{i+1}) = t(a_i)$ for $1 \leq i \leq l-1$. By convention a vertex is said to be a trivial path of length 0.

The path coalgebra kQ is the k-space spanned by the paths of Q with counit and comultiplication maps defined by $\varepsilon(g) = 1$ and $\Delta(g) = g \otimes g$ for each $g \in Q_0$, and $\varepsilon(p) = 0$ and

$$\Delta(p) = p \otimes s(a_1) + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + t(a_n) \otimes p$$

for each nontrivial path $p = a_n \cdots a_1$. The length of paths gives a natural gradation on the path coalgebra. Let Q_n denote the set of paths of length n in Q. Then $kQ = \bigoplus_{n \geq 0} kQ_n$ and $\Delta(kQ_n) \subseteq \bigoplus_{n=i+j} kQ_i \otimes kQ_j$. Clearly kQ is pointed with the set of group-likes $G(kQ) = Q_0$, and has the coradical filtration

$$kQ_0 \subseteq kQ_0 \oplus kQ_1 \subseteq kQ_0 \oplus kQ_1 \oplus kQ_2 \subseteq \cdots.$$

Hence kQ is coradically graded.

Cibils and Rose [2002] call a quiver Q a Hopf quiver if the corresponding path coalgebra kQ admits a graded Hopf algebra structure. Hopf quivers can be determined by ramification data of groups. Let G be a group and $\mathscr{C}$ the set of conjugacy classes. A ramification datum R of the group G is a formal sum $\sum_{C \in \mathscr{C}} R_C C$ of conjugacy classes with coefficients in $\mathbb{N} = \{0, 1, 2, \dots\}$. The corresponding Hopf quiver $Q = Q(G, R)$ is defined so that the set of vertices Q_0 is G, and for each $x \in G$ and $c \in C$, there are R_C arrows going from x to cx.

A Hopf quiver $Q = Q(G, R)$ is connected if and only if the union of the conjugacy classes with nonzero coefficients in R generates G. We denote the unit element of G by e. If $R_{\{e\}} \neq 0$, then there are $R_{\{e\}}$-loops attached to each vertex; if the order of elements in a conjugacy class $C \neq \{e\}$ is n and $R_C \neq 0$, then corresponding to these data in Q there is a subquiver (n, R_C)-cycle (called basic n-cycle if $R_C = 1$), that is, the quiver having n vertices, indexed by the set of integers modulo n, and R_C arrows going from $\bar{i}$ to $\overline{i+1}$ for each i; if the order of elements in a conjugacy class C is ∞, then in Q there is a subquiver R_C-chain (called infinite linear quiver if $R_C = 1$), that is, a quiver having set of vertices indexed by the set of integral numbers, and R_C arrows going from j to $j+1$ for each j. Therefore, basic cycles (including loop as 1-cycle) and the infinite linear quiver are basic building blocks of the general Hopf quivers.

By [Cibils and Rosso 2002], for a given Hopf quiver Q, the set of graded Hopf structures on kQ is in bijection with the set of kQ_0-Hopf bimodule structures on kQ_1. The graded Hopf structures are obtained from Hopf bimodules via the quantum shuffle product [Rosso 1998], and can be restricted to sub-Hopf quivers; hence for the very local sub-Hopf structures it suffices to consider those arising from the Hopf quivers of form $Q(\langle g\rangle, g)$.

Let H be a pointed Hopf algebra. Denote its coradical filtration by $\{H_n\}_{n=0}^{\infty}$. Define

$$\mathrm{gr}(H) = H_0 \oplus H_1/H_0 \oplus H_2/H_1 \oplus \cdots$$

as the corresponding (coradically) graded coalgebra. Then $\mathrm{gr}(H)$ inherits from H a coradically graded Hopf algebra structure; see for example [Montgomery 1993]. Any generating set of $\mathrm{gr}(H)$ (as an algebra) can be lifted to one for H; this useful fact can be verified easily by induction.

Lemma 2.1. *Assume that $\mathscr{G} \subset \mathrm{gr}(H)$ is a generating set and $\widetilde{\mathscr{G}} \subset H$ an arbitrary set of its representatives. Then $\widetilde{\mathscr{G}}$ generates H.*

The procedure of going from H to $\mathrm{gr}\, H$ is called degeneration. The converse procedure is called deformation. According to Gerstenhaber and Schack [1990], a coalgebra-preserving deformation is called preferred. If we want to classify all the Hopf structures on the whole path coalgebra of a Hopf quiver, or the bialgebras of type one [Nichols 1978], then we only need to carry out preferred deformation procedure.

According to van Oystaeyen and Zhang [2004], if H is a coradically graded pointed Hopf algebra, then there exists a unique Hopf quiver $Q(H)$ such that H can be realized as a large sub-Hopf algebra of a graded Hopf structure on the path coalgebra $kQ(H)$. Here by "large" we mean H contains the subspace $kQ(H)_0 \oplus kQ(H)_1$. This Gabriel-type theorem allows us to classify pointed Hopf algebras exhaustively in the quiver setting. The combinatorial structure of Hopf quivers implies clearly a Cartier–Gabriel decomposition theorem (see for example [Sweedler 1969; Montgomery 1993]) for general pointed Hopf algebras as given by Montgomery [1995]. It suffices to study only Hopf structures on connected Hopf quivers.

3. Hopf structures on basic cycles

Let $G = \langle g \mid g^n = 1\rangle$ be a cyclic group of order n and let $\mathscr{Z}$ denote the Hopf quiver $Q(G, g)$. The quiver $\mathscr{Z}$ is a basic n-cycle and this is the only possible way it is realized as a Hopf quiver. If $n = 1$, then $\mathscr{Z}$ is the one-loop quiver, that is, it consists of one vertex and one loop. It is easy to see that such a quiver provides only the familiar divided power Hopf algebra in one variable; this algebra is isomorphic to the polynomial algebra in one variable.

From now on we assume $n > 1$ and fix a basic n-cycle $\mathfrak{Z}$. For each integer i modulo n, let a_i denote the arrow $g^i \to g^{i+1}$. Let p_i^l denote the path $a_{i+l-1} \cdots a_{i+1} a_i$ of length l. Then $\{p_i^l \mid 0 \le i \le n-1,\ l \ge 0\}$ is a basis of $k\mathfrak{Z}$.

Before moving on, we fix some notations of Gaussian binomials. For any $q \in k$ and integers $l, m \ge 0$, let

$$l_q = 1 + q + \cdots + q^{l-1}, \quad l!_q = 1_q \cdots l_q, \quad \binom{l+m}{l}_q = \frac{(l+m)!_q}{l!_q\, m!_q}.$$

When $1 \neq q \in k$ is an n-th root of unity of order d,

$$\binom{l+m}{l}_q = 0 \quad \text{if and only if} \quad [(l+m)/d] - [m/d] - [l/d] > 0,$$

where $[x]$ means the integer part of x.

We'll need the following fact about automorphisms of the path coalgebra $k\mathfrak{Z}$.

Lemma 3.1. *Let $d > 1$ be an integer and $k\mathfrak{Z}[d]$ the subcoalgebra $\bigoplus_{i=0}^{d} k\mathfrak{Z}_i$. For any $\lambda \in k$ the linear map*

$$f_\lambda^d(0) : k\mathfrak{Z}[d] \to k\mathfrak{Z}[d], \qquad \begin{aligned} & p_i^l \mapsto p_i^l \quad \textit{for all } i,\ 0 \le l \le d-1, \\ & p_0^d \mapsto p_0^d + \lambda(1 - g^d), \\ & p_i^d \mapsto p_i^d \quad \textit{for } 1 \le i \le n-1. \end{aligned}$$

defines a coalgebra automorphism of $k\mathfrak{Z}[d]$. There is a coalgebra automorphism $F_\lambda^d(0) : k\mathfrak{Z} \to k\mathfrak{Z}$ whose restriction to $k\mathfrak{Z}[d]$ is $f_\lambda^d(0)$. A similar map $f_\lambda^d(j)$ can be defined and extended to $F_\lambda^d(j)$ for any j. Also, any automorphism of the subcoalgebra $k\mathfrak{Z}[d]$ whose restriction to $k\mathfrak{Z}[d-1]$ is the identity is a finite composition of some of the $f_\lambda^d(j)$. Therefore all such automorphisms are extendable to automorphisms of the path coalgebra $k\mathfrak{Z}$.

Proof. The claim that $f_\lambda^d(0)$ is a coalgebra automorphism is obvious. For the second claim, define the map $F_\lambda^d(0) : k\mathfrak{Z} \to k\mathfrak{Z}$ by $p_i^l \mapsto p_i^l$ for all i and $0 \le l \le d-1$, by $p_0^d \mapsto p_0^d + \lambda(1 - g^d)$, by $p_i^d \mapsto p_i^d$ for $1 \le i \le n-1$, and, for $l > d$, by

$$p_i^l \mapsto \begin{cases} p_0^l - \lambda p_d^{l-d} & \text{for } i = 0,\ l \neq d \pmod n, \\ p_0^l + \lambda p_0^{l-d} - \lambda p_d^{l-d} & \text{for } i = 0,\ l = d \pmod n, \\ p_i^l + \lambda p_i^{l-d} & \text{for } 1 \le i \le n-1,\ i + l = d \pmod n, \\ p_i^l & \text{for } 1 \le i \le n-1,\ i + l \neq d \pmod n. \end{cases}$$

It is straightforward (but a bit tedious) to verify that $F_\lambda^d(0)$ is the desired coalgebra automorphism of $k\mathfrak{Z}$. The remaining claims are easy. □

First we recall the graded Hopf structures on $k\mathcal{Z}$. By [Cibils and Rosso 2002], they are in bijection with the kG-module structures on ka_0, and in turn with the set of n-th roots of unity. For each $q \in k$ with $q^n = 1$, let $g \,.\, a_0 = qa_0$ define a kG-module. The corresponding kG-Hopf bimodule is $kG \otimes_{kG} ka_0 \otimes kG = ka_0 \otimes kG$. We identify $a_i = a_0 \otimes g^i$. This is how we view $k\mathcal{Z}_1$ as a kG-Hopf bimodule. The path multiplication formula

$$p_i^l \cdot p_j^m = q^{im} \binom{l+m}{l}_q p_{i+j}^{l+m}. \tag{3-1}$$

was given in [Cibils and Rosso 2002] by induction. In particular,

$$g \cdot p_i^l = q^l p_{i+1}^l, \quad p_i^l \cdot g = p_{i+1}^l, \quad a_0^l = l_q!\, p_0^l. \tag{3-2}$$

For each q, the corresponding graded Hopf algebra is denoted by $k\mathcal{Z}(q)$.

We consider in the following lemma the algebraic side of $k\mathcal{Z}(q)$. The facts are our starting point of the preferred deformation process.

Lemma 3.2. *As an algebra, $k\mathcal{Z}(q)$ can be presented by generators with relations as follows*:

(1) *When $q = 1$, the generators are g and a_0, and the relations are $g^n = 1$ and $ga_0 = a_0 g$.*

(2) *When $\operatorname{ord}(q) = d > 1$, the generators are g, a_0 and p_0^d, and the relations are $g^n = 1$, $ga_0 = qa_0 g$, $a_0^d = 0$, $a_0 p_0^d = p_0^d a_0$ and $gp_0^d = p_0^d g$.*

Proof. The claim about the generators and the relations they satisfy is direct consequence of (3-1) and (3-2). In particular, for the case $\operatorname{ord}(q) = d > 1$, we have

$$(p_0^d)^l = p_0^{dl} \quad \text{and} \quad p_0^{dl} a_0^j = j!_q\, p_0^{j+dl}. \tag{3-3}$$

It suffices to prove conversely that the relations are enough to define $k\mathcal{Z}(q)$.

Let $H(q)$ denote the algebra defined in the lemma. To avoid confusion, we use new notations for the generators: change g to h, a_0 to a, and p_0^d to p. The relations are obtained by substituting the old notations with the new ones.

For the case $q = 1$, by the well-known diamond lemma [Bergman 1978] we know that $\{a^l h^i \mid 0 \le i \le n-1,\ l \ge 0\}$ is a basis of $H(1)$. Now define a linear map $f : H(1) \to k\mathcal{Z}(1)$, $a^l h^i \mapsto l!\, p_i^l$. Evidently this is a linear isomorphism. It remains to check that it respects the multiplication. This is again direct consequence of (3-1) and (3-2):

$$f((a^l h^i)(a^m h^j)) = (l+m)!\, p_{i+j}^{l+m} = (l!\, p_i^l)(m!\, p_j^m) = f(a^l h^i) f(a^m h^j).$$

For the case $\operatorname{ord}(q) = d > 1$, the set $\{p^l a^j h^i \mid 0 \le i \le n-1,\ 0 \le j \le d-1,\ l \ge 0\}$ is a basis of $H(q)$, again by the diamond lemma. Define a linear map

$$f : H(q) \to k\mathcal{Z}(q), \quad p^l a^j h^i \mapsto j!_q\, p_i^{j+dl}.$$

Similarly one can verify by direct computation with the help of (3-1) and (3-3) that this is an algebra isomorphism:

$$\begin{aligned} f((p^l a^j h^i)(p^{l'} a^{j'} h^{i'})) &= q^{ij'}(j+j')!_q\, p_{i+i'}^{j+j'+d(l+l')} \\ &= (j!_q\, p_i^{j+dl})(j'!_q\, p_{i'}^{j'+dl'}) \\ &= f(p^l a^j h^i) f(p^{l'} a^{j'} h^{i'}). \end{aligned}$$

□

Now we are ready to state the main result of this section. We classify all the (nongraded) Hopf structures on the path coalgebra $k\mathfrak{Z}$.

Theorem 3.3. *Let H be a Hopf structure on $k\mathfrak{Z}$ with* $\operatorname{gr} H \cong k\mathfrak{Z}(q)$. *Then as algebra, it can be presented by generators and relations as follows*:

(1) *If $q=1$, the generators are g and a_0; the relations are $g^n=1$ and $ga_0=a_0g$. In particular, the Hopf algebra H is isomorphic to $k\mathfrak{Z}(1)$.*

(2) *If* $\operatorname{ord}(q)=n$, *the generators are g, a_0 and p_0^n; the relations are $g^n=1$, $a_0^n=0$, $ga_0=qa_0g$, $gp_0^n=p_0^ng$ and $a_0p_0^n-p_0^na_0=\lambda a_0$ with some $\lambda\in k$.*

(3) *If* $1<\operatorname{ord}(q)=d<n$ *with $n\neq 2d$, the generators are g, a_0 and p_0^d; the relations are $g^n=1$, $a_0^d=0$, $ga_0=qa_0g$, $gp_0^d=p_0^dg$ and $a_0p_0^d-p_0^da_0=0$. In other words, the Hopf algebra H is isomorphic to $k\mathfrak{Z}(q)$.*

(4) *If $n=2d$ is even and* $\operatorname{ord}(q)=d$, *the generators are g, a_0 and p_0^n; the relations are $g^n=1$, $a_0^d=\mu(1-g^d)$, $ga_0=qa_0g$, $gp_0^d=p_0^dg$ and $a_0p_0^d-p_0^da_0=(\mu(1-q)/(d-1)_q)a_0(1+g^d)$ with some $\mu\in k$.*

Proof. The proof will be separated into several steps. The main idea is to determine all the possible preferred deformations from the graded ones, with help from the quiver.

Part (1): $q=1$. In this case H is generated by g and a_0 by Lemma 3.2(1). We only need to give all the possible relations that involve them. It suffices to consider all the possible preferred deformations of the graded generating relations. In this situation, we need to determine the lower terms, that is, $= ga_0g^{-1}-a_0$. Since $\Delta(g\cdot a_0\cdot g^{-1})=\Delta(g)\Delta(a_0)\Delta(g^{-1})=g\cdot a_0\cdot g^{-1}\otimes 1+g\otimes g\cdot a_0\cdot g^{-1}$, we can conclude that $g\cdot a_0\cdot g^{-1}\in {}^{g}(k\mathfrak{Z})^1$; hence $g\cdot a_0\cdot g^{-1}-a_0=\lambda(1-g)$ for some $\lambda\in k$. The relation $g^n=1$ is stable under deformation. Note that

$$a_0=g^n\cdot a_0\cdot g^{-n}=a_0+n\lambda(1-g).$$

This forces $\lambda=0$. Hence there are no nontrivial preferred deformations for $k\mathfrak{Z}(1)$.

Part (2): $\operatorname{ord}(q)=n$. If $\operatorname{gr} H\cong k\mathfrak{Z}(q)$, then H is generated by g, a_0 and p_0^n by Lemma 3.2(2). We again need to determine all the possible preferred deformations of the graded generating relations in Lemma 3.2(2).

First, the relation $ga_0 = qa_0 g$ might be deformed to $ga_0 g^{-1} = qa_0 + \alpha(1-g)$ for some $\alpha \in k$. Let $\widetilde{a_0} = a_0 - (\alpha/(1-q))(1-g)$; then we have $g\widetilde{a_0}g^{-1} = q\widetilde{a_0}$. Set $\lambda = \alpha/(1-q)$ and f_λ^1 as in Lemma 3.1. It follows that the map f_λ^1 can be extended to a coalgebra automorphism F_λ^1 of $k\mathscr{Z}$. Now the original Hopf structure can be transported through F_λ^1 to one with $ga_0 = qa_0 g$. By iterative application of the lemma, we can have through coalgebra automorphism (or base change)

$$a_0 g^i = a_i, \quad a_0^l g^i = l_q!\, p_i^l \quad \text{for } i = 0, 1, \dots, n-1,\ l = 1, \dots, n-1. \tag{3-4}$$

Note that under such coalgebra automorphisms, the new elements g, a_0 and p_0^n are generators of H all the same according to Lemma 3.2.

Second, consider the relation $a_0^n = 0$. To see to what it may be deformed to, we should look at $\Delta(a_0^n)$. By the Gaussian binomial formula (see for example [Kassel 1995, Proposition IV.2.2]), we have

$$\Delta(a_0^n) = (\Delta(a_0))^n = (a_0 \otimes 1 + g \otimes a_0)^n = \sum_{i=0}^{n} \binom{n}{i}_q a_0^i g^{n-i} \otimes a_0^{n-i} = a_0^n \otimes 1 + 1 \otimes a_0^n.$$

It follows that $a_0^n = 0$ since in $k\mathscr{Z}$ there is no loop attached to 1.

Finally we consider the relations involved p_0^n. Similarly, by

$$\Delta(gp_0^n - p_0^n g) = (gp_0^n - p_0^n g) \otimes g + g \otimes (gp_0^n - p_0^n g), \tag{3-5}$$

we have $gp_0^n - p_0^n g = 0$. By (3-4) and (3-5) we have

$$\begin{aligned}
\Delta(a_0 p_0^n) = \Delta(a_0)\Delta(p_0^n) &= (a_0 \otimes 1 + g \otimes a_0) \sum_{l=0}^{n} p_l^{n-l} \otimes p_0^l \\
&= \sum_{l=0}^{n} a_0 p_l^{n-l} \otimes p_0^l + \sum_{l=0}^{n} g p_l^{n-l} \otimes a_0 p_0^l \\
&= \sum_{l=1}^{n} p_l^{n+1-l} \otimes p_0^l + a_0 p_0^n \otimes 1 + g^{n+1} \otimes a_0 p_0^n, \\
\Delta(p_0^n a_0) = \Delta(p_0^n)\Delta(a_0) &= \Big(\sum_{l=0}^{n} p_l^{n-l} \otimes p_0^l\Big)(a_0 \otimes 1 + g \otimes a_0) \\
&= \sum_{l=0}^{n} p_l^{n-l} a_0 \otimes p_0^l + \sum_{l=0}^{n} p_l^{n-l} g \otimes p_0^l a_0 \\
&= \sum_{l=1}^{n} p_l^{n+1-l} \otimes p_0^l + p_0^n a_0 \otimes 1 + g^{n+1} \otimes p_0^n a_0.
\end{aligned}$$

Let $[a_0, p_0^n] = a_0 p_0^n - p_0^n a_0^n$. Then the previous equations give rise to $\Delta([a_0, p_0^n]) = [a_0, p_0^n] \otimes 1 + g \otimes [a_0, p_0^n]$. Now from the structure of the space of $(1, g)$-primitive elements of $k\mathcal{Z}$, we have $[a_0, p_0^n] = \lambda a_0 + \mu(1-g)$ for some $\lambda, \mu \in k$. By induction, one gets $a_0^n p_0^n = p_0^n a_0^n + n\lambda a_0^n + n\mu a_0^{n-1}$. With (3-4) and (3-5), this forces $\mu = 0$.

Part (3): $1 < \operatorname{ord}(q) = d < n$. Now assume $\operatorname{gr} H \cong k\mathcal{Z}(q)$, so that H is generated by g, a_0 and p_0^d. Repeating the argument that proved Theorem 3.3(2), we can assume without loss of generality for $i = 0, 1, \ldots, n-1$ and $l = 1, \ldots, d-1$ that

(3-6) $$ga_0 = qa_0 g, \quad a_0 g^i = a_i, \quad a_0^l g^i = l_q!\, p_i^l.$$

Consider $\Delta(a_0^d)$. By the Gaussian binomial formula again, we have

$$\Delta(a_0^d) = (\Delta(a_0))^d = (a_0 \otimes 1 + g \otimes a_0)^d$$
$$= \sum_{i=0}^{d} \binom{d}{i}_q a_0^i g^{d-i} \otimes a_0^{d-i} = a_0^d \otimes 1 + g^d \otimes a_0^d.$$

Since in $k\mathcal{Z}$ there is no arrow going from 1 to g^d, it follows that

(3-7) $$a_0^d = \mu(1-g^d) \quad \text{for some } \mu \in k.$$

We continue to consider the relations involving p_0^d. By

(3-8) $$\Delta(gp_0^d - p_0^d g) = (gp_0^d - p_0^d g) \otimes g + g^{d+1} \otimes (gp_0^d - p_0^d g),$$

it follows that $gp_0^d - p_0^d g = \nu(g - g^{d+1})$ for some $\nu \in k$. By induction we have $g^n p_0^d - p_0^d g^n = n\nu(1-g^d)$. Since $g^n = 1$, we conclude that $\nu = 0$ and $gp_0^d = p_0^d g$.

Finally we consider $\Delta([a_0, p_0^d])$. By (3-6), (3-7) and (3-8) we have

$$\Delta(a_0 p_0^d) = \Delta(a_0)\Delta(p_0^d)$$
$$= (a_0 \otimes 1 + g \otimes a_0)\Big(\sum_{l=0}^{d} p_l^{d-l} \otimes p_0^l\Big)$$
$$= \sum_{l=0}^{d} a_0 p_l^{d-l} \otimes p_0^l + \sum_{l=0}^{d} g p_l^{d-l} \otimes a_0 p_0^l$$
$$= \sum_{l=1}^{d} p_l^{d+1-l} \otimes p_0^l + a_0 p_0^d \otimes 1 + g^{d+1} \otimes a_0 p_0^d + a_0 p_1^{d-1} \otimes a_0 + g p_{d-1}^1 \otimes a_0 p_0^{d-1}$$
$$= \sum_{l=1}^{d} p_l^{d+1-l} \otimes p_0^l + a_0 p_0^d \otimes 1 + g^{d+1} \otimes a_0 p_0^d + \frac{\mu}{(d-1)_q!}(g - g^{d+1}) \otimes a_0 + \frac{q\mu}{(d-1)_q!} p_d^1 \otimes (g - g^{d+1})$$

and

$$\begin{aligned}\Delta(p_0^d a_0) &= \Delta(p_0^d)\Delta(a_0)\\ &= \Big(\sum_{l=0}^{d} p_l^{d-l}\otimes p_0^l\Big)(a_0\otimes 1+g\otimes a_0)\\ &= \sum_{l=0}^{d} p_l^{d-l}a_0\otimes p_0^l+\sum_{l=0}^{d} p_l^{d-l}g\otimes p_0^l a_0\\ &= \sum_{l=1}^{d} p_l^{d+1-l}\otimes p_0^l + p_0^d a_0\otimes 1+g^{d+1}\otimes p_0^d a_0\\ &\qquad\qquad + p_1^{d-1}a_0\otimes a_0+p_{d-1}^1 g\otimes p_0^{d-1}a_0\\ &= \sum_{l=1}^{d} p_l^{d+1-l}\otimes p_0^l + a_0 p_0^d\otimes 1+g^{d+1}\otimes a_0 p_0^d\\ &\qquad\qquad +\frac{q\mu}{(d-1)_q!}(g-g^{d+1})\otimes a_0+\frac{\mu}{(d-1)_q!}p_d^1\otimes(g-g^{d+1}).\end{aligned}$$

These equations lead to

$$\begin{aligned}&\Delta\Big([a_0,p_0^d]-\frac{\mu(1-q)}{(d-1)_q!}(a_0+p_d^1)\Big)\\ &=\Big([a_0,p_0^d]-\frac{\mu(1-q)}{(d-1)_q!}(a_0+p_d^1)\Big)\otimes 1+g^{d+1}\otimes\Big([a_0,p_0^d]-\frac{\mu(1-q)}{(d-1)_q!}(a_0+p_d^1)\Big).\end{aligned}$$

It follows as before that

$$[a_0,p_0^d]=\frac{\mu(1-q)}{(d-1)_q!}a_0(1+g^d)+\lambda(1-g^{d+1})\quad \text{for some } \lambda\in k.$$

Again by induction we have

$$a_0^d p_0^d = p_0^d a_0^d + d\frac{\mu^2(1-q)}{(d-1)_q!}(1-g^{2d})+d\lambda a_0^{d-1}.$$

So $\mu=\lambda=0$ if $n\neq 2d$, and $\lambda=0$ otherwise. Now we can conclude that

$$[a_0,p_0^d]=\begin{cases}\frac{\mu(1-q)}{(d-1)_q!}a_0(1+g^d) & \text{if } n=2d,\\ 0 & \text{otherwise.}\end{cases}\tag{3-9}$$

Remaining cases. So far we have proved that the Hopf structures on $k\mathfrak{Z}$ must be generated by g, a_0 and p_0^d (where $d=\operatorname{ord}(q)$) and must satisfy the relations presented in Theorem 3.3. To complete the proof, it suffices to verify that these relations are enough to define Hopf structures on $k\mathfrak{Z}$. We only need to prove the cases of $\operatorname{ord}(q)=n$ and $\operatorname{ord}(q)=n/2$, since otherwise the Hopf structures are graded, as treated in Lemma 3.2.

The check is sort of routine, and similar to that of the graded case. We only prove the case $\text{ord}(q) = n$. The case $\text{ord}(q) = n/2$ can be done similarly. Assume an algebra $\mathscr{C}(q, \lambda)$ is defined by generators h, a and p with relations

$$h^n = 1, \quad a^n = 0, \quad ha = qah, \quad hp = ph, \quad ap - pa = \lambda a.$$

By the diamond lemma, the algebra has

$$\{p^k a^j h^i \mid 0 \le i, j \le n-1,\ k \ge 0\}$$

as a basis. Since the Hopf algebra H has a similar basis

$$\{(p_0^n)^k a_0^j g^i \mid 0 \le i, j \le n-1,\ k \ge 0\},$$

we can define a linear isomorphism $F : \mathscr{C}(q, \lambda) \to H$ by sending $p^k a^j h^i$ to $(p_0^n)^k a_0^j g^i$. It is an algebra map by direct calculation. □

We summarize in the following all the Hopf algebra structures living on $k\mathfrak{Z}$. We denote by $k\mathfrak{Z}(n, q, \lambda)$ the Hopf algebra defined by Theorem 3.3(2) and by $k\mathfrak{Z}(n/2, q, \mu)$ the Hopf algebra defined by Theorem 3.3(4). The check of the statements about Hopf algebra isomorphism is routine and omitted.

Theorem 3.4. *Let $\mathfrak{Z}$ be a basic n-cycle and $k\mathfrak{Z}$ the associated path coalgebra.*

(1) *If n is odd, then the graded Hopf structure on $k\mathfrak{Z}$ is given by $k\mathfrak{Z}(q)$, and the nongraded one is $k\mathfrak{Z}(n, q, \lambda)$. We have the Hopf algebra isomorphisms $k\mathfrak{Z}(q) \cong k\mathfrak{Z}(q')$ if and only if $q = q'$, and $k\mathfrak{Z}(n, q, \lambda) \cong k\mathfrak{Z}(n, q', \lambda')$ if and only if $q = q'$ and there exists some $0 \ne \zeta \in k$ such that $\lambda = \zeta^n \lambda'$.*

(2) *If n is even, then the graded Hopf structure on $k\mathfrak{Z}$ is given by $k\mathfrak{Z}(q)$, and the nongraded ones are $k\mathfrak{Z}(n, q, \lambda)$ and $k\mathfrak{Z}(n/2, q, \mu)$. We have the Hopf algebra isomorphism $k\mathfrak{Z}(n/2, q, \mu) \cong k\mathfrak{Z}(n/2, q', \mu')$ if and only if $q = q'$ and there exists some $0 \ne \zeta \in k$ such that $\mu = \zeta^{n/2} \mu'$.*

4. Hopf structures on the infinite linear quiver

Let $G = \langle g \rangle$ be an infinite cyclic group and let $\mathscr{A}$ be the Hopf quiver $Q(G, g)$. Then $\mathscr{A}$ is the infinite linear quiver. This is the only possible way to view it as a Hopf quiver. Let e_i denote the arrow $g^i \to g^{i+1}$ and p_i^l the path $e_{i+l-1} \cdots e_i$ of length $l \ge 1$, for each $i \in \mathbb{Z}$. The notation p_i^0 is understood as e_i.

As in the case of the basic cycle, we need a lemma to make an appropriate base change in a later argument. The proof, omitted, is almost identical to that of Lemma 3.1.

Lemma 4.1. *Let $k\mathscr{A}[d]$ be the subcoalgebra $\bigoplus_{i=0}^{d} k\mathscr{A}_i$. For any $\lambda \in k$, the linear map*

$$f_\lambda^d : k\mathscr{A}[d] \to k\mathscr{A}[d], \qquad \begin{array}{l} p_i^l \mapsto p_i^l \quad \text{for all } i,\ 0 \le l \le d-1, \\ p_0^d \mapsto p_0^d + \lambda(1-g^d), \\ p_i^d \mapsto p_i^d \quad \text{for } i \neq 0. \end{array}$$

defines a coalgebra automorphism of $k\mathscr{A}[d]$. There exists a coalgebra automorphism $F_\lambda^d : k\mathscr{A} \to k\mathscr{A}$ whose restriction to $k\mathscr{A}[d]$ is f_λ^d.

We next collect some useful results about graded Hopf structures on $k\mathscr{A}$. These are in bijection with the left kG-module structures on ke_0, in turn with nonzero elements of k. Assume $g \,.\, e_0 = qe_0$ for some $0 \neq q \in k$. The corresponding kG-Hopf bimodule is $ke_0 \otimes kG$. We identify e_i and $e_0 \otimes g^i$, and in this way we have a kG-Hopf bimodule structure on $k\mathscr{A}_1$. We denote the corresponding graded Hopf algebra by $k\mathscr{A}(q)$. The next lemma gives the presentation of $k\mathscr{A}(q)$ by generators with relations. Its proof is like that of Lemma 3.2.

Lemma 4.2. *The algebra $k\mathscr{A}(q)$ can be presented via generators with relations*:

(1) *If $q = 1$, the generators are g, g^{-1} and e_0; the relations are $gg^{-1} = 1 = g^{-1}g$ and $ge_0 = e_0 g$.*

(2) *If $q \neq 1$ is not a root of unity, generators are g, g^{-1} and e_0; the relations are $gg^{-1} = 1 = g^{-1}g$ and $ge_0 = qe_0 g$.*

(3) *If $q \neq 1$ is a root of unity of order d, the generators are g, g^{-1}, e_0 and p_0^d; the relations are $gg^{-1} = 1 = g^{-1}g$, $e_0^d = 0$, $ge_0 = qe_0 g$, $gp_0^d = p_0^d g$ and $e_0 p_0^d = p_0^d e_0$.*

With the algebraic characterization of $k\mathscr{A}(q)$, we can proceed to the possible preferred deformations. The Hopf structures on $k\mathscr{A}$ are classified as follows.

Theorem 4.3. *Let H be a Hopf structure on $k\mathscr{A}$ with $\operatorname{gr} H \cong k\mathscr{A}(q)$. Then as algebra, it can be presented by generators and relations*:

(1) *If $q = 1$, the generators are g, g^{-1} and e_0; the relations are $gg^{-1} = 1 = g^{-1}g$ and $ge_0 g^{-1} = e_0 + \lambda(1-g)$ with $\lambda \in \{0, 1\}$.*

(2) *If $q \neq 1$ and is not a root of unity, the generators are g, g^{-1} and e_0; the relations are $gg^{-1} = 1 = g^{-1}g$ and $ge_0 = qe_0 g$. In particular, H is isomorphic to $k\mathscr{A}(q)$.*

(3) *If $q \neq 1$ is a root of unity of order d, the generators are g, g^{-1}, e_0 and p_0^d; the relations are $gg^{-1} = 1 = g^{-1}g$, $e_0^d = 0$, $ge_0 = qe_0 g$, $e_0 p_0^d = p_0^d e_0$ and $gp_0^d - p_0^d g = \lambda(g - g^{d+1})$ with $\lambda \in k$.*

Proof. The idea of the proof is the same as in the basic cycle case.

Part (1): $q=1$. Assume that H is a Hopf algebra on $k\mathscr{A}$ with $\operatorname{gr} H \cong k\mathscr{A}(1)$. Then as algebra, it is generated by g, g^{-1} and e_0 according to Lemmas 2.1 and 4.2. So in order to get the defining relations, all we need to do is determine the deformations of $ge_0g^{-1}=e_0$. By

$$\Delta(ge_0g^{-1}) = (ge_0g^{-1}) \otimes 1 + g \otimes (ge_0g^{-1}),$$

we have $ge_0g^{-1} = e_0 + \lambda(1-g)$ for some $\lambda \in k$. If $\lambda \neq 0$, then letting $E := e_0/\lambda$, we have $gEg^{-1} = E + (1-g)$. Note that H is generated by g and E; therefore through the coalgebra automorphism

$$F: k\mathscr{A} \to k\mathscr{A}, \quad \begin{array}{ll} g^i \mapsto g^i & \text{for all } i \in \mathbb{Z}, \\ e_0 \mapsto e_0/\lambda, & e_i \mapsto e_i \quad \text{for all } i \neq 0, \\ p_i^l \mapsto \lambda^t p_i^l & \text{if } e_0 \text{ appears } t \text{ times in } p_i^l, \text{ for all } i \in \mathbb{Z} \text{ and } l \geq 2, \end{array}$$

we can always reduce the relation in H to the equation $ge_0g^{-1} = e_0 + (1-g)$ when $\lambda \neq 0$.

On the contrary, as in the argument of the remaining cases on page 326, it is not difficult to verify that the relations in Theorem 4.3(1) are actually enough to define the algebra structure of H.

Part (2): $q \neq 1$ *is not a root of unity.* Assume that H is a Hopf algebra on $k\mathscr{A}$ with $\operatorname{gr} H \cong k\mathscr{A}(q)$. Again, as algebra, it is generated by g, g^{-1} and e_0 according to Lemmas 2.1 and 4.2. So we need to determine the deformation of $ge_0g^{-1} = qe_0$ to get defining relations for H. Similarly to the previous argument, we have $ge_0g^{-1} = qe_0 + \lambda(1-g)$ for some $\lambda \in k$. Now let $\widetilde{e_0} = e_0 + \lambda/(1-q)$; then we have $g\widetilde{e_0}g^{-1} = q\widetilde{e_0}$. By Lemma 4.1, there is a coalgebra isomorphism for the coalgebra $k\mathscr{A}$ that sends e_0 to $\widetilde{e_0}$. Now under the isomorphism, the original Hopf structure can be transported to a new one with relation $ge_0g^{-1} = qe_0$. So in this case, the Hopf structures are graded, and we are done with part (2) of the theorem.

Part (3): $q \neq 1$ *is a root of unity of order* d. Assume that H is a Hopf algebra on $k\mathscr{A}$ with $\operatorname{gr} H \cong k\mathscr{X}(q)$. In this situation, the Hopf algebra H is generated by g, g^{-1}, e_0 and p_0^d. By a similar argument we have in the first place

$$ge_0 = qe_0g, \quad e_0^d = \lambda(1-g^d), \quad gp_0^dg^{-1} = p_0^d + \alpha(1-g^d),$$
$$[e_0, p_0^d] = \frac{\lambda(1-q)}{(d-1)_q!} e_0(1+g^d) + \mu(1-g^{d+1})$$

for some $\lambda, \alpha, \mu \in k$. By induction we have

$$e_0^d p_0^d = p_0^d e_0^d + d\frac{\lambda(1-q)}{(d-1)_q!} e_0(1+g^d) + d\mu e_0^{d-1}.$$

Combining these with the previous equalities, we have

$$\lambda d\alpha(g^d - g^{2d}) + d\frac{\lambda^2(1-q)}{(d-1)_q!}(1-g^{2d}) + d\mu e_0^{d-1} = 0.$$

It follows from this equality that $\lambda = \mu = 0$. By a similar argument we can prove the relations are enough defining relations for H. □

Finally, we summarize all the Hopf structures arising from the infinite linear quiver $\mathscr{A}$. Let $k\mathscr{A}(1,\lambda)$ be the Hopf algebra defined by Theorem 4.3(1), and $k\mathscr{A}(d,q,\lambda)$ the Hopf algebra defined by Theorem 4.3(3). The condition of isomorphism is also given.

Theorem 4.4. *Let $\mathscr{A}$ be the infinite linear quiver and $k\mathscr{A}$ be the associated path coalgebra. All the Hopf algebra structures are given by $k\mathscr{A}(q)$ with $0 \neq q \in k$ an arbitrary element* (*graded*), *$k\mathscr{A}(1,\lambda)$ and $k\mathscr{A}(d,q,\lambda)$ with $q \neq 1$ a primitive d-th root of unity* (*nongraded*). *We have the Hopf algebra isomorphism $k\mathscr{A}(q) \cong k\mathscr{A}(q')$ if and only if $q = q'$; the isomorphism $k\mathscr{A}(1,\lambda) \cong k\mathscr{A}(1,\lambda')$ if and only if $\lambda = \lambda'$; and $k\mathscr{A}(d,q,\lambda) \cong k\mathscr{A}(d',q',\lambda')$ if and only if $d = d',\ q = q'$ and $\lambda = \lambda'$.*

5. Applications

In this section, we directly apply our classification results to bialgebras of type one of Nichols [1978], and simple-pointed Hopf algebras of Radford [1999].

Recall that the bialgebras of type one in the sense of Nichols are pointed Hopf algebras that are generated as algebras by group-like and skew-primitive elements. In the quiver terminology, such Hopf algebras live in Hopf quivers and as algebras are generated by vertices and arrows. We are going to investigate all the possible bialgebras of type one living in the Hopf quivers of form $Q(\langle g\rangle, g)$. Not all pointed Hopf algebras are bialgebras of type one. Later on we will see that in general quiver Hopf algebras are not so either.

The case of loop quiver is trivial. The quiver Hopf algebra is generated by the only vertex and arrow, and hence is a bialgebra of type one. For the cases of basic n-cycles ($n \geq 2$) and the infinite linear quiver, things turn out to be very different. The idea of classifying bialgebras of type one is similar to those of quiver Hopf algebras. First we classify the graded ones, and then determine all the possible deformations.

We deal with the basic cycle case first. Keep the notations of Section 3. The graded bialgebras of type one are the graded sub-Hopf algebras on Hopf quivers generated by vertices and arrows, so the classification of such algebras can be obtained as a direct consequence of Lemma 3.2.

Lemma 5.1. *Let $B\mathfrak{Z}(q)$ denote the sub-Hopf algebra of $k\mathfrak{Z}(q)$ generated by vertices and arrows.*

(1) *If* $q = 1$, *then* $B\mathscr{Z}(q) \cong k\mathscr{Z}(1)$.

(2) *If* $\mathrm{ord}(q) = d > 1$, *then* $B\mathscr{Z}(q)$ *can be presented by generators* g *and* a_0 *with relations* $g^n = 1,\ a_0^d = 0$ *and* $ga_0 = qa_0g$.

When q is a nontrivial root of unity, the bialgebra of type one $B\mathscr{Z}(q)$ is a very interesting Hopf algebra. When $\mathrm{ord}(q) = n$, it is the well-known Taft algebra [1971]. It also appears as the Borel subalgebra of Lusztig's small quantum $\mathfrak{sl}_2$ [1990]. For general $\mathrm{ord}(q) = d$, the algebra $B\mathscr{Z}(q)$ is a generalization of the Taft algebra.

It follows directly from Lemmas 3.2 and 5.1 that the only finite-dimensional subcoalgebras of $k\mathscr{Z}$ that admit Hopf algebra structures are $k\mathscr{Z}[d]$ with $d = \mathrm{ord}(q)$, where q is a nontrivial n-th root of unity. Hence d is a factor of n. This is the result [Chen et al. 2004, Theorem 3.1], which plays an important role in classifying the monomial Hopf algebras. The argument in this paper simplifies the old one.

For the nongraded bialgebras of type one living in the Hopf quiver $\mathscr{Z}$, it suffices to determine the preferred deformations of $B\mathscr{Z}(q)$. This actually was done before, though not in terms of deformation. It turns out that these are all the connected monomial Hopf algebras. For completeness, we include the result here.

Theorem 5.2 [Chen et al. 2004, Theorem 3.6]. *All the possible preferred deformations of* $B\mathscr{Z}(q)$ *can be presented by generators* g *and* a, *with relations*

$$g^n = 1, \quad a^d = \mu(1 - g^d), \quad ga = qag,$$

where $\mu \in \{0,\ 1\}$.

Now we consider the case of the infinite linear quiver. Keep the notation of Section 4. First by Lemma 4.2, we can classify the graded bialgebras of type one.

Lemma 5.3. *Let* $B\mathscr{A}(q)$ *denote the sub-Hopf algebra of* $k\mathscr{A}(q)$ *generated by vertices and arrows.*

(1) *If* q *is not a nontrivial root of unity*, *then* $B\mathscr{A}(q) \cong k\mathscr{A}(q)$.

(2) *If* q *is a root of unity with order* $\mathrm{ord}(q) = d > 1$, *then* $B\mathscr{A}(q)$ *can be presented by generators* $g,\ g^{-1}$ *and* e_0 *with relations* $gg^{-1} = 1 = g^{-1}g,\ e_0^d = 0$ *and* $ge_0 = qe_0g$.

Next we consider the possible deformations of $B\mathscr{A}(q)$. In the case that q is not a nontrivial root of unity, this was done in Theorem 4.3. When q is a root of unity with order $\mathrm{ord}(q) = d > 1$, it suffices to deform the relations $e_0^d = 0$ and $ge_0 = qe_0g$. By the argument from the proofs of Theorem 3.3(2) and Theorem 4.3(2), we can always preserve the relation $ge_0 = qe_0g$, while deforming $e_0^d = 0$ to $e_0^d = \lambda(1 - g^d)$. In this situation, as coalgebra $B\mathscr{A}(q)$ is identical to the subcoalgebra $k\mathscr{A}[d-1]$ of the path coalgebra $k\mathscr{A}$, namely the subcoalgebra spanned by paths of length strictly less than d. We record the results as follows.

Theorem 5.4. *If H is a bialgebra of type one with $Q(H) = \mathscr{A}$, then either*

(1) $H = k\mathscr{A}(q)$, *where q is not a nontrivial root of unity*, *or*

(2) *H can be presented by generators g, g^{-1} and e with relations $gg^{-1} = 1 = g^{-1}g$, $ge = qeg$ and $e^d = \mu(1-g^d)$, where q is a root of unity of order $d > 1$ and $\mu \in \{0, 1\}$.*

When q is not a root of unity, the Hopf algebra $k\mathscr{A}(q)$ is the well-known Borel subalgebra of the quantum group $U_\nu(\mathfrak{sl}_2)$, where $\nu = \sqrt{q}$. When q is a nontrivial root of unity, the Hopf algebra in (2), denoted by $B\mathscr{A}(q, \mu)$, is closely related to the (Borel subalgebra of) the quantum group $U_\nu(\mathfrak{sl}_2)$ of De Concini and Kac [1990] at roots of unity.

By comparing the previous classification of bialgebras of type one with the classification of quiver Hopf algebras, it is clear that there is no hope to extend the Gabriel-type theorem of van Oystaeyen and Zhang to nongraded pointed Hopf algebras. For example, when q is a nontrivial root of unity, the Hopf algebras $B\mathfrak{L}(q)$ and $B\mathscr{A}(q)$ do have nontrivial deformations, while the quiver Hopf algebras $k\mathfrak{L}(q)$ and $k\mathscr{A}(q)$ do not in general. In other words, these Hopf algebras living in a proper subcoalgebra of $k\mathfrak{L}$ and $k\mathscr{A}$ cannot be extended to the whole path coalgebras; hence they are not sub-Hopf algebras of any Hopf algebra structures on the path coalgebras of the corresponding Hopf quivers.

Now we apply our results to simple-pointed Hopf algebras. A Hopf algebra H is said to be simple pointed if it is pointed and not cocommutative, and if L being a proper sub-Hopf algebra of H implies $L \subseteq kG(H)$. Very naturally the "simpleness" of such Hopf algebras can be visualized by their corresponding Hopf quivers. The definition of simple-pointed Hopf algebras adopted here is from [Zhang 2006], which includes the infinite-dimensional situation. Zhang obtained the complete classification by methods different from ours.

Theorem 5.5. *A Hopf algebra H is simple pointed if and only if its graded version* gr H *is simple pointed, if and only if it is a bialgebra of type one living in either the Hopf quiver $\mathfrak{L}$ or $\mathscr{A}$. Hence it is either $k\mathfrak{L}(1)$, $B\mathfrak{L}(q, \mu)$ with q a root of unity of order greater than* 1 *and $\mu \in \{0, 1\}$, $k\mathscr{A}(q)$ with q not a nontrivial root of unity, $k\mathscr{A}(1, 1)$, or $B\mathscr{A}(q, \mu)$ with q a root of unity of order greater than* 1 *and $\mu \in \{0, 1\}$.*

Proof. Assume that H is simple-pointed. Then the Hopf quiver $Q(H)$ must be connected. It is not hard to deduce from the definition of simple-pointed Hopf algebras that there is exactly one arrow going from the unit of the group $G(H)$ to some nonunit element. So by the definition of Hopf quiver it follows at once that $G(H)$ must be a cyclic group and $Q(H)$ must be either $\mathfrak{L}$ or $\mathscr{A}$. Now the theorem follows directly from Lemma 5.1, Theorem 5.2, Lemma 5.3, and Theorem 5.4. □

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